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A NOTE ON EQUILIBRIUM GLAUBER AND KAWASAKI DYNAMICS FOR FERMION POINT PROCESSES

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Abstract. We construct two types of equilibrium dynamics of infinite particle systems in a locally compact Polish space X, for which certain fermion point processes are invariant. The Glauber dynamics is a birth-and-death process in X, while in the case of the Kawasaki dynamics interacting particles randomly hop over X. We establish conditions on generators of both dynamics under which corresponding conservative Markov processes exist.

1. Introduction

Let X be a locally compact Polish space. Let ν be a Radon measure on X and let K be a linear, Hermitian, locally trace class operator on $L^2(X, \nu)$ for which $\mathbf{0} \leq K \leq \mathbf{1}$. Then K is an integral operator and we denote by $K(\cdot, \cdot)$ the integral kernel of K.

Let $\Gamma = \Gamma_X$ denote the space of all locally finite subsets (configurations) in X. A fermion point process (also called determinantal point process) corresponding to K is a probability measure on Γ whose correlation functions are given by

$$k_\mu^{(n)}(x_1, \dots, x_n) = \det(K(x_i, x_j))_{i,j=1}^n. \tag{1.1}$$

Fermion point processes were introduced by Macchi [20] (see also Girard [8] and Menikoff [21]). These processes naturally arise in quantum mechanics, statistical mechanics, random matrix theory, and representation theory, see e.g. [4, 24, 25, 27] and the references therein.

In [28], Spohn investigated a diffusion dynamics on the configuration space $\Gamma_{\mathbb{R}}$ for which the fermion process corresponding to the Dyson (sine) kernel

$$K(x, y) = \sin(x - y)/(x - y)$$

is an invariant measure.

In the case where the operator K satisfies $K < \mathbf{1}$, Georgii and Yoo [7] (see also [30]) investigated Gibbsianness of fermion point processes. In particular, they proved that every fermion process with K as above possesses Papangelou (conditional) intensity.

Using Gibbsianness of fermion point processes, Yoo [29] constructed an equilibrium diffusion dynamics on the configuration space over $\mathbb{R}^d$, which has the fermion process as invariant measure. This Markov process is an analog of the gradient stochastic dynamics which has the standard Gibbs measure corresponding to a potential of pair interaction as invariant measure (see e.g. [1]).

On the other hand, in the case of a standard Gibbs measure, one considers further classes of equilibrium processes on the configuration space: the so-called Glauber and Kawasaki dynamics in continuum.

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The generator of the Glauber dynamics for a continuous particle system in $\mathbb{R}^d$, which is a birth-and-death process, is informally given by the formula

$$(H_{\mathrm{G}}F)(\gamma) = \sum_{x\in\gamma} d(x, \gamma\setminus x)(D_x^- F)(\gamma) + \int_{\mathbb{R}^d} b(x,\gamma)(D_x^+ F)(\gamma)\, dx, \tag{1.2}$$

where

$$(D_x^- F)(\gamma) = F(\gamma\setminus x) - F(\gamma), \quad (D_x^+ F)(\gamma) = F(\gamma\cup x) - F(\gamma). \tag{1.3}$$

Here and below, for simplicity of notations, we just write x instead of $\{x\}$. The coefficient $d(x, \gamma\setminus x)$ describes the rate at which the particle x of the configuration γ dies, while $b(x,\gamma)$ describes the rate at which, given the configuration γ, a new particle is born at x.

The Kawasaki dynamics of continuous particles is a process in which particles randomly hop over the space $\mathbb{R}^d$. The generator of such a process is then informally given by

$$(H_{\mathrm{K}}F)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x, y, \gamma\setminus x)(D_{xy}^{-+} F)(\gamma)\, dy, \tag{1.4}$$

where

$$(D_{xy}^{-+} F)(\gamma) = F(\gamma\setminus x\cup y) - F(\gamma) \tag{1.5}$$

and the coefficient $c(x, y, \gamma\setminus x)$ describes the rate at which the particle x of the configuration γ jumps to y.

Glauber and Kawasaki dynamics of continuous particle systems in infinite volume which have a standard Gibbs measure as symmetrizing measure were constructed in [15, 16], see also [3, 9, 10, 12, 22, 31]. For further studies on Glauber and Kawasaki dynamics, we refer to [5, 11, 14, 17].

The aim of the present note is to prove the existence of Glauber and Kawasaki dynamics of a continuous particle system which have a fermion point process as invariant measure. Our choice of dynamics seems to be natural for a fermion system. We recall that Shirai and Yoo [26] already constructed a Glauber dynamics on the lattice which has a fermion point process on the lattice as invariant measure.

Using the theory of Dirichlet forms (see e.g. [19]), we will construct conservative Markov processes on Γ with *cadlag* paths which have a fermion measure μ as symmetrizing, hence invariant measure. Furthermore, we will discuss the explicit form of the $L^2(\mu)$-generators of these process on a set of cylinder functions. These generators will have the form (1.2) in the case of Glauber dynamics, and (1.4) in the case of Kawasaki dynamics (with $\mathbb{R}^d$ replaced by a general topological space X). Since we significantly use the Papangelou intensity of the fermion point process, our study here is restricted by the assumption that $K < \mathbf{1}$.

Throughout the paper, we formulate our results for both dynamics, and give the proofs only in the case of the Kawasaki dynamics. The reason is that the proofs in the Glauber case are quite similar to, and simpler than the corresponding proofs for the Kawasaki dynamics.

Finally, let us mention some open problems, which will be topics of our further research:

(1) Construction of Glauber and Kawasaki dynamics for fermion point processes in the case where 1 belongs to the spectrum of the operator K (which is, e.g., the case when K has Dyson kernel).
(2) Finding a core for the generator of the dynamics (compare with [15]).
(3) Deciding whether the generator of the Glauber dynamics for a fermion point process has a spectral gap. (Recall that, in the case of a standard Gibbs measure

with a positive potential of pair interaction, there is a Glauber dynamics whose generator has a spectral gap [15]).

(4) Study of different types of scalings of Glauber and Kawasaki dynamics, in particular, suffusion approximation for Kawasaki dynamics (compare with [5, 11, 14]).

2. Fermion (determinantal) point processes

Let X be a locally compact, second countable Hausdorff topological space. Recall that such a space is known to be Polish. We denote by $\mathcal{B}(X)$ the Borel σ-algebra in X, and by $\mathcal{B}_0(X)$ the collection of all sets from $\mathcal{B}(X)$ which are relatively compact. We fix a Radon, non-atomic measure ν on $(X, \mathcal{B}(X))$.

