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INVERSE SPECTRAL PROBLEM FOR A STAR GRAPH OF STIELTJES STRINGS

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On the occasion of 70-th anniversary of M. L. Gorbachuk.

Abstract. We solve the inverse spectral problem for a star graph of Stieltjes strings (these are threads bearing a finite number of point masses) with the pendant ends fixed, i.e., we recover the masses and lengths of the intervals between them from the spectra of small transverse vibrations of the graph together with the spectra of the Dirichlet problems on the edges and the total lengths of the edges.

1. Introduction

It is well known that two spectra of boundary problems describing small transverse vibrations of a string, together with its length, uniquely determine the density of the string for a very large class of strings. This result was stated by M. G. Krein and proved by L. de Branges (see [1] p. 252). If the string is smooth such that its density belongs to the Sobolev space W_2^2 then the equation for the amplitude function of small vibrations of the string can be reduced by means of the Liouville transformation [2] (p. 292) to the Sturm-Liouville equation. The corresponding inverse problem of recovering the potential by two spectra of boundary problems was completely solved in [3].

For the opposite case of an extremely non-smooth string, known as a Stieltjes string which is a thread bearing point masses, with a finite number of point masses, the inverse problem was solved in [4] and [5]. It should be mentioned that the model of a massless string with point masses is used in many engineering studies [6], [7], [8], e.g. in electrical engineering [9].

The first example of an inverse problem for three spectra is the following: to find the potential for a given spectrum of the Dirichlet problem generated by a Sturm-Liouville equation with a real potential on an interval $[0, a]$, and spectra of the Dirichlet problems generated by the same equation on the subintervals $[0, a/2]$ and $[a/2, a]$. This problem was solved in [10] (see also [11], [12], where uniqueness of the solution was investigated).

This problem was generalized in [13] to the case of more general boundary conditions and potentials, in [14] to the case of coupled oscillating systems and in [15] for Jacobi matrices.

These three spectral problem can be considered as a problem on a star graph with two edges. Thus, it admits a generalization to three edged star graph [16] and to a q-edged star graph [17].

Here we consider the inverse problem for a star graph consisting of q Stieltjes strings joined at an interior vertex. The ends of the string are assumed to be fixed. At the interior vertex we assume the continuity condition to be satisfied together with the Kirchhoff condition. For $q = 2$ this problem has been solved in [18].

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Throughout the paper we assume the total lengths of the strings as well as the number of the point masses to be finite.

In Section 2 we describe relations between the spectra: the spectrum of the whole graph of Stieltjes strings and the spectra of its q parts obtained by clamping the graph (imposing Dirichlet boundary conditions) at the interior vertex, namely, we show that the spectrum of the whole graph and the union of the spectra of the parts interlace in a certain sense. It follows from the observation that the ratio of the product of the characteristic polynomials of the problems on the edges and the characteristic polynomial of the problem on the whole graph appears to be an S-function (a Nevanlinna function positive on $(-\infty, 0)$).

In Section 3 we solve the corresponding inverse problem: to recover values of the point masses, the lengths of the intervals between them (the subintervals) from given $q+1$ spectra mentioned above and the lengths of q strings. We use the method of continued fractions of [4] based on the results of [19]. We show that if the $q+1$ spectra do not intersect, then they, together with the lengths of the parts of the string, uniquely determine values of the point masses and the lengths of the subintervals. A procedure of recovering the values of the masses and the lengths of the subintervals is proposed.

2. Direct spectral problem

We consider a plane star graph of q edges. Each edge is a Stieltjes string (a threads, i.e., a string of zero density) bearing a finite number of point masses. Every string has one end joined at an interior vertex of the star graph. The j-th edge consists of n_j+1 ($n_j \geq 1$) intervals $l_k^{(j)}$ ($k = 0, 1, \ldots, n_j$) with point masses $m_k^{(j)}$ ($k = 1, 2, \ldots, n_j$) separating them ($l_{k-1}^{(j)}$ lies to the exterior vertex from $m_k^{(j)}$ and $l_k^{(j)}$ lies to the interior one). The lengths of the strings we denote by l_j, $l_j = \sum_{k=0}^{n_j} l_k^{(j)}$. The strings are stretched and the pendant ends fixed.

