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ABOUT NILPOTENT C_0-SEMIGROUPS OF OPERATORS IN THE HILBERT SPACES AND CRITERIA FOR SIMILARITY TO THE INTEGRATION OPERATOR

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Abstract. In the paper, we describe a class of operators A that have empty spectrum and satisfy the nilpotency property of the generated C_0-semigroup $U(t) = \exp\{-iAt\}$, $t \geqslant 0$, and such that the operatorA^{-1} is similar to the integration operator on the corresponding space $L_2(0,a)$.

Let $U(t)$, $t \geqslant 0$, be a nilpotent semigroup of C_0-class in the separable Hilbert space $\mathfrak{H}$. We denote by a the nilpotency index, i.e., $U(a) = 0$ and $U(a_1) \neq 0$ for each $a_1 < a$. In what follows, the formula $U(t) = \exp\{-iAt\}$ will mean that the operator $-iA$ is a generator of the C_0-semigroup $U(t)$. In such a way the resolvent representation

$$-i(A - zI)^{-1} = \int\limits_0^a e^{izt} U(t)\, dt \tag{1}$$

holds. From this equality, and also from the Wiener-Paley theorem it follows that if $-iA$ generates a C_0-semigroup $U(t)$, it will be nilpotent if and only if $\sigma(A) = \varnothing$ and the entire operator-valued function $(A - zI)^{-1}$ is of the finite exponential type.

We'll get the most simple example of the nilpotent C_0-semigroup $\tilde{U}(t) = \exp\{-iAt\}$ if we assume $\mathfrak{H} = L_2(0,a)$ and the inverse to the operator A will be given by formula

$$\left(A^{-1}h\right)(x) = i\int\limits_0^x h(t)\, dt, \quad h \in L_2(0,a).$$

Then $\tilde{U}$ is a translation semigroup,

$$\tilde{U}(t)h = \tilde{h}(x-t), \quad t \geqslant 0, \quad x \in [0,a],$$

where $\tilde{h}$, by the function h, is defined by

$$\tilde{h}(x) = h(x),\ x \in [0,a], \quad \tilde{h}(x) = 0, \quad x \leqslant 0.$$

In this article, we consider the problem of finding conditions on the operator A so that the semigroup $U(t) = \exp\{-iAt\}$ is similar to the semigroup $\tilde{U}(t)$ in the space $L_2(0,a)$. In another words, when there is an isomorphism S from $\mathfrak{H}$ onto $L_2(0,a)$ such that

$$A^{-1} = S^{-1}JS, \quad (Jh)(x) := i\int\limits_0^x h(t)\, dt. \tag{2}$$

Considerations in the paper are based on perturbations of the one-dimensional operator A^{-1} [2] and on the theory of the de Branges space [3].

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1. *We denote by $\mathcal{N}$ the set of operators A on the separable Hilbert space $\mathfrak{H}$, which satisfy the conditions: 1) $\sigma(A)=\varnothing$; 2) the exponential type of the resolvent $(A-zB)^{-1}$ is finite.* In what follows we'll use the notation $\mathrm{Im}A^{-1}=\left(A^{-1}-(A^{-1})^*\right)/2i$. If the operator $A^{-1}(A\in\mathcal{N})$ is dissipative (i.e. $\mathrm{Im}A^{-1}\geqslant 0$) then the semigroup $U(t)$ is contractive and nilpotent. In the paper [2], criteria for a dissipative operator A^{-1} to be similar to the integration operator are obtained. In this article, we consider the case where the operator A^{-1} is non-dissipative.

We denote by $\mathcal{N}_1$ the set of operators $A\in\mathcal{N}$ such that for $\mathrm{Im}A^{-1}$ *the representation*

$$\mathrm{Im}A^{-1}h=\lambda(h,u)u+Qh,\quad \lambda>0,\quad Q\leqslant 0, \tag{3}$$

holds with a some vector $u\in\mathfrak{H}$ ($\|u\|=1$). For example, this representation takes place if the positive spectrum $\mathrm{Im}A^{-1}$ consists of a unique point λ to which there corresponds a one-dimensional eigen subspace. Let us recall that the operator J is defined by the formula (2).