The configuration space $\Gamma := \Gamma_X$ over X is defined as the set of all subsets of X which are locally finite

$$\Gamma := \big\{\gamma \subset X : |\gamma_\Lambda| < \infty \text{ for each } \Lambda \in \mathcal{B}_0(X)\big\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(X)$, where ε_x is the Dirac measure with mass at x, $\sum_{x\in\varnothing} \varepsilon_x$:=zero measure, and $\mathcal{M}(X)$ stands for the set of all positive Radon measures on $\mathcal{B}(X)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps

$$\Gamma \ni \gamma \mapsto \langle f, \gamma\rangle := \int_X f(x)\,\gamma(dx) = \sum_{x\in\gamma} f(x), \quad f \in C_0(X),$$

are continuous. Here, $C_0(X)$ is the space of all continuous, real-valued functions on X with compact support. We will denote the Borel σ-algebra on Γ by $\mathcal{B}(\Gamma)$. A point process μ is a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$.

A point process μ is said to have correlation functions if, for any $n \in \mathbb{N}$, there exists a non-negative, measurable, symmetric function $k_\mu^{(n)}$ on X^n such that, for any measurable, symmetric function $f^{(n)} : X^n \to [0, +\infty]$

$$\begin{aligned}\int_\Gamma \sum_{\{x_1,\dots,x_n\}\subset\gamma} & f^{(n)}(x_1,\dots,x_n)\,\mu(d\gamma)\\ &= \frac{1}{n!}\int_{X^n} f^{(n)}(x_1,\dots,x_n) k_\mu^{(n)}(x_1,\dots,x_n)\,\nu(dx_1)\cdots\nu(dx_n).\end{aligned}$$

Let K be a linear, bounded, Hermitian operator on the space $L^2(X, \nu)$ (real or complex) which satisfies the following assumptions:

(1) K is locally of trace class, i.e.,

$$\mathrm{Tr}(P_\Lambda K P_\Lambda) < \infty \quad \text{for all} \quad \Lambda \in \mathcal{B}_0(X),$$

where P_Λ denotes the operator of multiplication by the indicator function χ_Λ of the set Λ.

(2) We have $\mathbf{0} \le K \le \mathbf{1}$.

Under the above assumptions, K is an integral operator, and its kernel can be chosen as

$$K(x,y) = \int_X K_1(x,z) K_1(z,y)\,\nu(dz),$$

where $K_1(\cdot,\cdot)$ is any version of the kernel of the integral operator $\sqrt{K}$, [18] (see also [7, Lemma A.4]).

A point process μ having correlation functions

$$k_\mu^{(n)}(x_1,\dots,x_n) = \det(K(x_i,x_j))_{i,j=1}^n.$$

is called the fermion (or determinantal) point process corresponding to the operator K. Under the above assumptions on K, such a point process μ exists and is unique, see e.g. [18, 20, 24, 27].

Using the definition of a fermion process, it can be easily checked that μ has all local moments finite, i.e.,

$$\int_\Gamma \langle f,\gamma\rangle^n\,\mu(d\gamma)<\infty,\quad f\in C_0(X),\quad f\ge 0,\quad n\in\mathbb{N}.$$

In what follows, we will always assume that the operator K is strictly less than $\mathbf{1}$, i.e., 1 does not belong to the spectrum of K. Then, as has been shown in [7], the fermion process μ has Papangelou (conditional) intensity. That is, there exists a measurable function $r: X\times\Gamma\to[0,+\infty]$ such that

$$\int_\Gamma \mu(d\gamma)\int_X \gamma(dx)F(x,\gamma)=\int_\Gamma\mu(d\gamma)\int_X \nu(dx)\,r(x,\gamma)F(x,\gamma\cup x) \tag{2.1}$$

for any measurable function $F: X\times\Gamma\to[0,+\infty]$.

Remark 2.1. Let us briefly explain the construction of the Papangelou intensity $r(x,\gamma)$, following [7].

For each $\Lambda\in\mathcal{B}_0(X)$, consider $K_\Lambda:=P_\Lambda K P_\Lambda$ as an operator in $L^2(\Lambda,\nu)$ and let $J_{[\Lambda]}:=K_\Lambda(\mathbf{1}-K_\Lambda)^{-1}$. Denote by $J_{[\Lambda]}(\cdot,\cdot)$ the kernel of the operator $J_{[\Lambda]}$ (chosen analogously to the kernel of K). For any $\gamma\in\Gamma$, set

$$\det J_{[\Lambda]}(\gamma_\Lambda,\gamma_\Lambda):=\det\left[J_{[\Lambda]}(x_i,x_j)\right]_{i,j=1}^m,$$

with $\gamma_\Lambda=\{x_1,\dots,x_m\}$ being any numeration of points of γ_Λ (in the case $\gamma_\Lambda=\varnothing$, set $\det J_{[\Lambda]}(\varnothing,\varnothing):=0$). Now, for any $x\in\Lambda$ and $\gamma\in\Gamma$, set

$$r_\Lambda(x,\gamma_\Lambda):=\frac{\det J_{[\Lambda]}(x\cup\gamma_\Lambda,x\cup\gamma_\Lambda)}{\det J_{[\Lambda]}(\gamma_\Lambda,\gamma_\Lambda)}\,,$$

where the expression on the right hand side is assumed to be zero if $\det J_{[\Lambda]}(\gamma_\Lambda,\gamma_\Lambda)=0$.

Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathcal{B}_0(X)$ that increases to X. Then $r(x,\gamma)$ is a $\nu\otimes\mu$-a.e. limit of $r_{\Lambda_n}(x,\gamma_{\Lambda_n})$ as $n\to\infty$.

Set $J:=K(\mathbf{1}-K)^{-1}$. The operator J is integral and we choose its kernel $J(\cdot,\cdot)$ analogously to choosing the kernel of K. Note that

$$\operatorname{Tr}(P_\Lambda J P_\Lambda)=\int_\Lambda J(x,x)\,\nu(dx)<\infty.$$

The following proposition is a direct corollary of [7, Theorem 3.7 and Lemma A.1].

Proposition 2.1. *We have, for $\nu\otimes\mu$-a.e. $(x,\gamma)\in X\times\Gamma$:*

$$r(x,\gamma)\le J(x,x).$$

3. Equilibrium dynamics

In what follows, we will consider a fermion point process μ corresponding to an operator K as defined in Section 2. We introduce the set $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ of all functions of the form

$$\Gamma\ni\gamma\mapsto F(\gamma)=g(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle),$$

where $N\in\mathbb{N}$, $\varphi_1,\dots,\varphi_N\in C_0(X)$ and $g\in C_{\mathrm{b}}(\mathbb{R}^N)$. Here, $C_{\mathrm{b}}(\mathbb{R}^N)$ denotes the set of all continuous, bounded functions on $\mathbb{R}^N$.

