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INTERPOLATION WITH A FUNCTION PARAMETER AND REFINED SCALE OF SPACES

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ABSTRACT. The interpolation of couples of separable Hilbert spaces with a function parameter is studied. The main properties of the classical interpolation are proved. Some applications to the interpolation of isotropic Hörmander spaces over a closed manifold are given.

1. Introduction

In this paper we study the interpolation of couples of separable Hilbert spaces with a functional parameter. We generalize the classical theorems on interpolation with a power parameter with the index $\theta \in (0,1)$ to a maximal class of functions.

As an application, we consider the interpolation of isotropic Hörmander spaces over a closed manifold

$$H^{s,\varphi} := H_2^{\langle\cdot\rangle^s\,\varphi(\langle\cdot\rangle)}, \quad \langle\xi\rangle := \bigl(1+|\xi|^2\bigr)^{1/2}. \tag{1.1}$$

Here, $s \in \mathbb{R}$ and φ is a functional parameter slowly varying at $+\infty$ in Karamata's sense. In particular, every standard function

$$\varphi(t) = (\log t)^{r_1}(\log\log t)^{r_2}\ldots(\log\ldots\log t)^{r_n}, \quad \{r_1, r_2, \ldots, r_n\} \subset \mathbb{R}, \quad n \in \mathbb{N},$$

is admissible. This scale was introduced and investigated by the authors in [1, 2]. It contains the Sobolev scale $\{H^s\} \equiv \{H^{s,1}\}$ and is attached to it by the number parameter s and being considerably finer.

Spaces of form (1.1) arise naturally in different spectral problems: convergence of spectral expansions of self-adjoint elliptic operators almost everywhere, in the norm of the spaces L_p with $p > 2$ or C (see survey [3]); spectral asymptotics of general self-adjoint elliptic operators in a bounded domain, the Weyl formula, a sharp estimate of the remainder in it (see [4, 5]) and others. They may be expected to be useful in other "fine" questions. Due to their interpolation properties, the spaces $H^{s,\varphi}$ occupy a special position among the spaces of a generalized smoothness which are actively investigated and used today (see survey [6], recent articles [7, 8] and the bibliography given there).

One of the main results of the article is a description of the refined scale by means of regularly varying functions of a positive elliptic pseudodifferential operator. The related questions were studied in [9, 10] and by the authors in [11–20].

2. An interpolation with a function parameter

2.1. **A definition of the interpolation.**

Definition 2.1. An ordered couple $[X_0, X_1]$ of complex Hilbert spaces X_0 and X_1 is called *admissible* if the spaces X_0, X_1 are separable and the continuous embedding $X_1 \hookrightarrow X_0$ holds.

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Let an admissible couple $X = [X_0, X_1]$ of Hilbert spaces be given. It is known [21, Ch. 1, Sec. 2.1], [22, Ch. IV, Sec. 9.1] that for this couple X there exists an isometric isomorphism $J : X_1 \leftrightarrow X_0$ such that J is a self-adjoint positive operator on the space X_0 with the domain X_1. The operator J is called a *generating* one for the couple X. This operator is uniquely determined by the couple X. Indeed, assume that J_1 is also a generating operator for the couple X. Then the operators J and J_1 are metrically equal, i.e., $\|Ju\|_{X_0} = \|u\|_{X_1} = \|J_1 u\|_{X_0}$ for any $u \in X_1$. Moreover, these operators are positive. Hence, they are equal.

We denote by $\mathcal{B}$ the set of all functions $\psi : (0, +\infty) \to (0, +\infty)$ such that

a) ψ is Borel measurable on the semiaxis $(0, +\infty)$;
b) ψ is bounded on each closed interval $[a, b]$, where $0 < a < b < +\infty$;
c) $1/\psi$ is bounded on each set $[r, +\infty)$, where $r > 0$.

Let $\psi \in \mathcal{B}$. Generally, the unbounded operator $\psi(J)$ is defined on the space X_0 as a function of J. We denote by $[X_0, X_1]_\psi$ or, simply, by X_ψ the domain of the operator $\psi(J)$, equipped with the inner product $(u, v)_{X_\psi} := (\psi(J)u, \psi(J)v)_{X_0}$ and the corresponding norm $\| u \|_{X_\psi} = (u, u)^{1/2}_{X_\psi}$.

The space X_ψ is Hilbert separable and, moreover, the continuous dense embedding $X_\psi \hookrightarrow X_0$ is fulfilled. Indeed, we have $\operatorname{Spec} J \subseteq [r, +\infty)$ and $\psi(t) \geq c$ for $t \geq r$, where r, c are some positive numbers. Hence, $\operatorname{Spec} \psi(J) \subseteq [c, +\infty)$, that implies the isometric isomorphism $\psi(J) : X_\psi \leftrightarrow X_0$. It follows that the space X_ψ is complete and separable as well as that the function $\| \cdot \|_{X_\psi}$ is positive definite, so this is a norm. Next, since the operator $\psi^{-1}(J)$ is bounded on the space X_0, a bounded embedding operator $I = \psi^{-1}(J)\psi(J) : X_\psi \to X_0$ exists. The embedding $X_\psi \hookrightarrow X_0$ is dense because the domain of the operator $\psi(J)$ is a dense linear manifold in the space X_0.

Further, it is useful to note the following. Let functions $\varphi, \psi \in \mathcal{B}$ be such that $\varphi \asymp \psi$ in a neighborhood of $+\infty$. Then, by the definition of the set $\mathcal{B}$, we have $\varphi \asymp \psi$ on $\operatorname{Spec} J$. Hence, $X_\varphi = X_\psi$ up to equivalent norms.

Definition 2.2. A function $\psi \in \mathcal{B}$ is called an *interpolation parameter* if the following condition is satisfied for all admissible couples $X = [X_0, X_1]$, $Y = [Y_0, Y_1]$ of Hilbert spaces and an arbitrary linear mapping T given on X_0: if the restriction of the mapping T to the space X_j is a bounded operator $T : X_j \to Y_j$ for each $j = 0, 1$, then the restriction of the mapping T to the space X_ψ is also a bounded operator $T : X_\psi \to Y_\psi$.

Otherwise speaking, ψ is an interpolation parameter if and only if the mapping $X \mapsto X_\psi$ is an interpolation functor given on the category of all admissible couples X of Hilbert spaces [23, Sec. 1.2.2], [24, Sec. 2.4]. In the case where ψ is an interpolation parameter, we will say that the space X_ψ is obtained by the *interpolation with the function parameter* ψ of the admissible couple X.

Further we will investigate the main properties of the mapping $X \mapsto X_\psi$.

2.2. **Embeddings of spaces.**

Theorem 2.1. *Let $\psi \in \mathcal{B}$ be an interpolation parameter and $X = [X_0, X_1]$ be an admissible couple of Hilbert spaces. Then the continuous dense embeddings $X_1 \hookrightarrow X_\psi \hookrightarrow X_0$ hold.*

Proof. According to Subsection 2.1 it only remains to prove the continuous dense embedding $X_1 \hookrightarrow X_\psi$. Let us consider two bounded embedding operators $I : X_1 \to X_0$ and $I : X_1 \to X_1$. Since ψ is an interpolation parameter, these operators imply the bounded embedding operator $I : X_1 \to X_\psi$. Thus, the continuous embedding $X_1 \hookrightarrow X_\psi$ is valid. We will prove that it is dense. For an arbitrary $u \in X_\psi$, we have $v := (1+\psi^2(J))^{1/2}\,u \in X_0$. Since X_1 is dense in X_0, there is a sequence $(v_k) \subset X_1$ such that $v_k \to v$ in X_0 as $k \to \infty$. From this and from (1.1) it follows that

$$u_k := (1+\psi^2(J))^{-1/2}\,v_k \to u \quad \text{in} \quad X_\psi \quad \text{for} \quad k \to \infty.$$

It remains to note that

$$u_k = (1+\psi^2(J))^{-1/2} J^{-1} J v_k = J^{-1}(1+\psi^2(J))^{-1/2} J v_k \in X_1.$$

Theorem 2.1 is proved. □

Theorem 2.2. *Let functions $\psi, \chi \in \mathcal{B}$ be given such that the function ψ/χ is bounded in a neighborhood of $+\infty$. Then, for each admissible couple $X = [X_0, X_1]$ of Hilbert spaces, the continuous and dense embedding $X_\chi \hookrightarrow X_\psi$ holds. If the embedding $X_1 \hookrightarrow X_0$ is compact and $\psi(t)/\chi(t) \to 0$ as $t \to +\infty$, then the embedding $X_\chi \hookrightarrow X_\psi$ is also compact.*

Proof. Let J be a generating operator for the couple X. Let us note that $\operatorname{Spec} J \subseteq [r, +\infty)$ for some number $r > 0$. According to the condition of the theorem, we have $\psi(t)/\chi(t) \le c$ for $t \ge r$. Therefore

$$X_\chi = \operatorname{Dom} \chi(J) \subseteq \operatorname{Dom} \psi(J) = X_\psi, \quad \|\psi(J)\,u\|_{X_0} \le c\,\|\chi(J)\,u\|_{X_0}.$$

From these formulae and from the definition of the spaces X_χ, X_ψ we obtain the continuous embedding $X_\chi \hookrightarrow X_\psi$. Let us prove its density.

We consider the isometric isomorphisms $\psi(J) : X_\psi \leftrightarrow X_0$ and $\chi(J) : X_\chi \leftrightarrow X_0$. For any given $u \in X_\psi$, we have $\psi(J)\,u \in X_0$. Since the space X_χ is densely embedded into X_0, a sequence $(v_k) \subset X_\chi$ such that $v_k \to \psi(J)\,u$ in X_0 as $k \to \infty$ exists. Hence, $\psi^{-1}(J)\,v_k \to u$ in X_ψ as $k \to \infty$, where

$$\psi^{-1}(J)\,v_k = \psi^{-1}(J)\,\chi^{-1}(J)\,\chi(J)\,v_k = \chi^{-1}(J)\,\psi^{-1}(J)\,\chi(J)\,v_k \in X_\chi.$$

Thus, we have proved the density of the embedding $X_\chi \hookrightarrow X_\psi$.

Now let us assume that the embedding $X_1 \hookrightarrow X_0$ is compact and $\psi(t)/\chi(t) \to 0$ as $t \to +\infty$. We will prove the compactness of the embedding $X_\chi \hookrightarrow X_\psi$. Let (u_k) be an arbitrary bounded sequence belonging to X_χ. Since the sequence of elements $w_k := J^{-1}\,\chi(J)\,u_k$ is bounded in X_1, we can select a subsequence of elements $w_{k_n} = J^{-1}\,\chi(J)\,u_{k_n}$ being the Cauchy sequence in X_0. We will show that (u_{k_n}) is the Cauchy sequence in X_ψ.