Theorem 1. *Let $A\in\mathcal{N}_1$ and the exponential type of $(A-zI)^{-1}$ be equal to a. The semigroup $U(t)=\exp\{-iAt\}$, $t\geqslant 0$ belongs to the C_0-class if and only if the operator A^{-1} is similar to the integration operator J in the space $L_2(0,a)$.*

Proof. Let $A\in\mathcal{N}_1$, and the corresponding semigroup $U(t)$ belong to the C_0-class. We consider, on $\mathfrak{H}$, the operator K_α of the type

$$K_\alpha h=Bh+i\alpha(h,u)u,\quad B:=A^{-1}, \tag{4}$$

where the vector u is contained in the representation (3), and α is a complex parameter. Let us choose α in such a way that the following conditions hold:

$$\ker K_\alpha=\ker K_\alpha^*=\{0\},\quad \mathrm{Im}K_\alpha\leqslant 0. \tag{5}$$

Indeed, $\ker K_\alpha\neq\{0\}$ only if $u\in\mathcal{D}(A)$, moreover, the equality $1+i\alpha(Au,u)=0$ takes place. Analogously, $\ker K_\alpha^*\neq\{0\}$ only in the case where $u\in\mathcal{D}(A^*)$ and $1-i\bar{\alpha}(A^*u,u)=0$. Further, if $\mathrm{Re}\alpha\leqslant-\lambda$ (λ is contained in the representation (3)) then with regard for (3) we'll get

$$\begin{aligned}\mathrm{Im}(K_\alpha h,h)&=\mathrm{Im}(Bh,h)+\mathrm{Re}\alpha|(h,u)|^2=(\mathrm{Im}A^{-1}h,h)+\mathrm{Re}\alpha|(h,u)|^2\\&=(\lambda+\mathrm{Re}\alpha)|(h,u)|^2+(Qh,h)\leqslant 0,\quad h\in\mathfrak{H}.\end{aligned}$$

So, for the choosen α, a densely defined operator $L_\alpha:=K_\alpha^{-1}$ exists. It is dissipative,

$$\mathrm{Im}(L_\alpha f,f)\geqslant 0,\quad f\in\mathcal{D}(L_\alpha).$$

From the formula

$$L_\alpha^{-1}h=Bh+i\alpha(h,u)u,\quad h\in\mathfrak{H} \tag{6}$$

one easily deduces that $\sigma(L_\alpha)$ is the same as the set of zeros of an entire function of the exponential type,

$$\varphi_\alpha(z):=1-iz\alpha((I-zB)^{-1}u,u).$$

We'll show that $\alpha\in\mathbb{C}$ can be selected in such a way that the conditions (5) hold, together with the equalities

$$h(\pi/2;\varphi_\alpha)=0,\quad h(-\pi/2;\varphi_\alpha)=a, \tag{7}$$

where $h(\theta;\varphi_\alpha)$ is the growth indicator of the function φ_α,

$$h(\theta;\varphi_\alpha):=\limsup_{r\to\infty} r^{-1}\log\left|\varphi_\alpha(re^{i\theta})\right|,\quad -\pi<\theta\leqslant\pi.$$

If we assume the notation $B = A^{-1}$, then it follows from the equality (1) that $h(\pi/2, \varphi_\alpha) \leqslant 0$. Further, the inequality $h(\pi/2, \varphi_\alpha) < 0$ is possible only for a unique value of the parameter α, which can be determined from the equation

$$1 + \alpha \lim_{y \to +\infty} y\left((I - iyB)^{-1}u, u\right) = 0.$$

So, we assume that the first equality (7) takes place. To prove the second equality (7), we'll make sure that

$$\mathfrak{H} = \operatorname{closspan}\left\{(B^*)^n u \,:\, n \geqslant 0\right\} = \operatorname{closspan}\left\{B^n u \,:\, n \geqslant 0\right\}. \tag{8}$$