For a function $F:\Gamma\to\mathbb{R}$, $\gamma\in\Gamma$, $x,y\in X$, we introduce the notations $(D_x^-F)(\gamma)$, $(D_x^+F)(\gamma)$, and $(D_{xy}^{-+}F)(\gamma)$ by (1.3) and (1.5), respectively. We consider measurable mappings

$$X\times\Gamma\ni(x,\gamma)\mapsto d(x,\gamma)\in[0,\infty),$$

$$X\times X\times\Gamma\ni(x,y,\gamma)\mapsto c(x,y,\gamma)\in[0,\infty).$$

Assume that

$$c(x,y,\gamma)=c(x,y,\gamma)\chi_{\{r(x,\gamma)>0,\,r(y,\gamma)>0\}},\quad x,y\in X,\quad \gamma\in\Gamma. \tag{3.1}$$

Remark 3.1. As we will see below, the coefficient $c(x,y,\gamma\setminus x)$ describes the rate of the jump of particle $x\in\gamma$ to y. For each $\gamma\in\Gamma$ and $x\in\mathbb{R}^d\setminus\gamma$, we interpret $r(x,\gamma)$ as $\exp[-E(x,\gamma)]$, where $E(x,\gamma)$ is the relative energy of interaction between configuration γ and particle x. Hence, if $r(y,\gamma\setminus x)=0$, then the relative energy of interaction between the configuration $\gamma\setminus x$ and particle y is $+\infty$. Hence, it is intuitively clear that the particle x cannot jump to y, i.e., $c(x,y,\gamma\setminus x)$ should be equal to zero. A symmetry reason also implies that we should have $c(x,y,\gamma\setminus x)=0$ if $r(x,\gamma\setminus x)=0$, i.e., if the relative energy of interaction between $x\in\gamma$ and the rest of configuration is ∞.

Further, we assume that, for each $\Lambda\in\mathcal{B}_0(X)$,

$$\int_\Gamma\mu(d\gamma)\int_\Lambda\gamma(dx)\,d(x,\gamma\setminus x)<\infty, \tag{3.2}$$

$$\int_\Gamma\mu(d\gamma)\int_X\gamma(dx)\int_X\nu(dy)c(x,y,\gamma\setminus x)(\chi_\Lambda(x)+\chi_\Lambda(y))<\infty. \tag{3.3}$$

We define bilinear forms

$$\mathcal{E}_{\mathrm{G}}(F,G):=\int_\Gamma\mu(d\gamma)\int_X\gamma(dx)\,d(x,\gamma\setminus x)(D_x^-F)(\gamma)(D_x^-G)(\gamma), \tag{3.4}$$

$$\mathcal{E}_{\mathrm{K}}(F,G):=\int_\Gamma\mu(d\gamma)\int_X\gamma(dx)\int_X\nu(dy)\,c(x,y,\gamma\setminus x)(D_{xy}^{-+}F)(\gamma)(D_{xy}^{-+}G)(\gamma), \tag{3.5}$$

where $F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$.

We note that, for any $F\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$, there exist $\Lambda\in\mathcal{B}_0(X)$ and $C>0$ such that

$$|(D_x^-F)(\gamma)|\le C\chi_\Lambda(x),\quad |(D_{xy}^{-+}F)(\gamma)|\le C(\chi_\Lambda(x)+\chi_\Lambda(y)),\quad \gamma\in\Gamma,\quad x,y\in X.$$

Therefore, by assumptions (3.2), (3.3) the right-hand sides of formulas (3.4) and (3.5) are well-defined and finite.

Using (2.1) and (3.1), we have, for any $F\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$:

$$\begin{aligned}\mathcal{E}_{\mathrm{K}}(F)&=\int_\Gamma\mu(d\gamma)\int_X\nu(dx)\int_X\nu(dy)\,r(x,\gamma)c(x,y,\gamma)\\&\quad\times\chi_{\{r(y,\gamma)>0)\}}\frac{r(y,\gamma)}{r(y,\gamma)}(F(\gamma\cup y)-F(\gamma\cup x))^2\\&=\int_\Gamma\mu(d\gamma)\int_X\nu(dx)\int_X\gamma(dy)\,r(x,\gamma\setminus y)c(x,y,\gamma\setminus y)\\&\quad\times\chi_{\{r(y,\gamma\setminus y)>0\}}\frac{1}{r(y,\gamma\setminus y)}(D_{yx}^{-+}F)^2(\gamma).\end{aligned}$$

Here, we used the notation

$$\mathcal{E}_{\mathrm{K}}(F):=\mathcal{E}_{\mathrm{K}}(F,F).$$

Therefore, for any $F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$,

$$\mathcal{E}_{\mathrm{K}}(F,G)=\int_\Gamma\mu(d\gamma)\int_X\gamma(dx)\int_X\nu(dy)\tilde c(x,y,\gamma\setminus x)(D_{xy}^{-+}F)(\gamma)(D_{xy}^{-+}G)(\gamma),$$

where

$$\tilde c(x,y,\gamma):=\frac12\left(c(x,y,\gamma)+c(y,x,\gamma)\chi_{\{r(x,\gamma)>0\}}\frac{r(y,\gamma)}{r(x,\gamma)}\right).$$

Note that, by (3.1), we have $\tilde{\tilde{c}}(x,y,\gamma)=\tilde{c}(x,y,\gamma)$. Therefore, without loss of generality, in what follows we will assume that $\tilde{c}(x,y,\gamma)=c(x,y,\gamma)$, i.e.,

$$r(x,\gamma)c(x,y,\gamma)=r(y,\gamma)c(y,x,\gamma). \tag{3.6}$$

Lemma 3.1. *We have $\mathcal{E}_\sharp(F,G)=0$ for all $F,G\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $F=0$ μ-a.e., $\sharp=\mathrm{G},\mathrm{K}$.*

Proof. It suffices to show that, for $F\in\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $F=0$ μ-a.e., we have $(D^{-+}_{x,y}F)(\gamma)=0$ $\tilde{\mu}$-a.e. Here, $\tilde{\mu}$ is the measure on $X\times X\times\Gamma$ defined by

$$\tilde{\mu}(dx,dy,d\gamma):=c(x,y,\gamma\setminus x)\gamma(dx)\,\nu(dy)\,\mu(d\gamma). \tag{3.7}$$

For any F as above, we evidently have that $F(\gamma)=0$ $\tilde{\mu}$-a.e. Next, by (2.1) and (3.1)

$$\begin{aligned}
&\int_\Gamma\mu(d\gamma)\int_\Lambda\gamma(dx)\int_\Lambda\nu(dy)|F(\gamma\setminus x\cup y)|c(x,y,\gamma\setminus x)\\
&\quad=\int_\Gamma\mu(d\gamma)\int_\Lambda\nu(dx)\int_\Lambda\nu(dy)r(x,\gamma)|F(\gamma\cup y)|c(x,y,\gamma)\chi_{\{r(y,\gamma)>0\}}\frac{r(y,\gamma)}{r(y,\gamma)}\\
&\quad=\int_\Gamma\mu(d\gamma)\int_\Lambda\nu(dx)\int_\Lambda\gamma(dy)|F(\gamma)|c(x,y,\gamma\setminus y)\,\frac{r(x,\gamma\setminus y)}{r(y,\gamma\setminus y)}\,\chi_{\{r(y,\gamma\setminus y)>0\}}\\
&\quad=\int_\Gamma\mu(d\gamma)\int_\Lambda\nu(dx)\int_\Lambda\gamma(dy)|F(\gamma)|c(x,y,\gamma\setminus y)\frac{r(x,\gamma\setminus y)}{r(y,\gamma\setminus y)}.
\end{aligned}\tag{3.8}$$

Since F is bounded, by (3.3) the integral in (3.8) is finite. Therefore,

$$|F(\gamma)|\,\frac{r(x,\gamma\setminus y)}{r(y,\gamma\setminus y)}<\infty\quad\text{for } \tilde{\mu}\text{-a.a.}\quad(x,y,\gamma)\in X\times X\times\Gamma. \tag{3.9}$$

Because $F=0$ $\tilde{\mu}$-a.e., by (3.8) and (3.9), $F(\gamma\setminus x\cup y)=0$ $\tilde{\mu}$-a.e. □

By Lemma 3.1, the bilinear forms $(\mathcal{E}_{\mathrm{G}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ and $(\mathcal{E}_{\mathrm{K}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ are well defined on $L^2(\Gamma,\mu)$.

