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ON SOLUTIONS OF PARABOLIC AND ELLIPTIC TYPE DIFFERENTIAL EQUATIONS ON $(-\infty, \infty)$ IN A BANACH SPACE

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To my father.

Abstract. We show that every classical solution of a parabolic or elliptic type homogeneous differential equation on $(-\infty, \infty)$ in a Banach space may be extended to an entire vector-valued function. The description of all the solutions is given, and necessary and sufficient conditions for a solution to be continued to a finite order and finite type entire vector-valued function are presented.

1. Let $\mathfrak{B}$ be a complex Banach space with norm $\|\cdot\|$. Denote by $E(\mathfrak{B})$ $(L(\mathfrak{B}))$ the set of all densely defined closed linear operators (bounded linear operators) in $\mathfrak{B}$. We also denote by $I, \mathcal{D}(A), \rho(A)$ and $R_A(\cdot)$ the identity operator, the domain, the resolvent set, and the resolvent of the operator A. In what follows, by $\{e^{tA}\}_{t\geq 0}$ we mean a C_0-semigroup of bounded linear operators in $\mathfrak{B}$ with generator A (for a C_0-semigroup theory we refer to [1, 2]). Recall that a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ is called bounded analytic of angle $\theta \in \left(0, \frac{\pi}{2}\right]$ if e^{tA} admits an extension to an $L(\mathfrak{B})$-valued function e^{zA}, analytic inside the sector $\Sigma_\theta = \{z \in \mathbb{C} : |\arg z| < \theta\}$, strongly continuous at 0 on each ray of this sector, and for any $\theta' < \theta$ there exists a constant $c_{\theta'}$ such that $\|e^{zA}\| \leq c_{\theta'}$ as $z \in \overline{\Sigma_{\theta'}} = \{z \in \mathbb{C} : |\arg z| \leq \theta'\}$.

For an operator $A \in E(\mathfrak{B})$ and a number $\beta \geq 0$, we put

$$\mathfrak{G}_{\{\beta\}}(A) = \bigcup_{\alpha>0} \mathfrak{G}_\beta^\alpha(A), \quad \mathfrak{G}_{(\beta)}(A) = \bigcap_{\alpha>0} \mathfrak{G}_\beta^\alpha(A),$$

where

$$\mathfrak{G}_\beta^\alpha(A) = \{x \in C^\infty(A) \big| \exists c = c(x) > 0, \forall k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} : \|A^k x\| \leq c\alpha^k k^{k\beta}\}$$

is a Banach space with norm

$$\|x\|_{\mathfrak{G}_\beta^\alpha(A)} = \sup_{k\in\mathbb{N}_0} \frac{\|A^k x\|}{\alpha^k k^{k\beta}},$$

$C^\infty(A) = \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(A^n)$. In $\mathfrak{G}_{\{\beta\}}(A)$ $(\mathfrak{G}_{(\beta)}(A))$, the topology of inductive (projective) limit of the spaces $\mathfrak{G}_\beta^\alpha(A)$ is introduced

$$\mathfrak{G}_{\{\beta\}}(A) = \operatorname*{ind\,lim}_{\alpha\to\infty} \mathfrak{G}_\beta^\alpha(A), \quad \mathfrak{G}_{(\beta)}(A) = \operatorname*{proj\,lim}_{\alpha\to 0} \mathfrak{G}_\beta^\alpha(A).$$

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The spaces $\mathfrak{G}_{\{1\}}(A)$ and $\mathfrak{G}_{(1)}(A)$ are the spaces of analytic and entire vectors, respectively, for the operator A. It is not hard to see that if $\beta_1 < \beta_2$, then the dense and continuous embeddings

$$\mathfrak{G}_{(\beta_1)}(A) \subseteq \mathfrak{G}_{\{\beta_1\}}(A) \subseteq \mathfrak{G}_{(\beta_2)}(A) \subseteq \mathfrak{G}_{\{\beta_2\}}(A)$$

hold.

If the operator A is bounded, then for any $\beta > 0$

$$\mathfrak{G}_{\{0\}}(A) = \mathfrak{G}_{(\beta)}(A) = \mathfrak{G}_{\{\beta\}}(A) = \mathfrak{B}.$$

It is also easily shown that for an arbitrary β, one can choose an unbounded operator A so that the space $\mathfrak{G}_{\{\beta\}}(A)$ (and all the more $\mathfrak{G}_{(\beta)}(A)$) consists only of zero vector. But if an operator A is the generator of a bounded analytic C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ with angle θ, then, as was proved in [3], $\overline{\mathfrak{G}_{(\beta)}(A)} = \mathfrak{B}$ if $\beta > 1 - \frac{2\theta}{\pi}$. For $\beta = 1 - \frac{2\theta}{\pi}$, the cases are possible when $\mathfrak{G}_{\{\beta\}}(A) = \{0\}$.