Indeed, the subspace $\mathcal{L}$ is orthogonal to all the vectors $(B^*)^n u$, $n \in \mathbb{Z}$, $n \geqslant 0$, and it is invariant with respect to the operator B. If we assume that $\mathcal{L} \neq \{0\}$, then from the resolvent formula

$$(L_\alpha - zI)^{-1}h = B(I - zB)^{-1}h + i\alpha\varphi_\alpha^{-1}(z)\left((I - zB)^{-1}h, u\right)(I - zB)^{-1}u,$$

it follows that $\mathcal{L}$ is invariant with respect to $(L_\alpha - zI)^{-1}$, $z \notin \sigma(L_\alpha)$ and the equality

$$(L_\alpha - zI)^{-1}\big|\,\mathcal{L} = B(I - zB)^{-1}\big|\,\mathcal{L} = B_1(I - zB_1)^{-1}, \quad B_1 := B|\,\mathcal{L} \tag{9}$$

holds.

It follows from the formula (1) that $\left\|B_1(I - zB_1)^{-1}\right\| \leqslant C(\varepsilon)$ in each half-plane $\mathrm{Im}z \leqslant -\varepsilon$, $\varepsilon > 0$. On the other hand, since L_α is dissipative, from (9) we get the estimate $\left\|B_1(I - zB_1)^{-1}\right\| \leqslant C(\varepsilon)$, $\mathrm{Im}z \leqslant -\varepsilon$, which contradicts the Liouville theorem. Thus, the first equality (8) is proved. The second equality (8) is proved analogously.

We return to the second equality (7). From the formula (1) we conclude that $h(-\pi/2; \varphi_\alpha) \leqslant a$, where a is an exponential type resolvent growth of $(A - zI)^{-1}$. If we assume that $h(-\pi/2; \varphi_\alpha) = a_1 < a$, then the representation

$$\left(B(I - zB)^{-1}u, u\right) = i\int_0^{a_1} e^{izt}(U(t)u, u)\,dt \tag{10}$$

follows from the Wiener-Paley theorem. For each positive integer $n \geqslant 2$, the formula

$$\left(B^n(I - zB)^{-1}u, u\right) = z^{-1}\left((B^{n-1}(I - zB)^{-1}u, u) - (B^{n-1}u, u)\right)$$

is true. This premits to express $\left(B^n(I - zB)^{-1}u, u\right)$ in terms of $\left(B(I - zB)^{-1}u, u\right)$. Therefore, it follows from (10) that

$$\left(B^n(I - zB)^{-1}u, u\right) = i\int_0^{a_1} e^{izt}\,(U(t)B^n u, u)\;dt$$

and also

$$\left(B(I - zB)^{-1}B^p u, (B^*)^q u\right) = i\int_0^{a_1} e^{izt}\,(U(t)B^p u, (B^*)^q u)\;dt$$

for each positive integers p, q. With regard to (8), we conclude that the entire function exponential types $\left(B(I - zB)^{-1}f, g\right)$, $f, g \in \mathfrak{H}$ do not overestimate $a_1 < a$, which is contradicting to the hypothesis of the theorem.

So, if the semigroup $U(t) = \exp\{-iAt\}$ belongs to the C_0-class, under a suitable selection of the parameter $\alpha \in \mathbb{C}$ the operator L_α given by the formula (6) generates a C_0-semigroup $\exp\{iL_\alpha t\}$ and the function φ_α satisfies the equalities (7). In the paper [2] (Theorem 4.4) it is proved that under this conditions the operator B is similar to the operator J on space $L_2(0, a)$. □

Corollary. *Let A belong to the $\mathcal{N}_1$ class. If A^{-1} is similar to some dissipative operator, then it is similar to the integration operator J on the space $L_2(0,a)$, where a is the exponential type of the resolvent $(A-zI)^{-1}$.*

2. The earlier mentioned Theorem 4.4 [2] gives an additional information about similarity of the operator A^{-1} to the integration operator. Let us use it for strengthening of the formulation of Theorem 1.

