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RECURSION RELATION FOR ORTHOGONAL POLYNOMIALS ON THE COMPLEX PLANE

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Dedicated to dear M. L. Gorbachuk on the occasion of his 70th birthday.

Abstract. The article deals with orthogonal polynomials on compact infinite subsets of the complex plane. Orthogonal polynomials are treated as coordinates of generalized eigenvector of a normal operator A. It is shown that there exists a recursion that gives the possibility to reconstruct these polynomials. This recursion arises from generalized eigenvalue problem and, actually, this means that every generalized eigenvector of A is also a generalized eigenvector of A^* with the complex conjugated eigenvalue.

If the subset is actually the unit circle, it is shown that the presented algorithm is a generalization of the well-known Szegő recursion from OPUC theory.

1. Orthogonal polynomials on $\mathbb{R}$. Classical Jacobi matrices

We start with a brief overview of the corresponding results from OPRL (Orthogonal Polynomials on the Real Line) and classical theory of Jacobi matrices.

Let us have a probability Borel measure $\rho : \mathfrak{B}(\mathbb{R}) \to [0;1]$ with infinite compact support. Consider the following sequence of functions

$$1, \lambda, \lambda^2, \dots, \tag{1}$$

belonging to $L^2(\mathbb{R}, d\rho(\lambda))$. Construct an orthonormal basis $P(\lambda) = (P_n(\lambda))_{n=0}^{\infty}$ from this sequence using the standard Gramm-Schmidt orthogonalization procedure. Now construct the matrix of the operator of multiplication by the independent variable in this basis,

$$a_n = \int_{\mathbb{R}} \lambda P_n(\lambda) P_{n+1}(\lambda)\, d\rho(\lambda), \quad b_n = \int_{\mathbb{R}} \lambda (P_n(\lambda))^2 d\rho(\lambda), \\ n \in \mathbb{N}_0 = \{0, 1, \dots\} = \{0\} \cup \mathbb{N}. \tag{2}$$

It is easy to see that $a_n > 0, n \in \mathbb{N}_0$. Thus we have a self-adjoint three-diagonal Jacobi matrix with uniformly bounded elements and non-zero coefficients on the adjoint diagonals,

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

This matrix generates a Hermitian operator A in ℓ_2. Its domain is the set of finite sequences $f \in \ell_{\text{fin}} \subset \ell_2$; denote the selfadjoint closure of this operator by the same letter A. The sequence of polynomials $P(\lambda) = \left(P_n(\lambda)\right)_{n=0}^{\infty}$ is a generalized eigenvector of A and

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corresponds to the eigenvalue $\lambda \in \mathbb{R}$ and ρ is a spectral measure of A. The following Fourier transform (after extending by continuity) gives a unitary mapping between ℓ_2 (where the initial operator A acts) and $L^2(\mathbb{R}, d\rho(\lambda))$ (where its image, the operator of multiplication by λ acts),

$$l_2 \supset l_{\text{fin}} \ni f = (f_n)_{n=0}^{\infty} \mapsto \widehat{f}(\lambda) := \sum_{n=0}^{\infty} f_n P_n(\lambda) \in L^2(\mathbb{R}, d\rho(\lambda)). \tag{4}$$

The polynomials $P_n(\lambda)$ can be recovered as solutions of the equation $JP(\lambda) = \lambda P(\lambda)$. That is, $\forall n \in \mathbb{N}_0$

$$a_{n-1}P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda P_n(\lambda), \quad P_0(\lambda) = 1, \quad P_{-1}(\lambda) = 0. \tag{5}$$

Here $a_n > 0, n \in \mathbb{N}_0$, so it is easy to construct a two-terms recursion that recovers all $P_n(\lambda)$ step by step.

A similar situation takes place for normal operators A (see [2]). Instead of $\ell_2 = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$, it is necessary to use $\mathbf{l}_2 = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^4 \oplus \cdots$. the matrix J has, in this case, a block three-diagonal structure and is normal. The corresponding polynomials $P_n(z)$ of the variable $z \in \mathbb{C}$ (actually of z and $\bar{z}$) constitute an orthonormal basis of $L^2(\mathbb{C}, d\rho(z))$. They cannot be reconstructed from (5), because the matrix coefficients are non-invertible in this case. Now it is necessary to use two corresponding equations, from which we can find $P_n(z)$ step by step. Let us explain the situation in more details. In the normal case, the sequence $P(z) = \big(P_n(z)\big)_{n=0}^{\infty}$, similar to $P(\lambda)$, is also a generalized eigenvector of the operator A and corresponds to the eigenvalue z. But the operator A is normal, therefore $P(z)$ is also a generalized eigenvector for its adjoint operator A^*, with the eigenvalue $\bar{z}$. As a result, in this case we have two analogical equations for A and A^*, instead of one equation (5). The details can be found in [2], but the proof in this article of the corresponding result (Lemma 7) is only an outline. In Section 2 we give a complete proof of this Lemma (see Theorem 1 below).

If the spectral measure ρ is concentrated on the unit circle, then the corresponding operator A will be unitary. In this case, the polynomials $P_n(z)$ become orthonormal on the unit circle (see [1]), which plays an important role in the OPUC theory. In Section 3, we show that the well-known Szegő recursion [4, 5, 6] (which recovers orthogonal polynomials on the unit circle in OPUC) is actually the two relations of type (5) that is written for a unitary operator.

