Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 1, pp. 32–53

GENERALIZED STOCHASTIC DERIVATIVES ON A SPACE OF REGULAR GENERALIZED FUNCTIONS OF MEIXNER WHITE NOISE

N. A. KACHANOVSKY

Abstract. We introduce and study generalized stochastic derivatives on a Kondratiev-type space of regular generalized functions of Meixner white noise. Properties of these derivatives are quite analogous to the properties of the stochastic derivatives in the Gaussian analysis. As an example we calculate the generalized stochastic derivative of the solution of some stochastic equation with a Wick-type nonlinearity.

0. Introduction

Let $\mathcal{S}'$ be the Schwartz distributions space, μ be the Gaussian measure on $\mathcal{S}'$. As is well known, every square integrable function $f \in L^2(\mathcal{S}', \mu)$ can be presented in the form

$$f = \sum_{n=0}^{\infty} \langle H_n, f^{(n)} \rangle, \tag{0.1}$$

where $\{\langle H_n, f^{(n)}\rangle\}_{n=0}^{\infty}$ are the generalized Hermite polynomials, $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}$, $\mathcal{H}$ (in the simplest case) is the complexification of $L^2(\mathbb{R})$, $\widehat{\otimes}$ denotes a symmetric tensor product. A stochastic derivative $\mathcal{D} : L^2(\mathcal{S}', \mu) \to \mathcal{L}(\mathcal{H}, L^2(\mathcal{S}', \mu))$ can be defined on its domain, $\{f \in L^2(\mathcal{S}', \mu) : \sum_{n=1}^{\infty} n!n|f^{(n)}|^2_{\mathcal{H}^{\otimes n}} < \infty\}$, by the formula

$$(\mathcal{D}f)(g^{(1)}) := \sum_{n=1}^{\infty} n \langle H_{n-1}, \langle f^{(n)}, g^{(1)} \rangle \rangle \quad \forall g^{(1)} \in \mathcal{H},$$

where $\langle f^{(n)}, g^{(1)}\rangle \in \mathcal{H}^{\widehat{\otimes} n-1}$ is defined by

$$\langle \langle f^{(n)}, g^{(1)} \rangle, h^{(n-1)} \rangle = \langle f^{(n)}, g^{(1)} \widehat{\otimes} h^{(n-1)} \rangle \quad \forall h^{(n-1)} \in \mathcal{H}^{\widehat{\otimes} n-1}$$

(here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathcal{H}^{\widehat{\otimes} n}$).

In the paper [4] Fred E. Benth extended the derivative $\mathcal{D}$ on the Kondratiev space of nonregular generalized functions $(\mathcal{S})^{-1}$ (elements of $(\mathcal{S})^{-1}$ can be presented in the similar to (0.1) form, but the kernels $\{f^{(n)}\}_{n=0}^{\infty}$ are singular). This generalization is useful for different applications. For example, as distinct from $L^2(\mathcal{S}', \mu)$, in the space $(\mathcal{S})^{-1}$ one can introduce the Wick product $\Diamond$ by setting for the Hermite polynomials $\langle H_n, f^{(n)}\rangle \Diamond \langle H_m, g^{(m)}\rangle := \langle H_{n+m}, f^{(n)} \widehat{\otimes} g^{(m)}\rangle$, and $\mathcal{D}$ is a differentiation with respect to $\Diamond$: for all $F, G \in (\mathcal{S})^{-1}$ $\mathcal{D}(F \Diamond G) = (\mathcal{D}F) \Diamond G + F \Diamond (\mathcal{D}G)$. Using this result (and another properties of $\mathcal{D}$) one can study properties of solutions of stochastic equations with Wick type nonlinearity. Another possible applications are connected with the fact that the stochastic derivative is the adjoint operator to the extended (Skorohod) stochastic integral.

2000 *Mathematics Subject Classification.* Primary 60H07; Secondary 60H05, 46F05.

Key words and phrases. Meixner measure, Meixner polynomials, Wick product, Kondratiev space, stochastic derivative.

In the papers [15, 16] the author generalized the results of [4] to the spaces of generalized functions of the so-called Gamma white noise analysis (i.e., instead of the Gaussian measure the introduced in [23] Gamma measure γ on $\mathcal{S}'$ was used). Since the Gamma measure has no the so-called Chaotic Representation Property (i.e., a function $f \in L^2(\mathcal{S}', \gamma)$ can not be presented in complete analogy with (0.1), generally speaking) and has some another peculiarities, the corresponding spaces have a more complicated than in the Gaussian analysis structure; nevertheless a natural and rich in content analog of the Gaussian theory is possible.

A next natural step consists in the construction of a theory of stochastic differentiation on the generalized functions spaces of the so-called Meixner analysis. In fact, the (introduced in [29]) generalized Meixner measure μ on the Schwartz distributions space D' (the base measure of the Meixner analysis) is a direct generalization of "classical" measures on D', such as the Gaussian, Poisson and Gamma measures. This measure is very general, but still has some "classical" properties (for example, the orthogonal polynomials in $L^2(D', \mu)$ are Schefer (generalized Appell in another terminology) ones), therefore a constructive theory is still possible.

In the paper [20] the author constructed and studied generalized stochastic derivatives on the Kondratiev-type (finite order) spaces of *nonregular* generalized functions ($\mathcal{H}_{-\tau}$) of Meixner white noise. The obtained results are very general (actually, they can be rewritten for the generalized functions spaces of the so-called "biorthogonal analysis", see, e.g., [3, 2, 24, 18, 22, 19, 5, 8]); but this merit simultaneously is a lack. In fact, an analysis on the very general spaces ($\mathcal{H}_{-\tau}$) can not "take into account" all characteristics of the generalized Meixner measure μ. As a result, the natural generalized stochastic derivatives on these spaces have no some "classical" properties. For example, these derivatives are not adjoint to the extended stochastic integral, their restrictions on $L^2(D', \mu)$ do not coincide with the natural stochastic derivatives on this space, etc. Moreover, the kernels from the natural orthogonal decompositions of elements of ($\mathcal{H}_{-\tau}$) belong to the distributions spaces without "good" description. All these peculiarities are inconvenient for applications. Therefore there is a good motivation to make a next step: to construct and study generalized stochastic derivatives on the Kondratiev-type space of *regular* generalized functions $(L^2)^{-1}$. In fact, this space (cf. [14]) "takes into account" characteristics of μ, and there are no the mentioned problems with generalized stochastic derivatives and orthogonal decompositions on $(L^2)^{-1}$. Moreover, it turned out that almost all results of [20] can be transferred on this "regular" case. The realization of this "next step" is a main aim of the present paper.

The paper is organized in the following manner. In the first section we give a necessary information about the generalized Meixner measure, spaces of test and generalized functions, the stochastic integration and the Wick calculus. In the second section we introduce and study generalized stochastic derivatives on $(L^2)^{-1}$. In the third section we discuss some properties of stochastic derivatives on $L^2(D', \mu)$.

1. Preliminaries

Let σ be a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ (here $\mathcal{B}$ denotes the Borel σ-algebra) satisfying the following assumptions:

1) σ is absolutely continuous with respect to the Lebesgue measure and the density is an infinite differentiable function on $\mathbb{R}_+$;
2) σ is a nondegenerate measure, i.e., for each nonempty open set $O \subset \mathbb{R}_+$ $\sigma(O) > 0$.

Remark 1.1. Note that these assumptions are the "simplest *sufficient* ones" for our considerations; actually it is possible to consider a much more general σ.

By D denote the set of all real-valued infinite differentiable functions on $\mathbb{R}_+$ with compact supports. This set can be naturally endowed with a (projective limit) topology of a nuclear space (by analogy with, e.g., [7]): $D = \text{pr}\lim_{\tau\in T}\mathcal{H}_\tau$, where T is the set of all pairs $\tau = (\tau_1, \tau_2)$, $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that $\tau_2(t) \geq 1\ \forall t \in \mathbb{R}_+$; $\mathcal{H}_\tau = \mathcal{H}_{(\tau_1,\tau_2)}$ is the Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., the scalar product in $\mathcal{H}_\tau$ is given by the formula

$$(f, g)_\tau := (f, g)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+}\Big(f(t)g(t) + \sum_{k=1}^{\tau_1} f^{(k)}(t)g^{(k)}(t)\Big)\tau_2(t)\sigma(dt).$$

Hence in what follows, we understand D as the corresponding *topological space*.

Let us consider the (nuclear) chain (the rigging of $L^2(\mathbb{R}_+, \sigma)$ that is the space of real-valued functions on $\mathbb{R}_+$ square integrable with respect to σ)

$$D' = \operatorname*{ind\,lim}_{\tau'\in T}\mathcal{H}_{-\tau'} \supset \mathcal{H}_{-\tau} \supset L^2(\mathbb{R}_+, \sigma) =: \mathcal{H} \supset \mathcal{H}_\tau \supset \operatorname*{pr\,lim}_{\tau'\in T}\mathcal{H}_{\tau'} = D,$$

where $\mathcal{H}_{-\tau}$, D' are the spaces dual to $\mathcal{H}_\tau$, D with respect to $\mathcal{H}$, correspondingly. By $|\cdot|_0$ denote the norm in $\mathcal{H}$. Let $\langle\cdot,\cdot\rangle$ be the dual pairing between elements of D' and D, generated by the scalar product in $\mathcal{H}$ (and also $\mathcal{H}_{-\tau}$ and $\mathcal{H}_\tau$). The notation $|\cdot|_0$ and $\langle\cdot,\cdot\rangle$ will be preserved for tensor powers and complexifications of spaces.

Remark 1.2. Note that *all scalar products and pairings in this paper are real*, i.e., they are *bilinear* functionals.

Let us fix arbitrary functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{C}$ that are smooth and satisfy

$$\theta := -\alpha - \beta \in \mathbb{R}, \quad \eta := \alpha\beta \in \mathbb{R}_+, \tag{1.1}$$

θ and η are bounded on $\mathbb{R}_+$ (note that in a sense η is a "key parameter" and will be mentioned very often below). Further, let $\widetilde{\upsilon}(\alpha, \beta, ds)$ be a probability measure on $\mathbb{R}$ that is defined by its Fourier transform

$$\int_{\mathbb{R}} e^{ius}\widetilde{\upsilon}(\alpha, \beta, ds) = \exp\Big\{ - iu(\alpha+\beta) + 2\alpha\beta\sum_{m=1}^{\infty}\frac{(\alpha\beta)^{m-1}}{m}\Big[\sum_{n=2}^{\infty}\frac{(-iu)^n}{n!}(\beta^{n-2} + \beta^{n-3}\alpha + \cdots + \alpha^{n-2})\Big]^m\Big\},$$

$\upsilon(\alpha, \beta, ds) := \frac{1}{s^2}\widetilde{\upsilon}(\alpha, \beta, ds)$.

Definition 1.1. We say that the probability measure μ on the measurable space $(D', \mathcal{F})$ (here $\mathcal{F}$ is the σ-algebra on D' generated by cylinder sets) with the Fourier transform

$$\int_{D'} e^{i\langle x,\xi\rangle}\mu(dx) = \exp\Big\{\int_{\mathbb{R}_+}\sigma(dt)\int_{\mathbb{R}}\upsilon(\alpha(t), \beta(t), ds)(e^{is\xi(t)} - 1 - is\xi(t))\Big\}$$

(here $\xi \in D$) is called the *generalized Meixner measure*.

Let us denote by a subindex $\mathbb{C}$ complexifications of spaces.

Theorem 1.1. ([29]) *The generalized Meixner measure μ is a generalized stochastic process with independent values in the sense of* [13]. *The Laplace transform of μ is given in a neighborhood of zero $\mathcal{U}_0 \subset D_{\mathbb{C}}$ by the following formula:*

$$l_\mu(\lambda) = \int_{D'} e^{\langle x,\lambda\rangle}\mu(dx) = \exp\Big\{\int_{\mathbb{R}_+}\sum_{m=1}^{\infty}\frac{(\alpha(t)\beta(t))^{m-1}}{m} \times \Big(\sum_{n=2}^{\infty}\frac{(-\lambda)^n}{n!}(\beta(t)^{n-2} + \beta(t)^{n-3}\alpha(t) + \cdots + \alpha(t)^{n-2})\Big)^m\sigma(dt)\Big\}, \quad \lambda \in \mathcal{U}_0.$$

Remark 1.3. According to the classical classification [27] (see also [26, 29]) for $\alpha = \beta = 0$ (here and below all such equalities we understand σ-a.e.) μ is the Gaussian measure; for $\alpha \neq 0$ (here and below $a(\cdot) \neq b(\cdot)$ means that $a - b \neq 0$ on some measurable set M such that $\sigma(M) > 0$), $\beta = 0$, μ is the centered Poissonian measure; for $\alpha = \beta \neq 0$, μ is the centered Gamma measure; for $\alpha \neq \beta$, $\alpha\beta \neq 0$, $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$, μ is the centered Pascal measure; for $\alpha = \overline{\beta}$, $\mathrm{Im}(\alpha) \neq 0$, μ is the centered Meixner measure.

It was established in [21] that there exists $\widetilde{\tau} \in T$ such that the generalized Meixner measure is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$.

Now by $(L^2) = L^2(D', \mu)$ denote the space of complex-valued functions on D', square integrable with respect to μ. Let us construct orthogonal polynomials on (L^2).