It is proved in [2] that if for some operator $A \in \mathcal{N}_1$, the semigroup $\exp\{-iAt\}$, $t \geqslant 0$, belongs to the C_0-class, then the weight $w^2(x) := \left\|(I-xB)^{-1}u\right\|^2$, where $B = A^{-1}$, satisfies the A_2-Muckenhoupt condition [4] on the real axis. Therefore there exist outer in $\mathbb{C}_+(\mathbb{C}_-)$ functions $w_+(w_-)$ such that

$$|w_+(x+i0)|^2 \overset{\text{a.e.}}{=} |w_-(x-i0)|^2 \overset{\text{a.e.}}{=} w^2(x), \quad x \in \mathbb{R}. \tag{11}$$

We denote by $w_+(x-i0)\,(w_-(x-i0))$ the existing almost everywhere non-tangent limit values of the functions $w_+(z)$, $z \in \mathbb{C}_+(w_-(z), z \in \mathbb{C}_-)$. Let us consider the entire function

$$F(z) = \left((I-zB)^{-1}u, (I-\bar{z}B)^{-1}u\right),$$

which belongs to the Cartwright class [1, p. 324] if the integral

$$\int_{-\infty}^{+\infty} \frac{|F(x)|}{1+x^2} dx < \infty$$

converges. Since $F(x) > 0$, $x \in \mathbb{R}$, from the Akhiezer theorem [1, p. 567] we get the representation

$$\left((I-zB)^{-1}u, (I-\bar{z}B)^{-1}u\right) = E(z)E^*(z), \quad h(\pi/2; E) = a/2, \tag{12}$$

where $E^*(z) := \overline{E(z)}$ and roots of the function E lie in the lower half-plane of $\mathbb{C}_-$. Since the function E is determined to within an exponential factor $e^{i\gamma z}(\gamma \in \mathbb{R})$, the condition $h(\pi/2; E) = a/2$ has a normalization nature. We note that the function E also belongs to the Cartwright class. Because of the Krein theorem [5], E is a function of bounded type in $\mathbb{C}_+$ and also the function $\Phi(z) := e^{i\frac{a}{2}z}E(z)(z-z_0)^{-1}$, where z_0 is the zero of the function E, is such. Therefore [3, p. 32] Φ belongs to the Hardy class H_+^2 in the upper half-plane, and the function $e^{i\frac{a}{2}z}E(z)$ is outer in $\mathbb{C}_+$ [4]. Also it follows from (12) that

$$w^2(x) = \left|e^{i\frac{a}{2}x}E(x)\right|^2, \quad x \in \mathbb{R}.$$

Taking into account (11) we come to the formulas

$$w_+(z) = e^{i\frac{a}{2}z}E(z), \quad w_-(z) = e^{-i\frac{a}{2}z}E^*(z). \tag{13}$$

It is proved in the paper [2, p. 885] that the outer function w_- assumes a special integral representation. Taking into account the second formula (13) it can be written as

$$e^{-i\frac{a}{2}z}E^*(z) = z\int_0^\infty e^{-izt}y_E(t)dt, \quad z \in \mathbb{C}_-, \tag{14}$$

where $y_E \in L_2^{\text{loc}}(\mathbb{R}_+)$. Then we can strengthen the formuation of Theorem 1. *If for an operator $A \in \mathcal{N}_1$, the semigroup $U(t) = \exp\{-iAt\}$ belongs to the C_0-class then there is an isomorphism S from $\mathfrak{H}$ onto $L_2(0,a)$ such that*

$$A^{-1} = S^{-1}JS, \quad Su = y_E^a.$$

Here the vector u is contained in the representation (3) and y_E^a denotes the restriction of the function y_E to $[0,a]$. The function y_E is determined by the equality (14), [2, p. 915].

The function E that solves the factorization problem (12) assumes the multiplicative representation

$$E(z) = \text{v.p.} \prod (1 - z/\mu_k) e^{i\gamma z},$$

where $\mu_k \in \mathbb{C}_-$, $\gamma \leqslant 0$. So in the upper half-plane the equality

$$E^*(z)/E(z) = e^{-2i\gamma z}\mathcal{B}(z)$$

is correct. Here the Blaschke product with zeroes on the sequence $\{\bar{\mu}_k\}$ is denoted by $\mathcal{B}$. Note, the convergence of $\mathcal{B}$ follows from the fact that the function E belongs to the Cartwright class. In such a way, the inequality