2. Orthogonal polynomials on the complex plane. Three diagonal block Jacobi-type normal matrices

Let ρ be a probability Borel measure on $\mathbb{C}$ with compact support and $L^2(\mathbb{C}, d\rho(z))$ be the space of square integrable complex-valued functions defined on $\mathbb{C}$. We suppose that the support of this measure is an infinite set such that the functions $\mathbb{C} \ni z \mapsto z^m \bar{z}^n$, $m, n \in \mathbb{N}_0$ are linearly independent in $L^2(\mathbb{C}, d\rho(z))$. Let

$$P_0(z, \bar{z});\ P_{1;0}(z, \bar{z}), P_{1;1}(z, \bar{z});\ P_{2;0}(z, \bar{z}), P_{2;1}(z, \bar{z}), P_{2;2}(z, \bar{z}); \ldots \tag{6}$$

be polynomials obtained by using the standard Gramm-Schmidt orthogonalization procedure in $L^2(\mathbb{C}, d\rho(z))$ applied to the system of functions

$$1;\ z^1 \bar{z}^0, z^0 \bar{z}^1;\ z^2 \bar{z}^0, z^1 \bar{z}^1, z^0 \bar{z}^2; \ldots \tag{7}$$

These polynomials have the form $P_{n;\alpha}(z) := P_{n;\alpha}(z, \bar{z}) = k_{n;\alpha} z^{n-\alpha} \bar{z}^{\alpha} + \ldots$, $n \in \mathbb{N}_0$, $\alpha = 0, \ldots, n$; $k_{n;\alpha} > 0$. The support $\operatorname{supp} \rho$ is a compact set. This implies that the family (7) is total in $L^2(\mathbb{C}, d\rho(z))$. Thus polynomials (6) make an orthonormal basis in the space $L^2(\mathbb{C}, d\rho(z))$.

According to [2], Theorem 5, the bounded normal operator of multiplication by the variable z in the space $L^2(\mathbb{C}, d\rho(z))$, with respect to basis (6), has a three diagonal block

Jacobi-type normal matrix $J=(a_{j,k})_{j,k=0}^{\infty}$. This matrix generates a normal operator A in $\mathbf{l}_2=\mathcal{H}_0\oplus\mathcal{H}_1\oplus\mathcal{H}_2\oplus\dots$, $\mathcal{H}_n=\mathbb{C}^{n+1}$, $n\in\mathbb{N}_0$. The norms of all operators $a_{j,k}\colon\mathcal{H}_k\to\mathcal{H}_j$ are uniformly bounded with respect to $j,k\in\mathbb{N}_0$, where $a_{j,k}$ is a $(j+1)\times(k+1)$-matrix and

$$a_{j,k;\alpha,\beta}=\int_{\mathbb{C}} zP_{k;\beta}(z)\overline{P_{j;\alpha}(z)}\,d\rho(z),\quad \alpha=0,\dots,j,\quad \beta=0,\dots,k.$$

Let $a_n:=a_{n+1,n}\colon\mathcal{H}_n\to\mathcal{H}_{n+1}$, $b_n:=a_{n,n}\colon\mathcal{H}_n\to\mathcal{H}_n$ and $c_n:=a_{n,n+1}\colon\mathcal{H}_{n+1}\to\mathcal{H}_n$ then J has the form

$$J=\begin{pmatrix} b_0 & c_0 & 0 & 0 & \dots\\ a_0 & b_1 & c_1 & 0 & \dots\\ 0 & a_1 & b_2 & c_2 & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\tag{8}$$

and matrices a_n, c_n have the form
(9)

$$a_n=\begin{pmatrix} a_{n;\,0,0} & * & \dots & a_{n;0,n}\\ 0 & a_{n;\,1,1} & \dots & a_{n;1,n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_{n;\,n,n}\\ 0 & 0 & \dots & 0 \end{pmatrix},\quad c_n=\begin{pmatrix} c_{n;\,0,0} & c_{n;\,0,1} & 0 & \dots & 0\\ c_{n;\,1,0} & c_{n;\,1,1} & c_{n;\,1,2} & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ c_{n;\,n,0} & c_{n;\,n,1} & c_{n;\,n,2} & \dots & c_{n;\,n,n+1} \end{pmatrix},$$

where $a_{n;\,\alpha,\alpha}>0$, $c_{n;\,\alpha,\alpha+1}>0$, $\alpha=0,\dots,n$. The adjoint operator A^* is also constructed by a similar three-diagonal block Jacobi type matrix J^+ in the basis (6). The matrices J,J^+ act as follows: $\forall f=(f_n)_{n=0}^{\infty}\in\mathbf{l}_2$,

$$\begin{aligned}(Jf)_n&=a_{n-1}f_{n-1}+b_nf_n+c_nf_{n+1},\\ (J^+f)_n&=c^*_{n-1}f_{n-1}+b^*_nf_n+a^*_nf_{n+1},\quad n\in\mathbb{N}_0,\quad f_{-1}=0,\quad a^*_{j,k;\alpha,\beta}=\overline{a}_{k,j;\beta,\alpha}.\end{aligned}\tag{10}$$

The following result is contained in [1], Lemma 7; we give here a complete full proof.