Definition 1.2. We define a so-called *Wick exponential* (a generating function of the orthogonal polynomials) by setting

(1.2)

$$
\begin{aligned}
&: \exp(x; \lambda) : \\
&\stackrel{def}{=} \exp\Big\{ - \int_{\mathbb{R}_+} \Big(\frac{\lambda(t)^2}{2} + \sum_{n=3}^{\infty} \frac{\lambda(t)^n}{n} (\alpha(t)^{n-2} + \alpha(t)^{n-3}\beta(t) + \cdots + \beta(t)^{n-2}) \Big) \sigma(dt) \\
&+ \Big\langle x, \lambda + \sum_{n=2}^{\infty} \frac{\lambda^n}{n} (\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}) \Big\rangle \Big\},
\end{aligned}
$$

where $\lambda \in \mathcal{U}_0 \subset D_{\mathbb{C}}$, $x \in D'$, $\mathcal{U}_0$ is some neighborhood of $0 \in D_{\mathbb{C}}$.

Remark 1.4. It was proved in [29] that

$$: \exp(x; \lambda) := \frac{e^{\langle x, \Psi(\lambda) \rangle}}{l_\mu(\Psi(\lambda))}$$

with $\Psi(\lambda) = \lambda + \sum_{n=2}^{\infty} \frac{\lambda^n}{n} (\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1})$, therefore $: \exp(x; \cdot) :$ is a generating function of the so-called *Schefer polynomials* (or the *generalized Appell polynomials* in another terminology). This fact gives us the possibility to use in our considerations well-known results of the so-called "biorthogonal analysis" (see, e.g., [3, 2, 24, 18, 22, 19, 5, 8] and references therein).

It is clear (see also [29]) that $: \exp(x; \cdot) :$ is a function on $D_{\mathbb{C}}$ holomorphic at zero for each $x \in D'$. Therefore using the Cauchy inequalities (see, e.g., [12]) and the kernel theorem (see, e.g., [7]) one can obtain the representation

$$: \exp(x; \lambda) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n(x), \lambda^{\otimes n} \rangle, \quad P_n(x) \in D_{\mathbb{C}}'^{\widehat{\otimes} n}, \quad x \in D', \quad \lambda \in D_{\mathbb{C}}.$$

Here (and below) $\widehat{\otimes}$ denotes a symmetric tensor product, $\lambda^{\otimes 0} = 1$ even for $\lambda \equiv 0$.

Remark 1.5. It follows from the given in [29] recurrence formula for $P_n(x)$ that actually $P_n(x) \in D'^{\widehat{\otimes} n}$ for $x \in D'$. Moreover, if $\tau \in T$ is such that the Dirac delta-function $\delta_0 \in \mathcal{H}_{-\tau}$ (it means that $\delta_s \in \mathcal{H}_{-\tau}$ $\forall s \in \mathbb{R}_+$, see, e.g., [7]) then *for $x \in \mathcal{H}_{-\tau}$ we have $P_n(x) \in \mathcal{H}_{-\tau}^{\widehat{\otimes} n}$.*

Definition 1.3. We say that the polynomials $\langle P_n, f^{(n)} \rangle$, $f^{(n)} \in D_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, are called the *generalized Meixner polynomials*.

Remark 1.6. Depending on α and β in (1.2) the generalized Meixner polynomials can be the generalized Hermite polynomials ($\alpha = \beta = 0$); the generalized Charlier polynomials ($\alpha \neq 0, \beta = 0$); the generalized Laguerre polynomials ($\alpha = \beta \neq 0$); the Meixner polynomials ($\alpha \neq \beta$, $\alpha\beta \neq 0$, $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$); the Meixner-Pollaczek polynomials ($\alpha = \overline{\beta}$, $\mathrm{Im}(\alpha) \neq 0$) (see also Remark 1.3).

In order to formulate a statement on orthogonality of the generalized Meixner polynomials we need the following.

Definition 1.4. We define the scalar product $\langle\cdot,\cdot\rangle_{\text{ext}}$ on $D_{\mathbb{C}}^{\widehat{\otimes}n}$ ($n\in\mathbb{N}$) by the formula

(1.3)
$$\langle f^{(n)},g^{(n)}\rangle_{\text{ext}}:=\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}}\frac{n!}{l_1^{s_1}\dots l_k^{s_k}s_1!\dots s_k!}$$
$$\times\int_{\mathbb{R}_+^{s_1+\dots+s_k}}f^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1},\dots,t_{s_1}}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})$$
$$\times g^{(n)}(\underbrace{t_1,\dots,t_1}_{l_1},\dots,\underbrace{t_{s_1},\dots,t_{s_1}}_{l_1},\dots,\underbrace{t_{s_1+\dots+s_k},\dots,t_{s_1+\dots+s_k}}_{l_k})\eta(t_1)^{l_1-1}\dots\eta(t_{s_1})^{l_1-1}$$
$$\times\eta(t_{s_1+1})^{l_2-1}\dots\eta(t_{s_1+s_2})^{l_2-1}\dots\eta(t_{s_1+\dots+s_{k-1}+1})^{l_k-1}\dots\eta(t_{s_1+\dots+s_k})^{l_k-1}$$
$$\times\sigma(dt_1)\dots\sigma(dt_{s_1+\dots+s_k}).$$

Denote by $|\cdot|_{\text{ext}}$ the corresponding norm, i.e., $|f^{(n)}|_{\text{ext}}^2=\langle f^{(n)},\overline{f^{(n)}}\rangle_{\text{ext}}$. For $n=0$, $\langle f^{(0)},g^{(0)}\rangle_{\text{ext}}:=f^{(0)}g^{(0)}\in\mathbb{C}$, $|f^{(0)}|_{\text{ext}}=|f^{(0)}|$.

Example. It is easy to see that for $n=1$,
$$\langle f^{(1)},g^{(1)}\rangle_{\text{ext}}=\langle f^{(1)},g^{(1)}\rangle=\int_{\mathbb{R}_+}f^{(1)}(t)g^{(1)}(t)\sigma(dt).$$
Further, for $n=2$,
$$\langle f^{(2)},g^{(2)}\rangle_{\text{ext}}=\langle f^{(2)},g^{(2)}\rangle+\int_{\mathbb{R}_+}f^{(2)}(t,t)g^{(2)}(t,t)\eta(t)\sigma(dt).$$
If $\eta=0$ (this means that μ is the Gaussian or Poissonian measure, see Remark 1.3) then $\langle f^{(n)},g^{(n)}\rangle_{\text{ext}}=\langle f^{(n)},g^{(n)}\rangle$ for all $n\in\mathbb{Z}_+$; in general, $\langle f^{(n)},g^{(n)}\rangle_{\text{ext}}=\langle f^{(n)},g^{(n)}\rangle+\dots$.

Theorem 1.2. ([29]) *The generalized Meixner polynomials are orthogonal in (L^2) in the sense that*
$$\int_{D'}\langle P_n(x),f^{(n)}\rangle\langle P_m(x),g^{(m)}\rangle\mu(dx)=\delta_{mn}n!\langle f^{(n)},g^{(n)}\rangle_{\text{ext}}. \tag{1.4}$$

By $\mathcal{H}_{\text{ext}}^{(n)}$ ($n\in\mathbb{N}$) denote the closure of $D_{\mathbb{C}}^{\widehat{\otimes}n}$ with respect to the norm $|\cdot|_{\text{ext}}$, $\mathcal{H}_{\text{ext}}^{(0)}:=\mathbb{C}$. Of course, $\mathcal{H}_{\text{ext}}^{(n)}$, $n\in\mathbb{Z}_+$, are Hilbert spaces; for the scalar products in these spaces it is natural to preserve the notation $\langle\cdot,\cdot\rangle_{\text{ext}}$.

Remark 1.7. It is not difficult to prove by analogy with [6] that the space $\mathcal{H}_{\text{ext}}^{(n)}$ is, generally speaking, an orthogonal sum of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}\equiv L^2(\mathbb{R}_+,\sigma)_{\mathbb{C}}^{\widehat{\otimes}n}$ and some other Hilbert spaces (as a "limit case" one can consider $\eta=0$, in this case $\mathcal{H}_{\text{ext}}^{(n)}=\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$). In this sense $\mathcal{H}_{\text{ext}}^{(n)}$ *is an extension of* $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$.

One can give another explanation of the fact that $\mathcal{H}_{\text{ext}}^{(n)}$ is a more wide space than $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$. Namely, let $F^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ ($F^{(n)}$ is an equivalence class in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$). We select a representative (a function) $\widetilde{F}^{(n)}\in F^{(n)}$ with a "zero diagonal", i.e., $\widetilde{F}^{(n)}$ is such that $\widetilde{F}^{(n)}(t_1,\dots,t_n)=0$ if $t_i=t_j$ for $i\neq j$, where $i,j\in\{1,\dots,n\}$. This function generates the equivalence class $\widehat{F}^{(n)}$ in $\mathcal{H}_{\text{ext}}^{(n)}$ that can be identified with $F^{(n)}$ (see [21] for details).

Definition 1.5. For $F^{(n)}\in\mathcal{H}_{\text{ext}}^{(n)}$ ($n\in\mathbb{Z}_+$) we define $\langle P_n,F^{(n)}\rangle\in(L^2)$ as an (L^2)-limit
$$\langle P_n,F^{(n)}\rangle:=\lim_{k\to\infty}\langle P_n,f_k^{(n)}\rangle,$$

where $(f_k^{(n)} \in D_{\mathbb{C}}^{\widehat{\otimes} n})_{k=1}^{\infty}$ is a sequence of "smooth" functions such that $f_k^{(n)} \to F^{(n)}$ (as $k \to \infty$) in $\mathcal{H}_{\text{ext}}^{(n)}$.

The correctness of this definition was proved in [21].

The following statement follows from results of [29].

Theorem 1.3. *A function $F \in (L^2)$ if and only if there exists a sequence of kernels $(F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)})_{n=0}^{\infty}$ such that F can be presented in the form*

$$F = \sum_{n=0}^{\infty} \langle P_n, F^{(n)} \rangle, \tag{1.5}$$

where the series converges in (L^2), i.e., the (L^2)-norm of F

$$\|F\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{\text{ext}}^2 < \infty.$$

Furthermore, the system $\{\langle P_n, F^{(n)} \rangle,\ F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)},\ n \in \mathbb{Z}_+\}$ plays a role of an orthogonal basis in (L^2) in the sense that for $F, G \in (L^2)$

$$(F, G)_{(L^2)} = \sum_{n=0}^{\infty} n! \langle F^{(n)}, G^{(n)} \rangle_{\text{ext}},$$

where $F^{(n)}, G^{(n)}$ are the kernels from decompositions (1.5) *for F, G (in particular,* (1.4) *for $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $G^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$ holds true).*

Now let us introduce the Kondratiev-type spaces of *regular* test and generalized functions (cf. [14, 17, 16]). First we consider the set $\mathcal{P} := \{f = \sum_{n=0}^{N_f} \langle P_n, f^{(n)} \rangle,\ f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)},\ N_f \in \mathbb{Z}_+\} \subset (L^2)$ of polynomials and $\forall q \in \mathbb{N}$ introduce on this set the scalar product $(\cdot, \cdot)_q$ by setting for $f = \sum_{n=0}^{N_f} \langle P_n, f^{(n)} \rangle$, $g = \sum_{n=0}^{N_g} \langle P_n, g^{(n)} \rangle$

$$(f, g)_q := \sum_{n=0}^{\min(N_f, N_g)} (n!)^2 2^{qn} \langle f^{(n)}, g^{(n)} \rangle_{\text{ext}}.$$

Let $\|\cdot\|_q$ be the corresponding norm, $\|f\|_q = \sqrt{(f, \overline{f})_q} = \sqrt{\sum_{n=0}^{N_f} (n!)^2 2^{qn} |f^{(n)}|_{\text{ext}}^2}$.

Definition 1.6. We define the *Kondratiev-type spaces of ("regular") test functions* $(L^2)_q^1$ $(q \in \mathbb{N})$ as the closures of $\mathcal{P}$ with respect to the norms $\|\cdot\|_q$, $(L^2)^1 := \text{pr}\lim_{q \in \mathbb{N}} (L^2)_q^1$.

It is not difficult to see that $f \in (L^2)_q^1$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} \langle P_n, f^{(n)} \rangle \tag{1.6}$$

with

$$\|f\|_q^2 := \|f\|_{(L^2)_q^1}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_{\text{ext}}^2 < \infty,$$

and for $f, g \in (L^2)_q^1$ $(f, g)_q := (f, g)_{(L^2)_q^1} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} \langle f^{(n)}, g^{(n)} \rangle_{\text{ext}}$, where $f^{(n)}, g^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ are the kernels from decompositions (1.6) for f and g correspondingly. Therefore the generalized Meixner polynomials play a role of an orthogonal basis in $(L^2)_q^1$.

It was proved in [21] that for each $q \in \mathbb{N}$, $(L^2)_q^1$ is densely and continuously embedded in (L^2). Therefore one can consider the chain

$$(L^2)^{-1} = \operatorname*{ind\,lim}_{\widetilde{q} \in \mathbb{N}} (L^2)_{-\widetilde{q}}^{-1} \supset (L^2)_{-q}^{-1} \supset (L^2) \supset (L^2)_q^1 \supset (L^2)^1,$$

where $(L^2)^{-1}_{-q}$, $(L^2)^{-1}$ are the dual to $(L^2)^1_q$, $(L^2)^1$ with respect to (L^2) spaces correspondingly.