Lemma 3.2. 1) *The bilinear form $(\mathcal{E}_{\mathrm{G}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ is closable on $L^2(\Gamma,\mu)$ and its closure will be denoted by $(\mathcal{E}_{\mathrm{G}},D(\mathcal{E}_{\mathrm{G}}))$.*

2) *Assume that, for some $u\in\mathbb{R}$,*

$$\int_\Lambda\nu(dx)\int_\Lambda\gamma(dy)r(x,\gamma\setminus y)r(y,\gamma\setminus y)^u\chi_{\{r(y,\gamma\setminus y)>0\}}c(x,y,\gamma\setminus y)\in L^2(\Gamma,\mu) \tag{3.10}$$

for all $\Lambda\in\mathcal{B}_0(X)$. Then the bilinear form $(\mathcal{E}_{\mathrm{K}},\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ is closable on $L^2(\Gamma,\mu)$ and its closure will be denoted by $(\mathcal{E}_{\mathrm{K}},D(\mathcal{E}_{\mathrm{K}}))$.

Proof. Let $(F_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$ such that $\|F_n\|_{L^2(\Gamma,\mu)}\to0$ as $n\to\infty$ and

$$\mathcal{E}_{\mathrm{K}}(F_n-F_k)\to0\quad\text{as}\quad n,k\to\infty. \tag{3.11}$$

To prove the closability of $\mathcal{E}_{\mathrm{K}}$, it suffices to show that there exists a subsequence $\{F_{n_k}\}_{k=1}^\infty$ such that $\mathcal{E}_{\mathrm{K}}(F_{n_k})\to0$ as $k\to\infty$.

Since $\|F_n\|_{L^2(\Gamma,\mu)}\to0$ as $n\to\infty$, there exists a subsequence $(F_n^{(1)})_{n=1}^\infty$ of $(F_n)_{n=1}^\infty$ such that $F_n^{(1)}(\gamma)\to0$ for $\tilde{\mu}$-a.a. $(x,y,\gamma)\in X\times X\times\Gamma$. Next, by (3.10), we have, for any $\Lambda\in\mathcal{B}_0(X)$,

$$\begin{aligned}
&\int_\Gamma\mu(d\gamma)\int_\Lambda\gamma(dx)\int_\Lambda\nu(dy)c(x,y,\gamma\setminus x)r(y,\gamma\setminus x)^{u+1}\chi_{\{r(y,\gamma\setminus x)>0\}}|F_n^{(1)}(\gamma\setminus x\cup y)|\\
&\quad=\int_\Gamma\mu(d\gamma)\int_\Lambda\nu(dx)\int_\Lambda\nu(dy)r(x,\gamma)c(x,y,\gamma)r(y,\gamma)^{u+1}\chi_{\{r(y,\gamma)>0\}}|F_n^{(1)}(\gamma\cup y)|
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) r(x, \gamma \setminus y) r(y, \gamma \setminus y)^u \\
&\quad \times \chi_{\{r(y,\gamma\setminus y)>0\}} c(x, y, \gamma \setminus y) |F_n^{(1)}(\gamma)| \\
&\le \left(\int_\Gamma \mu(d\gamma) |F_n^{(1)}(\gamma)|^2 \right)^{1/2} \left(\int_\Gamma \mu(d\gamma) \left(\int_\Lambda \nu(dx) \int_\Lambda \gamma(dy) r(x, \gamma \setminus y) \right.\right. \\
&\quad \left.\left. \times r(y, \gamma \setminus y)^u \chi_{\{r(y,\gamma\setminus y)>0\}} c(x, y, \gamma \setminus y) \right)^2 \right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Therefore, there exists a subsequence $(F_n^{(2)})_{n=1}^\infty$ of $(F_n^{(1)})_{n=1}^\infty$ such that $F_n^{(2)}(\gamma \setminus x \cup y) \to 0$ as $n \to \infty$ for

$$
c(x, y, \gamma \setminus x) r(y, \gamma \setminus x)^u \chi_{\{r(y,\gamma\setminus x)>0\}} \gamma(dx) \nu(dy) \mu(d\gamma)\text{-a.e.} \quad (x, y, \gamma) \in X \times X \times \Gamma.
$$

By (3.1), the latter measure is equivalent to $\tilde{\mu}$, and therefore

$$
(D^{-+}_{x,y} F_n^{(2)})(\gamma) \to 0 \quad \text{for } \tilde{\mu}\text{-a.e.} \quad (x, y, \gamma) \in X \times X \times \Gamma. \tag{3.12}
$$

Now, by (3.12) and Fatou's lemma

$$
\begin{aligned}
\mathcal{E}_{\mathrm{K}}(F_n^{(2)}) &= \int (D^{-+}_{xy} F_n^{(2)})(\gamma)^2 \, \tilde{\mu}(dx, dy, d\gamma) \\
&= \int \left((D^{-+}_{xy} F_n^{(2)})(\gamma) - \lim_{m\to\infty} (D^{-+}_{xy} F_m^{(2)})(\gamma) \right)^2 \tilde{\mu}(dx, dy, d\gamma) \\
&\le \liminf_{m\to\infty} \int ((D^{-+}_{xy} F_n^{(2)})(\gamma) - (D^{-+}_{xy} F_m^{(2)})(\gamma))^2 \, \tilde{\mu}(dx, dy, d\gamma) \\
&= \liminf_{m\to\infty} \mathcal{E}_{\mathrm{K}}(F_n^{(2)} - F_m^{(2)}),
\end{aligned}
$$

which by (3.11) can be made arbitrarily small for n large enough. □

Before proceeding with our study of the bilinear forms $(\mathcal{E}_{\mathrm{G}}, D(\mathcal{E}_{\mathrm{G}}))$ and $(\mathcal{E}_{\mathrm{K}}, D(\mathcal{E}_{\mathrm{K}}))$, let us briefly recall some definitions from the theory of Dirichlet forms, see [6, 19] for details and further generality.

