Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 2, pp. 117–123

QUASILINEAR PARABOLIC EQUATIONS WITH A LÉVY LAPLACIAN FOR FUNCTIONS OF INFINITE NUMBER OF VARIABLES

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To dear Myroslav Gorbachuk on his 70th birthday.

Abstract. We construct solutions to initial, boundary and initial-boundary value problems for quasilinear parabolic equations with an infinite dimensional Lévy Laplacian Δ_L,

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x) + f_0(U(t,x)),$$

in fundamental domains of a Hilbert space. The solution is defined in the functional class where a solution of the corresponding problem for the heat equation $\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x)$ exists.

1. Introduction

In 1919 P. Lévy considered a quasilinear elliptic equation

$$\Delta_L U(x) = f(U(x)),$$

where $U(x)$ is a function defined on a Hilbert space H and $f(\xi)$ is a function of a scalar argument. P. Lévy showed in [1] that a general solution of this equation is given implicitly by the relation

$$\varphi(U(x)) - \frac{1}{2}\|x\|_H^2 = \Psi(x),$$

where $\varphi(\xi) = \int \frac{d\xi}{f(\xi)}$ and $\Psi(x)$ is an arbitrary harmonic function, and reduced solution of the Dirichlet problem for this equation in a bounded domain of the Hilbert space to the Dirichlet problem for the Lévy Laplace equation. Later these problems were described in the book by P. Lévy [2].

Solution of the Cauchy problem for a quasilinear parabolic equation with the Lévy Laplacian were constructed by M. Feller in paper [3]. One can find more references in the book by M. Feller [4].

In this article we construct a solution of a boundary value problem and an initial-boundary value problem for quasilinear parabolic equations with the Lévy Laplacian

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x) + f_0(U(t,x)),$$

(here $f_0(\xi)$ is a function defined on $\mathbb{R}^1$) for a fundamental domain in a real infinite dimensional Hilbert space. To make the article self-consistent we give the expression of the solution of the Cauchy problem for a quasilinear parabolic equation with the Lévy Laplacian.

2000 *Mathematics Subject Classification.* Primary 35R15, 46405.

Key words and phrases. Lévy Laplacian, quasilinear equations, parabolic equations, initial-boundary value problems.

2. Preliminaries

Let H be a real infinite dimensional Hilbert space.

The infinite dimensional Laplacian defined by P. Lévy can be described as follows. If a scalar function F defined on H is twice strongly differentiable at a point x_0 then the Lévy Laplacian of F at the point x_0 is defined (if it exists) by the formula

$$\Delta_L F(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(F''(x_0)f_k, f_k)_H, \tag{1}$$

where $F''(x)$ is the Hessian of $F(x)$, and $\{f_k\}_1^\infty$ is an orthonormal basis in H.

We shall recall an important property of the Lévy Laplacian described in [2] that we will need in the sequel. Assume that

$$F(x) = f(U_1(x), \dots, U_m(x)),$$

where $f(u_1, \dots, u_m)$ is a twice continuously differentiable function of m variables in a domain $\{U_1(x), \dots, U_m(x)\} \subset \mathbb{R}^m$, where $(U_1(x), \dots, U_m(x))$ is a vector of values of functions $U_1(x), \dots, U_m(x)$. We assume that $U_j(x)$ are uniformly continuous in a bounded domain $\Omega \subset H$ and strongly twice differentiable functions, and $\Delta_L U_j(x)$ exist $(j = 1, \dots, m)$. Then $\Delta_L F(x)$ exists and

$$\Delta_L F(x) = \sum_{j=1}^{m} \frac{\partial f(u)}{\partial u_j}\Big|_{u_j = U_j(x)} \Delta_L U_j(x). \tag{2}$$

We deduce two consequences from (2).

1) If the functions $U_k(x)$ are harmonic in some domain Ω $(k = 1, \dots, m)$, then the function $F(x)$ is also harmonic in Ω.

2) The Lévy Laplacian is a "derivative". Namely, for $F(x) = U_1(x)U_2(x)$ we have $\Delta_L[U_1(x)U_2(x)] = \Delta_L U_1(x) \cdot U_2(x) + U_1(x) \cdot \Delta_L U_2(x)$.