Definition 1.7. The spaces $(L^2)^{-1}_{-q}$, $(L^2)^{-1}$ are called the *Kondratiev-type spaces of regular generalized functions.*

Theorem 1.4. ([21]) *A regular generalized function $F \in (L^2)^{-1}_{-q}$ $(q \in \mathbb{N})$ if and only if there exists a sequence*

$$(F^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}})_{n=0}^{\infty} \tag{1.7}$$

such that F can be presented in form (1.5)*, where the series converges in $(L^2)^{-1}_{-q}$, i.e., the norm*

$$\|F\|^2_{-q} := \|F\|^2_{(L^2)^{-1}_{-q}} = \sum_{n=0}^{\infty} 2^{-qn} |F^{(n)}|^2_{\text{ext}} < \infty. \tag{1.8}$$

Furthermore, the system $\{\langle P_n, F^{(n)}\rangle \colon F^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}, n \in \mathbb{Z}_+\}$ plays a role of an orthogonal basis in $(L^2)^{-1}_{-q}$ in the sense that for $F, G \in (L^2)^{-1}_{-q}$,

$$(F, G)_{(L^2)^{-1}_{-q}} = \sum_{n=0}^{\infty} 2^{-qn} \langle F^{(n)}, G^{(n)}\rangle_{\text{ext}},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$ are the kernels from decompositions (1.5) *for F and G correspondingly.*

Remark 1.8. It is easy to see that $F \in (L^2)^{-1}$ if and only if there exists a sequence (1.7) such that F can be presented in form (1.5) with finite norm (1.8) for some $q \in \mathbb{N}$.

The generated by the scalar product in (L^2) (real) dual pairing between elements of $(L^2)^{-1}_{-q}$ and $(L^2)^1_q$ (in the same way as $(L^2)^{-1}$ and $(L^2)^1$) will be denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. It was proved in [21] that for a generalized function F of form (1.5) and a test function f of form (1.6),

$$\langle\!\langle F, f \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}, f^{(n)}\rangle_{\text{ext}}. \tag{1.9}$$

Remark 1.9. In [21, 20] the test function spaces $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau) = \operatorname{pr}\lim_{q\in\mathbb{N}}(\mathcal{H}_\tau)_q$, $(D) = \operatorname{pr}\lim_{q\in\mathbb{N},\tau\in T}(\mathcal{H}_\tau)_q$ and the corresponding dual to them with respect to (L^2) spaces of *nonregular* generalized functions $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$, $(D')'$ were introduced ($f \in (\mathcal{H}_\tau)_q$ if and only if f has form (1.6) with $\|f\|^2_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|^2_{\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}} < \infty$).

Now let us recall elements of the Wick calculus.

Definition 1.8. For $F \in (L^2)^{-1}$ we define an S-transform (SF) as a *formal* series

$$(SF)(\lambda) = \sum_{n=0}^{\infty} \langle F^{(n)}, \lambda^{\otimes n}\rangle_{\text{ext}}, \tag{1.10}$$

where $F^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$, $n \in \mathbb{Z}_+$ are the kernels from decomposition (1.5) for F. In particular, $(SF)(0) = F^{(0)}$, $S1 \equiv 1$.

Definition 1.9. For $F, G \in (L^2)^{-1}$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ we define the *Wick product* $F \lozenge G \in (L^2)^{-1}$ and the *Wick version of h* $h^{\lozenge}(F) \in (L^2)^{-1}$ by setting

$$F \lozenge G := S^{-1}(SF \cdot SG), \quad h^{\lozenge}(F) := S^{-1} h(SF).$$

The correctness of this definition and, moreover, the fact that the Wick multiplication is continuous in the topology of $(L^2)^{-1}$ were proved in [21].

Remark 1.10. It is easy to see that the Wick multiplication $\lozenge$ is commutative, associative and distributive (over the field $\mathbb{C}$). Further, if h from Definition 1.9 is presented in the form

$$h(u)=\sum_{n=0}^{\infty}h_n(u-(SF)(0))^n \tag{1.11}$$

then $h^{\lozenge}(F)=\sum_{n=0}^{\infty}h_n(F-(SF)(0))^{\lozenge n}$, where $F^{\lozenge n}:=\underbrace{F\lozenge\ldots\lozenge F}_{n\ times}$.

Let us write out the "coordinate form" of $F\lozenge G$ and $h^{\lozenge}(F)$ (this form is necessary for calculations).

Lemma 1.1. ([21]) *Let $F^{(n)}\in\mathcal{H}_{\mathrm{ext}}^{(n)}$, $G^{(m)}\in\mathcal{H}_{\mathrm{ext}}^{(m)}$, $n,m\in\mathbb{Z}_+$. We define the element $F^{(n)}\diamond G^{(m)}\in\mathcal{H}_{\mathrm{ext}}^{(n+m)}$ as follows. Let $\dot F^{(n)}\in F^{(n)}$, $\dot G^{(m)}\in G^{(m)}$ be some representatives (functions) from the equivalence classes $F^{(n)}$, $G^{(m)}$. Set*

$$\begin{aligned}&(\widetilde{F^{(n)}\dot{G}^{(m)}})(t_1,\ldots,t_n;t_{n+1},\ldots,t_{n+m})\\ &:=\begin{cases}\dot F^{(n)}(t_1,\ldots,t_n)\dot G^{(m)}(t_{n+1},\ldots,t_{n+m}), & \text{if } \ {}^{\forall i\in\{1,\ldots,n\},}_{\forall j\in\{n+1,\ldots,n+m\}}\ t_i\neq t_j,\\ 0, \quad \text{in other cases}\end{cases}\end{aligned} \tag{1.12}$$

$\widehat{F^{(n)}\dot{G}^{(m)}}:=\mathrm{Pr}\widetilde{F^{(n)}\dot{G}^{(m)}}$, *where* Pr *is the symmetrization operator. Then $F^{(n)}\diamond G^{(m)}$ is the generated by $\widehat{F^{(n)}\dot{G}^{(m)}}$ equivalence class in $\mathcal{H}_{\mathrm{ext}}^{(n+m)}$, this class is well-defined and does not depend on a choice of representatives $\dot F^{(n)}$, $\dot G^{(m)}$. Moreover,*

$$|F^{(n)}\diamond G^{(m)}|_{\mathrm{ext}}\le|F^{(n)}|_{\mathrm{ext}}|G^{(m)}|_{\mathrm{ext}}. \tag{1.13}$$

Remark 1.11. Note that non-strictly speaking $F^{(n)}\diamond G^{(m)}$ is the symmetrization of the function

$$\begin{aligned}&\widetilde{F^{(n)}G^{(m)}}(t_1,\ldots,t_n;t_{n+1},\ldots,t_{n+m})\\ &:=\begin{cases}F^{(n)}(t_1,\ldots,t_n)G^{(m)}(t_{n+1},\ldots,t_{n+m}), & \text{if } \ {}^{\forall i\in\{1,\ldots,n\},}_{\forall j\in\{n+1,\ldots,n+m\}}\ t_i\neq t_j,\\ 0, \quad \text{in other cases}\end{cases}\end{aligned}$$

with respect to $n+m$ variables.

It is obvious that the "multiplication" $\diamond$ is commutative, associative and distributive (over the field $\mathbb{C}$).

Remark 1.12. Note that for $\eta=0$ $F^{(n)}\diamond G^{(m)}=F^{(n)}\widehat{\otimes}G^{(m)}$ (we recall that in this case $\mathcal{H}_{\mathrm{ext}}^{(n)}=\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ for each $n\in\mathbb{Z}_+$). In a general case let us denote by ${D'_{\mathbb{C}}}^{(n)}$ and ${D'_{\mathbb{C}}}^{\widehat{\otimes}n}$ the spaces that are dual to ${D_{\mathbb{C}}}^{\widehat{\otimes}n}$ with respect to $\mathcal{H}_{\mathrm{ext}}^{(n)}$ and $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ correspondingly; and let $U_n:{D'_{\mathbb{C}}}^{(n)}\to{D'_{\mathbb{C}}}^{\widehat{\otimes}n}$ $(n\in\mathbb{Z}_+)$ be the natural isomorphisms that are defined by the formulas

$$\langle U_nF^{(n)},f^{(n)}\rangle=\langle F^{(n)},f^{(n)}\rangle_{\mathrm{ext}} \tag{1.14}$$

for all $F^{(n)}\in{D'_{\mathbb{C}}}^{(n)}$ and $f^{(n)}\in{D_{\mathbb{C}}}^{\widehat{\otimes}n}$. It was proved in [21] that

$$F^{(n)}\diamond G^{(m)}=U_{n+m}^{-1}(U_nF^{(n)}\widehat{\otimes}U_mG^{(m)}). \tag{1.15}$$

It was shown in [21] (see also [16]) that

$$F\lozenge G=\sum_{n=0}^{\infty}\big\langle P_n,\sum_{k=0}^{n}F^{(k)}\diamond G^{(n-k)}\big\rangle, \tag{1.16}$$

$$h^{\Diamond}(F) = h_0 + \sum_{n=1}^{\infty} \Big\langle P_n, \sum_{k=1}^{n} h_k \sum_{m_1,\dots,m_k\in\mathbb{N}: m_1+\dots+m_k=n} F^{(m_1)} \diamond \dots \diamond F^{(m_k)} \Big\rangle, \tag{1.17}$$

where $F^{(k)}, G^{(k)} \in \mathcal{H}_{\rm ext}^{(k)}$ are the kernels from decompositions (1.5) for F and G correspondingly, $h_k \in \mathbb{C}$ ($k \in \mathbb{Z}_+$) are the coefficients from decomposition (1.11) for h.

Remark 1.13. It follows from (1.16) that, in particular,

$$\begin{aligned} \langle P_n, F^{(n)}\rangle \Diamond \langle P_m, G^{(m)}\rangle &= \langle P_{n+m}, F^{(n)} \diamond G^{(m)}\rangle, \\ F \Diamond \langle P_m, G^{(m)}\rangle &= \sum_{n=0}^{\infty} \langle P_{n+m}, F^{(n)} \diamond G^{(m)}\rangle. \end{aligned} \tag{1.18}$$

The first formula can be used in order to *define* the Wick product (and then the Wick version of a holomorphic function as a series) without the S-transform. Formulas (1.16) and (1.17) also can be used as definitions.

Finally, we recall the construction of the extended stochastic integral in the Meixner analysis (see [21] for a detailed presentation).

Let $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$. It follows from Theorem 1.3 that F can be presented in the form

$$F(\cdot) = \sum_{n=0}^{\infty} \langle P_n, F_{\cdot}^{(n)}\rangle, \quad F_{\cdot}^{(n)} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}} \tag{1.19}$$

with

$$\|F\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} n! |F_{\cdot}^{(n)}|^2_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty.$$

Lemma 1.2. ([21]) *For given $F_{\cdot}^{(n)} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ and $t \in [0, +\infty]$ we construct the element $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\rm ext}^{(n+1)}$ by the following way. Let $\dot{F}_{\cdot}^{(n)} \in F_{\cdot}^{(n)}$ be some representative (function) from the equivalence class $F_{\cdot}^{(n)}$. We set*

$$\tilde{\dot{F}}_{[0,t)}^{(n)}(u_1,\dots,u_n,u) := \begin{cases} \dot{F}_u^{(n)}(u_1,\dots,u_n) 1_{[0,t)}(u), & \text{if} \quad u \neq u_1,\dots,u\neq u_n, \\ 0, & \text{in other cases} \end{cases} \tag{1.20}$$

(here and below 1_A is the indicator of a set A), $\hat{\dot{F}}_{[0,t)}^{(n)} := {\rm Pr}\tilde{\dot{F}}_{[0,t)}^{(n)}$, where Pr *is the symmetrization operator. Let $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\rm ext}^{(n+1)}$ be the generated by $\hat{\dot{F}}_{[0,t)}^{(n)}$ equivalence class in $\mathcal{H}_{\rm ext}^{(n+1)}$. This class is well-defined, does not depend on the representative $\dot{F}_{\cdot}^{(n)}$, and the estimate*

$$|\widehat{F}_{[0,t)}^{(n)}|_{\rm ext} \le |F_{\cdot}^{(n)} 1_{[0,t)}(\cdot)|_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} \le |F_{\cdot}^{(n)}|_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}}$$

is valid.

Let $\{M_s := \langle P_1, 1_{[0,s)}\rangle\}_{s\ge 0}$ be the Meixner random process (this process is a locally square integrable normal martingale with independent increments, see [21, 29] for more details).

Definition 1.10. Let $t \in [0, +\infty]$ and $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ be such that

$$\sum_{n=0}^{\infty} (n+1)! |\widehat{F}_{[0,t)}^{(n)}|^2_{\rm ext} < \infty,$$

where $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\rm ext}^{(n+1)}$ ($n \in \mathbb{Z}_+$) are constructed in Lemma 1.2 starting from the kernels $F_{\cdot}^{(n)} \in \mathcal{H}_{\rm ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (1.19) for F. We define the extended stochastic integral with respect to the Meixner process $\int_0^t F(s)\widehat{d}M_s \in (L^2)$ by setting

$$\int_0^t F(s)\widehat{d}M_s := \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{F}^{(n)}_{[0,t)} \rangle. \tag{1.21}$$

Since $\| \int_0^t F(s)\widehat{d}M_s\|^2_{(L^2)} = \sum_{n=0}^{\infty}(n+1)!|\widehat{F}^{(n)}_{[0,t)}|^2_{\mathrm{ext}} < \infty$, this definition is correct. □

It was shown in [21] that $\int_0^t \circ(s)\widehat{d}M_s$ is a generalization of the Itô stochastic integral.