Theorem 1. *Let $A \in E(\mathfrak{B})$. Then for an arbitrary $x \in \mathfrak{G}_{\{\beta\}}(A)$ ($x \in \mathfrak{G}_{(\beta)}(A)$), the vector-valued function*

$$\exp(zA) = \sum_{k=0}^{\infty} \frac{z^k A^k x}{k!}$$

is entire in the space $\mathfrak{G}_{\{\beta\}}(A)$ as $\beta < 1$ (in the space $\mathfrak{G}_{(\beta)}(A)$ as $\beta \leq 1$). The collection $\{\exp(zA)\}_{z\in\mathbb{C}}$ forms a C_0-group of linear continuous operators in these spaces.

If A is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ in $\mathfrak{B}$, then

$$\forall x \in \mathfrak{G}_{(1)}(A), \quad \forall t \geq 0: \quad \exp(tA)x = e^{tA}x.$$

In the case where the semigroup $\{e^{tA}\}_{t\geq 0}$ is bounded analytic one, the latter relation is true for all $t \in \mathbb{R}^1$ (if $t < 0, e^{tA} := (e^{-tA})^{-1}$).

Proof. It is evident that if $x \in \mathfrak{G}_{(1)}(A)$, then the series $\sum_{k=0}^{\infty} \frac{z^k A^k x}{k!}$ converges in $\mathfrak{B}$ for any $z \in \mathbb{C}$, and it defines an entire $\mathfrak{B}$-valued function.

Now, let $x \in \mathfrak{G}_{(\beta)}(A)$ with $\beta \leq 1$, that is,

$$\forall \alpha > 0, \quad \exists c = c(x, \alpha) > 0, \quad \forall n \in \mathbb{N}_0: \quad \|A^n x\| \leq c\alpha^n n^{n\beta} \quad (\beta \leq 1).$$

Then, for an arbitrary $m \in \mathbb{N}_0$,

$$\left\| A^n \left(\exp(zA)x - \sum_{k=0}^{m} \frac{z^k A^k x}{k!} \right) \right\| = \left\| A^n \sum_{k=m+1}^{\infty} \frac{z^k A^k x}{k!} \right\| \leq \sum_{k=m+1}^{\infty} \frac{|z|^k \|A^{n+k}x\|}{k!}$$
$$\leq c \sum_{k=m+1}^{\infty} \frac{|z|^k}{k!} \alpha^{n+k} (n+k)^{(n+k)\beta} = c\alpha^n n^{n\beta} \sum_{k=m+1}^{\infty} \frac{|z|^k}{k!} k^{k\beta} \left(1 + \frac{k}{n}\right)^{n\beta} \left(1 + \frac{n}{k}\right)^{k\beta}.$$

The inequalities

$$\left(1 + \frac{k}{n}\right)^{n\beta} \leq \left(1 + \frac{k}{n}\right)^{n} \leq e^k$$

and

$$\left(1 + \frac{n}{k}\right)^{k\beta} \leq \left(1 + \frac{n}{k}\right)^{k} \leq e^n$$

imply that

$$\left\| A^n \left(\exp(zA)x - \sum_{k=0}^{m} \frac{z^k A^k x}{k!} \right) \right\| \leq c_m (\alpha e)^n n^{n\beta},$$

where

$$c_m = \sum_{k=m+1}^{\infty} \frac{|z\alpha e|^k}{k!} k^{k\beta}.$$

Set $m = 0$. Then, for any fixed $z \in \mathbb{C}$, we have the inclusion $\exp(zA)x \in \mathfrak{G}_{(\beta)}(A)$. Moreover, whatever large $\delta > 0$ is taken, the series $\sum_{k=0}^{\infty} \frac{z^k}{k!} A^k x$ converges in the space $\mathfrak{G}_{\beta}^{\alpha}(A)$ in the disk $|z| < \delta$ for any $\alpha < \frac{1}{e^2\delta}$. So, this series defines an entire vector-valued function in $\mathfrak{G}_{\beta}^{\alpha}(A),\ \alpha \in \left(0, \frac{1}{e^2\delta}\right)$, and, therefore, in $\mathfrak{G}_{(\beta)}(A)$.