Let E_t, $t \ge r$, be a resolution of the unity in X_0, corresponding to the self-adjoint operator J. We can write

$$\begin{aligned} \|u_{k_n} - u_{k_m}\|^2_{X_\psi} &= \|\psi(J)\,(u_{k_n} - u_{k_m})\|^2_{X_0} = \|\psi(J)\,\chi^{-1}(J)\,J\,(w_{k_n} - w_{k_m})\|^2_{X_0} \\ &= \int_r^{+\infty} \psi^2(t)\,\chi^{-2}(t)\,t^2\,d\,\|E_t(w_{k_n} - w_{k_m})\|^2_{X_0}. \end{aligned} \tag{2.1}$$

Let us choose an arbitrary number $\varepsilon > 0$. There is a number $\rho = \rho(\varepsilon) > r$ such that

$$\psi(t)/\chi(t) \le (2c_0)^{-1}\varepsilon \quad \text{for} \quad t \ge \rho \quad \text{and} \quad c_0 := \sup\{\,\|w_k\|_{X_1} : k \in \mathbb{N}\,\} < \infty.$$

Hence, for all indices n, m we have

$$\begin{aligned} &\int_\rho^{+\infty} \psi^2(t)\,\chi^{-2}(t)\,t^2\,d\,\|E_t(w_{k_n} - w_{k_m})\|^2_{X_0} \\ &\le (2c_0)^{-2}\,\varepsilon^2 \int_\rho^{+\infty} t^2\,d\,\|E_t(w_{k_n} - w_{k_m})\|^2_{X_0} \\ &\le (2c_0)^{-2}\,\varepsilon^2\,\|J\,(w_{k_n} - w_{k_m})\|^2_{X_0} = (2c_0)^{-2}\,\varepsilon^2\,\|w_{k_n} - w_{k_m}\|^2_{X_1} \le \varepsilon^2. \end{aligned} \tag{2.2}$$

In addition, by the inequality $\psi(t)/\chi(t) \le c$ for $t \ge r$, we can write the following:

$$\begin{aligned} &\int_r^{\rho} \psi^2(t)\,\chi^{-2}(t)\,t^2\,d\,\|E_t(w_{k_n} - w_{k_m})\|^2_{X_0} \le c^2\rho^2 \int_r^{\rho} d\,\|E_t(w_{k_n} - w_{k_m})\|^2_{X_0} \\ &\le c^2\rho^2\,\|w_{k_n} - w_{k_m}\|^2_{X_0} \to 0 \quad \text{as} \quad n, m \to \infty. \end{aligned} \tag{2.3}$$

Now formulae (2.1)–(2.3) imply the inequality $\|u_{k_n} - u_{k_m}\|_{X_\psi} \le 2\varepsilon$ for sufficiently large n, m. Therefore (u_{k_n}) is the Cauchy sequence in the space X_ψ which means the compactness of the embedding $X_\chi \hookrightarrow X_\psi$. Theorem 2.2 is proved. □

2.3. **Reiteration.**

Theorem 2.3. *Let functions $f, g, \psi \in \mathcal{B}$ be given. Suppose that the function f/g is bounded in a neighborhood of $+\infty$. Then $[X_f, X_g]_\psi = X_\omega$ holds with the equality of norms for each admissible couple X of Hilbert spaces. Here the function $\omega \in \mathcal{B}$ is given by the formula $\omega(t) := f(t)\,\psi(g(t)/f(t))$ for $t > 0$. If f, g, ψ are interpolation parameters, so is ω.*

Proof. Since the function f/g is bounded in a neighborhood of $+\infty$, the couple $[X_f, X_g]$ is admissible by the Theorem 2.2 and, in addition, $\omega \in \mathcal{B}$. So, the spaces $[X_f, X_g]_\psi$ and X_ω are well defined. We will prove them to be equal.

Let an operator J be generating for the couple $X = [X_0, X_1]$, where $\operatorname{Spec} J \subseteq [r, +\infty)$ for some number $r > 0$. We have three isometric isomorphisms

$$f(J) : X_f \leftrightarrow X_0, \quad g(J) : X_g \leftrightarrow X_0, \quad B := f^{-1}(J)\, g(J) : X_g \leftrightarrow X_f.$$

Let us consider B as a closed operator on the space X_f, defined on X_g. The operator B is generating for the couple $[X_f, X_g]$ because B is positive and self-adjoint on X_f. The positiveness of B follows from the condition $f(t)/g(t) \le c$ for $t \ge r$ which implies

$$(Bu, u)_{X_f} = (g(J)\,u, f(J)\,u)_{X_0} \ge c^{-1}\,(f(J)\,u, f(J)\,u)_{X_0} = c^{-1}\,\|u\|^2_{X_f}.$$

The self-adjointness follows from the fact that 0 is a regular point for the operator B.

Using the spectral theorem, we reduce the self-adjoint on X_0 operator J to the form of multiplication by a function: $J = I^{-1}(\alpha \cdot I)$. Here, $I : X_0 \leftrightarrow L_2(U, d\mu)$ is an isometric isomorphism, (U, μ) is a space with a finite measure, $\alpha : U \to [r, +\infty)$ is a measurable function. The isometric isomorphism $I f(J) : X_f \leftrightarrow L_2(U, d\mu)$ reduces the self-adjoint in X_f operator B to the form of multiplication by the function $(g/f) \circ \alpha$:

$$I f(J)\, B\, u = I g(J)\, u = (g \circ \alpha)\, I u = (g \circ \alpha)\, I f^{-1}(J) f(J)\, u = ((g/f) \circ \alpha)\, I f(J)\, u, \quad u \in X_g.$$

Therefore, for an arbitrary $u \in X_\omega$, we have

$$\|\psi(B)\, u\|_{X_f} = \|(\psi \circ (g/f) \circ \alpha) \cdot (I f(J)\, u)\|_{L_2(U,d\mu)} = \|(\omega \circ \alpha) \cdot (Iu)\|_{L_2(U,d\mu)} = \|\omega(J)\, u\|_{X_0}.$$

Let us note that the function f/ω is bounded in a neighborhood of $+\infty$. Hence, $X_\omega \hookrightarrow X_f$ and the expression $f(J)\,u$ is well defined. Thus, the equality $[X_f, X_g]_\psi = X_\omega$ is proved.

We now assume that f, g, ψ are interpolation parameters. We will show that ω is also an interpolation parameter. Let arbitrary admissible couples $X = [X_0, X_1]$, $Y = [Y_0, Y_1]$ and a linear mapping T be the same as those in Definition 2.2. We have the bounded operators $T : X_f \to Y_f$ and $T : X_g \to Y_g$ which imply the boundedness of the operator $T : [X_f, X_g]_\psi \to [Y_f, Y_g]_\psi$. We have already proved that $[X_f, X_g]_\psi = X_\omega$ and $[Y_f, Y_g]_\psi = Y_\omega$. So, a bounded operator $T : X_\omega \to Y_\omega$ exists. It means that ω is an interpolation parameter. Theorem 2.3 is proved. □

2.4. **The interpolation of dual spaces.** Let H be a Hilbert space. We denote by H' the space dual to H. Thus, H' is the Banach space of all linear continuous functionals $l : H \to \mathbb{C}$. By the Riesz theorem, the mapping $S : v \mapsto (\,\cdot\,, v)_H$, where $v \in H$, establishes the antilinear isometric isomorphism $S : H \leftrightarrow H'$. This implies that H' is the Hilbert space with respect to the inner product $(l, m)_{H'} := (S^{-1}l, S^{-1}m)_H$. We emphasize that we do not identify H' as H by means of the isomorphism S.

Theorem 2.4. *Let $\psi \in \mathcal{B}$ be such that the function $\psi(t)/t$ is bounded in a neighborhood of $+\infty$. Then, for each admissible couple $[X_0, X_1]$ of Hilbert spaces, the equality of the spaces $[X_1', X_0']_\psi = [X_0, X_1]'_\chi$ with the equality of norms hold. Here the function $\chi \in \mathcal{B}$*

is given by the formula $\chi(t) := t/\psi(t)$ for $t > 0$. If ψ is an interpolation parameter, so is χ.

Proof. Note that the couple $[X_1', X_0']$ is admissible, provided that we naturally identify functionals from X_0' as their restrictions to the space X_1. From the condition of the theorem it follows that $\varphi \in \mathcal{B}$. Thus, the spaces $[X_1', X_0']_\psi$ and $[X_0, X_1]_\chi'$ are well defined. Let us prove these spaces to be equal.

Let $J : X_1 \leftrightarrow X_0$ be a generating operator for the couple $[X_0, X_1]$. Let us consider the isometric isomorphisms $S_j : X_j \leftrightarrow X_j'$, $j = 0, 1$, which appear in the Riesz theorem. The operator J', being adjoint to J, satisfies the equality $J' = S_1 J^{-1} S_0^{-1}$. This results from the following:

$$(J'l)u = l(Ju) = (Ju, S_0^{-1}l)_{X_0} = (u, J^{-1}S_0^{-1}l)_{X_1} = (S_1 J^{-1} S_0^{-1} l)u \quad \text{for each} \quad l \in X_0',\ u \in X_1.$$

Thus, the isometric isomorphism

$$J' = S_1 J^{-1} S_0^{-1} : X_0' \leftrightarrow X_1' \tag{2.4}$$

exists.

Let us note that the equalities

$$\begin{aligned}(u, JS_1^{-1}l)_{X_0} &= (J^{-1}u, S_1^{-1}l)_{X_1} = l(J^{-1}u),\\ (u, J^{-1}S_0^{-1}l)_{X_0} &= (J^{-1}u, S_0^{-1}l)_{X_0} = l(J^{-1}u),\end{aligned}$$

where $l \in X_0' \hookrightarrow X_1'$, $u \in X_0$, imply the property

$$JS_1^{-1}l = J^{-1}S_0^{-1}l \in X_1 \quad \text{for each} \quad l \in X_0'. \tag{2.5}$$

Let us consider J' as a closed operator on the space X_1' with the domain X_0'. The operator J' is generating for the couple $[X_1', X_0']$ because J' is positive and self-adjoint on X_1'. The positiveness of J' results from the positiveness of the operator J on the space X_0 and from (2.5) in the following way:

$$\begin{aligned}(J'l, l)_{X_1'} &= (S_1 J^{-1} S_0^{-1} l, l)_{X_1'} = (J^{-1}S_0^{-1}l, S_1^{-1}l)_{X_1} = (JJ^{-1}S_0^{-1}l, JS_1^{-1}l)_{X_0}\\ &= (JJS_1^{-1}l, JS_1^{-1}l)_{X_0} \geq c\,\|JS_1^{-1}l\|^2_{X_0} = c\,\|S_1^{-1}l\|^2_{X_1} = c\,\|l\|^2_{X_1'}.\end{aligned}$$

Here the number $c > 0$ does not depend on $l \in X_0'$. The operator J' is self-adjoint because 0 is its regular point for the operator J' (see (2.4)). Let us reduce the operator J to the form of multiplication by a function: $J = I^{-1}(\alpha \cdot I)$ as it has been done in the proof of Theorem 2.3. The isometric isomorphism

$$IJS_1^{-1} : X_1' \leftrightarrow L_2(U, d\mu) \tag{2.6}$$

reduces the operator J' to the form of multiplication by the same function α

$$(IJS_1^{-1})J'l = IS_0^{-1}l = IJJ^{-1}S_0^{-1}l = \alpha \cdot IJ^{-1}S_0^{-1}l = \alpha \cdot IJS_1^{-1}l \quad \text{for each} \quad l \in X_0'.$$

The last equality follows from (2.5).