Let Ω be a bounded domain in the Hilbert space H (that is a bounded open set in H), and $\overline{\Omega} = \Omega \cup \Gamma$ be a domain in H with boundary Γ.

Consider the domain Ω in H with boundary Γ of the form

$$\Omega = \{x \in H : 0 \le Q(x) < R^2\}, \quad \Gamma = \{x \in H : Q(x) = R^2\},$$

where $Q(x)$ is a twice strongly differentiable function such that $\Delta_L Q(x) = \gamma$, where $\gamma > 0$ is a positive constant. Such domains are called fundamental domains.

Examples of fundamental domains are

1) a ball $\overline{\Omega} = \{x \in H : \|x\|_H^2 \le R^2\}$,

2) an ellipsoid $\overline{\Omega} = \{x \in H : (Bx, x)_H \le R^2\}$, where $B = \gamma E + S(x)$, E is the identity operator and $S(x)$ is a compact operator acting in H.

Consider the function

$$T(x) = \frac{R^2 - Q(x)}{\gamma},$$

that will be used in the sequel. The function $T(x)$ possesses the following properties:

$$0 < T(x) \le \frac{R^2}{\gamma}, \quad \Delta_L T(x) = -1 \quad \text{if} \quad x \in \Omega,$$

$$T(x) = 0 \quad \text{if} \quad x \in \Gamma.$$

3. The Cauchy problem

Consider the Cauchy problem

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x) + f_0(U(t,x)), \tag{3}$$

$$U(0,x) = U_0(x), \tag{4}$$

where $U(t,x)$ is a function on $[0,\mathfrak{T}] \times H$, $f_0(\xi)$ is a given function of one variable, $U_0(x)$ is a given function defined on H.

Theorem 1. *Let $f_0(\xi)$ be a differentiable function in the domain $\{U(t,x)\}$ ($\{U(t,x)\}$ is a domain in $\mathbb{R}^1$, where the function $U(t,x)$ takes its values).*

Assume that there exists a primitive $\varphi(\xi) = \int \frac{d\xi}{f_0(\xi)}$ and the inverse function φ^{-1}.

Assume in addition that in a certain functional class $\mathcal{F}$ there exists a solution of the Cauchy problem for the heat equation

$$\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x), \quad V(0,x) = U_0(x). \tag{5}$$

Then the solution $U(t,x)$ of the Cauchy problem (3), (4) in the same class $\mathcal{F}$ is given by the equation

$$\varphi(U(t,x)) = t + \varphi(V(t,x)), \tag{6}$$

i.e., $U(t,x) = \varphi^{-1}(t + \varphi(V(t,x)))$.

Proof. We deduce from (6) that

$$\varphi'_\xi(U(t,x)) \frac{\partial U(t,x)}{\partial t} = 1 + \varphi'_\xi(V(t,x)) \frac{\partial V(t,x)}{\partial t}.$$

For $m = 1$, we deduce from (6) using (2) that

$$\varphi'_\xi(U(t,x)) \Delta_L U(t,x) = \varphi'_\xi(V(t,x)) \Delta_L V(t,x).$$

But $\varphi'_\xi(\xi) = \frac{1}{f_0(\xi)}$, hence

$$\frac{\partial U(t,x)}{\partial t} = f_0(U(t,x)) + \frac{f_0(U(t,x))}{f_0(V(t,x))}\, \frac{\partial V(t,x)}{\partial t}, \tag{7}$$

$$\Delta_L U(t,x) = \frac{f_0(U(t,x))}{f_0(V(t,x))}\, \Delta_L V(t,x). \tag{8}$$

Substituting (7) and (8) into (3), we derive

$$f_0(U(t,x)) + \frac{f_0(U(t,x))}{f_0(V(t,x))}\, \frac{\partial V(t,x)}{\partial t} = \frac{f_0(U(t,x))}{f_0(V(t,x))} \Delta_L V(t,x) + f_0(U(t,x)),$$

i.e., (6) satisfies equation (3).