$$|E^*(z)| < |E(z)|, \quad z \in \mathbb{C}_+,$$

holds and, therefore, the function E generates the de Branges space $\mathcal{H}(E)$ [3]. Let us recall that the de Branges space $\mathcal{H}(E)$ consists of entire functions f such that

$$\frac{f}{E} \in H^2_+, \quad \frac{f^*}{E} \in H^2_+ \tag{14a}$$

and is a Hilbert space with respect to the inner product

$$\langle f, g \rangle := \int_{-\infty}^{+\infty} f(x) g^*(x) |E(x)|^{-2} dx.$$

Comparing the formulas (13), (14a) we conclude that the function $f \in \mathcal{H}(E)$ is entire if and only if the function $f_a(z) := e^{i\frac{a}{2}z} f(z)$ satisfies the conditions

$$\frac{f_a(z)}{w_+(z)} \in H^2_+, \quad \frac{f_a(z) e^{-iaz}}{w_-(z)} \in H^2_-, \tag{15}$$

where H^2_- denotes the Hardy class in $\mathbb{C}_-$. Entire functions satisfying the conditions (15) assume the integral representations in form [2, p. 887]

$$f_a(z) = \int_0^a y_E(z,t)(Tf)(t)\, dt, \quad f \in \mathcal{H}(E), \tag{16}$$

where T is an isomorphism of space $\mathcal{H}(E)$ onto space $L_2(0,a)$. In this formula the kernel $y_E(z,t)$ solves the integral equation

$$y_E(z,t) - iz \int_0^t y_E(z,s)\, ds = y_E(t), \quad t \geqslant 0, \tag{17}$$

the right-hand side of which is determined by formula (14). Let us now consider an operator K on $\mathcal{H}(E)$ given by the formula

$$(Kf)(z) = z^{-1} \left(f(z) - e^{-i\frac{a}{2}z} f(0) \right), \quad f \in \mathcal{H}(E).$$

The operator K is compact and its spectrum consists of the unique point 0. Under the action of the isomorphism T, the operator K is transformed as

$$(Kf)_a(z) = e^{i\frac{a}{2}z}(Kf)(z) = z^{-1}\left(e^{i\frac{a}{2}z} f(z) - f(0)\right) = z^{-1}\left(f_a(z) - f_a(0)\right).$$

Now if we take into account the equality (16) then these calculations can be continued,

$$\begin{aligned} (Kf)_a(z) &= z^{-1}\Bigg(\int_0^a y_E(z,t)(Tf)(t)\, dt - \int_0^a y_E(0,t)(Tf)(t)\, dt \Bigg) \\ &= \int_0^a z^{-1}\left(y_E(z,t) - y_E(0,t)\right)(Tf)(t)\, dt. \end{aligned}$$

We note that by (17), the equality

$$z^{-1}\left(y_E(z,t)-y_E(0,t)\right)=i\int_0^t y_E(z,s)\,ds$$

holds and, therefore,

$$(Kf)_a(z)=\int_0^a y_E(z,s)(\tilde{J}Tf)(s)\,ds.$$

The operator $\tilde{J}$ is given by the formula

$$(\tilde{J}h)(t):=i\int_t^a h(s)\,ds,\quad h\in L_2(0,a).$$

In such a way we come to the equality $TK=\tilde{J}T$ and if we take into account that $\tilde{J}$ is unitarily equivalent to the operator J, we get that the operator K is similar to the integration operator J on $L_2(0,a)$.

Finally, we'll formulate an essential strengthening of Theorem 1, we mentioned at the beginning of the subsection. Let us recall that, in the following formulation, $A\in\mathcal{N}_1$, the vector u is contained in the representation (3), and $B:=A^{-1}$, the function E solves the factorization problem (12). Further, we denote by y_E^a the restriction of the function y_E to $[0,a]$, which is given by the equality (14).