By Theorem 2.2, two continuous dense embeddings $X_0' \hookrightarrow [X_1', X_0']_\psi$ and $[X_0, X_1]_\chi \hookrightarrow X_0$ hold. The second embedding implies the continuous dense embedding $X_0' \hookrightarrow [X_0, X_1]_\chi'$. Let us show that the norms in the spaces $[X_1', X_0']_\psi$ and $[X_0, X_1]_\chi'$ are equal on the dense subset X_0'. For each $l \in X_0'$, $u \in [X_0, X_1]_\chi$, we can write

$$l(u) = (u, S_0^{-1}l)_{X_0} = (\chi(J)u, \chi^{-1}(J)S_0^{-1}l)_{X_0} = (v, \chi^{-1}(J)S_0^{-1}l)_{X_0}$$

with $v := \chi(J)u \in X_0$. It implies the following:

$$\begin{aligned}\|l\|_{[X_0,X_1]'_\chi} &= \sup\{\,|l(u)|\,/\,\|u\|_{[X_0,X_1]_\chi} : u \in [X_0,X_1]_\chi,\ u \neq 0\,\}\\ &= \sup\{\,|(v,\chi^{-1}(J)S_0^{-1}l)_{X_0}|\,/\,\|v\|_{X_0} :\ v \in X_0,\ v \neq 0\,\}\\ &= \|\chi^{-1}(J)S_0^{-1}l\|_{X_0} = \|I\chi^{-1}(J)S_0^{-1}l\|_{L_2(U,d\mu)} = \|(\chi^{-1}\circ\alpha)\cdot IS_0^{-1}l\|_{L_2(U,d\mu)}.\end{aligned}$$

On the other hand, using isomorphisms (2.6), (2.4), we have

$$\begin{aligned}\|l\|_{[X_1',X_0']_\psi} &= \|\psi(J')l\|_{X_1'} = \|\chi^{-1}(J')J'l\|_{X_1'} = \|(IJS_1^{-1})\chi^{-1}(J')J'l\|_{L_2(U,d\mu)}\\ &= \|(\chi^{-1}\circ\alpha)\cdot(IJS_1^{-1})J'l\|_{L_2(U,d\mu)} = \|(\chi^{-1}\circ\alpha)\cdot IS_0^{-1}l\|_{L_2(U,d\mu)}.\end{aligned}$$

Thus, norms in the spaces $[X_1',X_0']_\psi$ and $[X_0,X_1]'_\chi$ are equal on the dense subset X_0'. So, these spaces coincide.

Now suppose ψ to be an interpolation parameter. We will show that so is χ. Let admissible couples $X = [X_0,X_1]$, $Y = [Y_0,Y_1]$ and a linear mapping T be the same as those in Definition 2.2. Passing to the adjoint operator T', we get the bounded operators $T' : Y_j' \to X_j'$, $j = 0, 1$. Since ψ is an interpolation parameter, a bounded operator $T' : [Y_1',Y_0']_\psi \to [X_1',X_0']_\psi$ exists. As we have already proved, $[X_1',X_0']_\psi = [X_0,X_1]'_\chi$ and $[Y_1',Y_0']_\psi = [Y_0,Y_1]'_\chi$ with equalities of norms. Therefore a bounded operator $T' : [Y_0,Y_1]'_\chi \to [X_0,X_1]'_\chi$ exists. Thus, passing to the second adjoint operator T'', we get the bounded operator $T'' : [X_0,X_1]''_\chi \to [Y_0,Y_1]''_\chi$. It remains to identify the second dual spaces with original spaces which leads us to the bounded operator $T : [X_0,X_1]_\chi \to [Y_0,Y_1]_\chi$. This means that χ is an interpolation parameter. Theorem 2.4 is proved. □

2.5. **The interpolation of direct products of spaces.**

Theorem 2.5. *Let a finite or countable set of admissible couples of Hilbert spaces $X^{(k)} := [X_0^{(k)}, X_1^{(k)}]$, $k \in \omega$ be given. Suppose that the set of norms of the embedding operators $X_1^{(k)} \hookrightarrow X_0^{(k)}$, $k \in \omega$, is bounded. Then, for an arbitrary function $\psi \in \mathcal{B}$, the equality of the spaces*

$$\Bigl[\,\prod_{k\in\omega} X_0^{(k)},\ \prod_{k\in\omega} X_1^{(k)}\Bigr]_\psi = \prod_{k\in\omega}\Bigl[X_0^{(k)}, X_1^{(k)}\Bigr]_\psi$$

and the equality of norms in them hold.

Proof. We assume that $\omega = \mathbb{N}$ (the case of finite set ω is treated analogously and easier). The spaces $X_0 := \prod_{k=1}^\infty X_0^{(k)}$, $X_1 := \prod_{k=1}^\infty X_1^{(k)}$ are Hilbert and separable ones. The continuous embedding $X_1 \hookrightarrow X_0$ is evident due to the condition of the theorem. Let $u := (u_1, u_2, \ldots) \in X_0$. For all indices n, k an element $v_{n,k} \in X_1^{(k)}$ such that $\|u_k - v_{n,k}\|_{X_0^{(k)}} < 1/n$ exists. Let us form a sequence of vectors $v^{(n)} := (v_{n,1}, \ldots, v_{n,n}, 0, 0, \ldots) \in X_1$. We have

$$\begin{aligned}\|u - v^{(n)}\|_{X_0}^2 &= \sum_{k=1}^n \|u_k - v_{n,k}\|_{X_0}^2 + \sum_{k=n+1}^\infty \|u_k\|_{X_0}^2\\ &\le \frac{n}{n^2} + \sum_{k=n+1}^\infty \|u_k\|_{X_0}^2 \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Thus, the couple $X := [X_0, X_1]$ is admissible.

Let us denote by J_k a generating operator for the couple $X^{(k)}$. An operator $J := (J_1, J_2, \ldots)$ is generating for the couple X which may be proved directly. Moreover, it is natural to expect that $\psi(J) = (\psi(J_1), \psi(J_2), \ldots)$ and $\operatorname{Dom}\psi(J) = \prod_{k=1}^\infty X_\psi^{(k)}$. Now we will prove these equalities. Let us reduce the operator J_k to the form of multiplication by a function: $I_k J_k = \alpha_k \cdot I_k$. Here $I_k : X_0^{(k)} \leftrightarrow L_2(V_k, d\mu_k)$ is an isometric isomorphism, V_k is a space with a finite measure μ_k and $\alpha_k : V_k \to (0, +\infty)$ is a measurable function. We may consider the sets V_k to be mutually disjoint. Let us set $V := \bigcup_{k=1}^{\infty} V_k$. We call $\Omega \subseteq V$ a measurable set if, for every index k, the set $\Omega \cap V_k$ is μ_k-measurable. On the σ-algebra of all measurable sets $\Omega \subseteq V$, we introduce the σ-finite measure $\mu(\Omega) := \sum_{k=1}^{\infty} \mu_k(\Omega \cap V_k)$. Further, for a vector $u := (u_1, u_2, \ldots) \in X_0$, we consider the measurable functions Iu and α, defined on the set V by the formulae $(Iu)(\lambda) := (I_k u_k)(\lambda)$ and $\alpha(\lambda) := \alpha_k(\lambda)$ with $\lambda \in V_k$. Now we have the isometric isomorphism $I : X_0 \leftrightarrow L_2(V, d\mu)$. It reduces the operator J to the form of multiplication by the function α because

$$(IJu)(\lambda) = (I_k J_k u_k)(\lambda) = \alpha_k(\lambda)(I_k u_k)(\lambda) = \alpha(\lambda)(Iu)(\lambda) \quad \text{for} \quad u \in X_1, \quad \lambda \in V_k.$$

Hence, we can write down the following:

$$\begin{aligned} X_\psi = \operatorname{Dom}\psi(J) &= \{\, u \in X_0 : (\psi \circ \alpha) \cdot (Iu) \in L_2(V, d\mu) \,\} \\ &= \Big\{\, u \in X_0 : \sum_{k=1}^{\infty} \|(\psi \circ \alpha_k) \cdot (I_k u_k)\|^2_{L_2(V_k, d\mu_k)} < \infty \Big\} \\ &= \Big\{\, u : u_k \in \operatorname{Dom}\psi(J_k), \ \sum_{k=1}^{\infty} \|\psi(J_k)u_k\|^2_{X_0^{(k)}} < \infty \Big\} = \prod_{k=1}^{\infty} X_\psi^{(k)}. \end{aligned}$$

Furthermore, for each $u \in \operatorname{Dom}\psi(J)$, we have

$$\begin{aligned} (I\psi(J)u)(\lambda) &= \psi(\alpha(\lambda))\,(Iu)(\lambda) = \psi(\alpha_k(\lambda))\,(I_k u_k)(\lambda) \\ &= (I_k \psi(J_k) u_k)(\lambda) = \big(I(\psi(J_1)u_1, \psi(J_2)u_2, \ldots)\big)(\lambda) \quad \text{for} \ \ \lambda \in V_k. \end{aligned}$$

Therefore $\psi(J)u = (\psi(J_1)u_1, \psi(J_2)u_2, \ldots)$ which implies

$$\|u\|^2_{X_\psi} = \|\psi(J)u\|^2_{X_0} = \sum_{k=1}^{\infty} \|\psi(J_k)u_k\|^2_{X_0^{(k)}} = \sum_{k=1}^{\infty} \|u_k\|^2_{X_\psi^{(k)}}.$$

Theorem 2.5 is proved. □

2.6. **An operator norm in interpolation spaces.**

Theorem 2.6. *For given interpolation parameter $\psi \in \mathcal{B}$ and number $m > 0$, there is a number $c = c(\psi, m) > 0$ such that*

$$\|T\|_{X_\psi \to Y_\psi} \le c \max\{\, \|T\|_{X_j \to Y_j} : j = 0, 1 \,\}.$$

Here $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ are admissible couples of Hilbert spaces for which the norms of the embedding operators $X_1 \hookrightarrow X_0$ and $Y_1 \hookrightarrow Y_0$ do not exceed the number m, and T is any linear mapping defined on the space X_0 and establishing the bounded operators $T : X_j \to Y_j$ with $j = 0, 1$.

Proof. Let us suppose the contrary. Then we can write

$$\|T_k\|_{X_\psi^{(k)} \to Y_\psi^{(k)}} > k\, m_k \quad \text{for each index} \ \ k. \tag{2.7}$$

Here, $X^{(k)} := [X_0^{(k)}, X_1^{(k)}]$ and $Y^{(k)} := [Y_0^{(k)}, Y_1^{(k)}]$ are some admissible couples of Hilbert spaces for which the norms of the embedding operators $X_1^{(k)} \hookrightarrow X_0^{(k)}$ and $Y_1^{(k)} \hookrightarrow Y_0^{(k)}$ do not exceed the number m. Furthermore, T_k is a certain linear mapping defined on the space $X_0^{(k)}$ and establishing the bounded operators $T_k : X_j^{(k)} \to Y_j^{(k)}$ with $j = 0, 1$. We also use the notation

$$m_k := \max\{\, \|T_k\|_{X_0^{(k)} \to Y_0^{(k)}}, \ \|T_k\|_{X_1^{(k)} \to Y_1^{(k)}} \,\} > 0.$$

Now let us consider the bounded operators

$$\begin{gathered} T : u = (u_1, u_2, \ldots) \mapsto (m_1^{-1}\, T_1 u_1, m_2^{-1}\, T_2 u_2, \ldots), \\ T : \prod_{k=1}^{\infty} X_j^{(k)} \to \prod_{k=1}^{\infty} Y_j^{(k)}, \quad j = 0, 1. \end{gathered} \tag{2.8}$$

Their boundedness results from the following inequalities:

$$\sum_{k=1}^{\infty} \|m_k^{-1}\, T_k u_k\|^2_{Y_j^{(k)}} \leq \sum_{k=1}^{\infty} m_k^{-2}\, \|T_k\|^2_{X_j^{(k)} \to Y_j^{(k)}}\, \|u_k\|^2_{X_j^{(k)}} \leq \sum_{k=1}^{\infty} \|u_k\|^2_{X_j^{(k)}}.$$

Since ψ is an interpolation parameter, the boundedness of operators (2.8) implies the existence of the bounded operator

$$T:\ \Big[\prod_{k=1}^{\infty} X_0^{(k)}, \prod_{k=1}^{\infty} X_1^{(k)}\Big]_\psi \to \Big[\prod_{k=1}^{\infty} Y_0^{(k)}, \prod_{k=1}^{\infty} Y_1^{(k)}\Big]_\psi$$

which by Theorem 2.5 means the boundedness of the operator

$$T:\ \prod_{k=1}^{\infty} X_\psi^{(k)} \to \prod_{k=1}^{\infty} Y_\psi^{(k)}.$$

Let c_0 be the norm of the last operator. For every index k we consider a vector $u^{(k)} := (u_1, \ldots, u_k, \ldots)$ such that $u_k \in X_\psi^{(k)}$ and $u_j = 0$ for $j \neq k$. We have

$$\|T_k u_k\|_{Y_\psi^{(k)}} = m_k\, \|T u^{(k)}\|_{\prod_{j=1}^{\infty} Y_\psi^{(j)}} \leq m_k\, c_0\, \|u^{(k)}\|_{\prod_{j=1}^{\infty} X_\psi^{(j)}} = m_k\, c_0\, \|u_k\|_{X_\psi^{(k)}}$$

for each $u_k \in X_\psi^{(k)}$. Hence,

$$\|T_k\|_{X_\psi^{(k)} \to Y_\psi^{(k)}} \leq c_0\, m_k \quad \text{for every index } k,$$

contrary to inequality (2.7). Thus, our supposition is false and the theorem is true. □

2.7. **A criterion for a function to be an interpolation parameter.** Using Peetre's results [25], [24, Sec. 5.4] (see also [27]), we prove the following criterion.