Theorem 1. *Let $\varphi(z)=(\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z)\in\mathcal{H}_n$, $z\in\mathbb{C}$, be a generalized eigenvector of the operator $\hat{A}$. Then $\varphi(z)$ is a solution, which lies in $(\mathbf{l}_{\mathrm{fin}})'$, of the two equations*

$$\begin{aligned}(J\varphi(z))_n&=a_{n-1}\varphi_{n-1}(z)+b_n\varphi_n(z)+c_n\varphi_{n+1}(z)=z\varphi_n(z),\\ (J^+\varphi(z))_n&=c^*_{n-1}\varphi_{n-1}(z)+b^*_n\varphi_n(z)+a^*_n\varphi_{n+1}(z)=\overline{z}\varphi_n(z),\quad \varphi_{-1}(z)=0\end{aligned}\tag{11}$$

with the initial condition $\varphi_0\in\mathbb{C}$. The vector $\varphi_n(z)$ has the form

$$\begin{aligned}\varphi_n(z)&=Q_n(z)\varphi_0=(Q_{n;0}(z),Q_{n;1}(z),\dots,Q_{n;n}(z))\varphi_0,\\ Q_{n;\alpha}(z)&=l_{n;\alpha}\overline{z}^{n-\alpha}z^{\alpha}+q_{n;\alpha}(z,\overline{z}),\quad \alpha=1,\dots,n,\end{aligned}\tag{12}$$

where $l_{n;\alpha}>0$ and $q_{n;\alpha}(z)$ is a linear combination of $\overline{z}^jz^k$, $0\le j+k\le n-1$, and $\overline{z}^{n-(\alpha-1)}z^{\alpha-1}$.

Moreover, the following equality holds true:

$$Q_{n;\alpha}(z)=\overline{P_{n;\alpha}(z)},\quad \forall z\in\mathbb{C},\quad \alpha=1,\dots,n,\quad n\in\mathbb{N}_0.\tag{13}$$

Proof. For $n=0$, system (11) has the form

$$\begin{aligned}b_0\varphi_0+c_0\varphi_1&=z\varphi_0,\\ b^*_0\varphi_0+a^*_0\varphi_1&=\overline{z}\varphi_0,\end{aligned}$$

where $\forall n\in\mathbb{N}$, $\varphi_n(z)=(\varphi_{n;0}(z),\dots,\varphi_{n;n}(z))\in\mathcal{H}_n$; $\varphi_0=\varphi_{0;0}$. From (9) we obtain

$$\begin{aligned}\varphi_{1;0}(z)&=\frac{1}{a_{0;\,0,0}}(\overline{z}-\overline{b}_{0;\,0,0})\varphi_0=Q_{1;0}(z)\varphi_0,\\ \varphi_{1;1}(z)&=\big(r_1(\overline{z}-\overline{b}_{0;\,0,0})+r_2(z-b_{0;\,0,0})+r_3\big)\varphi_0=Q_{1;1}(z)\varphi_0,\end{aligned}$$

where $r_1>0$, r_2 and r_3 are some constants. Therefore, solutions have the form (12).

Suppose, by induction, that for $n \in \mathbb{N}$ the coordinates $\varphi_{n-1}(z)$ and $\varphi_n(z)$ of our generalized eigenvector $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$ have the form (12), and prove that $\varphi_{n+1}(z)$ has the same form (12).

The eigenvector $\varphi(z)$ satisfies system (11). This system is overdetermined. According to (9), a_n^* and c_n act on $\psi_{n+1} \in \mathcal{H}_n$ as follows:

$$a_n^*\psi_{n+1}(z) = \begin{pmatrix} \bar{a}_{n;\,0,0} & 0 & \dots & 0 & 0 \\ \bar{a}_{n;\,1,0} & \bar{a}_{n;\,1,1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{a}_{n;\,n,0} & \bar{a}_{n;\,n,1} & \dots & \bar{a}_{n;\,n,n} & 0 \end{pmatrix} \psi_{n+1}(z),$$

$$c_n\psi_{n+1}(z) = \begin{pmatrix} c_{n;\,0,0} & c_{n;\,0,1} & 0 & \dots & 0 \\ c_{n;\,1,0} & c_{n;\,1,1} & c_{n;\,1,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ c_{n;\,n,0} & c_{n;\,n,1} & c_{n;\,n,2} & \dots & c_{n;\,n,n+1} \end{pmatrix} \psi_{n+1}(z).$$

Construct a $(n+2) \times (n+2)$-matrix $\triangle_n$ of the form

$$\triangle_n = \begin{pmatrix} a_{n;\,0,0} & 0 & \dots & 0 \\ c_{n;\,0,0} & c_{n;\,0,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n;\,n,0} & c_{n;\,n,1} & \dots & c_{n;\,n,n+1} \end{pmatrix}. \tag{14}$$

The matrix in (14) is invertible, thus it is possible to find $\varphi_{n+1}(z)$. Rewrite identities (11) as

$$\begin{aligned} a_n^*\varphi_{n+1}(z) &= \bar{z}\varphi_n(z) - c_{n-1}^*(z) - b_n^*\varphi_n(z), \\ c_n\varphi_{n+1}(z) &= z\varphi_n(z) - a_{n-1}(z) - b_n\varphi_n(z), \quad n \in \mathbb{N}. \end{aligned} \tag{15}$$