In the same way, it is established that $\exp(zA)x,\ x \in \mathfrak{G}_{\{\beta\}}(A)$ $(\beta < 1)$, is an entire vector-valued function in $\mathfrak{G}_{\{\beta\}}(A)$.

The group property of $\{\exp(zA)\}_{z\in\mathbb{C}}$ is checked as in the same was in the scalar case. □

Obviously, the vector-valued function $\exp(zA)x,\ x \in \mathfrak{G}_{(\beta)}(A)$, is a solution of the Cauchy problem

$$\begin{cases} y'(t) &= Ay(t), \quad t \in (-\infty, \infty), \\ y(0) &= x. \end{cases}$$

2. Consider the equations

$$y'(t) - Ay(t) = 0, \quad t \in (-\infty, \infty), \tag{1}$$

and

$$y'(t) + Ay(t) = 0, \quad t \in (-\infty, \infty), \tag{2}$$

where A is the generator of a bounded analytic C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ in $\mathfrak{B}$. The equation (1) is an abstract parabolic equation while equation (2) is an inverse abstract parabolic one.

Examples. Let $\mathfrak{B}$ is one of the spaces $L^p(\mathbb{R}^n)(1 \leq p < \infty)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ ($BUC(\mathbb{R}^n)$) is the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity (bounded uniformly continuous functions on $\mathbb{R}^n$) with the supremum norm. Define in these spaces the operator A in the following way:

$$Au(x) = \Delta u(x),\ x \in \mathbb{R}^n; \quad \mathcal{D}(A) = \{u \in \mathfrak{B} : \Delta u \in \mathfrak{B}\}$$

(Δ is taken in the distribution sense).

The operator A generates a bounded analytic C_0-semigroup of angle $\frac{\pi}{2}$ in $\mathfrak{B}$, namely,

$$(e^{tA} f)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} f(x-s) e^{-|s|^2/4t}\, ds, \quad t > 0, \quad f \in \mathfrak{B}, \quad x \in \mathbb{R}^n,$$

(see [4]). In this case, equation (1) is the classical heat one.

If $A \leq 0$ is a selfadjoint operator in a Hilbert space, then A generates a bounded analytic C_0-semigroup of angle $\frac{\pi}{2}$, too.

By a solution (classical) of equation (1) or equation (2) on $(-\infty, \infty)$ we mean a strongly continuously differentiable vector-valued function $y(t) : (-\infty, \infty) \mapsto \mathcal{D}(A)$ satisfying (1) or (2), respectively.

Theorem 2. *Let A be the generator of a bounded analytic C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ in $\mathfrak{B}$. A vector-valued function $y(t) : (-\infty, \infty) \mapsto \mathcal{D}(A)$ is a solution of equation (1) on $(-\infty, \infty)$ if and only if it may be represented in the form*

$$y(t) = \exp(tA)g, \quad g \in \mathfrak{G}_{(1)}(A), \quad t \in (-\infty, \infty). \tag{3}$$

So, every solution $y(t)$ of equation (1) on $(-\infty, \infty)$ admits an extension to an entire vector-valued function in the space $\mathfrak{G}_{(1)}(A)$.

Proof. Suppose $y(t)$ to be a solution of equation (1) on $(-\infty, \infty)$. Since $y(t)$ is a solution of this equation on $[0, \infty)$, we have (see [1])

$$y(t) = e^{tA} f = \exp(tA) f, \quad f \in \mathcal{D}(A), \quad t \in [0, \infty).$$

Put $z(t) = y(-t),\ t \geq 0$. The vector-valued function $z(t)$ is a solution of equation (2) on $[0, \infty)$. As was shown in [5],

$$y(-t) = z(t) = \exp(-tA)g, \quad g \in \mathfrak{G}_{(1)}(A), \quad t \in [0, \infty).$$

Taking into account the continuity of $y(t)$ at 0, we obtain $f = g$. Thus, $y(t)$ is represented in the form (3) on the whole real axis. As was remarked in Theorem 1, such a vector-valued function is entire in the space $\mathfrak{G}_{(1)}(A)$. □

Note that the fact that the values of a solution $y(t)$ of equation (1) belong to the space $\mathfrak{G}_{(1)}(A)$, for the heat equation means that its solutions are entire functions not only in t, but in x as well.