The extended stochastic integral can be generalized to the spaces $(L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ $\forall q \in \mathbb{N}$, $(L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$. More exactly, it follows from Theorem 1.4 that for $F \in (L^2)^{-1}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ (or $(L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$) decomposition (1.19) holds true. The corresponding extended stochastic integral can be defined by formula (1.21).

Moreover, $\int_0^t \circ(s)\widehat{d}M_s$ can be generalized on $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, $q \in \mathbb{N}$, $\tau \in T$, $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{\mathbb{C}}$, $(D')' \otimes \mathcal{H}_{\mathbb{C}}$ (see Remark 1.9), the details are described in [21, 20].

Let $\{M'_s := \langle P_1, \delta_s \rangle \in (\mathcal{H}_{-\tau})\}_{s \geq 0}$ (here δ is the Dirac delta-function) be the Meixner white noise (the mentioned in Theorem 1.1 generalized stochastic process). The interconnection between the Wick calculus and the extended stochastic integration is described by the following

Theorem 1.5. ([21]) *For all $t \in [0, +\infty]$ and $F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ $\int_0^t F_s \Diamond M'_s \sigma(ds)$ can be considered as a linear continuous functional on $(L^2)^1$ that coincides with $\int_0^t F(s)\widehat{d}M_s$, i.e.,*

$$\int_0^t F(s) \Diamond M'_s \sigma(ds) = \int_0^t F(s)\widehat{d}M_s \in (L^2)^{-1}. \tag{1.22}$$

2. Generalized stochastic derivatives on $(L^2)^{-1}$

We begin from some "technical preparation". For $F^{(n)} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$ and $f^{(m)} \in \mathcal{H}^{(m)}_{\mathrm{ext}}$ $(n > m)$ we define a "pairing" $\langle F^{(n)}, f^{(m)} \rangle_{\mathrm{ext}} \in \mathcal{H}^{(n-m)}_{\mathrm{ext}}$ by the formula

$$\langle \langle F^{(n)}, f^{(m)} \rangle_{\mathrm{ext}}, g^{(n-m)} \rangle_{\mathrm{ext}} = \langle F^{(n)}, f^{(m)} \diamond g^{(n-m)} \rangle_{\mathrm{ext}} \quad \forall g^{(n-m)} \in \mathcal{H}^{(n-m)}_{\mathrm{ext}}. \tag{2.1}$$

Since (see (1.13))

$$|\langle F^{(n)}, f^{(m)} \diamond g^{(n-m)} \rangle_{\mathrm{ext}}| \leq |F^{(n)}|_{\mathrm{ext}} |f^{(m)} \diamond g^{(n-m)}|_{\mathrm{ext}} \leq |F^{(n)}|_{\mathrm{ext}} |f^{(m)}|_{\mathrm{ext}} |g^{(n-m)}|_{\mathrm{ext}},$$

this definition is correct and

$$|\langle F^{(n)}, f^{(m)} \rangle_{\mathrm{ext}}|_{\mathrm{ext}} \leq |F^{(n)}|_{\mathrm{ext}} |f^{(m)}|_{\mathrm{ext}}.$$

In order to define a generalized stochastic derivative on $(L^2)^{-1}$ we need the following statement.

Lemma 2.1. ([21]) *For given $F^{(n)} \in \mathcal{H}^{(n)}_{\mathrm{ext}}$ $(n \in \mathbb{N})$ we construct the element $F^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}$ by the following way. Let $\dot{F}^{(n)} \in F^{(n)}$ be some representative (function) from the equivalence class $F^{(n)}$. We consider $\dot{F}^{(n)}(\cdot)$ (i.e., separate a one argument of $\dot{F}^{(n)}$). Let $F^{(n)}(\cdot) \in \mathcal{H}^{(n-1)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}$ be the generated by $\dot{F}^{(n)}(\cdot)$ equivalence class in $\mathcal{H}^{(n-1)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}$. This class is well-defined, does not depend on the representative $\dot{F}^{(n)}$, and*

$$|F^{(n)}(\cdot)|_{\mathcal{H}^{(n-1)}_{\mathrm{ext}} \otimes \mathcal{H}_{\mathbb{C}}} \leq |F^{(n)}|_{\mathrm{ext}}. \tag{2.2}$$

Remark 2.1. Note that for each $f^{(1)} \in \mathcal{H}_{\mathbb{C}}$

$$\int_{\mathbb{R}_+} F^{(n)}(s) f^{(1)}(s) \sigma(ds) = \langle F^{(n)}, f^{(1)} \rangle_{\mathrm{ext}} \in \mathcal{H}^{(n-1)}_{\mathrm{ext}}. \tag{2.3}$$

In fact, for each $g^{(n-1)} \in \mathcal{H}_{\mathrm{ext}}^{(n-1)}$ and for $F^{(n)} \in D_{\mathbb{C}}^{\widehat{\otimes} n} \subset \mathcal{H}_{\mathrm{ext}}^{(n)}$

$$\begin{aligned}\Big\langle \int_{\mathbb{R}_+} F^{(n)}(s) f^{(1)}(s)\sigma(ds), g^{(n-1)}\Big\rangle_{\mathrm{ext}} &= \int_{\mathbb{R}_+} \langle F^{(n)}(s), g^{(n-1)}\rangle_{\mathrm{ext}} f^{(1)}(s)\sigma(ds)\\ &= \int_{\mathbb{R}_+} \langle F^{(n)}(s), U_{n-1}g^{(n-1)}\rangle f^{(1)}(s)\sigma(ds) = \langle F^{(n)}, (U_{n-1}g^{(n-1)})\widehat{\otimes} f^{(1)}\rangle\\ &= \langle F^{(n)}, g^{(n-1)} \diamond f^{(1)}\rangle_{\mathrm{ext}} = \langle\langle F^{(n)}, f^{(1)}\rangle_{\mathrm{ext}}, g^{(n-1)}\rangle_{\mathrm{ext}}\end{aligned}$$

(see (1.14), (1.15)); in a general case the result can be obtained by the corresponding passing to a limit.

Definition 2.1. Let $F \in (L^2)^{-1}$. We define the generalized stochastic derivative $\partial_{\cdot} F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ by setting

$$\partial_{\cdot} F := \sum_{n=1}^{\infty} n\langle P_{n-1}, F^{(n)}(\cdot)\rangle \equiv \sum_{n=0}^{\infty} (n+1)\langle P_n, F^{(n+1)}(\cdot)\rangle, \tag{2.4}$$

where the kernels $F^{(n)}(\cdot) \in \mathcal{H}_{\mathrm{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ are constructed in Lemma 2.1 starting from the kernels $F^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ from decomposition (1.5) for F.

Since $F \in (L^2)^{-1}$, there exists $q \in \mathbb{N}$ such that $F \in (L^2)^{-1}_{-(q-1)}$. We have

$$\begin{aligned}\|\partial_{\cdot} F\|^2_{(L^2)^{-1}_{-q}\otimes\mathcal{H}_{\mathbb{C}}} &= \sum_{n=1}^{\infty} 2^{-q(n-1)} n^2 |F^{(n)}(\cdot)|^2_{\mathcal{H}_{\mathrm{ext}}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}}\\ &\le 2^q \sum_{n=1}^{\infty} [n^2 2^{-n}] 2^{-(q-1)n} |F^{(n)}|^2_{\mathrm{ext}} \le 9\cdot 2^{q-3} \|F\|^2_{-(q-1)} < \infty\end{aligned}$$

(we used the equality $\max_{n\in\mathbb{N}}[n^2 2^{-n}] = 9/8$). Hence $\partial_{\cdot}$ is a well-defined linear *continuous* operator acting from $(L^2)^{-1}$ to $(L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$. □

Remark 2.2. Note that Definition 2.1 is a *direct generalization* of the definition of the stochastic derivative $\partial_{\cdot}$ on (L^2), see [21] and Section 3. In the Gaussian analysis such stochastic derivative is called the *Hida derivative*. Simultaneously we note that the restriction on $(L^2)^{-1}$ of introduced in [20] stochastic derivative $\partial_{\cdot}$ on $(\mathcal{H}_{-\tau})$ (see Remark 1.9) *differs* from derivative (2.4) if $\eta \neq 0$, this is a consequence of properties of the generalized Meixner measure.

It was shown in [21] that the extended stochastic integral on $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$ is connected with the generalized stochastic derivative on (L^2) by the formula

$$\langle\!\langle \int_0^t F(s)\widehat{d}M_s, G\rangle\!\rangle = \int_0^t \langle\!\langle F(s), \partial_s G\rangle\!\rangle \sigma(ds)$$

(also this result holds true for a generalized function F). The operator $\partial_{\cdot}$ on $(L^2)^{-1}$ has a similar property. More exactly, we have

Theorem 2.1. *Let $t \in [0, +\infty]$, $F \in (L^2)^{-1}$, $f \in (L^2)^1 \otimes \mathcal{H}_{\mathbb{C}}$. Then*

$$\mathbf{E}\big[F \int_0^t f(s)\widehat{d}M_s\big] \equiv \langle\!\langle F, \int_0^t f(s)\widehat{d}M_s\rangle\!\rangle = \int_0^t \langle\!\langle \partial_s F, f(s)\rangle\!\rangle \sigma(ds) \equiv \int_0^t \mathbf{E}[(\partial_s F) f(s)]\sigma(ds) \tag{2.5}$$

(here and below by $\mathbf{E}$ *an expectation is denoted).*

Proof. Since $\partial_{\cdot} F \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ and, as it was shown in [21], $\int_0^t f(s)\widehat{d}M_s \in (L^2)^1$, all terms in (2.5) are well-defined. The equality in (2.5) can be proved in the same manner as in the case $F \in (L^2)$, see the proof of Theorem 3.2 in [21]. □

Let $\partial^*_\cdot : (L^2)^1 \otimes \mathcal{H}_\mathbb{C} \to (L^2)^1$ be the adjoint to $\partial_\cdot$ operator. Since $\forall F \in (L^2)^{-1}$, $\forall f \in (L^2)^1 \otimes \mathcal{H}_\mathbb{C}$

$$(\partial_\cdot F, f)_{(L^2)\otimes\mathcal{H}_\mathbb{C}} \equiv \int_{\mathbb{R}_+} \langle\!\langle \partial_s F, f(s)\rangle\!\rangle \sigma(ds) = \langle\!\langle F, \partial^*_\cdot f\rangle\!\rangle,$$

it is natural to write *formally*

$$\int_{\mathbb{R}_+} \langle\!\langle \partial_s F, f(s)\rangle\!\rangle \sigma(ds) = \int_{\mathbb{R}_+} \langle\!\langle F, \partial_s^\dagger f(s)\rangle\!\rangle \sigma(ds) = \langle\!\langle F, \int_{\mathbb{R}_+} \partial_s^\dagger f(s)\sigma(ds)\rangle\!\rangle,$$

where we accepted the notation $\int_{\mathbb{R}_+} \partial_s^\dagger f(s)\sigma(ds) := \partial^*_\cdot f$ (cf. $\partial_x^\dagger$ in [29]). Also we denote $\int_0^t \partial_s^\dagger f(s)\sigma(ds) := \int_{\mathbb{R}_+} \partial_s^\dagger f(s) 1_{[0,t)}(s)\sigma(ds) = \partial^*_\cdot(f 1_{[0,t)})$ (here $t \in [0,+\infty]$).

Remark 2.3. *Formally* one can understand $\partial_s^\dagger$ ($s \in \mathbb{R}_+$) as the adjoint to ∂_s with respect to the scalar product in (L^2) operator. Strictly speaking, if we consider $\partial_\cdot$ on $(L^2)^{-1}$ (or even on (L^2)) then such a "definition" of $\partial_s^\dagger$ is incorrect because for $f^{(n+1)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_\mathbb{C}$ $f^{(n+1)}(s)$ is not determined and therefore ∂_s is not determined. But for the "Hida derivative" $\partial_\cdot$ on $(\mathcal{H}_\tau)_q$ (see Remark 1.9) ∂_s is a linear continuous operator in $(\mathcal{H}_\tau)_q$ for each $s \in \mathbb{R}_+$, therefore $\forall s \in \mathbb{R}_+$ $\partial_s^\dagger$ is well-defined as a linear continuous operator in $(\mathcal{H}_{-\tau})_{-q}$, see [21] for more details.

Corollary. *Let* $t \in [0,+\infty]$ *and* $f \in (L^2)^1 \otimes \mathcal{H}_\mathbb{C}$. *Then*

$$\int_0^t f(s)\widehat{d}M_s = \partial^*_\cdot(f 1_{[0,t)}) = \int_0^t \partial_s^\dagger f(s)\sigma(ds),$$

and, therefore,

$$\Big(\int_0^t \circ \widehat{d}M\Big)^*_\cdot = 1_{[0,t)}(\cdot)\partial_\cdot$$

(here and below by $(\int_0^t \circ\widehat{d}M)^*_\cdot$ *the operator that is adjoint to the extended stochastic integral on* $[0,t)$ *is denoted), the last equality can be accepted as a definition of* $\partial_\cdot$.