Choosing $t = 0$ in (6) and taking into account that $V(0,x) = U_0(x)$, we deduce that $U(0,x) = U_0(x)$. □

Example 1. Let us construct a solution of the Cauchy problem

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x) - U^3(t,x)), \tag{9}$$

$$U(0,x) = h\Big(\frac{1}{2}\|x\|_H^2\Big), \tag{10}$$

where $h(\lambda)$ is a smooth function on $(-\infty < \lambda < \infty)$.

Note that in (9) $f_0(\xi) = -\xi^3$ that yields $\varphi(\xi) = -\int \frac{d\xi}{\xi^3} = \frac{1}{2\xi^2}$, and hence $\varphi^{-1}(z) = \frac{1}{\sqrt{2z}}$.

A solution of the Cauchy problem for the heat equation

$$\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x), \quad V(0,x) = h\Big(\frac{1}{2}\|x\|_H^2\Big)$$

has the form

$$V(t,x) = h\Big(t + \frac{1}{2}\|x\|_H^2\Big).$$

Thus, by (6) we obtain a solution of the problem (9), (10) in the form

$$U(t,x) = \frac{1}{\sqrt{2\Big[t + \frac{1}{2}[h(t+\frac{1}{2}\|x\|_H^2)]^{-2}\Big]}}.$$

4. Boundary value problem

Consider a boundary value problem

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x)) + f_0(U(t,x)) \quad \text{in} \quad \Omega, \tag{11}$$

$$U(t,x) = G(t,x) \quad \text{on} \quad \Gamma, \tag{12}$$

where $U(t,x)$ is a function on $[0,\mathfrak{T}] \times H, f_0(\xi)$ is a given function of one dimensional argument and $G(t,x)$ is a given function.

Theorem 2. *Let $f_0(\xi)$ be a differentiable function in the domain $\{U(t,x)\}$.*

Let there exist both the primitive $\varphi(\xi) = \int \frac{d\xi}{f_0(\xi)}$ and its inverse function φ^{-1}.

Let, in addition, $\overline{\Omega}$ be a fundamental domain.

Assume that in a certain functional class $\mathcal{F}$ there exists a solution of the boundary value problem for the heat equation,

$$\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x) \quad \textit{in} \quad \Omega, \quad V(t,x)\Big|_\Gamma = G(t,x). \tag{13}$$

Then, in this class $\mathcal{F}$, a solution of the boundary value problem (11), (12) is defined by the formula

$$\varphi(U(t,x)) = T(x) + \varphi(V(t,x)), \tag{14}$$

i.e., $U(t,x) = \varphi^{-1}(T(x) + \varphi(V(t,x)))$. Here $T(x) = \frac{R^2 - Q(x)}{\gamma}$ (see p. 2).

Proof. We deduce from (14) the relation

$$\varphi'_\xi(U(t,x))\frac{\partial U(t,x)}{\partial t} = \varphi'_\xi(V(t,x))\frac{\partial V(t,x)}{\partial t}.$$

For $m=1$, we deduce from (14) using (2) the relation

$$\varphi'_\xi(U(t,x))\Delta_L U(t,x) = \Delta_L T(x) + \varphi'_\xi(V(t,x))\Delta_L V(t,x).$$

Since $\Delta_L T(x) = -1$ and $\varphi'_\xi(\xi) = \frac{1}{f_0(\xi)}$, we get

$$\frac{\partial U(t,x)}{\partial t} = \frac{f_0(U(t,x))}{f_0(V(t,x))}\,\frac{\partial V(t,x)}{\partial t}, \tag{15}$$

$$\Delta_L U(t,x) = -f_0(U(t,x)) + \frac{f_0(U(t,x))}{f_0(V(t,x))}\,\Delta_L V(t,x). \tag{16}$$

Substituting (15) and (16) into (11), we derive

$$\frac{f_0(U(t,x))}{f_0(V(t,x))}\,\frac{\partial V(t,x)}{\partial t} = -f_0(U(t,x)) + \frac{f_0(U(t,x))}{f_0(V(t,x))}\,\Delta_L V(t,x) + f_0(U(t,x)),$$

i.e., (14) satisfies equation (11).