In a way analogous to that used for equation (1), one can prove that a vector-valued function $y(t) : (-\infty, \infty) \mapsto \mathcal{D}(A)$ is a solution of equation (2) on $(-\infty, \infty)$ if and only if

$$y(t) = \exp(-tA)g, \quad g \in \mathfrak{G}_{(1)}(A), \quad t \in (-\infty, \infty). \tag{4}$$

By Theorem 1, a vector-valued function of the form (4) is also entire in the space $\mathfrak{G}_{(1)}(A)$.

3. Now we pass to the second-order equation

$$y''(t) - By(t) = 0, \quad t \in (-\infty, \infty), \tag{5}$$

where B is a weakly positive operator in $\mathfrak{B}$, that is, $B \in E(\mathfrak{B}),\ \rho(B) \supset (-\infty, 0)$, and there exists a constant $M > 0$ such that

$$\forall \lambda > 0 : \|R_B(-\lambda)\| \leq \frac{M}{\lambda}.$$

If, in addition, $0 \in \rho(B)$, then the operator B is called positive.

As was shown in [1], for a weakly positive operator B, the powers $B^\alpha, 0 < \alpha < 1$, are defined, and $A = -B^{1/2}$ is a generating operator of a bounded analytic C_0-semigroup in $\mathfrak{B}$.

Under a solution (classical) of equation (5) on $(-\infty, \infty)$ we mean a twice continuously differentiable function $y(t) : (-\infty, \infty) \mapsto \mathcal{D}(B)$ satisfying (5) on $(-\infty, \infty)$.

Theorem 3. *Let B be a weakly positive operator in $\mathfrak{B}$. A function $y(t) : (-\infty, \infty) \mapsto \mathcal{D}(B)$ is a solution of equation (5) on $(-\infty, \infty)$ if and only if it admits a representation in the form*

$$y(t) = \exp(tA)f + \frac{\sinh(tA)}{A}g, \quad f, g \in \mathfrak{G}_{(1)}(A), \tag{6}$$

where $A = -B^{1/2}$,

$$\frac{\sinh(zA)}{A} = \int_0^z \coth(zA)\, dz = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} A^{2k},$$

$$\coth(zA) = \frac{1}{2}[\exp(zA) + \exp(-zA)] = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} A^{2k}.$$

So, every solution of equation (5) on $(-\infty, \infty)$ is an entire vector-valued function in the space $\mathfrak{G}_{(1)}(A)$.

Proof. Suppose that $y(t)$ is a solution of equation (5) on $(-\infty, \infty)$. The equation (5) may be written as

$$\left(\frac{d}{dt} + A\right)\left(\frac{d}{dt} - A\right)y(t) = 0.$$

Put $z(t) = \left(\frac{d}{dt} - A\right) y(t)$. Obviously, $z(t)$ is a solution of equation (2) on $(-\infty, \infty)$ with $A = -B^{1/2}$ which is the generator of a bounded analytic C_0-semigroup in $\mathfrak{B}$. As we have proved above,

$$z(t) = \exp(-tA)g, \quad g \in \mathfrak{G}_{(1)}(A), \quad t \in (-\infty, \infty).$$

Hence, $y(t)$ is a solution on $(-\infty, \infty)$ of the equation

$$\left(\frac{d}{dt} - A\right) y(t) = \exp(-tA)g.$$

Set

$$z_0(t) = y(t) - \frac{\sinh(tA)}{A} g.$$

Then

$$\left(\frac{d}{dt} - A\right) z_0(t) = \exp(-tA)g - \left(\frac{d}{dt} - A\right) \frac{\sinh(tA)}{A} g = 0,$$

i.e. $z_0(t)$ is a solution of equation (1) on $(-\infty, \infty)$. Therefore,

$$z_0(t) = \exp(tA)f, \quad f \in \mathfrak{G}_{(1)}(A), \quad t \in (-\infty, \infty),$$

whence

$$y(t) = \exp(tA)f + \frac{\sinh(tA)}{A} g, \quad f, g \in \mathfrak{G}_{(1)}(A),$$

which, in view of Theorem 1 and the fact that the vector-valued function $\coth(zA)$ is entire in $\mathfrak{G}_{(1)}(A)$, enables to conclude that $y(t)$ can be extended to an entire vector-valued function $y(z)$ in $\mathfrak{G}_{(1)}(A)$.