Definition 2.3. Let a function $\psi : (0, +\infty) \to (0, +\infty)$ and a number $r \geq 0$ be given. The function ψ is called *pseudoconcave* on the semiaxis $(r, +\infty)$ if a concave function $\psi_1 : (r, +\infty) \to (0, +\infty)$ such that $\psi(t) \asymp \psi_1(t)$ for $t > r$ exists. The function ψ is called pseudoconcave in a neighborhood of $+\infty$ if it is pseudoconcave on a certain semiaxis $(r, +\infty)$, where r is a sufficiently large number.

Theorem 2.7. *A function $\psi \in \mathcal{B}$ is an interpolation parameter if and only if it is pseudoconcave in a neighborhood of $+\infty$.*

To prove this theorem we need two lemmas.

Lemma 2.1. *Let a function ψ belong to the set $\mathcal{B}$ and be pseudoconcave in a neighborhood of $+\infty$. Then there is a concave function $\psi_0 : (0, +\infty) \to (0, +\infty)$ such that for every number $\varepsilon > 0$ it holds $\psi(t) \asymp \psi_0(t)$ with $t \geq \varepsilon$.*

Proof is evident. □

Lemma 2.2. *Let a function $\psi \in \mathcal{B}$ and a number $r \geq 0$ be given. The function ψ is pseudoconcave on the semiaxis $(r, +\infty)$ if and only if there is a number $c > 0$ such that*

$$\psi(t)/\psi(s) \leq c \max\{1,\, t/s\} \quad \textit{for each} \quad t, s > r.$$

Proof. In the case where $r = 0$ this lemma was proved by J. Peetre [25], [24, Lemma 5.4.3] (the condition $\psi \in \mathcal{B}$ being superfluous). In the case where $r > 0$ the sufficiency can be proved analogously. The necessity is be reduced to the case $r = 0$ with the help of Lemma 2.1. Indeed, let us put $\varepsilon = r$ in this lemma. Then we have a function ψ_0 such that

$$\psi(t)/\psi(s) \asymp \psi_0(t)/\psi_0(s) \leq c_0 \max\{1,\, t/s\} \quad \text{for each} \quad t, s > r.$$

(In fact, $c_0 = 1$ for a concave function ψ_0 [25]). Lemma 2.2 is proved. □

Proof of theorem 2.7. **Sufficiency.** Let us suppose that a function $\psi \in \mathcal{B}$ is pseudoconcave in a neighborhood of $+\infty$. We need to prove that ψ is an interpolation parameter.

Let admissible couples $X = [X_0, X_1]$, $Y = [Y_0, Y_1]$ and a linear mapping T be the same as those in Definition 2.2. In addition, let operators $J_X : X_1 \leftrightarrow X_0$ and $J_Y : Y_1 \leftrightarrow Y_0$ be generating once for the couples X and Y respectively. Using the spectral theorem we reduce these operators, self-adjoint in X_0 and in Y_0 respectively, to the form of multiplication by a function

$$J_X = I_X^{-1}\,(\alpha \cdot I_X) \quad \text{and} \quad J_Y = I_Y^{-1}\,(\beta \cdot I_Y). \tag{2.9}$$

Here, $I_X : X_0 \leftrightarrow L_2(U, d\mu)$ and $I_Y : Y_0 \leftrightarrow L_2(V, d\nu)$ are certain isometric isomorphisms, (U, μ) and (V, ν) are spaces with finite measures and $\alpha : U \to (0, +\infty)$ and $\beta : V \to (0, +\infty)$ are some measurable functions. Since the operators $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ are bounded, so are the operators

$$I_Y\, T\, I_X^{-1} :\, L_2(U, d\mu) \to L_2(V, d\nu), \tag{2.10}$$

$$I_Y\, J_Y\, T\, J_X^{-1}\, I_X^{-1} :\, L_2(U, d\mu) \to L_2(V, d\nu). \tag{2.11}$$

By virtue of (2.9), we can write

$$I_Y\, J_Y\, T\, J_X^{-1}\, I_X^{-1} = (\beta \cdot I_Y)\, T\, (\alpha^{-1} \cdot I_X^{-1}).$$

Hence (2.11) implies the boundedness of the operator

$$I_Y\, T\, I_X^{-1} = \beta^{-1} \cdot (I_Y\, J_Y\, T\, J_X^{-1}\, I_X^{-1}) \cdot \alpha :\, L_2(U, \alpha^2 d\mu) \to L_2(V, \beta^2 d\nu). \tag{2.12}$$

Let a concave function $\psi_0 : (0, +\infty) \to (0, +\infty)$ be the same as that in Lemma 2.1. Let us note that $\psi_0 \in \mathcal{B}$ and (see Subsection 2.1)

$$X_\psi = X_{\psi_0}, \quad Y_\psi = Y_{\psi_0} \quad \text{with equivalence of norms.} \tag{2.13}$$

J. Peetre [25], [24, Theorem 5.4.4] proved that a positive function is pseudoconcave on $(0, +\infty)$ if and only if it is an interpolation function in the sense of the definition stated in [24, Definition 5. 4.2]. Hence, for the function ψ_0, the boundedness of operators (2.10), (2.12) implies the existence of a bounded operator

$$I_Y\, T\, I_X^{-1} :\, L_2(U, (\psi_0 \circ \alpha^2)\, d\mu) \to L_2(V, (\psi_0 \circ \beta^2)\, d\nu). \tag{2.14}$$

Let us pass from (2.14) to the operator $T : X_{\psi_0} \to Y_{\psi_0}$ with the help of the isometric isomorphisms $\psi_0(J_X) :\, X_{\psi_0} \leftrightarrow X_0$ and $\psi_0(J_Y) :\, Y_{\psi_0} \leftrightarrow Y_0$. We reduce these isomorphisms (which are self-adjoint operators in X_0 and Y_0 respectively) to the form of the multiplication by a function

$$I_X\, \psi_0(J_X) = (\psi_0 \circ \alpha) \cdot I_X :\, X_{\psi_0} \leftrightarrow L_2(U, d\mu),$$
$$I_Y\, \psi_0(J_Y) = (\psi_0 \circ \beta) \cdot I_Y :\, Y_{\psi_0} \leftrightarrow L_2(V, d\nu).$$

We get the isometric isomorphisms

$$I_X = (\psi_0^{-1} \circ \alpha) \cdot (I_X\, \psi_0(J_X)) :\, X_{\psi_0} \leftrightarrow L_2(U, (\psi^2 \circ \alpha)\, d\mu),$$
$$I_Y = (\psi_0^{-1} \circ \beta) \cdot (I_Y\, \psi_0(J_Y)) :\, Y_{\psi_0} \leftrightarrow L_2(V, (\psi^2 \circ \beta)\, d\nu).$$

From this and (2.14) the existence of the bounded operator

$$T = I_Y^{-1}(I_Y\, T\, I_X^{-1}) I_X :\, X_{\psi_0} \to Y_{\psi_0}$$

follows.

Thus, due to equations (2.13) we have

$$(T : X_j \to Y_j,\, j = 0, 1) \,\Rightarrow\, (T : X_{\psi_0} \to Y_{\psi_0}) \,\Rightarrow\, (T : X_\psi \to Y_\psi),$$

where the linear operators are bounded. So, by Definition 2.2 the function ψ is an interpolation parameter. Sufficiency is proved.

Necessity. Now we suppose that a function $\psi \in \mathcal{B}$ is an interpolation parameter. We need to prove that ψ is pseudoconcave in a neighborhood of $+\infty$. The proof is similar to [25], [24, Sec. 5.4].

Let us consider the space $L_2(U, d\mu)$ with $U = \{0, 1\}$, $\mu(\{0\}) = \mu(\{1\}) = 1$ and define on it the linear mapping T by the formula $(Tu)(0) := 0$, $(Tu)(1) := u(0)$, where $u \in L_2(U, d\mu)$. Let us choose arbitrary numbers $s, t > 1$ and put $\omega(0) := s^2$, $\omega(1) := t^2$. We have the admissible couple $X := [L_2(U, d\mu), L_2(U, \omega\, d\mu)]$ and bounded operators

$$T : L_2(U, d\mu) \to L_2(U, d\mu) \quad \text{and} \quad T : L_2(U, \omega\, d\mu) \to L_2(U, \omega\, d\mu)$$

with norms 1 and t/s respectively. From this, since ψ is an interpolation parameter, it follows that the bounded operator $T : X_\psi \to X_\psi$ exists. By Theorem 2.6 with $Y = X$ and $m = 1$ we conclude that the norm of this operator satisfies the inequality

$$\|T\|_{X_\psi \to X_\psi} \leq c \max\{1, t/s\}. \tag{2.15}$$

Here, the number $c > 0$ does not depend on $t, s > 1$.

It is not difficult to calculate the norm in the space X_ψ. Indeed, the operator J of multiplication by the function $\omega^{1/2}$ is a generating one for the couple X. Hence, since $\psi(J)$ is the operator of multiplication by the function $\psi \circ \omega^{1/2}$, we can write

$$\big\|u\big\|^2_{X_\psi} = \big\|(\psi \circ \omega^{1/2})\, u\big\|^2_{L_2(U,d\mu)} = \psi^2(s)\,|u(0)|^2 + \psi^2(t)\,|u(1)|^2, \quad \|Tu\|^2_{X_\psi} = \psi^2(t)\,|u(0)|^2.$$

It follows that

$$\|T\|_{X_\psi \to X_\psi} = \psi(t)/\psi(s). \tag{2.16}$$

Now relations (2.15), (2.16) imply the inequality

$$\psi(t) \leq c \max\{1, t/s\}\, \psi(s) \quad \text{for each} \quad t, s > 1.$$

According to Theorem 2.2, the last statement is equivalent to the quasiconcavity of the function ψ on the semiaxis $(1, +\infty)$. Necessity is proved. □

3. A refined scale of spaces

3.1. Quasiregularly varying functions. We recall the following:

Definition 3.1. A positive function ψ defined on a semiaxis $[b, +\infty)$ is called a function *regularly varying* at $+\infty$ with the index $\theta \in \mathbb{R}$ if ψ is Borel measurable on $[b_0, +\infty)$ for some number $b_0 \geq b$ and

$$\lim_{t \to +\infty} \psi(\lambda t)/\psi(t) = \lambda^\theta \quad \text{for each} \quad \lambda > 0.$$

A function regularly varying at $+\infty$ with the index $\theta = 0$ is called *slowly varying* at $+\infty$.