Let E be a Polish space, let $\mathcal{B}(E)$ be the Borel σ-algebra on E, and let m be a Radon measure on $(E, \mathcal{B}(E))$. Let $(\mathcal{E}, D(\mathcal{E}))$ be a closed, symmetric, non-negative, bilinear form on $L^2(E, m)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is called a Dirichlet form if, for each $F \in D(\mathcal{E})$, we have $(F \vee 0) \wedge 1 \in D(\mathcal{E})$ and

$$
\mathcal{E}((F \vee 0) \wedge 1) \le \mathcal{E}(F).
$$

Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(E, m)$. For a subset $A \subset E$, we define

$$
D(\mathcal{E})_A := \{F \in D(\mathcal{E}) \mid F = 0 \text{ on } E \setminus A\}.
$$

A sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$-nest if

$$
\bigcup_{n\in\mathbb{N}} D(\mathcal{E})_{A_n}
$$

is dense in $D(\mathcal{E})$ with respect to the norm

$$
\|\cdot\|_{D(\mathcal{E})} := \big(\mathcal{E}(\cdot) + \|\cdot\|_{L^2(E,m)}\big)^{1/2}.
$$

A subset $N \subset E$ is called $\mathcal{E}$-exceptional if

$$
N \subset \bigcap_{n\in\mathbb{N}} A_n^c
$$

for some $\mathcal{E}$-nest $(A_n)_{n\in\mathbb{N}}$. Note that every Borel $\mathcal{E}$-exceptional set has m measure zero. A property of points in E holds $\mathcal{E}$-quasi-everywhere (abbreviated $\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set.

Assume that there exists a subset $\mathcal{F}$ of $D(\mathcal{E})\cap C(E)$ which is dense in $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_{D(\mathcal{E})}$ and such that the functions from $\mathcal{F}$ separate points of E. Then, the Dirichlet form $(\mathcal{E},D(\mathcal{E}))$ is called quasi-regular if there exists an $\mathcal{E}$-nest $(A_n)_{n\in\mathbb{N}}$ consisting of compact sets in E.

Now, completely analogously to [16, Lemma 3.3], we get

Lemma 3.3. *$(\mathcal{E}_{\mathrm{G}},D(\mathcal{E}_{\mathrm{G}}))$ and $(\mathcal{E}_{\mathrm{K}},D(\mathcal{E}_{\mathrm{K}}))$ are Dirichlet forms on $L^2(\Gamma,\mu)$.*

Remark 3.2. We note that the central idea of the proof of Lemma 3.3 is that, for each function $F:\Gamma\to\mathbb{R}$ and any $\gamma\in\Gamma$, $x\in\gamma$, and $y\in\mathbb{R}^d\setminus\gamma$,

$$|(D_x^-(F\vee 0)\wedge 1))(\gamma)|\le|(D_x^-F)(\gamma)|,\quad |(D_{xy}^{-+}(F\vee 0)\wedge 1))(\gamma)|\le|(D_{xy}^{-+}F)(\gamma)|.$$

We now need the bigger space $\ddot\Gamma$ consisting of all $\mathbb{Z}_+\cup\{\infty\}$-valued Radon measures on X, which is Polish, see e.g. [13]. Since $\Gamma\subset\ddot\Gamma$ and $\mathcal{B}(\ddot\Gamma)\cap\Gamma=\mathcal{B}(\Gamma)$, we can consider μ as a measure on $(\ddot\Gamma,\mathcal{B}(\ddot\Gamma))$ and correspondingly $(\mathcal{E}_{\mathrm{G}},D(\mathcal{E}_{\mathrm{G}}))$ and $(\mathcal{E}_{\mathrm{K}},D(\mathcal{E}_{\mathrm{K}}))$ as bilinear forms on $L^2(\ddot\Gamma,\mu)$.

Lemma 3.4. *Under the assumption of Lemma 3.2, $(\mathcal{E}_{\mathrm{G}},D(\mathcal{E}_{\mathrm{G}}))$ and $(\mathcal{E}_{\mathrm{K}},D(\mathcal{E}_{\mathrm{K}}))$ are quasi-regular Dirichlet forms on $L^2(\ddot\Gamma,\mu)$.*

The proof of Lemma 3.4 is analogous to that of [16, Lemmas 3.3 and 3.4], so we omit it.

Lemma 3.5. *The set $\ddot\Gamma\setminus\Gamma$ is exceptional for both $(\mathcal{E}_{\mathrm{G}},D(\mathcal{E}_{\mathrm{G}}))$ and $(\mathcal{E}_{\mathrm{K}},D(\mathcal{E}_{\mathrm{K}}))$.*

Proof. We fix any metric on X which generates the topology on X. For any $a\in X$ and $r>0$, we denote by $B(a,r)$ the closed ball in X, with center at x and radius r. It suffices to prove the lemma locally, i.e., to show that, for any fixed $a\in X$, there exists $r>0$ such that

$$N_a:=\{\gamma\in\ddot\Gamma:\sup_{x\in B(a,r)}\gamma(\{x\})\ge 2\}$$

is $\mathcal{E}_{\mathrm{K}}$-exceptional. So, we fix any $a\in X$ and choose $r>0$ so that $B(a,2r)\in\mathcal{B}_0(X)$.

By [23, Lemma 1], we need to prove that there exists a sequence $u_n\in D(\mathcal{E}_{\mathrm{K}})$, $n\in\mathbb{N}$, such that each u_n is a continuous function on $\ddot\Gamma$, $u_n\to\chi_{N_a}$ pointwise as $n\to\infty$, and $\sup_{n\in\mathbb{N}}\mathcal{E}_{\mathrm{K}}(u_n)<\infty$.

Fix any $n\in\mathbb{N}$ such that

$$2/n<r. \tag{3.13}$$

Let

$$\{B(a_k,1/n)\mid k=1,\dots,K_n\},$$

with $a_k\in B(a,r)$, $k=1,\dots,K_n$, be a finite covering of $B(a,r)$. Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(u):=(1-|u|)\vee 0$.

For each $k=1,\dots,K_n$, we define a continuous function $f_k^{(n)}:X\to\mathbb{R}$ by

$$f_k^{(n)}(x):=f\big(n\,\mathrm{dist}(x,B(a_k,1/n))\big),\quad x\in X.$$

Here, $\mathrm{dist}(x,B)$ denotes the distance from a point $x\in X$ to a set $B\subset X$. We evidently have:

$$\chi_{B(a_k,1/n)}\le f_k^{(n)}\le\chi_{B(a_k,2/n)}. \tag{3.14}$$

Let $\psi\in C_{\mathrm{b}}^1(\mathbb{R})$ be such that $\chi_{[2,\infty)}\le\psi\le\chi_{[1,\infty)}$ and

$$0\le\psi'\le 2\,\chi_{(1,\infty)}. \tag{3.15}$$

We define a continuous function

$$\ddot{\Gamma} \ni \gamma \mapsto u_n(\gamma) := \psi\left(\sup_{k\in\{1,\dots,K_n\}} \langle f_k^{(n)}, \gamma\rangle\right),$$

whose restriction to Γ belongs to $\mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$. Evidently, $u_n \to \chi_{N_a}$ pointwise as $n\to\infty$.