Theorem 2. *Let an operator A belong to the class $\mathcal{N}_1$. Then the next conditions are equivalent:*

1) *the semigroup $U(t)=\exp\{-iAt\}$ belongs to the C_0-class;*
2) *the operator A^{-1} is similar to some dissipative operator;*
3) *the operator A^{-1} is similar to the operator $(Jh)(t)=i\int_0^t h(s)\,ds$, $h\in L_2(0,a)$, where a is the exponential type of the resolvent;*
4) *the operator A^{-1} is similar to the operator J in the space $L_2(0,a)$ and there is an isomorphism S that*
$$A^{-1}=S^{-1}JS,\quad Su=y_E^a;$$
5) *the operator A^{-1} is similar to the operator K given by the formula*
$$(Kf)(z)=z^{-1}\left(f(z)-e^{-i\frac{a}{2}z}f(0)\right),\quad f\in\mathcal{H}(E),$$
in the de Branges space $\mathcal{H}(E)$ and the weight $|E(x)|^2$ satisfies the condition (A_2) on $\mathbb{R}$.

Recall remind that taking into account the definition of $\mathcal{N}_1$, we see that the C_0-semigroup $U(t)$ is nilpotent.

3. Let us add to the formulated results the following remarks. If we start from an operator $A\in\mathcal{N}_1$ and $\mathrm{Im}A^{-1}$ is nuclear, then we can give another expression for the exponential type a. Indeed,, the operator L_α^{-1} (reffer to formula (6)) has a complete system of root subspaces [2] under the condition (7). If $\mathrm{Im}A^{-1}$ is a nuclear operator, then the trace formula is true [6],

$$\mathrm{SpIm}L_\alpha^{-1}=\sum\mathrm{Im}\lambda_k^{-1}. \tag{18}$$

Here the sequence $\{\lambda_k\}$ is the same as the function φ_α root set taking into account the multiplicity. If the equalities (7) hold then, since φ_α belongs to the Cartwright class, we have the representation

$$\varphi_\alpha(z)=e^{i\frac{a}{2}z}\mathrm{v.p.}\prod(1-z/\lambda_k).$$

Taking into account the definition of φ_α we obtain

$$i\alpha(u,u) = -\varphi'_\alpha(0) = -\varphi'_\alpha(0)/\varphi_\alpha(0) = -\left(\log\varphi_\alpha(z)\right)'|_{z=0} = -a/2 + \text{v.p.}\sum\lambda_k^{-1}.$$

Now comparing the formulas (6) and (18) we conclude that

$$\sum \mathrm{Im}\lambda_k^{-1} = \mathrm{SpIm}L_\alpha^{-1} = \mathrm{SpIm}A^{-1} + \mathrm{Im}(i\alpha(u,u)) = \mathrm{SpIm}A^{-1} - a/2 + \sum \mathrm{Im}\lambda_k^{-1},$$

i.e., $a = 2\mathrm{SpIm}A^{-1}$. So, the next statement holds.

Theorem 3. *Let $A \in \mathcal{N}_1$ and let the exponential type of resolvent $(A - zI)^{-1}$ be equal to a. If the semigroup $U(t) = \exp\{-iAt\}$ belongs to the C_0-class and the operator $\mathrm{Im}A^{-1}$ is nuclear, then $a = 2\mathrm{SpIm}A^{-1}$.*

Corollary. *Let $A \in \mathcal{N}_1$ and the operator $\mathrm{Im}A^{-1}$ be nuclear. If A^{-1} is similar to some dissipative operator, then $\mathrm{SpIm}A^{-1} > 0$.*

The second remark deals with operators with two-dimensional imaginary parts. Let us assume that $\sigma(A) = \varnothing$ and $\mathrm{Im}A^{-1}$ is a two-dimensional operator. Then its resolvent $(A - zI)^{-1}$ has a finite exponential type of growth [6]. Further, if the operator A^{-1} is non-dissipative, then the representation (3) with one-dimensional operator Q always takes place. In such a way, such operators belong to the $\mathcal{N}_1$ class and so all the previous theorems take place.

Theorem 4. *Let the spectrum of an operator A be an empty set, A^{-1} be non-dissipative and the operator $\mathrm{Im}A^{-1}$ be two-dimensional. If the semigroup $\exp\{-iAt\}$ belongs to the C_0-class, the operator A^{-1} is similar to the integration operator J in the space $L_2(0,a)$, $a = 2\mathrm{SpIm}A^{-1}$.*

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