The theory of regularly varying functions was founded by J. Karamata in the 1930s. These functions are closely related to the power functions and have numerous applications, mainly due to their special role in Tauberian-type theorems [26, 28, 29, 30]. A standard example of functions regularly varying at $+\infty$ with the index θ is

$$\psi(t) = t^\theta\, (\ln t)^{r_1}\, (\ln \ln t)^{r_2} \ldots (\ln \ldots \ln t)^{r_k} \quad \text{for} \quad t \gg 1,$$

where $r_1, r_2, \ldots, r_k \in \mathbb{R}$. In the case where $\theta = 0$ these functions form the *logarithmic multiscale* which has a number of applications in the theory of function spaces.

Definition 3.2. A positive function ψ defined on a semiaxis $[b, +\infty)$ is called a function *quasiregularly varying* at $+\infty$ with the index $\theta \in \mathbb{R}$ if there exist a number $b_1 \geq b$ and a function $\psi_1 : [b_1, +\infty) \to (0, +\infty)$ regular varying at $+\infty$ with the index θ such that $\psi(t) \asymp \psi_1(t)$ with $t \geq b_1$. A function quasiregularly varying at $+\infty$ with the index $\theta = 0$ is called *quasislowly varying* at $+\infty$.

Let us denote by QSV the set of all functions quasislowly varying at $+\infty$. It is evident that ψ is a function quasiregularly varying at $+\infty$ with the index θ if and only if $\psi(t) = t^\theta \varphi(t)$, $t \gg 1$, for some function $\varphi \in \mathrm{QSV}$. From the known [26, Theorem 1.2] integral representation of a slowly varying function it immediately results the following description of the set QSV.

Theorem 3.1. *Let* $\varphi \in \mathrm{QSV}$. *Then*

$$\varphi(t) = \exp\left(\beta(t) + \int_r^t \frac{\alpha(\tau)}{\tau}\, d\tau\right), \quad t \geq r, \tag{3.1}$$

for some number $r > 0$, *a continuous function* $\alpha : [r, +\infty) \to \mathbb{R}$ *approaching zero at* $+\infty$ *and a bounded function* $\beta : [r, +\infty) \to \mathbb{R}$. *The converse statement is also true: every function of form* (3.1) *belongs to the set* QSV.

Following interpolation property of quasiregularly varying functions will play a decisive role in further.

Theorem 3.2. *Let* $\psi \in \mathcal{B}$ *be a function quasiregularly varying at* $+\infty$ *with the index* θ *where* $0 < \theta < 1$. *Then* ψ *is an interpolation parameter.*

Proof. We can write $\psi(t) = t^{\theta}\varphi(t)$ for $t > 0$ with $\varphi \in \mathrm{QSV}$. According to Theorem 3.1, the function φ can be represented in form (3.1). Let us set $\varepsilon := \min\{\theta, 1-\theta\} > 0$ and choose a number $r_\varepsilon \geq r$ such that $|\alpha(t)| < \varepsilon$ for $t > r_\varepsilon$. For each $t, s > r_\varepsilon$, we have by virtue of (3.1) the following:

$$\frac{\varphi(t)}{\varphi(s)} = \exp\left(\beta(t) - \beta(s) + \int_s^t \frac{\alpha(\tau)}{\tau}\, d\tau\right) \leq c \exp\left|\int_s^t \frac{\varepsilon}{\tau}\, d\tau\right| = c \max\left\{(t/s)^{\varepsilon}, (s/t)^{\varepsilon}\right\}.$$

Here, the number $c > 0$ does not depend on t and s because the function β is bounded. From this and from the inequality $0 \leq \theta \pm \varepsilon \leq 1$ it follows that

$$\psi(t)/\psi(s) = (t^{\theta}\varphi(t))/(s^{\theta}\varphi(s)) \leq c \max\left\{(t/s)^{\theta+\varepsilon}, (t/s)^{\theta-\varepsilon}\right\} \leq c \max\{1, t/s\}.$$

Hence, by Theorem 2.2 the function $\psi \in \mathcal{B}$ is pseudoconcave in a neighborhood of $+\infty$. According to Theorem 2.7, this is equivalent to the statement that ψ is an interpolation parameter. Theorem 3.2 is proved. □

We need the following properties of the set QSV.

Theorem 3.3. *Let* $\varphi, \chi \in \mathrm{QSV}$. *The following assertions are true.*

(i) *There is a positive function* $\varphi_1 \in C^{\infty}((0; +\infty))$ *regularly varying at* $+\infty$ *such that* $\varphi(t) \asymp \varphi_1(t)$ *with* $t \gg 1$.
(ii) *For each* $\theta > 0$, *the limits* $t^{-\theta}\varphi(t) \to 0$ *and* $t^{\theta}\varphi(t) \to +\infty$ *as* $t \to +\infty$ *hold.*
(iii) *The functions* $\varphi + \chi$, $\varphi\,\chi$, φ/χ *and* φ^{σ}, *where* $\sigma \in \mathbb{R}$, *belong to the set* QSV.
(iv) *Let* $\theta \geq 0$, *in the case where* $\theta = 0$ *suppose that* $\varphi(t) \to \infty$ *as* $t \to +\infty$. *Then the composite function* $\chi(t^{\theta}\varphi(t))$ *of the argument* t *belongs to the set* QSV.

Proof. For regularly varying functions φ, χ these assertions are known [26, Sec. 1.5] (even with the strong equivalence being in assertion (i)). This implies immediately assertions (i), (ii), (iii) for the functions $\varphi, \chi \in \mathrm{QSV}$.

It remains to prove assertion (iv). Let $\lambda > 0$. Since $\varphi \in \mathrm{QSV}$, the functions $\varphi(\lambda t)/\varphi(t)$ and $\varphi(t)/\varphi(\lambda t)$ are bounded in a neighborhood of $+\infty$. Therefore applying the theorem [26, Sec. 1.2] on uniform convergence to a positive slowly varying function χ_1 such that $\chi_1(\tau) \asymp \chi(\tau)$ with $\tau \gg 1$, we can write

$$\chi_1\left((\lambda t)^{\theta}\varphi(\lambda t)\right)/\chi_1\left(t^{\theta}\varphi(t)\right) = \chi_1\left(\frac{\lambda^{\theta}\varphi(\lambda t)}{\varphi(t)}\, t^{\theta}\varphi(t)\right)\Big/\chi_1\left(t^{\theta}\varphi(t)\right) \to 1 \quad \text{as} \quad t \to +\infty.$$

Here we use the limit $t^{\theta}\varphi(t) \to \infty$ as $t \to +\infty$. Hence, the function $\chi_1(t^{\theta}\varphi(t))$ is slowly varying at $+\infty$. In addition, $\chi(t^{\theta}\varphi(t)) \asymp \chi_1(t^{\theta}\varphi(t))$ with $t \gg 1$. Thus, the function $\chi(t^{\theta}\varphi(t))$ belongs to the set QSV. Assertion (iv) is proved. □

3.2. **A refined scale over the Euclidean space.** Let $n \in \mathbb{N}$. As usual, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathcal{S}'(\mathbb{R}^n)$ denotes the linear topological Schwartz space of tempered distributions in $\mathbb{R}^n$. We use also the following notations: $\langle\xi\rangle = (1+\xi_1^2+\cdots+\xi_n^2)^{1/2}$ denotes the smoothed modulus of a vector $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$ and $\widehat{u}$ denotes the Fourier transform of the distribution $u \in \mathcal{S}'(\mathbb{R}^n)$. We will write an integral evaluated over the space $\mathbb{R}^n$ without limits.

Let $\mathcal{M}$ denote the set of all functions $\varphi : [1;+\infty) \to (0;+\infty)$ such that

a) φ is Borel measurable on the set $[1;+\infty)$;
b) functions φ and $1/\varphi$ are bounded on every closed interval $[1;b]$, where $1 < b < +\infty$;
c) $\varphi \in \mathrm{QSV}$.

Let $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$.

Definition 3.3. We denote by $H^{s,\varphi}(\mathbb{R}^n)$ the space of all distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that the Fourier transform $\widehat{u}$ is a function locally Lebesgue integrable on $\mathbb{R}^n$ which satisfies the inequality

$$\int \langle\xi\rangle^{2s}\varphi^2(\langle\xi\rangle)\,|\widehat{u}(\xi)|^2\,d\xi < \infty.$$

The inner product in the space $\mathrm{H}^{s,\varphi}(\mathbb{R}^n)$ is defined by the formula

$$(u,v)_{\mathrm{H}^{s,\varphi}(\mathbb{R}^n)} := \int \langle\xi\rangle^{2s}\varphi^2(\langle\xi\rangle)\,\widehat{u}(\xi)\,\overline{\widehat{v}(\xi)}\,d\xi$$

and generates the norm in the usual way.

The space $H^{s,\varphi}(\mathbb{R}^n)$ is a special isotropic Hilbert case of the spaces introduced by L. Hörmander [31, Sec. 2.2], [32, Sec. 10.1] and L. R. Volevich, B. P. Paneah [33, Sec. 2], [34, Sec. 1.4.2]. Let us note that this space is actually defined with the help of the function $\varphi_s(t) = t^s\varphi(t)$ regularly varying at $+\infty$ with the index s. However it is more convenient for us to represent the parameter φ_s as the couple of two parameters s and φ.

In the particular case where $\varphi \equiv 1$ the space $H^{s,\varphi}(\mathbb{R}^n)$ coincides with the Sobolev space $H^s(\mathbb{R}^n)$. In general, the following inclusions are true:

$$\bigcup_{\varepsilon>0} H^{s+\varepsilon}(\mathbb{R}^n) =: H^{s+}(\mathbb{R}^n) \subset H^{s,\varphi}(\mathbb{R}^n) \subset H^{s-}(\mathbb{R}^n) := \bigcap_{\varepsilon>0} H^{s-\varepsilon}(\mathbb{R}^n). \tag{3.2}$$

They result from assertion (ii) of Theorem 3.3 and from the definition of the set $\mathcal{M}$, according to which, for each $\varepsilon > 0$, there is a number $c_\varepsilon \geq 1$ such that $c_\varepsilon^{-1}t^{-\varepsilon} \leq \varphi(t) \leq c_\varepsilon t^\varepsilon$ for $t \geq 1$. Inclusions (3.2) mean that, in the collection of the spaces

$$\{H^{s,\varphi}(\mathbb{R}^n) : s \in \mathbb{R}, \varphi \in \mathcal{M}\}, \tag{3.3}$$

the function parameter φ *refines* the basic (power) s-smoothness. Therefore it is natural to give the following definition.

Definition 3.4. The collection of function spaces (3.3) is called a *refined* scale over $\mathbb{R}^n$ (with respect to the Sobolev scale).

Besides the properties inherent to the Hörmander spaces [31, Sec. 2.2], [32, Sec. 10.1] and the Volevich-Paneah spaces [33, Ch. I, II], [34, Sec. 1.4], the refined scale over $\mathbb{R}^n$ possesses the following fundamental interpolation property:

Theorem 3.4. *Let a function $\varphi \in \mathcal{M}$ and positive numbers ε, δ be given. Let $\psi(t) := t^{\varepsilon/(\varepsilon+\delta)}\,\varphi(t^{1/(\varepsilon+\delta)})$ for $t \geq 1$ and $\psi(t) := \varphi(1)$ for $0 < t < 1$. Then the following assertions are true:*

(i) *The function ψ belongs to the set $\mathcal{B}$ and is an interpolation parameter.*

(ii) *For an arbitrary $s \in \mathbb{R}$, the equality of spaces*

$$\left[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\right]_\psi = H^{s,\varphi}(\mathbb{R}^n)$$

and equality of norms in them hold.

Proof. Assertion (i). By virtue of assertions (ii), (iv) of Theorem 3.3, the function ψ belongs to the set $\mathcal{B}$ and is a function regular varying at $+\infty$ with the index $\theta = \varepsilon/(\varepsilon+\delta) \in (0, 1)$. Therefore ψ is an interpolation parameter because of Theorem 3.2. Assertion (i) is proved.