By (3.13), (3.14), (3.15), and the mean value theorem, we have, for each $\gamma\in\Gamma$, $x\in\gamma$, $y\in X\setminus\gamma$,

$$\begin{aligned}
(D_{xy}^{-+}u_n)^2(\gamma) &\le 4\left(\sup_{k\in\{1,\dots,K_n\}} \langle f_k^{(n)}, \gamma\setminus x\cup y\rangle - \sup_{k\in\{1,\dots,K_n\}} \langle f_k^{(n)}, \gamma\rangle\right)^2\\
&\le 4 \sup_{k\in\{1,\dots,K_n\}} |\langle f_k^{(n)}, \gamma\setminus x\cup y\rangle - \langle f_k^{(n)}, \gamma\rangle|^2\\
&\le 8\left(\sup_{k\in\{1,\dots,K_n\}} f_k^{(n)}(x)^2 + \sup_{k\in\{1,\dots,K_n\}} f_k^{(n)}(y)^2\right)\\
&\le 8\left(\sup_{k\in\{1,\dots,K_n\}} \chi_{B(a_k,2/n)}(x) + \sup_{k\in\{1,\dots,K_n\}} \chi_{B(a_k,2/n)}(y)\right)\\
&\le 8(\chi_{B(a,2r)}(x) + \chi_{B(a,2r)}(y)).
\end{aligned}$$

Hence, by (3.3),

$$\sup_n \mathcal{E}_{\mathrm{K}}(u_n) < \infty,$$

which implies the lemma. □

Let us briefly recall some notions appearing in Theorem 3.1 below, for further details see e.g. [6, 19]. Let

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\ge 0}, (\mathbf{\Theta}_t)_{t\ge 0}, (\mathbf{X}(t))_{t\ge 0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma})$$

be a Markov process Γ. Then $\mathbf{M}$ is called a conservative Hunt process if it satisfies the following additional properties:

- (*Normal property*) $\mathbf{P}_\gamma(\mathbf{X}(0)=\gamma)=1$ for all $\gamma\in\Gamma$.
- (*Infinite life-time*) $\mathbf{P}_\gamma(\mathbf{X}(t)\in\Gamma$ for all $t>0)=1$ for all $\gamma\in\Gamma$.
- (*Right continuity*) For each $\omega\in\mathbf{\Omega}$, $[0,\infty)\ni t\mapsto \mathbf{X}(t,\omega)\in\Gamma$ is right continuous.
- (*Strong Markov property*) For every $(\mathbf{F}_t)_{t\ge0}$-stopping time τ and every probability measure m on $(\Gamma,\mathcal{B}(\Gamma))$,
  $$\mathbf{P}_m(\mathbf{X}(\tau+t)\in A \mid \mathbf{F}_\tau) = \mathbf{P}_{\mathbf{X}(\tau)}(\mathbf{X}(t)\in A) \quad \mathbf{P}_m\text{-a.s.}$$
  for all $A\in\mathcal{B}(\Gamma)$ and $t\ge 0$. Here, $\mathbf{P}_m(\cdot) := \int_\Gamma m(d\gamma)\mathbf{P}_\gamma(\cdot)$.
- (*Left limits*) For every probability measure m on $(\Gamma,\mathcal{B}(\Gamma))$, $\lim_{s\uparrow t}\mathbf{X}(s)$ exists in Γ for all $t>0$ $\mathbf{P}_m$-a.s.
- (*quasi-left continuity*) For every probability measure m on $(\Gamma,\mathcal{B}(\Gamma))$, if τ, τ_n, $n\in\mathbb{N}$, are $(\mathbf{F}_t^{\mathbf{P}_m})_{t\ge0}$-stopping times such that $\tau_n\uparrow\tau$, then $\mathbf{X}(\tau_n)\to X(\tau)$ as $n\to\infty$ $\mathbf{P}_m$-a.s. Here, $\mathbf{F}_t^{\mathbf{P}_m}$ denotes the completion of the σ-algebra $\mathbf{F}_t$ with respect to the probability measure $\mathbf{P}_m$

Let $F : A\to\mathbb{R}$, $A\subset\Gamma$, and let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(\Gamma,\mu)$. Then the function F is called $\mathcal{E}$-quasi-continuous if there exists an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of Γ which form an $\mathcal{E}$-nest, $\bigcup_{n\in\mathbb{N}} A_n\subset A$, and for each $n\in\mathbb{N}$ the restriction of F to A_n is a continuous function.

Let $\mathbf{M}$ and $\mathbf{M}'$ be two Hunt processes with state space Γ, and denote by $(p_t)_{t\ge0}$, $(p'_t)_{t\ge0}$ their transition semigroups. Then $\mathbf{M}$ and $\mathbf{M}'$ are called μ-equivalent if there exists a set $S\in\mathcal{B}(\Gamma)$ such that

- $\mu(\Gamma \setminus S) = 0$;
- S is both $\mathbf{M}$-invariant and $\mathbf{M}'$-invariant;
- $(p_t F)(\gamma) = (p'_t F)(\gamma)$ for each $t > 0$, each bounded measurable function $F : \Gamma \to \mathbb{R}$, and each $\gamma \in S$.

Recall that S being $\mathbf{M}$-invariant means that there exists $\mathbf{\Omega}_{\Gamma\setminus S} \in \mathbf{F}$ such that

$$\mathbf{\Omega}_{\Gamma\setminus S} \supset \{\overline{\mathbf{X}_0^t} \cap (\Gamma \setminus S) \neq \varnothing \text{ for some } t \geq 0\}$$

and $\mathbf{P}_\gamma(\mathbf{\Omega}_{\Gamma\setminus S}) = 0$ for all $\gamma \in S$. Here, $\overline{\mathbf{X}_0^t}(\omega)$ is the closure of $\{\mathbf{X}(s,\omega) \mid s \in [0,t]\}$ in Γ.

We now ready to formulate the main result of this paper.