At the surface Γ we obtain $T(x) = 0$. Substituting $T(x) = 0$ into (14) and keeping in mind that $V(t,x)\Big|_\Gamma = G(t,x)$, we deduce the equality $U(t,x)\Big|_\Gamma = G(t,x)$. $\square$

Example 2. Let us construct a solution of the boundary value problem without initial data in the ball of the space H, $\overline{\Omega} = \{x \in H : \|x\|_H^2 \le R^2\}$,

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x) - U^3(t,x) \quad \text{in} \quad \Omega, \tag{17}$$

$$U(t,x)\Big|_{\|x\|^2 = R^2} = g\Big(t - \frac{1}{2}\|x\|_H^2\Big), \tag{18}$$

where $g(\lambda)$ is a smooth function on $(-\infty < \lambda < \infty)$.

For the considered domain $\Omega = \{x \in H : \|x\|_H^2 \le R^2\}$, we have $T(x) = \frac{R^2 - \|x\|_H^2}{2}$.

In the equation (17) we have $f_0(\xi) = -\xi^3$, which yields $\varphi(\xi) = \frac{1}{2\xi^2}$, $\varphi^{-1}(z) = \frac{1}{\sqrt{2z}}$.

A solution of the boundary value problem without initial data for the heat equation

$$\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x), \quad V(t,x)\Big|_{\|x\|_H^2 = R^2} = g\Big(t - \frac{1}{2}\|x\|_H^2\Big),$$

has the form

$$V(t,x) = g\Big(t + \frac{1}{2}\|x\|_H^2 - R^2\Big).$$

Hence by (14) we obtain a solution of the problem (17), (18) in the form

$$U(t,x) = \frac{1}{\sqrt{\Big[R^2 - \|x\|_H^2 + [g(t + \frac{1}{2}\|x\|_H^2 - R^2)]^{-2}\Big]}}.$$

5. Initial-boundary value problem

Consider the initial-boundary value problem

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x) + f_0(U(t,x)) \quad \text{in} \quad \Omega, \tag{19}$$

$$U(0,x) = U_0(x), \tag{20}$$

$$U(t,x) = G(t,x) \quad \text{on} \quad \Gamma, \tag{21}$$

where $U(t,x)$ is a function defined on $[0,\mathcal{T}] \times H$, $f_0(\xi)$ is a given function of one variable and $U_0(x), G(t,x)$ are given functions.

Theorem 3. *Let $f_0(\xi)$ be a differentiable function in the domain $\{U(t,x)\}$.*

Let there exist the primitive $\varphi(\xi) = \int \frac{d\xi}{f_0(\xi)}$ and the inverse function φ^{-1}.

Let the domain $\overline{\Omega}$ be fundamental.

Assume that, in a certain functional class $\mathcal{F}$, there exist solutions of the initial-boundary value problems

$$\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x) \quad \textit{in} \quad \Omega, \quad V(0,x) = U_0(x), \quad V(t,x)\Big|_\Gamma = G(t,x). \tag{22}$$

Then a solution of the initial-boundary value problem (19)–(21) in this class $\mathcal{F}$ is given by the formula

$$\varphi(U(t,x)) = \tau(t, T(x)) + \varphi(V(t,x)), \tag{23}$$

i.e., $U(t,x) = \varphi^{-1}(\tau(t,T(x)) + \varphi(V(t,x)))$, where $\tau(t,T(x)) = t - q(t - T(x))$, $q(\lambda) = \lambda$ if $\lambda \ge 0$ and $q(\lambda) = 0$ if $\lambda \le 0$, $T(x) = \frac{R^2 - Q(x)}{\gamma}$.

If the initial-boundary value problem (22) possesses a unique solution in the same functional class then the solution of the initial-boundary value problem (19)–(21) is unique in this class.

Proof. We deduce from (23) that

$$\varphi'_\xi(U(t,x))\frac{\partial U(t,x)}{\partial t} = \frac{\partial \tau(t,x)}{\partial t} + \varphi'_\xi(V(t,x))\frac{\partial V(t,x)}{\partial t}.$$

For $m = 1$, we deduce from (23) using (2) that

$$\varphi'_\xi(U(t,x))\Delta_L U(t,x) = \Delta_L \tau(t,x) + \varphi'_\xi(V(t,x))\Delta_L V(t,x).$$