It is not hard to check that a vector-valued function of the form (6) is a solution of equation (5). □

4. Denote by $\mathfrak{A}(\mathfrak{B})$ the set of all entire $\mathfrak{B}$-valued functions. We say that a vector-valued function $y(z) \in \mathfrak{A}(\mathfrak{B})$ is of finite growth order (finite order) if there exists a number $\gamma \geq 0$ such that

$$\|y(z)\| \leq e^{|z|^\gamma}$$

for sufficiently large $|z|$. The infimum $\rho(y)$ of such γ is called the order of $y(z)$.

Now let $\delta > 0$ be an arbitrary fixed number. By the degree of the function $y(z) \in \mathfrak{A}(\mathfrak{B})$ with respect to the number δ we mean the value

$$\sigma(y, \delta) = \overline{\lim_{r \to \infty}} \frac{\ln \max_{|z|=r} \|y(z)\|}{r^\delta}.$$

It is clear that if $y(z)$ has a finite order $\rho = \rho(y)$ and $\delta < \rho$, then $\sigma(y, \delta) = \infty$, but $\sigma(y, \delta) = 0$ for $\delta > \rho$. The number $\sigma(y) = \sigma(y, \rho)$ (the degree of $y(z)$ with respect to its order) is called the type of $y(z)$. It is usual to call a finite order vector-valued function $y \in \mathfrak{A}(\mathfrak{B})$ an exponential type vector-valued function if $\rho(y) \leq 1$ and $\sigma(y, 1) < \infty$.

For an arbitrary number $\rho > 0$, we denote by $\mathfrak{A}^\rho(\mathfrak{B})$ the set of all functions $y \in \mathfrak{A}(\mathfrak{B})$, whose orders do not exceed ρ, and of finite degrees with respect to this ρ. We also put

$$\mathfrak{A}^\rho_\alpha(\mathfrak{B}) = \{y \in \mathfrak{A}^\rho(\mathfrak{B}) \big| \exists c > 0, \forall z \in \mathbb{C} : \|y(z)\| \leq c e^{\alpha |z|^\rho}\},$$

where $0 < c = c(y) = \text{const}$. The set $\mathfrak{A}^\rho_\alpha(\mathfrak{B})$ is a Banach space with norm

$$\|y\|_{\mathfrak{A}^\rho_\alpha(\mathfrak{B})} = \sup_{r \geq 0} e^{-\alpha r^\rho} \max_{|z|=r} \|y(z)\|.$$

Evidently,

$$\mathfrak{A}^\rho(\mathfrak{B}) = \bigcup_{\alpha > 0} \mathfrak{A}^\rho_\alpha(\mathfrak{B}).$$

In the space $\mathfrak{A}^\rho(\mathfrak{B})$ we introduce the topology of inductive limit of the Banach spaces $\mathfrak{A}^\rho_\alpha(\mathfrak{B})$:

$$\mathfrak{A}^\rho(\mathfrak{B}) = \operatorname*{ind\,lim}_{\alpha \to \infty} \mathfrak{A}^\rho_\alpha(\mathfrak{B}).$$

The convergence $y_n \to y$ $(n \to \infty)$ in $\mathfrak{A}^\rho(\mathfrak{B})$ means the following: the sequence $\sigma(y_n, \rho)$ is bounded, and $\|y_n(z) - y(z)\| \to 0$ $(n \to \infty)$ uniformly on each compact set $K \subset \mathbb{C}$. It is easily seen that $\mathfrak{A}^1(\mathfrak{B})$ coincides with the space of exponential type entire $\mathfrak{B}$-valued functions.

It is reasonable to ask whether there exist solutions of equations (1), (2) or (5) on $(-\infty, \infty)$, admitting extensions to vector-valued functions from the class $\mathfrak{A}^\rho(\mathfrak{B})$, and if this is the case, then under what conditions, the set of such solutions of a corresponding equation is dense in the set of all its solutions, that is, for any solution $y(z)$ of equation (1), (2) or (5) there exists a sequence $y_n \in \mathfrak{A}^\rho(\mathfrak{B})$ converging uniformly to y on every compact set $K \subset \mathbb{C}$.