Theorem 3.1. *Let* (3.2), *respectively* (3.3) *and* (3.10), *hold. Let* $\sharp = \mathrm{G}, \mathrm{K}$. *Then we have*:

1) *There exists a conservative Hunt process*

$$\mathbf{M}^\sharp = (\mathbf{\Omega}^\sharp, \mathbf{F}^\sharp, (\mathbf{F}_t^\sharp)_{t\geq 0}, (\mathbf{\Theta}_t^\sharp)_{t\geq 0}, (\mathbf{X}^\sharp(t))_{t\geq 0}, (\mathbf{P}_\gamma^\sharp)_{\gamma\in\Gamma})$$

on Γ *which is properly associated with* $(\mathcal{E}_\sharp, D(\mathcal{E}_\sharp))$, *i.e., for all* ($\mu$-*versions of*) $F \in L^2(\Gamma,\mu)$ *and all* $t > 0$ *the function*

$$\Gamma \ni \gamma \mapsto p_t^\sharp F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}^\sharp(t))\, d\mathbf{P}_\gamma^\sharp \tag{3.16}$$

is an $\mathcal{E}_\sharp$-*quasi-continuous version of* $\exp(-tH_\sharp)F$, *where* $(H_\sharp, D(H_\sharp))$ *is the generator of* $(\mathcal{E}_\sharp, D(\mathcal{E}_\sharp))$. $\mathbf{M}^\sharp$ *is up to* μ-*equivalence unique. In particular,* $\mathbf{M}^\sharp$ *is* μ-*symmetric* (*i.e.,* $\int G\, p_t^\sharp F\, d\mu = \int F\, p_t^\sharp G\, d\mu$ *for all* $F, G : \Gamma \to \mathbb{R}_+$, $\mathcal{B}(\Gamma)$-*measurable*), *so has* μ *as an invariant measure.*

2) $\mathbf{M}^\sharp$ *from* 1) *is up to* μ-*equivalence unique between all Hunt processes* $\mathbf{M}' = (\mathbf{\Omega}', \mathbf{F}', (\mathbf{F}'_t)_{t\geq 0}, (\mathbf{\Theta}'_t)_{t\geq 0}, (\mathbf{X}'(t))_{t\geq 0}, (\mathbf{P}'_\gamma)_{\gamma\in\Gamma})$ *on* Γ *having* μ *as invariant measure and solving the martingale problem for* $(-H_\sharp, D(H_\sharp))$, *i.e., for all* $G \in D(H_\sharp)$

$$\widetilde{G}(\mathbf{X}'(t)) - \widetilde{G}(\mathbf{X}'(0)) + \int_0^t (H_\sharp G)(\mathbf{X}'(s))\, ds, \quad t \geq 0,$$

is an $(\mathbf{F}'_t)$-*martingale under* $\mathbf{P}'_\gamma$ *for* $\mathcal{E}_\sharp$-*q.e.* $\gamma \in \Gamma$. (*Here,* $\widetilde{G}$ *denotes an* $\mathcal{E}_\sharp$-*quasi-continuous version of* G.)

Remark 3.3. In Theorem 3.1, $\mathbf{M}^\sharp$ can be taken canonical, i.e., $\mathbf{\Omega}^\sharp$ is the set of all *cadlag* functions $\omega : [0,\infty) \to \Gamma$ (i.e., ω is right continuous on $[0,\infty)$ and has left limits on $(0,\infty)$), $\mathbf{X}^\sharp(t)(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}^\sharp$, $(\mathbf{F}_t^\sharp)_{t\geq 0}$ is the filtration generated by $(\mathbf{X}^\sharp(t))_{t\geq 0}$, $\mathbf{F}^\sharp$ is the minimal σ-algebra containing all $\mathbf{F}_t^\sharp$, $t \geq 0$, and $\mathbf{\Theta}_t^\sharp$, $t \geq 0$, are the corresponding natural time shifts.

Proof of Theorem 3.1. The first part of the theorem follows from Lemmas 3.4, 3.5, the fact that $1 \in D(\mathcal{E}_\sharp)$ and $\mathcal{E}_\sharp(1,1) = 0$, $\sharp = \mathrm{G}, \mathrm{K}$, and [19, Chap. IV, Theorem 3.5 and Chap. V, Proposition 2.15]. The second part follows directly from the proof of [2, Theorem 3.5]. □

Now we will derive explicit formulas for the generators of $\mathcal{E}_{\mathrm{G}}$ and $\mathcal{E}_{\mathrm{K}}$. However, for this, we will demand stronger conditions on the coefficients $d(x,\gamma)$ and $c(x,y,\gamma)$.

Theorem 3.2. 1) *Assume that, for each* $\Lambda \in \mathcal{B}_0(X)$,

$$\begin{gathered}\int_\Lambda \gamma(dx)\, d(x, \gamma\setminus x) \in L^2(\Gamma,\mu),\\ \int_\Lambda \nu(dx) b(x,\gamma) \in L^2(\Gamma,\mu),\end{gathered} \tag{3.17}$$

where

$$b(x,\gamma) := r(x,\gamma)\, d(x,\gamma), \quad x \in X, \quad \gamma \in \Gamma. \tag{3.18}$$

Then, for each $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$,

$$(H_{\mathrm{G}}F)(\gamma) = -\int_X \nu(dx)\, b(x,\gamma)(D_x^+F)(\gamma) - \int_X \gamma(dx)\, d(x,\gamma\backslash x)(D_x^-F)(\gamma) \quad \mu\text{-a.e.} \tag{3.19}$$

and $(H_{\mathrm{G}}, D(H_{\mathrm{G}}))$ *is the Friedrichs extension of* $(H_{\mathrm{G}}, \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ *in* $L^2(\Gamma,\mu)$.

2) *Assume that, for each* $\Lambda \in \mathcal{B}_0(X)$,

$$\int_X \gamma(dx) \int_X \nu(dy)\, c(x,y,\gamma\setminus x)(\chi_\Lambda(x) + \chi_\Lambda(y)) \in L^2(\Gamma,\mu). \tag{3.20}$$

Then, for each $F \in \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma)$,

$$(H_{\mathrm{K}}F)(\gamma) = -2\int_X \gamma(dx) \int_X \nu(dy) c(x,y,\gamma\setminus x)(D_{xy}^{-+}F)(\gamma) \quad \mu\text{-a.e.} \tag{3.21}$$

and $(H_{\mathrm{K}}, D(H_{\mathrm{K}}))$ *is the Friedrichs extension of* $(H_{\mathrm{K}}, \mathcal{F}C_{\mathrm{b}}(C_0(X),\Gamma))$ *in* $L^2(\Gamma,\mu)$.

Proof. By (2.1) and (3.6), the theorem easily follows from our assumptions (3.17), (3.20). □

3.1. **Examples.** For each $s \in [0,1]$, we define

$$c(x,y,\gamma) := a(x,y) r(x,\gamma)^{s-1} r(y,\gamma)^s \chi_{\{r(x,\gamma)>0,\, r(y,\gamma)>0\}}. \tag{3.22}$$

Here, the function $a : X^2 \to [0,\infty)$ is measurable, symmetric (i.e., $a(x,y) = a(y,x)$), bounded, and satisfies

$$\sup_{x\in X} \int_X a(x,y)\, \nu(dy) < \infty. \tag{3.23}$$

Assume also that there exists $\Lambda \in \mathcal{B}_0(X)$ such that

$$\sup_{x\in X\setminus\Lambda} J(x,x) < \infty. \tag{3.24}$$

Note that $c(x,y,\gamma)$ satisfies the balance condition (3.6)

Proposition 3.1. *Let the coefficient* $c(x,y,\gamma)$ *be given by* (3.22), *and let conditions* (3.23), (3.24) *hold. Then, for each* $s \in [0,1]$, *conditions* (3.3) *and* (3.10) *are satisfied, and therefore the conclusion of Theorem* 3.1 *holds for the corresponding Dirichlet form.*

Furthermore, for $s = 1$, *condition* (3.20) *is satisfied, and hence the conclusion of Theorem* 3.2 *holds for the corresponding generator* $(H_{\mathrm{K}}, D(H_{\mathrm{K}}))$.