In the classical Gaussian analysis the following result is well-known as the Clark-Ocone theorem (see, e.g., [9, 28, 25, 1]): if $F \in L^2(D',\gamma)$ (where γ is the Gaussian measure) and differentiable then

$$F = \mathbf{E}F + \int_{\mathbb{R}_+} \mathbf{E}(\partial_s F|_{\mathcal{F}_s})dB_s,$$

where $\{\mathcal{F}_s := \sigma(B_u : u \le s)\}_{s\ge 0}$ is the flow of the generated by the Brownian motion B full σ-algebras, $\mathbf{E}(\circ|_{\mathcal{F}_s})$ denotes a conditional expectation with respect to $\mathcal{F}_s$. In [11] this result was generalized to the case of a general Lévy process instead of B (this corresponds to *constants* α and β in our notation); but for $\eta \ne 0$ (see (1.1)), i.e., for not Gaussian and not Poissonian cases a special family of differential operators and integrators was used.

Now we'll formulate and prove a generalization of the Clark-Ocone theorem for $F \in (L^2)^{-1}$ in the Gaussian and Poissonian cases, and explain what kind of problems arise in a general Meixner analysis. A more general (but less convenient for applications) approach to the non-Gaussian Clark-Ocone theory is presented in [20].

Lemma 2.2. *Let* $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $t \in [0,+\infty]$. *We define* $F^{(n)}1_{[0,t]^n} \in \mathcal{H}_{\text{ext}}^{(n)}$ *as the equivalence class in* $\mathcal{H}_{\text{ext}}^{(n)}$ *that is generated by a function* $\dot{F}^{(n)}1_{[0,t]^n}$, *where* $\dot{F}^{(n)} \in F^{(n)}$. *This definition is correct and*

$$|F^{(n)}1_{[0,t]^n}|_{\text{ext}} \le |F^{(n)}|_{\text{ext}}. \tag{2.6}$$

Proof. It is obvious that

$$|\dot{F}^{(n)}1_{[0,t]^n}|_{\text{ext}} \le |\dot{F}^{(n)}|_{\text{ext}} = |F^{(n)}|_{\text{ext}} < \infty,$$

therefore $F^{(n)}1_{[0,t]^n}$ is well-defined as an element of $\mathcal{H}_{\mathrm{ext}}^{(n)}$ and (2.6) is valid. Let $\dot{F}_1^{(n)} \in F^{(n)}$ be another representative from $F^{(n)}$, $F_1^{(n)}1_{[0,t]^n}$ be the corresponding equivalence class in $\mathcal{H}_{\mathrm{ext}}^{(n)}$. Since

$$|F_1^{(n)}1_{[0,t]^n} - F^{(n)}1_{[0,t]^n}|_{\mathrm{ext}} = |\dot{F}_1^{(n)}1_{[0,t]^n} - \dot{F}^{(n)}1_{[0,t]^n}|_{\mathrm{ext}} = |(\dot{F}_1^{(n)} - \dot{F}^{(n)})1_{[0,t]^n}|_{\mathrm{ext}} \leq |\dot{F}_1^{(n)} - \dot{F}^{(n)}|_{\mathrm{ext}} = 0,$$

$F_1^{(n)}1_{[0,t]^n} = F^{(n)}1_{[0,t]^n}$. □

Definition 2.2. (cf. [11]) Let $F = \sum_{n=0}^{\infty}\langle P_n, F^{(n)}\rangle \in (L^2)^{-1}$, $t \in [0,+\infty)$. We define the conditional expectation $\mathbf{E}(F|_{\mathcal{F}_t}) \in (L^2)^{-1}$ (where $\mathcal{F}_t := \sigma(M_s : s \leq t)$ is the generated by M_s, $s \leq t$ full σ-algebra) by setting

$$\mathbf{E}(F|_{\mathcal{F}_t}) := \sum_{n=0}^{\infty}\langle P_n, F^{(n)}1_{[0,t]^n}\rangle.$$

The correctness of this definition from estimate (2.6) follows. □

Remark 2.4. It follows from results of [29] that $\forall n \in \mathbb{N}$ $\langle P_n, F^{(n)}\rangle$ is a "measurable combination" of polynomials of power 1 (for the Gaussian and Poissonian cases this fact is well-known). Therefore if $F = \sum_{n=0}^{\infty}\langle P_n, F^{(n)}\rangle$ then F is $\mathcal{F}_t$-measurable if and only if $\forall n \in \mathbb{N}$ $F^{(n)} = F^{(n)}1_{[0,t]^n}$. Hence for $F \in (L^2)$ the defined above $\mathbf{E}(F|_{\mathcal{F}_t})$ is $\mathcal{F}_t$-measurable, and $\forall A \in \mathcal{F}_t$ $\int_A F(x)\mu(dx) = \int_A \mathbf{E}(F(x)|_{\mathcal{F}_t})\mu(dx)$. Thus, our definition of a conditional expectation is natural. The reader can find a more detailed discussion in, e.g., [11].

Theorem 2.2. *Let $\eta = 0$ (see (1.1)), $F \in (L^2)^{-1}$. Then*

$$F = \mathbf{E}F + \int_{\mathbb{R}_+} \mathbf{E}(\partial_s F|_{\mathcal{F}_s})\widehat{d}M_s = \mathbf{E}F + \int_{\mathbb{R}_+} \mathbf{E}(\partial_s F|_{\mathcal{F}_s})\lozenge M_s'\sigma(ds). \tag{2.7}$$

Proof. Note that now $\mathcal{H}_{\mathrm{ext}}^{(n)} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$, therefore $F = \sum_{n=0}^{\infty}\langle P_n, F^{(n)}\rangle$, $F^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$.

Lemma 2.3. ([20]) *Let $F^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n \in \mathbb{N}$. Then* $\mathrm{Pr}(F^{(n)}(\cdot)1_{[0,\cdot]^{n-1}}) = \frac{1}{n}F^{(n)}$ *in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ (here $F^{(n)}(\cdot) \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n-1} \otimes \mathcal{H}_{\mathbb{C}}$ is obtained from $F^{(n)}$ by "separating a one argument",* Pr *is the symmetrization operator).*

Taking into consideration the result of this lemma, we obtain

$$\mathbf{E}F = F^{(0)} = \langle P_0, F^{(0)}\rangle,$$

$$\partial_\cdot F = \sum_{n=1}^{\infty} n\langle P_{n-1}, F^{(n)}(\cdot)\rangle,\ F^{(n)}(\cdot) \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n-1} \otimes \mathcal{H}_{\mathbb{C}},$$

$$\mathbf{E}(\partial_\cdot F|_{\mathcal{F}_\cdot}) = \sum_{n=1}^{\infty} n\langle P_{n-1}, F^{(n)}(\cdot)1_{[0,\cdot]^{n-1}}\rangle,$$

$$\int_{\mathbb{R}_+} \mathbf{E}(\partial_s F|_{\mathcal{F}_s})\widehat{d}M_s = \sum_{n=1}^{\infty} n\langle P_n, \widehat{F^{(n)}(\cdot)1_{[0,\cdot]^{n-1}}}\rangle$$
$$= \sum_{n=1}^{\infty} n\langle P_n, \mathrm{Pr}(F^{(n)}(\cdot)1_{[0,\cdot]^{n-1}})\rangle = \sum_{n=1}^{\infty}\langle P_n, F^{(n)}\rangle.$$

The second equality in (2.7) from Theorem 1.5 follows. □

Remark 2.5. If $\eta \neq 0$, i.e., if M is not a Gaussian or a Poissonian random process then a direct analog of (2.7) can not be obtained. In fact, now $\mathrm{Pr}(F^{(n)}(\cdot)1_{[0,\cdot]^{n-1}}) \neq \frac{1}{n}F^{(n)}$ even for $n = 2$, generally speaking (from the "proof of Lemma 2.3 point of view" (see [20]) we can not ignore processions with coinciding arguments). This is connected with "loss of an information" during construction of $F^{(n)}(\cdot)\in\mathcal{H}_{\rm ext}^{(n-1)}\otimes\mathcal{H}_{\mathbb{C}}$ starting from $F^{(n)}\in\mathcal{H}_{\rm ext}^{(n)}$, see Lemma 2.1: there are different $F^{(n)}$ with the same $F^{(n)}(\cdot)$. In order to "do not loss an information" one has to introduce *another differential operator*. Such an operator is introduced and studied in [20], here we note only that this "another $\partial.$" is not adjoint to the extended stochastic integral. Finally we notice that (2.7) holds true in the case $\eta\neq 0$ if $F\in(L^2)^{-1}$ is such that all kernels from decomposition (1.5) $F^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}\subset\mathcal{H}_{\rm ext}^{(n)}$ (the embedding in the sense of Remark 1.7).

By analogy with [4, 15, 16, 20] we consider now another stochastic differential operator (this new operator is closely connected with $\partial.$, see Proposition 2.1 below).

Definition 2.3. For each $f^{(n)}\in\mathcal{H}_{\rm ext}^{(n)}$ ($n\in\mathbb{Z}_+$) we define an operator $(\mathcal{D}^n\circ)(f^{(n)})\in\mathcal{L}((L^2)^{-1},(L^2)^{-1})$ (here and below $\mathcal{L}$ denotes the set of linear continuous operators) by setting for $F=\sum_{m=0}^{\infty}\langle P_m,F^{(m)}\rangle\in(L^2)^{-1}$

$$(\mathcal{D}^nF)(f^{(n)}):=\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}\langle P_m,\langle F^{(m+n)},f^{(n)}\rangle_{\rm ext}\rangle\in(L^2)^{-1}\tag{2.8}$$

(see (2.1)).

Since for each $F\in(L^2)^{-1}$ there exists $q\in\mathbb{N}$ such that $F\in(L^2)^{-1}_{-(q-2)}$, we have

$$\begin{aligned}\|(\mathcal{D}^nF)(f^{(n)})\|_{-q}^2&=\|\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}\langle P_m,\langle F^{(m+n)},f^{(n)}\rangle_{\rm ext}\rangle\|_{-q}^2\\&\le\sum_{m=0}^{\infty}2^{-qm}(\frac{(m+n)!}{m!})^2|F^{(m+n)}|_{\rm ext}^2|f^{(n)}|_{\rm ext}^2\\&\le(n!)^2|f^{(n)}|_{\rm ext}^2\sum_{m=0}^{\infty}2^{-qm}2^{2(m+n)}|F^{(m+n)}|_{\rm ext}^2\\&=2^{qn}(n!)^2|f^{(n)}|_{\rm ext}^2\sum_{m=0}^{\infty}2^{-(q-2)(m+n)}|F^{(m+n)}|_{\rm ext}^2\\&\le 2^{qn}(n!)^2|f^{(n)}|_{\rm ext}^2\|F\|_{-(q-2)}^2<\infty\end{aligned}$$

(we used the estimate $\frac{(m+n)!}{m!}=n!C_{m+n}^m\le n!2^{m+n}$), therefore this definition is correct and, moreover, for each $F\in(L^2)^{-1}$ $(\mathcal{D}^nF)(\circ)\in\mathcal{L}(\mathcal{H}_{\rm ext}^{(n)},(L^2)^{-1})$. □

Remark 2.6. We note that for $\eta\neq 0$ the restriction on $(L^2)^{-1}$ of introduced in [20] operator $(\mathcal{D}^n\circ)(f^{(n)})$ *differs* from operator (2.8), cf. Remark 2.2.

Remark 2.7. Let $\mathcal{D}F:=\mathcal{D}^1F$. It can be shown by analogy with the Gamma analysis (see [16]) that for $g_1^{(1)},g_2^{(1)},\dots,g_n^{(1)}\in\mathcal{H}_{\mathbb{C}}=\mathcal{H}_{\rm ext}^{(1)}$

$$(\underbrace{\mathcal{D}(\dots(\mathcal{D}((\mathcal{D}}_{n\ times}F)(g_1^{(1)})))(g_2^{(1)})\dots))(g_n^{(1)})=(\mathcal{D}^nF)(g_1^{(1)}\diamond g_2^{(1)}\diamond\dots\diamond g_n^{(1)}).$$

Theorem 2.3. *For each $F\in(L^2)^{-1}$ the kernels $F^{(n)}\in\mathcal{H}_{\rm ext}^{(n)}$, $n\in\mathbb{Z}_+$ from decomposition* (1.5) *can be presented in the form*

$$F^{(n)}=\frac{1}{n!}\mathbf{E}(\mathcal{D}^nF).\tag{2.9}$$

Proof. Using (2.8), for each $f^{(n)}\in\mathcal{H}_{\rm ext}^{(n)}$ we obtain

$$\mathbf{E}((\mathcal{D}^nF)(f^{(n)}))=\langle\!\langle(\mathcal{D}^nF)(f^{(n)}),1\rangle\!\rangle=n!\langle F^{(n)},f^{(n)}\rangle_{\rm ext},$$

this equality can be formally rewritten in form (2.9). □

Let us calculate the adjoint to $(\mathcal{D}^n\circ)(f^{(n)})$, $f^{(n)}\in\mathcal{H}_{\text{ext}}^{(n)}$ operator. For $F\in(L^2)^{-1}$ and $g\in(L^2)^1$ we have (see (2.8), (1.6), (1.9), (2.1), (1.5))

$$\begin{aligned}\langle\!\langle(\mathcal{D}^nF)(f^{(n)}),g\rangle\!\rangle&=\langle\!\langle\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}\langle P_m,\langle F^{(m+n)},f^{(n)}\rangle_{\text{ext}}\rangle,\sum_{k=0}^{\infty}\langle P_k,g^{(k)}\rangle\rangle\!\rangle\\&=\sum_{m=0}^{\infty}(m+n)!\langle\langle F^{(m+n)},f^{(n)}\rangle_{\text{ext}},g^{(m)}\rangle_{\text{ext}}\\&=\sum_{m=0}^{\infty}(m+n)!\langle F^{(m+n)},f^{(n)}\diamond g^{(m)}\rangle_{\text{ext}}\\&=\langle\!\langle\sum_{k=0}^{\infty}\langle P_k,F^{(k)}\rangle,\sum_{m=0}^{\infty}\langle P_{m+n},f^{(n)}\diamond g^{(m)}\rangle\rangle\!\rangle=\langle\!\langle F,(\mathcal{D}^ng)(f^{(n)})^*\rangle\!\rangle,\end{aligned}$$

therefore (see (1.18))

$$(\mathcal{D}^ng)(f^{(n)})^*=\sum_{m=0}^{\infty}\langle P_{m+n},f^{(n)}\diamond g^{(m)}\rangle=g\Diamond\langle P_n,f^{(n)}\rangle, \tag{2.10}$$

where $g^{(m)}\in\mathcal{H}_{\text{ext}}^{(m)}$, $m\in\mathbb{Z}_+$ are the kernels from decomposition (1.6) for g.