From (14), (15) we reconstruct $\varphi_{n+1}(z)$ using the formula

$$\begin{aligned} \triangle_n\varphi_{n+1}(z) = \Big(&\bar{z}Q_{n;0}(z) - (c_{n-1}^*Q_{n-1}(z))_{n;0} - (b_n^*Q_n(z))_{n;0}, \\ &zQ_{n;0}(z) - (a_{n-1}Q_{n-1}(z))_{n;0} - (b_nQ_n(z))_{n;0}, \dots, \\ &zQ_{n;n}(z) - (a_{n-1}Q_{n-1}(z))_{n;n} - (b_nQ_n(z))_{n;n}\Big)\varphi_0. \end{aligned} \tag{16}$$

From (14) and (16) we obtain

$$\begin{aligned} \varphi_{n+1;0}(z) = Q_{n+1;0}(z)\varphi_0 &= \frac{1}{a_{n;\,0,0}}\Big(\bar{z}Q_{n;0}(z) - (c_{n-1}^*Q_{n-1}(z))_{n;0} - (b_n^*Q_n(z))_{n;0}\Big)\varphi_0 \\ &= \frac{1}{a_{n;\,0,0}}\Big(\bar{z}(l_{n;0}\bar{z}^n + q_{n;0}(z)) - (c_{n-1}^*Q_{n-1}(z))_{n;0} - (b_n^*Q_n(z))_{n;0}\Big)\varphi_0, \end{aligned} \tag{17}$$

so the main summand in (17) is equal to $\frac{l_{n;0}}{a_{n;\,0,0}}\bar{z}^{n+1}z^0$, therefore it has the form (12).

Suppose, by induction, that $\varphi_{n+1;j}$ has the form (12) for any fixed $j = 0, \dots, n$. Let us show that $\varphi_{n+1;j+1}(z)$ has the same form. According to (14)–(16) we have that $\forall j = 0, \dots, n$,

$$\begin{aligned} \varphi_{n+1;j+1}(z) = \frac{1}{c_{j;\,j,j+1}}\Big(&z(l_{n;j}\bar{z}^{n-j}z^j + q_{n;j}(z,\bar{z})) - (a_{n-1}Q_{n-1}(z))_{n;j} \\ &- (b_nQ_n(z))_{n;j} - \sum_{l=0}^{j} c_{n;\,j,\,l}Q_{n+1;\,l}\Big)\varphi_0 = Q_{n+1;j+1}(z)\varphi_0. \end{aligned} \tag{18}$$

So the main summand in (18) is equal to $\frac{l_{n;j}}{c_{n;\,j,j+1}}\bar{z}^{n+1-(j+1)}z^{j+1}$ and $\varphi_{n+1;j+1}$ has the form as in (12), because the main summand in $Q_{n+1;j}$ is $\bar{z}^{n+1-j}z^{j} = \bar{z}^{n+1-((j+1)-1)}z^{(j+1)-1}$, and the rest of the terms are $\bar{z}^{m}z^{k}$, $0 \le m+k \le n$. So we prove by induction that $\varphi_{n+1}(z)$ is of the form (12). The equality (13) was proved in [2], p. 25–26. □

From Theorem 1 we obtain the following corollary. For the basis (6) the following identities take place: $\forall z \in \mathbb{C}$,

$$
\begin{aligned}
&a_{n-1}\overline{P_{n-1}(z)} + b_n\overline{P_n(z)} + c_n\overline{P_{n+1}(z)} = z\overline{P_n(z)},\\
&c^*_{n-1}\overline{P_{n-1}(z)} + b^*_n\overline{P_n(z)} + a^*_n\overline{P_{n+1}(z)} = \bar{z}\overline{P_n(z)}.
\end{aligned}
\tag{19}
$$

From the second identity we obtain

$$
\begin{aligned}
\bar{c}_{n-1;0,0}\overline{P_{n-1;0}(z)} + \bar{c}_{n-1;1,0}\overline{P_{n-1;1}(z)} + \cdots + \bar{c}_{n-1;n-1,0}\overline{P_{n-1;n-1}(z)}&\\
+ \bar{b}_{n;0,0}\overline{P_{n;0}(z)} + \cdots + \bar{b}_{n;n,0}\overline{P_{n;n}(z)} + a_{n;0,0}\overline{P_{n+1;0}(z)} &= \bar{z}\overline{P_{n;0}(z)}.
\end{aligned}
$$

Consider this equation and the first equation in (19). Let

$$
A_{n-1} := \begin{pmatrix}
\bar{c}_{n-1;\,0,0} & \bar{c}_{n-1;\,1,0} & \dots & \bar{c}_{n-1;\,n-1,0}\\
a_{n-1;\,0,0} & a_{n-1;\,0,1} & \dots & a_{n-1;\,0,n-1}\\
0 & a_{n-1;\,1,1} & \dots & a_{n-1;\,1,n-1}\\
\vdots & \vdots & \ddots & \\
 & & & a_{n-1;\,n-1,n-1}\\
0 & & \dots & 0
\end{pmatrix},
$$

$$
B_n := \begin{pmatrix}
\bar{b}_{n;\,0,0} & \bar{b}_{n;\,1,0} & \dots & \bar{b}_{n;\,n,0}\\
b_{n;\,0,0} & b_{n;\,0,1} & \dots & b_{n;\,0,n}\\
\vdots & \vdots & \ddots & \\
b_{n;\,n,0} & b_{n;\,n,1} & \dots & b_{n;\,n,n}
\end{pmatrix},
\quad
C_n := \begin{pmatrix}
a_{n;\,0,0} & 0 & \dots & 0\\
c_{n;\,0,0} & c_{n;\,0,1} & \dots & 0\\
 & & \ddots & \\
 & & \dots & c_{n;\,n,n+1}
\end{pmatrix},
$$