Proof. Let $s \in [0,1]$. We have, by (2.1), (3.23), (3.24) and Proposition 2.1,

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \nu(dy) c(x,y,\gamma\setminus x)(\chi_\Lambda(x) + \chi_\Lambda(y)) \\
&\quad = \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) a(x,y) r(x,\gamma)^s r(y,\gamma)^s \\
&\qquad \times \chi_{\{r(x,\gamma)>0,\, r(y,\gamma)>0\}}(\chi_\Lambda(x) + \chi_\Lambda(y)) \\
&\quad \le \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) a(x,y) \\
&\qquad \times (1 \wedge J(x,x))(1 \wedge J(y,y))(\chi_\Lambda(x) + \chi_\Lambda(y)) < \infty,
\end{aligned}$$

so that condition (3.3) is satisfied.

Next, setting $u=-s$, we see that in order to show that (3.10) is satisfied, it suffices to prove that, for each $\Lambda\in\mathcal B_0(X)$,

$$\int_\Lambda \nu(dx)\int_\Lambda \gamma(dy)a(x,y)r(x,\gamma\setminus y)^s\in L^2(\mu).$$

So, by Proposition 2.1, (2.1), (3.23), (3.24), and the boundedness of a, we have

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma)\bigg(\int_\Lambda \nu(dx)\int_\Lambda \gamma(dy)a(x,y)r(x,\gamma\setminus y)^s\bigg)^2\\
&\quad=\int_\Gamma\mu(d\gamma)\int_\Lambda \nu(dy)r(y,\gamma)\int_\Lambda\nu(dx_1)\int_\Lambda \nu(dx_2)a(x_1,y)a(x_2,y)r(x_1,\gamma)^s r(x_2,\gamma)^s\\
&\qquad+\int_\Gamma\mu(d\gamma)\int_\Lambda\nu(dy_1)\int_\Lambda\nu(dy_2)\int_\Lambda\nu(dx_1)\int_\Lambda\nu(dx_2)r(y_2,\gamma)r(y_1,\gamma\cup y_2)\\
&\qquad\times a(x_1,y_1)a(x_2,y_2)r(x_1,\gamma\cup y_2)^s r(x_2,\gamma\cup y_1)^s\\
&\quad\le\int_\Lambda\nu(dy)J(y,y)\int_\Lambda\nu(dx_1)\int_\Lambda\nu(dx_2)a(x_1,y)a(x_2,y)(1+J(x_1,x_1))(1+J(x_2,x_2))\\
&\quad+\int_\Lambda\nu(dy_1)\int_\Lambda\nu(dy_2)\int_\Lambda\nu(dx_1)\int_\Lambda\nu(dx_2)a(x_1,y_1)a(x_2,y_2)\\
&\qquad\times J(y_1,y_1)J(y_2,y_2)(1+J(x_1,x_1))(1+J(x_2,x_2))<\infty.
\end{aligned}$$

Now, let $s=1$. Analogously to the above, we have

$$\begin{aligned}
&\int_\Gamma\mu(d\gamma)\bigg(\int_X\gamma(dx)\int_X\nu(dy)c(x,y,\gamma\setminus x)(\chi_\Lambda(x)+\chi_\Lambda(y))\bigg)^2\\
&\quad=\int_\Gamma\mu(d\gamma)\int_X\nu(dx)r(x,\gamma)\int_X\nu(dy_1)\int_X\nu(dy_2)a(x,y_1)a(x,y_2)\\
&\qquad\times r(y_1,\gamma)r(y_2,\gamma)\chi_{\{r(x,\gamma)>0,\,r(y_1,\gamma)>0,\,r(y_2,\gamma)>0\}}(\chi_\Lambda(x)+\chi_\Lambda(y_1))(\chi_\Lambda(x)+\chi_\Lambda(y_2))\\
&\qquad+\int_\Gamma\mu(d\gamma)\int_X\nu(dx_1)\int_X\nu(dx_2)r(x_2,\gamma)r(x_1,\gamma\cup x_2)\\
&\qquad\times\int_X\nu(dy_1)\int_X\nu(dy_2)a(x_1,y_1)a(x_2,y_2)r(y_1,\gamma\cup x_2)r(y_2,\gamma\cup x_1)\\
&\qquad\times\chi_{\{r(x_1,\gamma\cup x_2)>0,\,r(x_2,\gamma\cup x_1)>0,\,r(y_1,\gamma\cup x_2)>0,\,r(y_2,\gamma\cup x_1)>0\}}\\
&\qquad\times(\chi_\Lambda(x_1)+\chi_\Lambda(y_1))(\chi_\Lambda(x_2)+\chi_\Lambda(y_2))\\
&\quad\le\int_X\nu(dx)\int_X\nu(dy_1)\int_X\nu(dy_2)a(x,y_1)a(x,y_2)\\
&\qquad\times J(y_1,y_1)J(y_2,y_2)(\chi_\Lambda(x)+\chi_\Lambda(y_1))(\chi_\Lambda(x)+\chi_\Lambda(y_2))\\
&\qquad+\int_X\nu(dx_1)\int_X\nu(dx_2)\int_X\nu(dy_1)\int_X\nu(dy_2)a(x_1,y_1)a(x_2,y_2)\\
&\qquad\times J(y_1,y_1)J(y_2,y_2)(\chi_\Lambda(x_1)+\chi_\Lambda(y_1))(\chi_\Lambda(x_2)+\chi_\Lambda(y_2))<\infty.
\end{aligned}$$

□

Next, for each $s\in[0,1]$, we define

$$d(x,\gamma):=r(x,\gamma)^{s-1}\chi_{\{r(x,\gamma)>0\}},\tag{3.25}$$

so that

$$b(x,\gamma):=r(x,\gamma)^s\chi_{\{r(x,\gamma)>0\}}.$$

Analogously to Proposition 3.1, we get

Proposition 3.2. *Let the coefficient* $d(x, \gamma)$ *be given by* (3.25)*. Then, for each* $s \in [0, 1]$*, condition* (3.2)*, is satisfied, and hence the conclusion of Theorem* 3.1 *holds for the corresponding Dirichlet form.*

Furthermore, for $s = 1$*, condition* (3.17) *is satisfied, and hence the conclusion of Theorem* 3.2 *holds for the corresponding generator* $(H_{\mathrm{G}}, D(H_{\mathrm{G}}))$.

We finally note that all our assumptions are trivially satisfied in the case of bounded coefficients $c(x, y, \gamma)$ and $d(x, \gamma)$, $b(x, \gamma)$, respectively.

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