Now we focus on the operator $\mathcal{D}=\mathcal{D}^1$. The interconnection between $\mathcal{D}$ and $\partial_\cdot$ is given by

Proposition 2.1. *For all* $F\in(L^2)^{-1}$, $f^{(1)}\in\mathcal{H}_{\mathbb{C}}$

$$\int_{\mathbb{R}_+}\partial_sF\cdot f^{(1)}(s)\sigma(ds)=(\mathcal{D}F)(f^{(1)}). \tag{2.11}$$

Proof. Using (2.4), (1.9), (2.3), (2.1) and (2.8) it is easy to show (by analogy with the proof of Proposition 2.2 in [16]) that $\forall g\in(L^2)^1$ $\langle\!\langle\int_{\mathbb{R}_+}\partial_sF\cdot f^{(1)}(s)\sigma(ds),g\rangle\!\rangle=\langle\!\langle(\mathcal{D}F)(f^{(1)}),g\rangle\!\rangle$, from where the result follows. □

Remark 2.8. Note that *formally* $\partial_\cdot\circ=(\mathcal{D}\circ)(\delta_\cdot)$, where δ is the Dirac delta-function.

Taking into consideration the result of Proposition 2.1, we preserve for $\mathcal{D}$ the name "a generalized stochastic derivative", cf. [4, 15, 16, 20].

Theorem 2.1 can be reformulated "in terms of $\mathcal{D}$" as follows:

Theorem 2.4. *For all* $F\in(L^2)^{-1}$, $f\in(L^2)^1$ *and* $g^{(1)}\in\mathcal{H}_{\mathbb{C}}$

$$\langle\!\langle F,\int_0^{\infty}f\cdot g^{(1)}(s)\widehat{d}M_s\rangle\!\rangle=\langle\!\langle F,f\Diamond\langle P_1,g^{(1)}\rangle\rangle\!\rangle=\langle\!\langle(\mathcal{D}F)(g^{(1)}),f\rangle\!\rangle. \tag{2.12}$$

Proof. The equality $\int_0^\infty f\cdot g^{(1)}(s)\widehat{d}M_s=f\Diamond\langle P_1,g^{(1)}\rangle$ follows from (1.18), (1.21) and the following

Lemma 2.4. *Let* $F^{(n)}\in\mathcal{H}_{\text{ext}}^{(n)}$, $g^{(1)}\in\mathcal{H}_{\mathbb{C}}$. *Then* $F^{(n)}\diamond g^{(1)}=\widehat{F}^{(n)}_{[0,+\infty)}\in\mathcal{H}_{\text{ext}}^{(n+1)}$, *where* $\widehat{F}^{(n)}_{[0,+\infty)}$ *is constructed in Lemma 1.2 starting from* $F^{(n)}\otimes g^{(1)}$ *and with* $t=+\infty$.

Proof. This result follows directly from Lemmas 1.1, 1.2, see (1.12), (1.20). □

The second equality in (2.12) follows from (2.10) (with $n=1$). □

Note that (2.12) can be used as a definition of $\mathcal{D}$.

As is well known, in the classical Gaussian and Poissonian analysis (the case $\eta=0$) the operator $\mathcal{D}$ is a pre-image of the directional derivative under the S-transform (see [15] for more details). In the Gamma analysis the situation is more complicated, but the similar result holds true (see [16]). Now the situation is very similar to the Gamma analysis, let us explain this explicitly.

Definition 2.4. We define a set of formal series B (a characterization set of $(L^2)^{-1}$ in terms of the S-transform) by setting $B := S\big((L^2)^{-1}\big) \equiv \big\{K|\exists F \in (L^2)^{-1} : K = SF\big\}$.

Definition 2.5. Let $g \in \mathcal{H}_{\mathbb{C}}$. We define a "directional derivative" $D_g^\diamond : B \to B$ by setting for $(SF)(\cdot) = \sum_{m=0}^{\infty}\langle F^{(m)}, \cdot^{\otimes m}\rangle_{\text{ext}} \in B$

(2.13)

$$(D_g^\diamond SF)(\cdot) := \sum_{m=1}^{\infty} m\langle F^{(m)}, \cdot^{\otimes(m-1)} \diamond g\rangle_{\text{ext}} \equiv \sum_{m=0}^{\infty}(m+1)\langle\langle F^{(m+1)}, g\rangle_{\text{ext}}, \cdot^{\otimes m}\rangle_{\text{ext}} \in B.$$

The correctness of this definition can be proved by analogy with the "Gamma case" (it is necessary to estimate the $(L^2)^{-1}_{-q}$-norm of $S^{-1}(D_g^\diamond SF)$, where $q \in \mathbb{N}$ is such that $F \in (L^2)^{-1}_{-q+1}$), see [16]. □

Remark 2.9. Note that if $\eta = 0$ then $D_g^\diamond$ is the usual directional derivative.

Theorem 2.5. *The generalized stochastic derivative $(\mathcal{D}\circ)(g)$ is a pre-image of the "directional derivative" $D_g^\diamond$ of $S\circ$ under the S-transform, i.e., for all $F \in (L^2)^{-1}$ and $g \in \mathcal{H}_{\mathbb{C}}$*

$$(\mathcal{D}F)(g) = S^{-1}(D_g^\diamond SF).$$

Proof. Using (2.8) (with $n = 1$) and the obvious formula $S^{-1}(D_g^\diamond SF) = \sum_{m=0}^{\infty}\langle P_m, (m+1)\langle F^{(m+1)}, g\rangle_{\text{ext}}\rangle$ (see (2.13), (1.10); $F^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$, $m \in \mathbb{Z}_+$ are the kernels from decomposition (1.5) for F) we obtain

$$(\mathcal{D}F)(g) = \sum_{m=0}^{\infty}\langle P_m, (m+1)\langle F^{(m+1)}, g\rangle_{\text{ext}}\rangle = S^{-1}(D_g^\diamond SF).$$

□

It was established in [4] that in the Gaussian analysis $\mathcal{D}$ is a differentiation with respect to the Wick product. In [15, 16] it was shown that this property of $\mathcal{D}$ holds true in the Poissonian and Gamma analysis (we remind that in the Gamma analysis (in the same way as in the Meixner one) on the spaces of nonregular and regular generalized functions the operators $\mathcal{D}$ are different; nevertheless the result is true for both operators). Therefore it is natural to expect that $\mathcal{D}$ is a differentiation in the Meixner analysis too. In fact, in [20] the corresponding result was established for $\mathcal{D}$ on the spaces of nonregular generalized functions; and now we have the following statement:

Theorem 2.6. *The generalized stochastic derivative $\mathcal{D}$ is a differentiation with respect to the Wick product, i.e., $\forall F, G \in (L^2)^{-1}$ we have*

$$\mathcal{D}(F\Diamond G) = (\mathcal{D}F)\Diamond G + F\Diamond(\mathcal{D}G). \tag{2.14}$$

Proof. Since the proof of this statement is completely analogous to the proof of the corresponding statement in the Gamma analysis (see the proof of Theorem 2.5 in [16]), we shall confine ourselves to the short description of main steps.

1. Applying the S-transform to the left- and right hand sides of (2.14) and taking into account the result of Theorem 2.5 one can show that it is sufficient to prove that

$$D_g^\diamond(SF \cdot SG) = D_g^\diamond(SF) \cdot SG + SF \cdot D_g^\diamond(SG). \tag{2.15}$$

   (Note that in the case $\eta = 0$ we can stop here because a *usual* directional derivative has this property.)

2. Using the definitions of the S-transform and $D_g^\diamond$ one can show that (2.15) is true if $\forall n,m\in\mathbb{Z}_+$

$$(n+m)\langle F^{(n)}\diamond G^{(m)},\lambda^{\otimes(n+m-1)}\diamond g\rangle_{\rm ext} = n\langle F^{(n)},\lambda^{\otimes(n-1)}\diamond g\rangle_{\rm ext}\langle G^{(m)},\lambda^{\otimes m}\rangle_{\rm ext} + m\langle F^{(n)},\lambda^{\otimes n}\rangle_{\rm ext}\langle G^{(m)},\lambda^{\otimes(m-1)}\diamond g\rangle_{\rm ext} \tag{2.16}$$

(here $F^{(n)}\in\mathcal{H}_{\rm ext}^{(n)}$, $G^{(m)}\in\mathcal{H}_{\rm ext}^{(m)}$ are the kernels from decompositions (1.5) for F and G correspondingly).
3. Formula (2.16) can be verified by the direct calculation that is based on (1.3).

□

By induction we obtain from Theorem 2.6

Corollary. *Let $n\in\mathbb{N}$, $F\in(L^2)^{-1}$, and $h:\mathbb{C}\to\mathbb{C}$ be a holomorphic at $(SF)(0)$ function. Then we have*

$$\mathcal{D}(F^{\Diamond n}) = nF^{\Diamond(n-1)}\Diamond(\mathcal{D}F),\qquad \mathcal{D}h^{\Diamond}(F) = {h'}^{\Diamond}(F)\Diamond(\mathcal{D}F), \tag{2.17}$$

where h' denotes the usual derivative of h.

Finally, let us calculate a commutator between the extended stochastic integral and the generalized stochastic derivative (known as a fundamental theorem of the Malliavin calculus, cf. [10]).

Theorem 2.7. *Let $F\in(L^2)^{-1}\otimes\mathcal{H}_{\mathbb{C}}$. Then $\forall t\in[0,+\infty]$*

$$(\mathcal{D}\int_0^t F_s\widehat{d}M_s)(\circ) = \int_0^t(\mathcal{D}F_s)(\circ)\widehat{d}M_s + \int_0^t F_s\circ(s)\sigma(ds). \tag{2.18}$$

Proof. Since, as above, the proof of this statement is completely analogous to the proof of the corresponding statement in the Gamma analysis (see the proof of Theorem 2.6 in [16]), again we shall confine ourselves to the short description of main steps.

1. Using the definitions of the extended stochastic integral and $\mathcal{D}$ one can show that in order to prove (2.18) it is sufficient to establish that $\forall n\in\mathbb{Z}_+$

$$(n+1)\langle\widehat{F}_{[0,t)}^{(n)},\circ\rangle_{\rm ext} = n\widehat{\langle F_\cdot^{(n)},\circ\rangle}_{{\rm ext}\,[0,t)} + \int_0^t F_s^{(n)}\circ(s)\sigma(ds),$$

where $F_\cdot^{(n)}\in\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}$ ($n\in\mathbb{Z}_+$) are the kernels from decomposition (1.19) for F; $\widehat{F}_{[0,t)}^{(n)}\in\mathcal{H}_{\rm ext}^{(n+1)}$, $\widehat{\langle F_\cdot^{(n)},\circ\rangle}_{{\rm ext}\,[0,t)}\in\mathcal{H}_{\rm ext}^{(n)}$ are constructed in Lemma 1.2 starting from $F_\cdot^{(n)}$, $\langle F_\cdot^{(n)},\circ\rangle_{\rm ext}$ correspondingly. Of course, this equality is true if $\forall g^{(1)}\in\mathcal{H}_{\mathbb{C}}$, $\forall f^{(n)}\in\mathcal{H}_{\rm ext}^{(n)}$

$$(n+1)\langle\langle\widehat{F}_{[0,t)}^{(n)},g^{(1)}\rangle_{\rm ext},f^{(n)}\rangle_{\rm ext} = n\langle\widehat{\langle F_\cdot^{(n)},g^{(1)}\rangle}_{{\rm ext}\,[0,t)},f^{(n)}\rangle_{\rm ext} + \Big\langle\int_0^t F_s^{(n)}g^{(1)}(s)\sigma(ds),f^{(n)}\Big\rangle_{\rm ext}. \tag{2.19}$$

2. Using the formula $\langle\widehat{F}_{[0,t)}^{(n)},f^{(n+1)}\rangle_{\rm ext}=\int_0^t\langle F_s^{(n)},f^{(n+1)}(s)\rangle_{\rm ext}\sigma(ds)$ (formula (3.22) in [21], $f^{(n+1)}(\cdot)\in\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}$ is obtained from $f^{(n+1)}\in\mathcal{H}_{\rm ext}^{(n+1)}$ in Lemma 2.1)

and the nonatomicity of σ we obtain (using the notation of Lemma 1.1)

$$
\begin{aligned}
&(n+1)\langle\langle \widehat{F}^{(n)}_{[0,t)}, g^{(1)}\rangle_{\rm ext}, f^{(n)}\rangle_{\rm ext} = (n+1)\langle \widehat{F}^{(n)}_{[0,t)}, f^{(n)} \diamond g^{(1)}\rangle_{\rm ext}\\
&\quad = (n+1)\int_0^t \langle F^{(n)}_s, (f^{(n)} \diamond g^{(1)})(s)\rangle_{\rm ext}\sigma(ds)\\
&\quad = \int_0^t \langle F^{(n)}_s, f^{(n)}(\widetilde{\cdot_1, \dots, \cdot_n}) g^{(1)}(s) + f^{(n)}(\widetilde{\cdot_2, \dots, s}) g^{(1)}(\cdot_1) + \cdots \qquad (2.20)\\
&\quad + f^{(n)}(\widetilde{s, \dots, \cdot_{n-1}}) g^{(1)}(\cdot_n)\rangle_{\rm ext}\sigma(ds) = \langle \int_0^t F^{(n)}_s g^{(1)}(s)\sigma(ds), f^{(n)}\rangle_{\rm ext}\\
&\quad + \int_0^t \langle F^{(n)}_s, f^{(n)}(\widetilde{\cdot_2, \dots, s}) g^{(1)}(\cdot_1) + \cdots + f^{(n)}(\widetilde{s, \dots, \cdot_{n-1}}) g^{(1)}(\cdot_n)\rangle_{\rm ext}\sigma(ds).
\end{aligned}
$$

Rewriting by analogy with (2.20) the right hand side of (2.19) we obtain the necessary result.