Assertion (ii). Let $s \in \mathbb{R}$. It follows from the properties of the Sobolev spaces that the couple $[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)]$ is admissible and the pseudodifferential operator with symbol $\langle\xi\rangle^{\varepsilon+\delta}$ is a generating operator J for this couple. Applying the Fourier transform $\mathcal{F} : H^{s-\varepsilon}(\mathbb{R}^n) \leftrightarrow L_2(\mathbb{R}^n, \langle\xi\rangle^{2(s-\varepsilon)}d\xi)$, we reduce the operator J to the form of multiplication by the function $\langle\xi\rangle^{\varepsilon+\delta}$ of $\xi \in \mathbb{R}^n$. Hence, the operator $\psi(J)$ is reduced to the form of multiplication by the function $\psi(\langle\xi\rangle^{\varepsilon+\delta}) = \langle\xi\rangle^{\varepsilon}\varphi(\langle\xi\rangle)$. This permits us to write the following in view of (3.2):

$$\begin{aligned}\left[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\right]_\psi &= \left\{u \in H^{s-\varepsilon}(\mathbb{R}^n) : \langle\xi\rangle^{\varepsilon}\varphi(\langle\xi\rangle)\ \widehat{u}(\xi) \in L_2(\mathbb{R}^n, \langle\xi\rangle^{2(s-\varepsilon)}d\xi)\right\} \\ &= \left\{u \in H^{s-\varepsilon}(\mathbb{R}^n) : \int \langle\xi\rangle^{2s}\varphi^2(\langle\xi\rangle)\,|\widehat{u}(\xi)|^2\,d\xi < \infty\right\} \\ &= H^{s-\varepsilon}(\mathbb{R}^n) \cap H^{s,\varphi}(\mathbb{R}^n) = H^{s,\varphi}(\mathbb{R}^n).\end{aligned}$$

In addition the norm in the space $\left[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\right]_\psi$ is equal to

$$\|\psi(J)u\|_{H^{s-\varepsilon}(\mathbb{R}^n)} = \left(\int |\langle\xi\rangle^{\varepsilon}\varphi(\langle\xi\rangle)\ \widehat{u}(\xi)|^2\,\langle\xi\rangle^{2(s-\varepsilon)}\ d\xi\right)^{1/2} = \|u\|_{H^{s,\varphi}(\mathbb{R}^n)}.$$

Assertion (ii) is proved. □

3.3. **A refined scale over a closed manifold.** Further let Γ be a closed (i.e., compact and without a boundary) infinitely smooth manifold of dimension n. We suppose that a certain C^∞-density dx is defined on Γ. We denote by $\mathcal{D}'(\Gamma)$ the linear topological space of all distributions on Γ, i.e., $\mathcal{D}'(\Gamma)$ is the space antidual to the space $C^\infty(\Gamma)$ with respect to the extension of the inner product in $L_2(\Gamma, dx) =: L_2(\Gamma)$ by continuity. This extension is denoted by $(f, w)_\Gamma$ for $f \in \mathcal{D}'(\Gamma)$, $w \in C^\infty(\Gamma)$.

The refined scale over the manifold Γ is constructed from scale (3.3) in the following way. We choose a finite atlas from the C^∞-structure on Γ consisting of the local charts $\alpha_j : \mathbb{R}^n \leftrightarrow U_j$, $j = 1, \dots, r$. Here, the open sets U_j form the finite covering of the manifold Γ. Let functions $\chi_j \in C^\infty(\Gamma)$, $j = 1, \dots, r$, form a partition of unity on Γ satisfying the condition $\operatorname{supp}\chi_j \subset U_j$. As before, $s \in \mathbb{R}, \varphi \in \mathcal{M}$.

Definition 3.5. We denote by $H^{s,\varphi}(\Gamma)$ the space of all distributions $f \in \mathcal{D}'(\Gamma)$ such that $(\chi_j f) \circ \alpha_j \in H^{s,\varphi}(\mathbb{R}^n)$ for each $j = 1, \dots, r$. Here $(\chi_j f) \circ \alpha_j$ is the representation of the distribution $\chi_j f$ in the local charts α_j. The inner product in the space $H^{s,\varphi}(\Gamma)$ is defined by the formula

$$(f, g)_{H^{s,\varphi}(\Gamma)} := \sum_{j=1}^{r} ((\chi_j f) \circ \alpha_j, (\chi_j\, g) \circ \alpha_j)_{H^{s,\varphi}(\mathbb{R}^n)}$$

and induces the norm in the usual way.

Definition 3.6. The collection of function spaces $\{H^{s,\varphi}(\Gamma) : s \in \mathbb{R}, \varphi \in \mathcal{M}\}$ is called a *refined* scale over the closed manifold Γ.

In the particular case where $\varphi \equiv 1$ the space $H^{s,\varphi}(\Gamma)$ coincides with the Sobolev space $H^s(\Gamma)$. Sobolev spaces are known [31, Sec. 2.6], [35, Sec. 7.5] to be complete and independent (up to equivalence of norms) of the choice of the atlas and the partition of unity. We will show that every space $H^{s,\varphi}(\Gamma)$ can be obtained by the interpolation of the proper couple of Sobolev's spaces. It implies that the space $H^{s,\varphi}(\Gamma)$ is Hilbert and independent of this choice.

Theorem 3.5. *Let a function $\varphi \in \mathcal{M}$ and positive numbers ε, δ be given. Then, for each $s \in \mathbb{R}$, the equality of spaces*

$$\big[H^{s-\varepsilon}(\Gamma),\ H^{s+\delta}(\Gamma)\big]_{\psi} = H^{s,\varphi}(\Gamma) \quad \text{with equivalence of norms} \tag{3.4}$$

hold. Here, ψ is the interpolation parameter from Theorem 3.4.

Proof. The couple of the Sobolev spaces on the left-hand side of equality (3.4) is admissible [35, Sec. 7.5, 7.6]. We deduce this equality from Theorem 3.4 with the help of the well known method of "rectification" and "sewing" of the manifold Γ. According to Definition 3.5, the linear mapping of "rectification"

$$T :\ f \mapsto (\,(\chi_1 f)\circ\alpha_1, \ldots, (\chi_r f)\circ\alpha_r\,), \quad f \in \mathcal{D}'(\Gamma),$$

defines the isometric operators

$$T :\ H^{\sigma}(\Gamma) \to (H^{\sigma}(\mathbb{R}^n))^r, \quad \sigma \in \mathbb{R}, \tag{3.5}$$

$$T :\ H^{s,\varphi}(\Gamma) \to (H^{s,\varphi}(\mathbb{R}^n))^r. \tag{3.6}$$

Since ψ is the interpolation parameter and operators (3.5) are bounded for $\sigma \in \{s-\varepsilon, s+\delta\}$, the bounded operator

$$T :\ \big[H^{s-\varepsilon}(\Gamma), H^{s+\delta}(\Gamma)\big]_{\psi} \to \big[\,(H^{s-\varepsilon}(\mathbb{R}^n))^r,\ (H^{s+\delta}(\mathbb{R}^n))^r\,\big]_{\psi} \tag{3.7}$$

exists. By virtue of Theorems 2.5, 3.4, the following equalities of spaces and norms in them are true:

$$\big[\,(H^{s-\varepsilon}(\mathbb{R}^n))^r,\ (H^{s+\delta}(\mathbb{R}^n))^r\,\big]_{\psi} = \Big(\,\big[H^{s-\varepsilon}(\mathbb{R}^n), H^{s+\delta}(\mathbb{R}^n)\big]_{\psi}\Big)^r = (H^{s,\varphi}(\mathbb{R}^n))^r. \tag{3.8}$$

Thus, since operator (3.7) is bounded, so is the operator

$$T :\ \big[H^{s-\varepsilon}(\Gamma), H^{s+\delta}(\Gamma)\big]_{\psi} \to (H^{s,\varphi}(\mathbb{R}^n))^r. \tag{3.9}$$

Now we construct for T the left inverse operator K of "sewing" of the manifold Γ. For each $j = 1, \ldots, r$ we choose a function $\eta_j \in C_0^{\infty}(\mathbb{R}^n)$ such that $\eta_j = 1$ on the set $\alpha_j^{-1}(\operatorname{supp}\chi_j)$. Let us consider the linear mapping

$$K :\ (h_1, \ldots, h_r) \mapsto \sum_{j=1}^{r} \Theta_j\left((\eta_j h_j)\circ\alpha_j^{-1}\right), \quad h_1, \ldots, h_r \in \mathcal{S}'(\mathbb{R}^n).$$

Here $(\eta_j h_j)\circ\alpha_j^{-1}$ is a distribution in the open set $U_j \subseteq \Gamma$ such that its representative in the local chart α_j has the form $\eta_j h_j$. In addition, Θ_j denotes the operator of extension by zero from U_j to Γ. This operator is well defined on distributions with support belonging to U_j. By the choice of the functions χ_j, η_j, we have

$$KTf = \sum_{j=1}^{r} \Theta_j\left((\eta_j\,((\chi_j f)\circ\alpha_j))\circ\alpha_j^{-1}\right) = \sum_{j=1}^{r} \Theta_j\left(\,(\chi_j f)\circ\alpha_j\circ\alpha_j^{-1}\right) = \sum_{j=1}^{r} \chi_j f = f,$$

i.e.,

$$KTf = f \quad \text{for each} \quad f \in \mathcal{D}'(\Gamma). \tag{3.10}$$

Let us show that the linear mapping K defines the bounded operator

$$K :\ (H^{s,\varphi}(\mathbb{R}^n))^r \to H^{s,\varphi}(\Gamma). \tag{3.11}$$

For an arbitrary vector $h=(h_1,\dots,h_r)$ from the space $(H^{s,\varphi}(\mathbb{R}^n))^r$, we write

$$\begin{aligned}\|Kh\|^2_{H^{s,\varphi}(\Gamma)} &= \sum_{l=1}^{r}\|(\chi_l\,Kh)\circ\alpha_l\|^2_{H^{s,\varphi}(\mathbb{R}^n)}\\ &= \sum_{l=1}^{r}\Big\|\Big(\chi_l\sum_{j=1}^{r}\ \Theta_j\big((\eta_j h_j)\circ\alpha_j^{-1}\big)\Big)\circ\alpha_l\Big\|^2_{H^{s,\varphi}(\mathbb{R}^n)} \qquad (3.12)\\ &= \sum_{l=1}^{r}\Big\|\sum_{j=1}^{r}(\eta_{j,l}\,h_j)\circ\beta_{j,l}\Big\|^2_{H^{s,\varphi}(\mathbb{R}^n)} \le \sum_{l=1}^{r}\Big(\sum_{j=1}^{r}\|(\eta_{j,l}\,h_j)\circ\beta_{j,l}\|_{H^{s,\varphi}(\mathbb{R}^n)}\Big)^2.\end{aligned}$$

Here, $\eta_{j,l}:=(\chi_l\circ\alpha_j)\,\eta_j\in C_0^\infty(\mathbb{R}^n)$ and $\beta_{j,l}:\mathbb{R}^n\leftrightarrow\mathbb{R}^n$ is a C^∞-diffeomorphism such that $\beta_{j,l}=\alpha_j^{-1}\circ\alpha_l$ in a neighborhood of $\operatorname{supp}\eta_{j,l}$ and $\beta_{j,l}(x)=x$ for all $x\in\mathbb{R}^n$ sufficiently large in modulus. The operator of multiplication by a function of the class $C_0^\infty(\mathbb{R}^n)$ and the operator of change of variables $u\mapsto u\circ\beta_{j,l}$ are known [36, Theorems B.1.7, B.1.8] to be bounded on every space $H^\sigma(\mathbb{R}^n)$ with $\sigma\in\mathbb{R}$. Therefore the linear operator $v\mapsto(\eta_{j,l}\,v)\circ\beta_{j,l}$ is bounded on the space $H^\sigma(\mathbb{R}^n)$. Then, boundedness of this operator on the space $H^{s,\varphi}(\mathbb{R}^n)$ follows from Theorem 3.3. Hence relations (3.12) imply the estimate

$$\|Kh\|^2_{H^{s,\varphi}(\Gamma)}\le c\sum_{j=1}^{r}\|h_j\|^2_{H^{s,\varphi}(\mathbb{R}^n)},$$

where the constant $c>0$ is independent of $h=(h_1,\dots,h_r)$. Thus, operator (3.11) is bounded for each $s\in\mathbb{R}$, $\varphi\in\mathcal{M}$.