$$
\omega(z) := \begin{pmatrix}
\bar{z} & 0 & \dots & 0\\
z & 0 & \dots & 0\\
0 & z & \dots & 0\\
 & & \ddots & \\
0 & & \dots & z
\end{pmatrix},
$$

where A_{n-1} is an $(n+2)\times n$-matrix, B_n is an $(n+2)\times(n+1)$-matrix, C_n is an $(n+2)\times(n+2)$-matrix and $\omega(z)$ is an $(n+2)\times(n+1)$-matrix. Then the identity $A_{n-1}\overline{P_{n-1}(z)} + B_n\overline{P_n(z)} + C_n\overline{P_{n+1}(z)} = \omega(z)\overline{P_n(z)}$ takes place. Since $\det C_n = a_{n;0,0}\prod_{i=0}^{n} c_{n;i,i+1} > 0$, C_n^{-1} exists for all $n \in \mathbb{N}_0$.

Therefore (11)*, i.e.* (19)*, can be rewritten in the form of one matrix equality (as the Szegő recursion),*

$$
\begin{aligned}
P_{n+1}(z) &= \overline{C_n^{-1}(\omega(z) - B_n)}P_n(z) - \overline{C_n^{-1}A_{n-1}}P_{n-1}(z),\\
P_{-1}(z) &= 0, \quad P_0(z) = 1, \quad n \in \mathbb{N}_0
\end{aligned}
\tag{20}
$$

(here the conjugation over a matrix means conjugation of each element of this matrix).

3. Orthogonal polynomials on the unit circle and Szegő recursion. Three diagonal block Jacobi type unitary matrices

Consider a special case of the situation described in Section 2. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, ρ a probability Borel measure on $\mathbb{T}$ such that its support is an infinite set. Thus, for $z \in \mathbb{T}$, we have $\bar{z} = \frac{1}{z}$ and $z = e^{i\theta}, \theta \in [0, 2\pi)$. Consider the following family of functions:

$$1; z, z^{-1}; z^2, z^{-2}; \dots \tag{21}$$

The power functions (21) belong to $L^2(\mathbb{T}, d\rho(z))$. Apply the standard Gramm-Schmidt orthogonalization procedure to this family. Since the system (21) is total in $L^2(\mathbb{T}, d\rho(z))$, we obtain an orthonormal basis in $L^2(\mathbb{T}, d\rho(z))$ of polynomials which we denote as follows:

$$\begin{aligned} P_0(z) = 1; P_{1;0}(z) = k_{1;0} z + \cdots, P_{1;1}(z) = k_{1;1} z^{-1} + \cdots; \cdots; \\ P_{n;0}(z) = k_{n;0} z^n + \cdots, P_{n;1}(z) = k_{n;1} z^{-n} + \cdots; \cdots, \end{aligned} \tag{22}$$

where $k_{n;0} > 0, k_{n;1} > 0$.

As is the normal case, the unitary operator A of multiplication by independent variable in space $L^2(\mathbb{T}, d\rho(z))$, in the basis (22), has the form of a three diagonal block Jacobi type unitary matrix $J = (a_{j,k})_{j,k=0}^{\infty}$, which acts in the space $\mathbf{l}_2$ of the previous type (see [1],Theorem 1), $\mathcal{H}_0 = \mathbb{C}, \mathcal{H}_n = \mathbb{C}^2, n \in \mathbb{N}$. The norms of all the operators $a_{j,k} \colon \mathcal{H}_k \to \mathcal{H}_j$ are uniformly bounded with respect to $j, k \in \mathbb{N}_0$. If a_n, b_n and c_n are defined as in Section 2, then J has the form (8), where b_0 is a scalar (a 1×1-matrix), $a_0 = (a_{0;\alpha})_{\alpha=0}^{1}$ (a 2×1-matrix), $c_0 = (c_{0;\alpha})_{\alpha=0}^{1}$ (a 1×2-matrix) and for $n \in \mathbb{N}$, the elements $a_n = (a_{n;\alpha,\beta})_{\alpha,\beta=0}^{1}$, $b_n = (b_{n;\alpha,\beta})_{\alpha,\beta=0}^{1}$, $c_n = (c_{n;\alpha,\beta})_{\alpha,\beta=0}^{1}$ are 2×2-matrices. Some elements of this matrix are positive or equal to zero, $a_{0;0} > 0$, $a_{0;1} = 0$, $c_{0;1} > 0$, $a_{n;1,0} = a_{n;1,1} = 0$, $a_{n;0,0} > 0$, $c_{n;0,0} = c_{n;0,1} = 0$, $c_{n;1,1} > 0$, $n \in \mathbb{N}$. A^* has the form of a three-diagonal block Jacobi type unitary matrix J^+ in the basis (22) where J, J^+ act on $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ according to (10).

For the matrices J, J^+, an analog of Theorem 1 ([1], Lemma 5) takes place. Namely, the system of equations

$$\begin{aligned} a_{n-1}\overline{P_{n-1}(z)} + b_n \overline{P_n(z)} + c_n \overline{P_{n+1}(z)} = z\overline{P_n(z)}, \\ c^*_{n-1}\overline{P_{n-1}(z)} + b^*_n \overline{P_n(z)} + a^*_n \overline{P_{n+1}(z)} = \bar{z}\overline{P_n(z)} \end{aligned} \tag{23}$$

recovers the orthonormal polynomials $P_{n;\alpha}(z), n \in \mathbb{N}_0, \alpha = 0, \dots, n$, where $P_{-1}(z) = 0$.