□

By analogy with [4, 15] as an application of our results we will calculate the generalized stochastic derivative of the solution of the stochastic equation

$$(L^2)^{-1} \ni F_t = F_0 + \int_0^t h^{\Diamond}(F_s)\widehat{d}M_s, \qquad (2.21)$$

where $h : \mathbb{C} \to \mathbb{C}$ is some entire function, $F_0 \in \mathbb{C}$. Under certain conditions on h a unique solution of (2.21) $F_t \in (L^2)^{-1}$ exists. Applying $\mathcal{D}$ to (2.21) and taking into account (2.18) and (2.17), for each $g \in \mathcal{H}_{\mathbb{C}}$ we obtain

$$
\begin{aligned}
(\mathcal{D}F_t)(g) &= (\mathcal{D}\int_0^t h^{\Diamond}(F_s)\widehat{d}M_s)(g)\\
&= \int_0^t h'^{\Diamond}(F_s)\Diamond(\mathcal{D}F_s)(g)\widehat{d}M_s + \int_0^t h^{\Diamond}(F_s)g(s)\sigma(ds).
\end{aligned} \qquad (2.22)
$$

Let $\phi^g_s(\lambda) := S((\mathcal{D}F_s)(g))(\lambda)$. Applying the S-transform to (2.22) and taking into account (1.22) we obtain

$$\phi^g_t(\lambda) = \int_0^t h'((SF_s)(\lambda))\phi^g_s(\lambda)\lambda(s)\sigma(ds) + \int_0^t h((SF_s)(\lambda))g(s)\sigma(ds).$$

The solution of this equation is

$$\phi^g_t(\lambda) = \int_0^t h((SF_s)(\lambda))g(s) \cdot \exp\Big\{\int_s^t h'((SF_u)(\lambda))\lambda(u)\sigma(du)\Big\}\sigma(ds).$$

By the inverse S-transform we obtain

$$(\mathcal{D}F_t)(g) = \int_0^t h^{\Diamond}(F_s)g(s)\Diamond\exp^{\Diamond}\Big\{\int_s^t h'^{\Diamond}(F_u)\widehat{d}M_u\Big\}\sigma(ds) \in (L^2)^{-1}. \qquad (2.23)$$

Remark 2.10. Let $\widetilde{\mathcal{D}}$ be the generalized stochastic derivative on $(\mathcal{H}_{-\tau})$, see Remark 1.9 and [20]. This operator can be defined by the formula $(\widetilde{\mathcal{D}}F)(g) := S^{-1}(D_g SF)$, where $F \in (\mathcal{H}_{-\tau})$, $g \in \mathcal{H}_{\tau,\mathbb{C}}$, D_g is the directional derivative operator (in the direction g). It is easy to see that the restriction of $\widetilde{\mathcal{D}}$ on $(L^2)^{-1}$ does not coincide with $\mathcal{D}$, generally speaking. Nevertheless, it follows from [20] and (2.23) that $(\widetilde{\mathcal{D}}F_t)(g) = (\mathcal{D}F_t)(g)$ for $F_t \in (L^2)^{-1}$ and $g \in \mathcal{H}_{\tau,\mathbb{C}}$. This strange at first sight result is connected with a form of F_t and properties of the extended stochastic integral: roughly speaking, when integrating we exclude a one "diagonal" of the integrand, but the difference between $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ is connected namely with this "diagonal".

3. Some properties of stochastic derivatives on (L^2)

The stochastic derivative $\partial_\cdot$ on (L^2) was considered in [21]. Here we study some properties of $\partial_\cdot$ and $\mathcal{D}^n$ that were not considered therein.

First we note that since (L^2) is embedded in $(L^2)^{-1}$, $\partial_\cdot$ and $(\mathcal{D}^n\circ)(f^{(n)})$ $(f^{(n)}\in\mathcal{H}_{\rm ext}^{(n)})$ are well-defined as operators acting from (L^2) to $(L^2)^{-1}$ (here we understand (L^2) as a *subset* of $(L^2)^{-1}$). But if we want to consider these operators on the *topological* space (L^2) then the situation is not so trivial because now $\partial_\cdot$ and $(\mathcal{D}^n\circ)(f^{(n)})$ are not continuous.

We begin our consideration from the operator $\partial_\cdot$.

Definition 3.1. We define the stochastic derivative $\partial_\cdot:(L^2)\to(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ with the domain ${\rm dom}\,\partial_\cdot=\{F\in(L^2):\|\partial_\cdot F\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}=\sum_{n=0}^\infty(n+1)!(n+1)|F^{(n+1)}(\cdot)|^2_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}}<\infty\}$ by formula (2.4) (here the kernels $F^{(n+1)}(\cdot)\in\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}$, $n\in\mathbb{Z}_+$ are obtained in Lemma 2.1 starting from the kernels $F^{(n+1)}\in\mathcal{H}_{\rm ext}^{(n+1)}$ from decomposition (1.5) for F).

Theorem 3.1. *For each* $t\in[0,+\infty]$

$$1_{[0,t)}(\cdot)\partial_\cdot=\Big(\int_0^t\circ\widehat{d}M\Big)^*_\cdot.$$

In particular, $1_{[0,t)}(\cdot)\partial_\cdot$ *(and, specifically,* $\partial_\cdot=1_{[0,+\infty)}(\cdot)\partial_\cdot$*) is a closed operator.*

Proof. It is sufficient to prove that ${\rm dom}(1_{[0,t)}(\cdot)\partial_\cdot)={\rm dom}(\int_0^t\circ\widehat{d}M)^*_\cdot$, then the result follows from formula (3.21) in [21]. Note that ${\rm dom}(1_{[0,t)}(\cdot)\partial_\cdot)=\{F\in(L^2):\sum_{n=0}^\infty(n+1)!(n+1)|F^{(n+1)}(\cdot)1_{[0,t)}(\cdot)|^2_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}}<\infty\}$ (here the kernels $F^{(n+1)}(\cdot)$, $n\in\mathbb{Z}_+$ are obtained in Lemma 2.1 from the kernels $F^{(n+1)}$ from decomposition (1.5) for F). Let us find ${\rm dom}(\int_0^t\circ\widehat{d}M)^*_\cdot$. By definition,

$$\begin{aligned}&\{F\in{\rm dom}(\int_0^t\circ\widehat{d}M)^*_\cdot\}\\&\Leftrightarrow\{(L^2)\otimes\mathcal{H}_{\mathbb{C}}\supset{\rm dom}\int_0^t\circ(s)\widehat{d}M_s\ni G\mapsto\langle\!\langle\int_0^tG(s)\widehat{d}M_s,F\rangle\!\rangle\\&\qquad\text{is a linear continuous functional}\}.\end{aligned}$$

By Riesz's theorem this is possible if and only if there exists $H\in(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ such that $\forall G\in{\rm dom}\int_0^t\circ(s)\widehat{d}M_s$ $\langle\!\langle\int_0^tG(s)\widehat{d}M_s,F\rangle\!\rangle=(G,H)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}$. For $G\in{\rm dom}\int_0^t\circ(s)\widehat{d}M_s$ and $F\in(L^2)$ we have (see (1.21), (1.5), (1.9))

$$\langle\!\langle\int_0^tG(s)\widehat{d}M_s,F\rangle\!\rangle=\langle\!\langle\sum_{n=0}^\infty\langle P_{n+1},\widehat{G}^{(n)}_{[0,t)}\rangle,\sum_{m=0}^\infty\langle P_m,F^{(m)}\rangle\rangle\!\rangle=\sum_{n=0}^\infty(n+1)!\langle\widehat{G}^{(n)}_{[0,t)},F^{(n+1)}\rangle_{\rm ext}.$$

On the other hand, using decompositions (1.19) for G and H, and the formula

$$\langle\widehat{G}^{(n)}_{[0,t)},F^{(n+1)}\rangle_{\rm ext}=\int_0^t\langle G_s^{(n)},F^{(n+1)}(s)\rangle_{\rm ext}\sigma(ds)=(G_\cdot^{(n)},F^{(n+1)}(\cdot)1_{[0,t)}(\cdot))_{\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}},$$

where $F^{(n+1)}(\cdot)\in\mathcal{H}_{\rm ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}$ is obtained from $F^{(n+1)}\in\mathcal{H}_{\rm ext}^{(n+1)}$ in Lemma 2.1 (see formula (3.22) in [21]), we can conclude that $(G,H)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}=\sum_{n=0}^\infty(n+1)!\langle\widehat{G}^{(n)}_{[0,t)},F^{(n+1)}\rangle_{\rm ext}$ if and only if $H(\cdot)=\sum_{m=0}^\infty\langle P_m,(m+1)F^{(m+1)}(\cdot)1_{[0,t)}(\cdot)\rangle$. This H belongs to $(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ if and only if $\|H\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}=\sum_{m=0}^\infty(m+1)!(m+1)|F^{(m+1)}(\cdot)1_{[0,t)}(\cdot)|^2_{\mathcal{H}_{\rm ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}}<\infty$, therefore

$$\begin{aligned}{\rm dom}\Big(\int_0^t\circ\widehat{d}M\Big)^*_\cdot&=\Big\{F\in(L^2):\sum_{m=0}^\infty(m+1)!(m+1)|F^{(m+1)}(\cdot)1_{[0,t)}(\cdot)|^2_{\mathcal{H}_{\rm ext}^{(m)}\otimes\mathcal{H}_{\mathbb{C}}}<\infty\Big\}\\&={\rm dom}(1_{[0,t)}(\cdot)\partial_\cdot).\end{aligned}$$

□

Clarc-Ocone formula (2.7) holds true for $F \in (L^2)$ and $\eta = 0$; but, generally speaking, now $\mathbf{E}(\partial_\cdot F|_{\mathcal{F}_\cdot}) \in (L^2)^{-1} \otimes \mathcal{H}_{\mathbb{C}}$. If $\sum_{n=0}^{\infty}(n+1)!(n+1)|F^{(n+1)}|_0^2 < \infty$ then $\partial_\cdot F, \mathbf{E}(\partial_\cdot F|_{\mathcal{F}_\cdot}) \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$. The statement of Remark 2.5 holds true for $F \in (L^2)$ (up to obvious modifications).

Now let us consider the operator $\mathcal{D}^n$.

Definition 3.2. For each $f^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ $(n \in \mathbb{Z}_+)$ we define an operator $(\mathcal{D}^n \circ)(f^{(n)}) : (L^2) \to (L^2)$ with the domain $\mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)}) = \{F \in (L^2) : \|(\mathcal{D}^n F)(f^{(n)})\|_{(L^2)}^2 = \sum_{m=0}^{\infty} \frac{((m+n)!)^2}{m!} |\langle F^{(m+n)}, f^{(n)}\rangle_{\mathrm{ext}}|_{\mathrm{ext}}^2 < \infty\}$ by formula (2.8) (here $F^{(m)} \in \mathcal{H}_{\mathrm{ext}}^{(m)}$, $m \in \mathbb{Z}_+$ are the kernels from decomposition (1.5) for F).

Theorem 3.2. *The operator $(\mathcal{D}^n \circ)(f^{(n)})$ ($f^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$, $n \in \mathbb{Z}_+$) is closed.*

Proof. It is clear that $\mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})$ is dense in (L^2), therefore there exists the adjoint to $(\mathcal{D}^n \circ)(f^{(n)})$ operator $(\mathcal{D}^n \circ)(f^{(n)})^* : (L^2) \to (L^2)$, this operator is defined by the formula $\langle\langle (\mathcal{D}^n F)(f^{(n)}), G \rangle\rangle = \langle\langle F, (\mathcal{D}^n G)(f^{(n)})^* \rangle\rangle$. By analogy with the calculation before (2.10) one can show that (see (1.18))

$$(\mathcal{D}^n G)(f^{(n)})^* = G \lozenge \langle P_n, f^{(n)} \rangle = \sum_{k=0}^{\infty} \langle P_{k+n}, G^{(k)} \diamond f^{(n)} \rangle, \tag{3.1}$$

where $G^{(k)} \in \mathcal{H}_{\mathrm{ext}}^{(k)}$, $k \in \mathbb{Z}_+$ are the kernels from decomposition (1.5) for G. Therefore $\mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})^* = \{G \in (L^2) : \|G \lozenge \langle P_n, f^{(n)} \rangle\|_{(L^2)}^2 = \sum_{k=0}^{\infty} (k+n)! |G^{(k)} \diamond f^{(n)}|_{\mathrm{ext}}^2 < \infty\}$.