In particular, the operators $K:(H^\sigma(\mathbb{R}^n))^r\to H^\sigma(\Gamma)$ with $\sigma\in\mathbb{R}$ are bounded. Let us choose $\sigma\in\{s-\varepsilon,\,s+\delta\}$ and use the interpolation with the parameter ψ. Due to equality (3.8), we obtain the bounded operator

$$K:(H^{s,\varphi}(\mathbb{R}^n))^r\to\big[H^{s-\varepsilon}(\Gamma),\,H^{s+\delta}(\Gamma)\big]_\psi. \qquad (3.13)$$

Now formulae (3.6), (3.13) and (3.10),imply that the identity mapping KT establishes the continuous embedding of the space $H^{s,\varphi}(\Gamma)$ into the interpolation space $[H^{s-\varepsilon}(\Gamma),$ $H^{s+\delta}(\Gamma)]_\psi$. Moreover, formulae (3.10) and (3.13) imply that the same mapping KT establishes also the inverse continuous embedding. Theorem 3.5 is proved. □

The following properties of the refined scale over the manifold Γ can be deduced from Theorem 3.5 and the interpolation properties established in Section 2.

Theorem 3.6. *Let $s\in\mathbb{R}$ and $\varphi,\varphi_1\in\mathcal{M}$. The following assertions are true.*

- (i) *The space $H^{s,\varphi}(\Gamma)$ is Hilbert separable and does not depend (up to equivalence of norms) on the choice of an atlas for Γ and partition of unity used in Definition* 3.5.
- (ii) *The set $C^\infty(\Gamma)$ is dense in the space $H^{s,\varphi}(\Gamma)$.*
- (iii) *For each $\varepsilon>0$, the compact and dense embedding $H^{s+\varepsilon,\varphi_1}(\Gamma)\hookrightarrow H^{s,\varphi}(\Gamma)$ holds.*
- (iv) *Suppose that the function φ/φ_1 is bounded in a neighborhood of $+\infty$. Then continuous dense embedding $H^{s+\varepsilon,\varphi_1}(\Gamma)\hookrightarrow H^{s,\varphi}(\Gamma)$ is valid. It is compact if $\varphi(t)/\varphi_1(t)\to0$ as $t\to+\infty$.*
- (v) *The spaces $H^{s,\varphi}(\Gamma)$ and $H^{-s,1/\varphi}(\Gamma)$ are mutually dual (up to equivalence of norms) with respect to the extension of the inner product in $L_2(\Gamma)$ by continuity.*

Proof. Assertion (i). The space $H^{s,\varphi}(\Gamma)$ is Hilbert and separable because, according to Theorem 3.5, this space is obtained by the interpolation of a certain couple of the Sobolev spaces. Let us consider two couples $\mathcal{A}_1$ and $\mathcal{A}_2$ each of which consists of an atlas and a partition of unity on Γ. We denote by $H^{s,\varphi}(\Gamma,\mathcal{A}_j)$ and $H^\sigma(\Gamma,\mathcal{A}_j)$ respectively the spaces from the refined scale and the Sobolev spaces which correspond to the couple $\mathcal{A}_j$, where $j=1,\,2$. For the Sobolev spaces, the identity mapping establishes the topological isomorphism $I : H^{\sigma}(\Gamma, \mathcal{A}_1) \leftrightarrow H^{\sigma}(\Gamma, \mathcal{A}_2)$ for each $\sigma \in \mathbb{R}$. Let us set $\sigma = s \mp 1$ and use the interpolation with the parameter ψ from Theorem 3.4. By virtue of Theorem 3.5 we arrive at the topological isomorphism $I : H^{s,\varphi}(\Gamma, \mathcal{A}_1) \leftrightarrow H^{s,\varphi}(\Gamma, \mathcal{A}_2)$. It means that the space $H^{s,\varphi}(\Gamma)$ is independent of the choice of the atlas and the unity partition mentioned above. Assertion (i) is proved.

Assertion (ii). By virtue of Theorems 2.1 and 3.5, we have the continuous dense embedding $H^{s+\delta}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma)$. Besides, the set $C^{\infty}(\Gamma)$ is dense in the Sobolev space $H^{s+\delta}(\Gamma)$ [35, Proposition 7.4]. These two facts imply assertion (ii).

Assertion (iii). Assume that $\varepsilon > 0$. By Theorem 3.5, there exist interpolation parameters $\chi, \eta \in \mathcal{B}$ such that the following equalities of spaces with equivalence of norms in them is true:

$$\left[H^{s+\varepsilon/2}(\Gamma), H^{s+2\varepsilon}(\Gamma)\right]_{\chi} = H^{s+\varepsilon,\varphi_1}(\Gamma) \quad \text{and} \quad \left[H^{s-\varepsilon}(\Gamma), H^{s+\varepsilon/3}(\Gamma)\right]_{\eta} = H^{s,\varphi}(\Gamma).$$

It implies by Theorem 2.1 the next chain of continuous embeddings

$$H^{s+\varepsilon,\varphi_1}(\Gamma) \hookrightarrow H^{s+\varepsilon/2}(\Gamma) \hookrightarrow H^{s+\varepsilon/3}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma).$$

Here, the central embedding of Sobolev spaces is compact [35, Theorem 7.4]. Therefore the embedding $H^{s+\varepsilon,\varphi_1}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma)$ is compact as well. This embedding is dense because of assertion (ii). Assertion (iii) is proved.

Assertion (iv). Let us assume that the function φ/φ_1 is bounded in a neighborhood of $+\infty$. By Theorem 3.5, we have the following equalities of spaces with equivalence of norms in them:

$$\left[H^{s-1}(\Gamma), H^{s+1}(\Gamma)\right]_{\psi} = H^{s,\varphi}(\Gamma) \quad \text{and} \quad \left[H^{s-1}(\Gamma), H^{s+1}(\Gamma)\right]_{\psi_1} = H^{s,\varphi_1}(\Gamma).$$

Here, the interpolation parameters $\psi, \psi_1 \in \mathcal{B}$ satisfy the condition $\psi(t)/\psi_1(t) = \varphi(t^{1/2})/\varphi_1(t^{1/2})$ for $t \geq 1$. Hence, the function ψ/ψ_1 is bounded in a neighborhood of $+\infty$ that, by Theorem 2.2, implies the continuous dense embedding $H^{s,\varphi_1}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma)$. Now suppose that $\varphi(t)/\varphi_1(t) \to 0$ as $t \to +\infty$. It implies the limit $\psi(t)/\psi_1(t) \to 0$ as $t \to +\infty$. In addition, we recall that the embedding of the Sobolev spaces $H^{s+1}(\Gamma) \hookrightarrow H^{s-1}(\Gamma)$ is compact. It follows from Theorem 2.2 that the embedding $H^{s,\varphi_1}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma)$ is compact as well. Assertion (iv) is proved.

Assertion (v) is known (see e.g. [35, Theorem 7.7]) in the case $\varphi \equiv 1$. From this the case of an arbitrary $\varphi \in \mathcal{M}$ can be obtained as follows. First let us note that $1/\varphi \in \mathcal{M}$ and therefore the space $H^{-s,1/\varphi}(\Gamma)$ is well defined. The Sobolev spaces $H^{s\pm1}(\Gamma)$ and $H^{-s\mp1}(\Gamma)$ are mutually dual with respect to the extension of the inner product in $L_2(\Gamma)$ by continuity. This means that the linear mapping $Q : w \mapsto (\,\cdot\,, \overline{w})_{\Gamma}$, $w \in C^{\infty}(\Gamma)$, is extended by continuity to the topological isomorphisms $Q : H^{s\mp1}(\Gamma) \leftrightarrow (H^{-s\pm1}(\Gamma))'$. Let us apply to them the interpolation with the parameter ψ from Theorem 3.5 in the case where $\varepsilon = \delta = 1$. We obtain one more topological isomorphism

$$Q : \left[H^{s-1}(\Gamma), H^{s+1}(\Gamma)\right]_{\psi} \leftrightarrow \left[(H^{-s+1}(\Gamma))', (H^{-s-1}(\Gamma))'\right]_{\psi}. \tag{3.14}$$

Here the left-hand interpolation space equals to $H^{s,\varphi}(\Gamma)$ and, by Theorem 2.4, the right-hand one can be written as

$$\left[(H^{-s+1}(\Gamma))', (H^{-s-1}(\Gamma))'\right]_{\psi} = \left[H^{-s-1}(\Gamma), H^{-s+1}(\Gamma)\right]'_{\chi} = (H^{-s,1/\varphi}(\Gamma))'.$$

Let us note that the last equality is valid because $\chi(t) := t/\psi(t) = t^{1/2}/\varphi(t^{1/2})$ for $t \geq 1$. Thus, (3.14) implies the topological isomorphism $Q : H^{s,\varphi}(\Gamma) \leftrightarrow (H^{-s,1/\varphi}(\Gamma))'$, which means the mutual duality of the spaces $H^{s,\varphi}(\Gamma)$ and $H^{-s,1/\varphi}(\Gamma)$ in the sense mentioned above. Assertion (v) is proved. □

The refined scale is closed with respect to the interpolation with a function parameter regular varying at $+\infty$.

Theorem 3.7. *Let $s_0, s_1 \in \mathbb{R}$, $s_0 \leq s_1$ and $\varphi_0, \varphi_1 \in \mathcal{M}$. In the case where $s_0 = s_1$ we suppose that the function φ_0/φ_1 is bounded in a neighborhood of $+\infty$. Let $\psi \in \mathcal{B}$ be a function regularly varying at $+\infty$ with the index θ, where $0 < \theta < 1$. By Theorem 3.2, ψ is an interpolation parameter. We represent it as $\psi(t) = t^\theta \chi(t)$ with $\chi \in$ QSV. Let us set $s := (1-\theta)s_0 + \theta s_1$ and*

$$\varphi(t) := \varphi_0^{1-\theta}(t)\, \varphi_1^{\theta}(t)\, \chi\left(t^{s_1 - s_0}\varphi_1(t)/\varphi_0(t)\right) \quad \textit{for} \quad t \geq 1.$$

Then $\varphi \in \mathcal{M}$ and

$$[\, H^{s_0,\varphi_0}(\Gamma), H^{s_1,\varphi_1}(\Gamma)\,]_\psi = H^{s,\varphi}(\Gamma) \quad \textit{with equivalence of norms.}$$

Proof. This theorem is a direct consequence of Theorems 3.5 and 2.3. □

Remark 3.1. Theorem 3.7 is true in the limiting case where $\theta = 0$ or $\theta = 1$ under additional supposition that the function ψ is pseudoconcave in a neighborhood of $+\infty$. Then, by Theorem 2.7, ψ is an interpolation parameter. For example, Theorem 3.7 is true for each of the functions $\psi(t) := \ln^r t$ and $\psi(t) := t/\ln^r t$, where $t \gg 1$ and $r > 0$.