Recall that $\varphi'_\xi(\xi) = \frac{1}{f_0(\xi)}$, which yields

$$\frac{\partial U(t,x)}{\partial t} = f_0(U(t,x))\frac{\partial \tau(t,x)}{\partial t} + \frac{f_0(U(t,x))}{f_0(V(t,x))}\,\frac{\partial V(t,x)}{\partial t}, \tag{24}$$

$$\Delta_L U(t,x) = f_0(U(t,x))\Delta_L \tau(t,x) + \frac{f_0(U(t,x))}{f_0(V(t,x))}\,\Delta_L V(t,x). \tag{25}$$

Substituting (24) and (25) into (19), we obtain

$$\begin{aligned} f_0(U(t,x))\,\frac{\partial \tau(t,x)}{\partial t} &+ \frac{f_0(U(t,x))}{f_0(V(t,x))}\,\frac{\partial V(t,x)}{\partial t} \\ &= f_0(U(t,x))\Delta_L \tau(t,x) + \frac{f_0(U(t,x))}{f_0(V(t,x))}\,\Delta_L V(t,x) + f_0(U(t,x)), \end{aligned}$$

i.e., (23) satisfies equation (19) (since $\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x)$, $\frac{\partial \tau(t,T)}{\partial t} = \Delta_L \tau(t,T) + 1$).

Setting $t = 0$ in (23) and taking into account that $V(0,x) = U_0(x)$, and $\tau(0,T) = 0$, we obtain the equality $U(0,x) = U_0(x)$.

Since $V(t,x) = G(t,x)$, $\tau(t,T) = 0$ on the surface Γ, we deduce from (23) that $U(t,x)\Big|_\Gamma = G(t,x)$.

The final statement of the theorem is obvious. $\square$

Example 3. Let us construct a solution of the initial-boundary value problem in the ball $\overline{\Omega} = \{x \in H : \|x\|_H^2 \le R^2\}$ of the space H,

$$\frac{\partial U(t,x)}{\partial t} = \Delta_L U(t,x) - U^3(t,x) \quad \text{in} \quad \Omega, \tag{26}$$

$$U(0,x) = h\Big(\frac{1}{2}\|x\|_H^2\Big), \tag{27}$$

$$U(t,x)\Big|_{\|x\|_H^2 = R^2} = g\Big(t - \frac{1}{2}\|x\|_H^2\Big), \tag{28}$$

where $h(\lambda)$ is a smooth function on the positive half axis with the support $[0, \frac{R^2}{2}]$, and $g(\lambda)$ is a smooth function such that $g(\lambda) = 0$ for $\lambda \le 0$ (note that comparing with Examples 1 and 2 we need some additional assumptions concerning the functions $h(\lambda)$ and $g(\lambda)$).

In the considered domain $\overline{\Omega} = \{x \in H : \|x\|_H^2 \le R^2\}$, we have $T(x) = \frac{R^2 - \|x\|_H^2}{2}$.

In equation (26), $f_0(\xi) = -\xi^3$ and, hence, $\varphi(\xi) = -\int \frac{d\xi}{\xi^3} = \frac{1}{\xi^2}$ and $\varphi^{-1}(z) = \frac{1}{\sqrt{2z}}$.

Consider the initial-boundary value problem for the heat equation in the Shilov class of functions

$$\frac{\partial V(t,x)}{\partial t} = \Delta_L V(t,x) \quad \text{in} \quad \Omega, \quad V(0,x) = h\Big(\frac{1}{2}\|x\|_H^2\Big),$$

$$V(t,x)\Big|_{\|x\|_H^2=R^2} = g\Big(t-\frac{1}{2}\|x\|_H^2\Big). \tag{29}$$

Its solution has the form

$$V(t,x) = h\Big(t+\frac{1}{2}\|x\|_H^2\Big)+g\Big(t+\frac{1}{2}\|x\|_H^2-R^2\Big).$$

It results from (23) that the solution of the problem (26)–(28) has the form

$$U(t,x) \quad = \frac{1}{\sqrt{2\Big\{t-q(t+\frac{\|x\|_H^2}{2}-\frac{R^2}{2})+\frac{1}{2}\Big[h(t+\frac{1}{2}\|x\|_H^2)+g(t+\frac{1}{2}\|x\|_H^2-R^2)\Big]^{-2}\Big\}}}.$$

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Received 20/12/2007