We will use this Jacobi block representation instead of the 5-diagonals representation of a unitary operator discovered in [3].

Let us define monic orthogonal polynomials $\Phi_n(z): \ \Phi_{-1} = 0, \forall n \in \mathbb{N}_0$

$$\Phi_n(z) = z^n + \omega_{n-1} z^{n-1} + \cdots, \qquad \int_{\mathbb{T}} z^{-j} \Phi_n(z)\, d\rho(z) = 0, \quad j = 0, 1, \dots, n-1,$$

and the anti-unitary map ${}^{*,n}: f^{*,n}(z) = z^n \overline{f(z)}$, $f \in L^2(\mathbb{T}, d\rho(z))$. For Φ_n, the Szegő recursion ([6], p. 5) takes place,

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \, \Phi^*_n(z), \tag{24}$$

where $\alpha_n, n \in \mathbb{N}$, are the Verblunsky coefficients

$$\alpha_n = -\overline{\Phi_{n+1}(0)} \quad \text{and} \quad \|\Phi_n\| = \prod_{j=0}^{n-1} (1 - |\alpha_j|^2)^{\frac{1}{2}}, \quad \Phi_0(z) = 1. \tag{25}$$

To simplify the notation, we use $*$ instead of ^{*,n} (this notation is standard; note that $*$ depends on n). Let $\phi_n(z) = \frac{\Phi_n(z)}{\|\Phi_n\|}$. Then $(\phi_n)_{n=0}^{\infty}$ is an orthogonal system in $L^2(\mathbb{T}, d\rho(z))$ but it is not a basis (see [6], §2). The Szegő recursion (24) can be rewritten in the following form [6], formula (2.28):

$$\begin{aligned} \begin{pmatrix} \phi_{n+1}(z) \\ \phi^*_{n+1}(z) \end{pmatrix} &= A(z, \alpha_n) \begin{pmatrix} \phi_n(z) \\ \phi^*_n(z) \end{pmatrix}, \\ A(z, \alpha_n) &= \rho_n^{-1} \begin{pmatrix} z & -\overline{\alpha}_n \\ -z\alpha_n & 1 \end{pmatrix}, \quad \rho_n = (1 - |\alpha_n|^2)^{1/2}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{26}$$

According to [6], Proposition 5.1,

$$\phi^*_{2n}(z) = z^n P_{n;1}(z), \quad \phi_{2n-1}(z) = z^{(n-1)} P_{n;0}(z), \quad n \in \mathbb{N}. \tag{27}$$

If we apply the transformation ^{*,2n} to the first equation and the transformation ${}^{*,2n-1}$ to the second equation, then (26) can be rewritten as

$$\begin{pmatrix} P^{*,l}_{l;1}(z) \\ z^l P_{l;1}(z) \end{pmatrix} = A(z, \alpha_{2l-1}) \begin{pmatrix} z^{l-1} P_{l;0}(z) \\ P^{*,l}_{l;0}(z) \end{pmatrix}, \quad l \in \mathbb{N}_0;$$

$$\begin{pmatrix} z^m P_{m+1;0}(z) \\ P^{*,m+1}_{m+1;0}(z) \end{pmatrix} = A(z, \alpha_{2m}) \begin{pmatrix} P^{*,m}_{m;1}(z) \\ z^m P_{m;1}(z) \end{pmatrix}, \quad m \in \mathbb{N}_0.$$

From these recurrence relations we obtain the following inverse recursion formulae:

$$P_{m+1;0}(z) = \frac{1}{\rho_{2m}} \big(z\overline{P_{m;1}(z)} - \overline{\alpha}_{2m} P_{m;1}(z) \big), \tag{28}$$

$$P_{m+1;1}(z) = \frac{1}{\rho_{2m+1}} \big(\overline{P_{m+1;0}(z)} - \alpha_{2m+1} P_{m+1;0}(z) \big), \quad m \in \mathbb{N}_0. \tag{29}$$

Theorem 2. *Systems of equations* (23) *and* (28), (29) *are equivalent.*

Proof. According to [6], Theorem 5.2, the operator A in $L^2(\mathbb{T}, d\rho(z))$, in the basis (22), has the form (8) in terms of the Verblunsky coefficients. Here a_n, b_n, c_n are found as

$$b_0 = \overline{\alpha}_0, \quad c_0 = (\overline{\alpha}_1 \rho_0,\ \rho_1 \rho_0), \quad a_0 = \begin{pmatrix} \rho_0 \\ 0 \end{pmatrix};$$

$$a_n = \begin{pmatrix} \rho_{2n}\rho_{2n-1} & -\rho_{2n}\alpha_{2n-1} \\ 0 & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} -\overline{\alpha}_{2n-1}\alpha_{2n-2} & -\rho_{2n-1}\alpha_{2n-2} \\ \overline{\alpha}_{2n}\rho_{2n-1} & -\overline{\alpha}_{2n}\alpha_{2n-1} \end{pmatrix}, \tag{30}$$

$$c_n = \begin{pmatrix} 0 & 0 \\ \overline{\alpha}_{2n+1}\rho_{2n} & \rho_{2n+1}\rho_{2n} \end{pmatrix}.$$