3.4. **An alternative definition of the refined scale.** Let A be an elliptic pseudodifferential operator on Γ with the index $m > 0$. We suppose that the operator $A : C^\infty(\Gamma) \to C^\infty(\Gamma)$ is positive on the space $L_2(\Gamma)$, i.e., there is a number $r > 0$ such that

$$(Lu, u)_\Gamma \geq r\,(u,u)_\Gamma \quad \text{for each} \quad u \in C^\infty(\Gamma). \tag{3.15}$$

In the present subsection, $(\,\cdot\,,\,\cdot\,)_\Gamma$ is the inner product in $L_2(\Gamma)$.

Let us denote by A_0 the closure of the operator $A : C^\infty(\Gamma) \to C^\infty(\Gamma)$ on the space $L_2(\Gamma)$. This closure exists and has the domain $H^m(\Gamma)$ because the operator A is elliptic on Γ [35, Corollary 8.3], [37, Theorem 2.3.5]. The pseudodifferential operator A is formally self-adjoint due to condition (3.15). Hence [35, Theorem 8.3], [37, Theorem 2.3.7], A_0 is an unbounded self-adjoint operator on the space $L_2(\Gamma)$ with $\operatorname{Spec} A_0 \subseteq [r, +\infty)$. In particular, we have $0 \notin \operatorname{Spec} A_0$, that implies the topological isomorphism

$$A:\ H^{s+m,\varphi}(\Gamma) \leftrightarrow H^{s,\varphi}(\Gamma) \quad \text{for each} \quad s \in \mathbb{R},\ \varphi \in \mathcal{M}. \tag{3.16}$$

In the Sobolev case where $\varphi \equiv 1$ this result is well known (see e.g. [35, Theorem 8.1, Proposition 8.5], [36, Theorem 19.2.1], [37, Sec. 2.3]). The general case of an arbitrary $\varphi \in \mathcal{M}$ follows immediately from the case $\varphi \equiv 1$ by virtue of Theorem 3.5.

Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. We set $\varphi_s(t) := t^{s/m}\varphi(t^{1/m})$ for $t \geq 1$ and, moreover, $\varphi_s(t) := \varphi(1)$ for $0 < t < 1$. Since the function φ is positive and Borel measurable on the semiaxis $(0, +\infty)$, a self-adjoint operator $\varphi_s(A_0)$ is well-defined on the space $L_2(\Gamma)$ as the function φ of A_0.

Lemma 3.1. *The following assertions are true:*

(i) *The domain of the operator $\varphi_s(A_0)$ contains the set $C^\infty(\Gamma)$.*

(ii) *The mapping*

$$f \;\mapsto\; \|\varphi_s(A_0)f\|_{L_2(\Gamma)}, \quad f \in C^\infty(\Gamma), \tag{3.17}$$

is a norm in the space $C^\infty(\Gamma)$.

Proof. Assertion (i). Let us choose an integer k such that $k > s/m$. Since $\varphi \in \mathcal{M}$, the function φ_s is bounded on every compact subset of the semiaxis $(0, +\infty)$ and, moreover, $t^{-k}\varphi_s(t) \to 0$ as $t \to +\infty$ because of assertions (ii), (iv) of Theorem 3.3. Hence, there is a number $c > 0$ such that $\varphi_s(t) \leq c\,t^k$ for $t \geq r$. Let us consider the unbounded operator A_0^k on the space $L_2(\Gamma)$. Since $A : C^\infty(\Gamma) \to C^\infty(\Gamma)$, we can write $C^\infty(\Gamma) \subset \operatorname{Dom} A_0^k \subset \operatorname{Dom} \varphi_s(A_0)$. Assertion (i) is proved.

Assertion (ii). According to assertion (i), mapping (3.17) is well-defined. For this mapping, all norm properties are evident except for the positive definiteness property.

Let us prove it. Applying the spectral theorem, we can write for an arbitrary function $f \in C^{\infty}(\Gamma)$

$$\|\varphi_s(A_0)f\|^2_{L_2(\Gamma)} = \int_r^{+\infty} \varphi_s^2(t)\, d(E_t f, f)_\Gamma \quad \text{and} \quad \|f\|^2_{L_2(\Gamma)} = \int_r^{+\infty} d(E_t f, f)_\Gamma. \tag{3.18}$$

Here E_t, $t \geq r$, is the resolution of identity in the space $L_2(\Gamma)$ which corresponds to the self-adjoint operator A_0. If $\|\varphi_s(A_0)f\|^2_{L_2(\Gamma)} = 0$, then from the first equality in (3.18) and from the inequality $\varphi_s > 0$ it follows that the measure $(E(\cdot)f, f)_\Gamma$ of the set $[r, +\infty)$ is equal to 0. Now the second equality in (3.18) implies that $f = 0$ on Γ. Assertion (ii) is proved. □

Definition 3.7. The space $H^{s,\varphi}_A(\Gamma)$ is a completion of the space $C^{\infty}(\Gamma)$ with respect to norm (3.17).

The space $H^{s,\varphi}_A(\Gamma)$ is Hilbert one because norm (3.17) is generated by the inner product $(\varphi_s(A_0)f, \varphi_s(A_0)g)_\Gamma$ of functions $f, g \in C^{\infty}(\Gamma)$.

Theorem 3.8. *For arbitrary $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$, the norms in the spaces $H^{s,\varphi}_A(\Gamma)$ and $H^{s,\varphi}(\Gamma)$ are equivalent on the dense linear manifold $C^{\infty}(\Gamma)$. Thus, $H^{s,\varphi}_A(\Gamma) = H^{s,\varphi}(\Gamma)$ up to equivalence of norms.*

Proof. At first suppose that $s > 0$. Let us choose $k \in \mathbb{N}$ such that that $k\,m > s$. Since the operator A_0^k is closed and positive on $L_2(\Gamma)$, its domain $\mathrm{Dom}\, A_0^k$ is Hilbert space with respect to the inner product $(A_0^k f, A_0^k g)_\Gamma$ of functions f, g. Let us note that the couple of spaces $[L_2(\Gamma), \mathrm{Dom}\, A_0^k]$ is admissible, and the operator A_0^k is a generating one for it. Moreover, since A_0^k is a closure of the elliptic pseudodifferential operator A^k on $L_2(\Gamma)$, the spaces $\mathrm{Dom}\, A_0^k$ and $H^{km}(\Gamma)$ are equal up to equivalent norms. Let a function ψ be the interpolation parameter from Theorems 3.3, 3.4 with $\varepsilon = s$ and $\delta = k\,m - s$. Then $\psi(t^k) = \varphi_s(t)$ for $t > 0$, so by Theorem 3.4 we can write

$$\begin{aligned}\|f\|_{H^{s,\varphi}(\Gamma)} &\asymp \|f\|_{[H^0(\Gamma),\, H^{km}(\Gamma)]_\psi} \asymp \|f\|_{[L_2(\Gamma),\, \mathrm{Dom}\, A_0^k\,]_\psi} \\ &= \|\psi(A_0^k)f\|_{L_2(\Gamma)} = \|\varphi_s(A_0)f\|_{L_2(\Gamma)},\end{aligned}$$

for each $f \in C^{\infty}(\Gamma)$.

Now let the number $s \in \mathbb{R}$ be arbitrary. Choose $k \in \mathbb{N}$ such that $s + k\,m > 0$. As has been proved,

$$\|g\|_{H^{s+km,\varphi}(\Gamma)} \asymp \|\varphi_{s+km}(A_0)\, g\|_{L_2(\Gamma)}, \quad g \in C^{\infty}(\Gamma). \tag{3.19}$$

The following topological isomorphism holds due to (3.16) :

$$A^k : H^{\sigma+km,\varphi}(\Gamma) \leftrightarrow H^{\sigma,\varphi}(\Gamma) \quad \text{for each} \quad \sigma \in \mathbb{R}. \tag{3.20}$$

Let us denote by A^{-k} the inverse operator to A^k. For every function $f \in C^{\infty}(\Gamma)$, we have $A^{-k}f \in C^{\infty}(\Gamma)$ and $A_0^k A^{-k} f = f$. Hence, by virtue of (3.20), (3.19), we can write

$$\begin{aligned}\|f\|_{H^{s,\varphi}(\Gamma)} &\asymp \|A^{-k}f\|_{H^{s+km,\varphi}(\Gamma)} \asymp \|\varphi_{s+km}(A_0)A^{-k}f\|_{L_2(\Gamma)} \\ &= \|\varphi_s(A_0)A_0^k A^{-k} f\|_{L_2(\Gamma)} = \|\varphi_s(A_0)f\|_{L_2(\Gamma)}, \quad f \in C^{\infty}(\Gamma).\end{aligned}$$

Theorem 3.8 is proved. □

Theorem 3.9. *Let $s \geq 0$ and $\varphi \in \mathcal{M}$. In the case where $s = 0$ we suppose that the function $1/\varphi$ is bounded in a neighborhood of $+\infty$. Then the space $H^{s,\varphi}(\Gamma)$ coincides with the domain of the operator $\varphi_s(A_0)$ and the norm in the space $H^{s,\varphi}(\Gamma)$ is equivalent to the graph norm of the operator $\varphi_s(A_0)$.*

Proof. The domain $\mathrm{Dom}\,\varphi_s(A_0)$ of the closed operator $\varphi_s(A_0)$ is Hilbert space with respect to the inner product of the graph of this operator. Let us prove that the norms in the spaces $\mathrm{Dom}\,\varphi_s(A_0)$ and $H^{s,\varphi}_A(\Gamma)$ are equivalent on the dense linear manifold $C^{\infty}(\Gamma)$.

By Theorem 3.8, it will imply the present theorem. According to the condition of the present theorem and by virtue of assertion (ii) of Theorem 3.3, there is a number $c > 0$ such that $\varphi_s(t) \geq c$ for $t > 0$. Therefore

$$\|\varphi_s(A_0)f\|_{L_2(\Gamma)} \geq c\,\|f\|_{L_2(\Gamma)} \quad \text{for each} \quad f \in C^\infty(\Gamma).$$

It yields the equivalence of norms mentioned above. It remains to prove the density of the set $C^\infty(\Gamma)$ in the space $\operatorname{Dom}\varphi_s(A_0)$.

Let $f \in \operatorname{Dom}\varphi_s(A_0)$. Since $\varphi_s(A_0)f \in L_2(\Gamma)$, there is a sequence of functions $h_j \in C^\infty(\Gamma)$ such that $h_j \to \varphi_s(A_0)f \quad L_2(\Gamma)$ as $j \to \infty$. Let us note that the operator $\varphi_s^{-1}(A_0)$ is bounded on the space $L_2(\Gamma)$ because $1/\varphi_s(t) \leq 1/c$ for $t > 0$. Hence, $f_j := \varphi_s^{-1}(A_0)h_j \to f$ and $\varphi_s(A_0)f_j = h_j \to \varphi_s(A_0)f$ in $L_2(\Gamma)$ as $j \to \infty$. In other words, $f_j \to f$ with respect to the graph norm of the operator $\varphi_s(A_0)$. Moreover, since $h_j \in C^\infty(\Gamma)$, then $f_j = A_0^{-k}\varphi_s^{-1}(A_0)A_0^k h_j \in H^{km}(\Gamma)$ for every $k \in \mathbb{N}$. Consequently, $f_j \in C^\infty(\Gamma)$ and the density of the set $C^\infty(\Gamma)$ in the space $\operatorname{Dom}\varphi_s(A_0)$ is established. Theorem 3.9 is proved. □

A significant example of the operator A investigated above is the operator $1 - \triangle_\Gamma$, where $\triangle_\Gamma$ is the Beltrami-Laplace operator on the Riemannian manifold Γ (then $m = 2$).

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