Theorem 4. *For a solution $y(z)$ of equations (1),(2) (or (5)) on $(-\infty, \infty)$ to belong to $\mathfrak{A}^\rho(\mathfrak{B})$, it is necessary and sufficient that $y(0) \in \mathfrak{G}_{\{\beta\}}(A)$ $(y(0), y'(0) \in \mathfrak{G}_{\{\beta\}}(A))$, where $\beta = \frac{\rho-1}{\rho}$. If this is the case, then $y(z) \in \mathfrak{G}_{\{\beta\}}(A)$ for any $z \in \mathbb{C}$. Under the condition that $\rho > \frac{\pi}{2\theta}$ (θ is the analyticity angle of the semigroup $\{e^{tA}\}_{t\geq 0}$), the set of solutions $y \in \mathfrak{A}^\rho(\mathfrak{B})$ of the corresponding equation is dense in the set of all its solutions.*

Proof. Let $y \in \mathfrak{A}^\rho(\mathfrak{B})$ be a solution of equation (1) on $(-\infty, \infty)$. Then $y(z)$ is represented in the form (3):

$$y(z) = \exp(zA)g, \quad g \in \mathfrak{G}_{(1)}(A).$$

In view of Theorem 2 from [6], $y(0) = g \in \mathfrak{G}_{\{\beta\}}(A)$, where $\beta = \frac{\rho-1}{\rho}$. By Theorem 1, $y(z) \in \mathfrak{G}_{\{\beta\}}(A)$ for an arbitrary $z \in \mathbb{C}$. The inverse assertion follows from the same theorem. The similar arguments are suitable for equation (2).

Now suppose $y \in \mathfrak{A}^\rho(\mathfrak{B})$ to be a solution of equation (5) on $(-\infty, \infty)$. By Theorem 2 from [6], $y(0), y'(0) \in \mathfrak{G}_{\{\gamma\}}(B)$ with $\gamma = 2\frac{\rho-1}{\rho}$. Since $\mathfrak{G}_{\{\gamma\}}(B) = \mathfrak{G}_{\{\frac{\gamma}{2}\}}(A)$, we have $y(0), y'(0) \in \mathfrak{G}_{\{\beta\}}(A)$. This and representation (6) imply the inclusions

$$y(0) = f \in \mathfrak{G}_{\{\beta\}}(A), \quad y'(0) = Af + g \in \mathfrak{G}_{\{\beta\}}(A)$$

Taking into account the embedding $A\mathfrak{G}_{\{\beta\}}(A) \subseteq \mathfrak{G}_{\{\beta\}}(A)$, we conclude that $g \in \mathfrak{G}_{\{\beta\}}(A)$. The Theorem 1 and the formula (6) guarantee the inclusion $y(z), y'(z) \in \mathfrak{G}_{\{\beta\}}(A)$ for any $z \in \mathbb{C}$.

In [3], it was shown that if $\beta > 1 - \frac{2\theta}{\pi}$ (i.e. $\rho = \frac{1}{1-\beta} > \frac{1}{1-\left(1-\frac{2\theta}{\pi}\right)} = \frac{\pi}{2\theta}$, then $\overline{\mathfrak{G}_{(\beta)}(A)} = \mathfrak{B}$. Since the solutions of equation (1) on $(-\infty, \infty)$ have the form $y(z) = \exp(zA)g, \quad g \in \mathfrak{G}_{(1)}(A)$, and $\overline{\mathfrak{G}_{\{\beta\}}(A)} = \mathfrak{G}_{(1)}(A)$, the vector g can be approximated in $\mathfrak{G}_{(1)}(A)$-topology by vectors $g_n \in \mathfrak{G}_{\{\beta\}}(A)$ $(n \in \mathbb{N})$. By Theorem 1, the sequence $y_n(z) = \exp(zA)g_n$ converges to $y(z)$ uniformly on each compact set $K \subset \mathbb{C}$. The similar arguments can be applicable for equations (2) and (5). □

As for $\rho = \frac{\pi}{2\theta}$, the considered equations may, generally, have no solutions on $(-\infty, \infty)$ in the class $\mathfrak{A}^\rho(\mathfrak{B})$ except for the trivial one. But (see [7]) under the conditions that $\theta = \frac{\pi}{2}$ and the inequality

$$\int_0^1 \ln\ln M(s)\, ds < \infty, \quad \text{where} \quad M(s) = \sup_{|\Im\lambda|\geq s} \|R_A(\lambda)\|$$

is fulfilled, the set of exponential type entire solutions is dense in the set of all solutions. This happens to be the case when, for example, A is a normal operator in a Hilbert space $\mathfrak{H}$, generating a bounded analytic C_0-semigroup, or B is a weakly positive normal operator in $\mathfrak{H}$.

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