Since $\mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})^*$ is dense in (L^2), it is possible to consider the adjoint to $(\mathcal{D}^n \circ)(f^{(n)})^*$ operator $(\mathcal{D}^n \circ)(f^{(n)})^{**} : (L^2) \to (L^2)$, this operator is defined by the formula $\langle\langle (\mathcal{D}^n F)(f^{(n)})^{**}, G \rangle\rangle = \langle\langle F, (\mathcal{D}^n G)(f^{(n)})^* \rangle\rangle$. Let us find $\mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})^{**}$. By definition

$$\big\{F \in \mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})^{**}\big\} \Leftrightarrow \big\{(L^2) \supset \mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})^* \ni G \mapsto \langle\langle F, (\mathcal{D}^n G)(f^{(n)})^* \rangle\rangle \text{ is a linear continuous functional}\big\}.$$

By Riesz's theorem the functional $G \mapsto \langle\langle F, (\mathcal{D}^n G)(f^{(n)})^* \rangle\rangle$ is continuous if and only if (see (1.5), (3.1) and (1.9))

$$\begin{aligned}\langle\langle F, (\mathcal{D}^n G)(f^{(n)})^* \rangle\rangle &= \langle\langle \sum_{m=0}^{\infty} \langle P_m, F^{(m)} \rangle, \sum_{k=0}^{\infty} \langle P_{k+n}, G^{(k)} \diamond f^{(n)} \rangle\rangle \\ &= \sum_{k=0}^{\infty} (k+n)! \langle F^{(k+n)}, G^{(k)} \diamond f^{(n)} \rangle_{\mathrm{ext}} = \langle\langle H, G \rangle\rangle\end{aligned}$$

with $H \in (L^2)$. It follows from Theorem 1.3 in [21], (1.9) and (2.1) that this H must have the form $H = \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \langle P_m, \langle F^{(m+n)}, f^{(n)} \rangle_{\mathrm{ext}} \rangle$, hence

$$\begin{aligned}&\big\{F \in \mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})^{**}\big\} \Leftrightarrow \big\{H \in (L^2)\big\} \\ &\Leftrightarrow \Big\{\|H\|_{(L^2)}^2 = \sum_{m=0}^{\infty} \frac{((m+n)!)^2}{m!} |\langle F^{(m+n)}, f^{(n)} \rangle_{\mathrm{ext}}|_{\mathrm{ext}}^2 < \infty\Big\} \\ &\Leftrightarrow \big\{F \in \mathrm{dom}(\mathcal{D}^n \circ)(f^{(n)})\big\}.\end{aligned}$$

But it means that $(\mathcal{D}^n \circ)(f^{(n)})^{**} = (\mathcal{D}^n \circ)(f^{(n)})$, therefore $(\mathcal{D}^n \circ)(f^{(n)})$ is closed as an adjoint operator. □

Remark 3.1. Let $M_n := \{F \in (L^2) : \sum_{m=0}^{\infty} \frac{((m+n)!)^2}{m!} |F^{(m+n)}|_{\mathrm{ext}}^2 < \infty\}$. For each $f^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)}$ we define the operator $(\mathcal{D}^n \circ)(f^{(n)}) : M_n \to (L^2)$ by formula (2.8). It follows from Theorem 3.2 that this operator (as an operator acting in the topological space (L^2)) is *closable*. Moreover, for each $F \in M_n$ the operator $(\mathcal{D}^n F)(\circ) : \mathcal{H}_{\text{ext}}^{(n)} \to (L^2)$ is continuous:

$$\|(\mathcal{D}^n F)(f^{(n)})\|_{(L^2)}^2 = \sum_{m=0}^{\infty} \frac{((m+n)!)^2}{m!} |\langle F^{(m+n)}, f^{(n)}\rangle_{\text{ext}}|_{\text{ext}}^2$$
$$\leq |f^{(n)}|_{\text{ext}} \cdot \sum_{m=0}^{\infty} \frac{((m+n)!)^2}{m!} |F^{(m+n)}|_{\text{ext}}^2.$$

Since the operators $\partial.$ and $\mathcal{D}^n$ on (L^2) are the restrictions of the corresponding operators acting on $(L^2)^{-1}$, we have the following results.

The interconnection between $\partial.$ and $\mathcal{D} \equiv \mathcal{D}^1$ on (L^2) is given by formula (2.11); now, of course, $F \in (L^2)$ must be from the domain of $(\mathcal{D}\circ)(g^{(1)})$, $g^{(1)} \in \mathcal{H}_{\mathbb{C}}$, if we need $(\mathcal{D}F)(g^{(1)}) \in (L^2)$. Generally speaking, in this case $\partial.F$ in the left hand side of (2.11) can be a generalized function. Note that $\partial.F$ and $(\mathcal{D}F)(g^{(1)})$ will be not generalized functions if, for example, $F \in M_1$ (see Remark 3.1); this statement follows from (2.2) and the definition of $(\mathcal{D}F)(g^{(1)})$.

Further, formula (2.12) holds true for $F \in \text{dom}(\mathcal{D}\circ)(g^{(1)})$, $f \in (L^2)$, $g^{(1)} \in \mathcal{H}_{\mathbb{C}}$. We note that now

$$\int_0^\infty f \cdot g^{(1)}(s) \widehat{d}M_s = f \Diamond \langle P_1, g^{(1)}\rangle = (\mathcal{D}f)(g^{(1)})^* \tag{3.2}$$

can be a generalized function if we do not accept the additional restriction $\sum_{n=0}^{\infty}(n+1)!|f^{(n)} \diamond g^{(1)}|_{\text{ext}}^2 < \infty$, here $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{Z}_+$ are the kernels from decomposition (1.5) for f.

Formula (3.2) can be generalized in the following sense. For a general $n \in \mathbb{N}$, $g^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $F \in (L^2)^{-1}$ one can **define** a *multiple* extended stochastic integral

$$\int_{\mathbb{R}_+^n} F \cdot g^{(n)}(u_1, \dots, u_n) \widehat{d}M_{u_1} \dots \widehat{d}M_{u_n} := F \Diamond \langle P_n, g^{(n)}\rangle = (\mathcal{D}^n F)(g^{(n)})^* \in (L^2)^{-1}$$

(see (3.1)). It is easy to see that for $g^{(n)} = g_1^{(1)} \diamond \dots \diamond g_n^{(1)}$, $g_1^{(1)}, \dots, g_n^{(1)} \in \mathcal{H}_{\mathbb{C}}$ this integral is a *repeated* extended stochastic one: $(\mathcal{D}^n F)(g_1^{(1)} \diamond \dots \diamond g_n^{(1)})^* = \int_0^\infty (\dots (\int_0^\infty (\int_0^\infty F \cdot g_1^{(1)}(u_1) \widehat{d}M_{u_1}) g_2^{(1)}(u_2) \widehat{d}M_{u_2}) \dots) g_n^{(1)}(u_n) \widehat{d}M_{u_n}$ (cf. Remark 2.7).

The result of Theorem 2.3 (formula (2.9)) holds true.

Finally, the result of Theorem 2.7 (formula (2.18)) holds true for $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$; and all terms in (2.18) are the operators acting from $\mathcal{H}_{\mathbb{C}}$ to (L^2) if $\sum_{n=0}^{\infty}(n+1)!(n+1)|\widehat{F}_{[0,t)}^{(n)}|_{\text{ext}}^2 < \infty$ (here $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$, $n \in \mathbb{Z}_+$ are the kernels constructed in Lemma 1.2 starting from $F_{\cdot}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (1.19) for F, $t \in [0, +\infty]$). The integrands in (2.18) can be generalized functions; in order to make these functions not generalized ones it is necessary to accept obvious (but not simply verifiable) additional restrictions.

References

1. K. Aase, B. Oksendal, N. Privault, J. Uboe, *White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance*, Finance Stochastics **4** (2000), 465–496.
2. S. Albeverio, Yu. L. Daletsky, Yu. G. Kondratiev, L. Streit, *Non-Gaussian infinite-dimensional analysis*, J. Funct. Anal. **138** (1996), no. 2, 311–350.
3. S. Albeverio, Yu. G. Kondratiev, L. Streit, *How to generalize white noise analysis to non-Gaussian spaces*, in Dynamics of Complex and Irregular Systems, Ph. Blanchard, L. Streit, M. Sirugue-Collin, and D. Testard, eds., World Scientific, Singapore, 1993, pp. 120–130.
4. F. E. Benth, *The Gross derivative of generalized random variables*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2** (1999), no. 3, 381–396.

5. Yu. M. Berezansky, *Infinite dimensional analysis related to generalized translation operators*, Ukrainian Math. J. **49** (1997), no. 3, 403–450.
6. Yu. M. Berezansky, D. A. Merzejewski, *The structure of the extended symmetric Fock space*, Methods Funct. Anal. Topology **6** (2000), no. 4, 1–13.
7. Yu. M. Berezansky, Z. G. Sheftel, G. F. Us, *Functional Analysis*, Vol. II, in Operator Theory: Advances and Applications, Birkhäuser, Basel, Vol. 86, 1996.
8. Yu. M. Berezansky, V. A. Tesko, *Spaces of test and generalized functions related to generalized translation operators*, Ukrainian Math. J. **55** (2003), no. 12, 1907–1979.
9. J. M. Clark, *The representation of functionals of Brownian motion by stochastic integrals*, Ann. Math. Statist. **41** (1970), 1282–1295.
10. G. Di Nunno, T. Meyer-Brandis, B. Oksendal, F. Proske, *Malliavin calculus and anticipative Itô formulae for Lévy processes*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (2005), no. 2, 235–258.
11. G. Di Nunno, B. Oksendal, F. Proske, *White noise analysis for Lévy processes*, J. Funct. Anal. **206** (2004), 109–148.
12. S. Dineen, *Complex Analysis in Locally Convex Spaces*, Math. Studies 57, North-Holland, Amsterdam, 1981.
13. I. M. Gelfand, N. Ya. Vilenkin, *Generalized Functions*, Vol. IV, Academic Press, New York—London, 1964.
14. M. Grothaus, Yu. G. Kondratiev, L. Streit, *Regular generalized functions in Gaussian analysis*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2** (1999), no. 3, 359–380.
15. N. A. Kachanovsky, *A generalized Malliavin derivative connected with the Poisson- and Gamma-measures*, Methods Funct. Anal. Topology **9** (2003), no. 3, 213–240.
16. N. A. Kachanovsky, *A generalized stochastic derivative on the Kondratiev-type space of regular generalized functions of Gamma white noise*, Methods Funct. Anal. Topology **12** (2006), no. 4, 363–383.
17. N. A. Kachanovsky, *Extended stochastic integral and Wick calculus on spaces of regular generalized functions connected with the Gamma measure*, Ukrainian Math. J. **57** (2005), no. 8, 1214–1248.
18. N. A. Kachanovsky, *Biorthogonal Appell-like systems in a Hilbert space*, Methods Funct. Anal. Topology **2** (1996), no. 3–4, 36–52.
19. N. A. Kachanovsky, *Dual Appell-like systems and finite order spaces in non-Gaussian infinite-dimensional analysis*, Methods Funct. Anal. Topology **4** (1998), no. 2, 41–52.
20. N. A. Kachanovsky, *Generalized stochastic derivatives on spaces of nonregular generalized functions of Meixner white noise*, Ukrainian Math. J. (to appear).
21. N. A. Kachanovsky, *On an extended stochastic integral and the Wick calculus on the connected with the generalized Meixner measure Kondratiev-type spaces*, Methods Funct. Anal. Topology **13** (2007), no. 4, 338–379.
22. Yu. G. Kondratiev, J. Luis da Silva, L. Streit, *Generalized Appell systems*, Methods Funct. Anal. Topology **3** (1997), no. 3, 28–61.
23. Yu. G. Kondratiev, J. Luis da Silva, L. Streit, G. F. Us, *Analysis on Poisson and Gamma spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1** (1998), no. 1, 91–117.
24. Yu. G. Kondratiev, L. Streit, W. Westerkamp, J. Yan, *Generalized functions in infinite-dimensional analysis*, Hiroshima Math. J. **28** (1998), 213–260.
25. A. Lokka, *Martingale Representation, Chaos Expansion and Clark-Ocone Formulas*, Research Report, Centre for Mathematical Physics and Stochastics, University of Aarhus, Denmark **22** (1999).
26. E. W. Lytvynov, *Polynomials of Meixner's type in infinite dimensions – Jacobi fields and orthogonality measures*, J. Funct. Anal. **200** (2003), 118–149.
27. J. Meixner, *Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion*, J. London Math. Soc. **9** (1934), no. 1, 6–13.
28. D. Ocone, *Malliavin's calculus and stochastic integral representation of functionals of diffusion processes*, Stochastics **12** (1984), no. 3-4, 161–185.
29. I. V. Rodionova, *Analysis connected with generalized functions of exponential type in one and infinite dimensions*, Methods Funct. Anal. Topology **11** (2005), no. 3, 275–297.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: nick2@zeos.net

Received 22/01/2007