Let us show that (28), (29) imply (23). From (29) we have $\overline{P_{m;1}(z)} = \frac{1}{\rho_{2m-1}} \big(P_{m;0}(z) - \overline{\alpha_{2m-1}} \overline{P_{m;0}(z)} \big)$. Substitute this expression into (28) to obtain

$$P_{m+1;0}(z) = \frac{1}{\rho_{2m}} \left(\frac{z}{\rho_{2m-1}} \left\{ P_{m;0}(z) - \overline{\alpha_{2m-1}} \overline{P_{m;0}(z)} \right\} - \overline{\alpha}_{2m} P_{m;1}(z) \right). \tag{31}$$

From (28) we get $\overline{P_{m;0}(z)} = \frac{1}{\rho_{2m-2}} \big(\bar{z} P_{m-1;1}(z) - \alpha_{2m-2} \overline{P_{m-1;1}(z)} \big)$; substituting this into (31) we obtain

$$\begin{aligned} P_{m+1;0}(z) = \frac{1}{\rho_{2m}} \bigg(\frac{z}{\rho_{2m-1}} \bigg\{ P_{m;0}(z) - \overline{\alpha_{2m-1}} \bigg[\frac{1}{\rho_{2m-2}} \big(\bar{z} P_{m-1;1}(z) \\ - \alpha_{2m-2} \overline{P_{m-1;1}(z)} \big) \bigg] \bigg\} - \overline{\alpha}_{2m} P_{m;1}(z) \bigg). \end{aligned}$$

Take ρ_{2m-1} out of brackets,

$$\begin{aligned} P_{m+1;0}(z) = \frac{1}{\rho_{2m}\rho_{2m-1}} \bigg(\bigg\{ z P_{m;0}(z) - z \overline{\alpha_{2m-1}} \bigg[\frac{1}{\rho_{2m-2}} \big(\bar{z} P_{m-1;1}(z) \\ - \alpha_{2m-2} \overline{P_{m-1;1}(z)} \big) \bigg] \bigg\} - \rho_{2m-1} \overline{\alpha}_{2m} P_{m;1}(z) \bigg). \end{aligned}$$

Now we use the fact that $z\bar{z} = 1$,

$$\begin{aligned} P_{m+1;0}(z) = \frac{1}{\rho_{2m}\rho_{2m-1}} \bigg(\bigg\{ z P_{m;0}(z) - \overline{\alpha_{2m-1}} \bigg[\frac{1}{\rho_{2m-2}} \big(P_{m-1;1}(z) \\ - \alpha_{2m-2} z \overline{P_{m-1;1}(z)} \big) \bigg] \bigg\} - \rho_{2m-1} \overline{\alpha}_{2m} P_{m;1}(z) \bigg). \end{aligned}$$

Use (28) again, $z\overline{P_{m-1;1}(z)} = \rho_{2m-2}P_{m;0}(z) + \overline{\alpha}_{2m-2}P_{m-1;1}(z)$, and therefore

$$P_{m+1;0}(z) = \frac{1}{\rho_{2m}\rho_{2m-1}}\Bigg(\Bigg\{zP_{m;0}(z) - \overline{\alpha_{2m-1}}\Bigg[\frac{1}{\rho_{2m-2}}\Bigg(P_{m-1;1}(z) - \alpha_{2m-2}\Bigg(\rho_{2m-2}P_{m;0}(z) + \overline{\alpha}_{2m-2}P_{m-1;1}(z)\Bigg)\Bigg)\Bigg]\Bigg\} - \rho_{2m-1}\overline{\alpha}_{2m}P_{m;1}(z)\Bigg).$$

The last step is to use the identity $\alpha_{2m-2}\overline{\alpha}_{2m-2} = |\alpha_{2m-2}|^2 = 1 - \rho_{2m-2}^2$. We get

$$P_{m+1;0}(z) = \frac{1}{\rho_{2m}\rho_{2m-1}}\big(\big\{zP_{m;0}(z) - \overline{\alpha_{2m-1}}\big[-\alpha_{2m-2}P_{m;0}(z) + \rho_{2m-2}P_{m-1;1}(z)\big]\big\} - \rho_{2m-1}\overline{\alpha}_{2m}P_{m;1}(z)\big).$$

This is just the one of the four equations in (23),

$$(J^{+}\overline{P})_{n;0} = \overline{z}\,\overline{P}_{n;0}. \tag{32}$$

Other three equations can be obtained in the same way. It is necessary to note that (23) is overdetermined.

Let us show that equations (28), (29) follow from (23). Consider the polynomial $P_{m+1;0}(z)$. It is obtained by orthogonalization of (21), so, by the definition, $P_{m+1;0}(z) \perp z^{\alpha}$, $\alpha = -m, \dots, m$. Similarly $P_{m;1}(z) \perp z^{\alpha}, \alpha = (-m+1), \dots, m$. Multiplication by z is a unitary operator in $L^2(\mathbb{T}, d\rho(z))$. Thus

$$0 = (P_{m;1}(z), z^{\alpha})_{L^2(\mathbb{T},d\rho(z))} = (\bar{z}^{\alpha}, \overline{P_{m;1}(z)})_{L^2(\mathbb{T},d\rho(z))} = (z\cdot\bar{z}^{\alpha}, z\overline{P_{m;1}(z)})_{L^2(\mathbb{T},d\rho(z))},$$
$$\alpha = (-m+1), \dots, m.$$

Finally we have $z\overline{P_{m;1}(z)} \perp z^{1-\alpha}, \alpha = (-m+1), \dots, m$. But this is the same as $z\overline{P_{m;1}(z)} \perp z^{\alpha}, \alpha = (-m+1), \dots, m$. These observations are necessary in order to obtain the following conclusion:

$$\left(\frac{1}{k_{m+1;0}}P_{m+1;0}(z) - z\frac{\overline{P_{m;1}(z)}}{k_{m;1}}\right) \perp \{z^m, \bar{z}^{m-1}, z^{m-1}, \dots\}.$$

It is easy to see that $\frac{1}{k_{m+1;0}}P_{m+1;0}(z) - z\frac{1}{k_{m;1}}\overline{P}_{m;1}(z) = \gamma_{m;1}P_{m;1}(z)$ (note that the polynomial $\frac{1}{k_{m+1;0}}P_{m+1;0}(z) - z\frac{1}{k_{m;1}}\overline{P}_{m;1}(z)$ is a linear combination of $z^{\alpha}, \alpha = -m, \dots, m$).

The last identity can be rewritten as

$$P_{m+1;0}(z) = \frac{k_{m+1;0}}{k_{m;1}}\big(z\overline{P_{m;1}(z)} + \gamma_{m;1}k_{m;1}P_{m;1}(z)\big), \tag{33}$$

where $\gamma_{m;1}$ is a complex constant. From (23) we obtain $k_{m+1;0} = \frac{1}{a_{m;0,0}}\cdot\frac{1}{a_{m-1;0,0}}\cdot\dots\cdot\frac{1}{a_{0;0}}$, $k_{m;1} = \frac{1}{c_{m-1;1,1}}\cdot\dots\cdot\frac{1}{c_{0;1}}$ and using (30) we get

$$\frac{k_{m+1;0}}{k_{m;1}} = \frac{c_{m-1;1,1}\cdot\dots\cdot c_{0;1}}{a_{m;0,0}\cdot\dots\cdot a_{0;0}} = \frac{\rho_{2m-1}\rho_{2m-2}\cdot\dots\cdot\rho_1\rho_0}{\rho_{2m}\rho_{2m-1}\rho_{2m-2}\rho_{2m-3}\cdot\dots\cdot\rho_0} = \frac{1}{\rho_{2m}}. \tag{34}$$

So if we show that $\gamma_{m;1}k_{m;1} = -\overline{\alpha}_{2m}$ we will finish the proof. From (25) and (27) we obtain $\alpha_{2m} = -\overline{(z^m P_{m+1;0}(z))\big|_{z=0}}\prod_{j=0}^{2m}\rho_j$. Using (33) we have $\overline{(z^m P_{m+1;0}(z))\big|_{z=0}} = \overline{\frac{k_{m+1;0}}{k_{m;1}}\gamma_{m;1}k_{m;1}^2} = \bar{\gamma}_{m;1}k_{m+1;0}k_{m;1}$. Note that $k_{m+1;0} = \frac{1}{\prod_{j=0}^{2m}\rho_j}$ (this follows from (25)). Now it becomes obvious that the identity holds true.

So we have shown that (28) can be obtained from (23). Let us show that from (23) one can find (29) too. In the same way as we did it before for $P_{m+1;0}(z)$, here we have

$$P_{m+1;1}(z) = \frac{k_{m+1;1}}{k_{m+1;0}}\big(\overline{P_{m+1;0}(z)} + \gamma_{m+1;0}k_{m+1;0}P_{m+1;0}(z)\big), \tag{35}$$

where $\gamma_{m+1;0}$ is a complex constant. Since $\alpha_{2m+1} = -\overline{(z^{m+1}\overline{P_{m+1;1}(z)})}\big|_{z=0} \prod_{j=0}^{2m+1} \rho_j$, we see that $-\alpha_{2m+1} = \gamma_{m+1;0} k_{m+1;0}$ and also $\frac{k_{m+1;1}}{k_{m+1;0}} = \frac{1}{\rho_{2m+1}}$. So (35) yields (29). □

Note that using (30) we can express the Verblunsky coefficients in terms of elements of the matrices a_n, b_n, c_n,

$$\frac{\overline{\alpha}_{2m}}{\rho_{2m}} = \frac{b_{m;1,0}}{a_{m;0,0}} = -\frac{\overline{b}_{m+1;0,1}}{c_{m;1,1}}, \quad m \in \mathbb{N}_0.$$

Therefore we can obtain several representations for α_{2m}. For example, $\alpha_{2m} = \frac{\overline{b}_{m;1,0}}{a_{m;0,0}}\big(1 - \frac{b_{m;1,0} b_{m+1;0,1}}{a_{m;0,0} c_{m;1,1}}\big)^{-\frac{1}{2}}$ or $\alpha_{2m} = \frac{\overline{b}_{m;1,0}}{\sqrt{a_{m;0,0}^2 + |b_{m;1,0}|^2}}$. Similarly,

$$\frac{\overline{\alpha}_{2m+1}}{\rho_{2m+1}} = \frac{c_{m;1,0}}{c_{m;1,1}} = -\frac{\overline{a}_{m+1;0,1}}{a_{m+1;0,0}}, \quad m \in \mathbb{N}_0.$$

Thus $\alpha_{2m+1} = \frac{\overline{c}_{m;1,0}}{\sqrt{c_{m;1,1}^2 + |c_{m;1,0}|^2}}$. The coefficient ρ_n can also be expressed using formulae of type (34). It is necessary to note that these representations are not unique.

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