Denote by $v_k^{(j)}(t)$ ($k = 1, 2, \ldots, n_j$; $j = 1, 2, \ldots, q$) the transverse displacement of the point mass $m_k^{(j)}$, which lies on the j-th string and is the k-th one counting from the pendant end, at the time t. We assume the threads to be stretched by the forces each equal to 1.

Then small transverse vibrations of the net are subject to the equation

$$\frac{v_k^{(j)}(t) - v_{k+1}^{(j)}(t)}{l_k^{(j)}} + \frac{v_k^{(j)}(t) - v_{k-1}^{(j)}(t)}{l_{k-1}^{(j)}} + m_k^{(j)} v_k^{(j)\prime\prime}(t) = 0$$

$$(k = 1, 2, \ldots, n_j, \quad j = 1, 2, \ldots, q).$$

Continuity of the string at the point joining the edges yields

$$v_{n_1+1}^{(1)}(t) = v_{n_2+1}^{(2)}(t) = \cdots = v_{n_q+1}^{(q)}(t).$$

Absence of a point mass or external force at the point of joining implies that

$$\sum_{j=1}^{q} \frac{v_{n_j+1}^{(j)}(t) - v_{n_j}^{(j)}(t)}{l_{n_j}^{(j)}} = 0. \tag{2.1}$$

We impose Dirichlet boundary conditions at the pendant vertices,

$$v_0^{(j)}(t) = 0, \quad j = 1, 2, \ldots, q,$$

which mean that the ends are fixed. Substituting $v_k^{(j)}(t) = u_k^{(j)} e^{i\lambda t}$ we obtain the following recurrences for the amplitudes $u_k^{(j)}$:

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}} - m_k^{(j)} \lambda^2 u_k^{(j)} = 0 \quad (k = 1, 2, \ldots, n_j, \quad j = 1, 2, \ldots, q), \tag{2.2}$$

$$u^{(1)}_{n_1+1} = u^{(2)}_{n_2+1} = \cdots = u^{(q)}_{n_q+1}, \tag{2.3}$$

$$\sum_{j=1}^{q} \frac{u^{(j)}_{n_j+1} - u^{(j)}_{n_j}}{l^{(j)}_{n_j}} = 0, \tag{2.4}$$

$$u^{(j)}_0 = 0, \quad j = 1, 2, \ldots, q. \tag{2.5}$$

According to [4],

$$u^{(j)}_k = R^{(j)}_{2k-2}(\lambda^2) u^{(j)}_1 \quad (k = 1, 2, \ldots, n_j),$$

where $R^{(j)}_{2k-2}(\lambda^2)$ are polynomials of degree $2k-2$ which can be obtained by solving (2.2). We set, by definition,

$$R^{(j)}_{2k-1}(\lambda^2) = \frac{R^{(j)}_{2k}(\lambda^2) - R^{(j)}_{2k-2}(\lambda^2)}{l^{(j)}_k}.$$

Due to (2.2), the polynomials $R^{(j)}_k$ satisfy the recurrence relations

$$R^{(j)}_{2k-1}(\lambda^2) = -\lambda^2 m^{(j)}_k R^{(j)}_{2k-2}(\lambda^2) + R^{(j)}_{2k-3}(\lambda^2),$$
$$R^{(j)}_{2k}(\lambda^2) = l^{(j)}_k R^{(j)}_{2k-1}(\lambda^2) + R^{(j)}_{2k-2}(\lambda^2)$$
$$(k = 1, 2, \ldots, n_j; \quad R^{(j)}_{-1}(\lambda^2) = \frac{1}{l^{(j)}_0}, \quad R^{(j)}_0(\lambda^2) = 1).$$

Due to (2.3) and (2.4), at the point of joining we have

$$R^{(1)}_{2n_1}(\lambda^2) u^{(1)}_1 = R^{(2)}_{2n_2}(\lambda^2) u^{(2)}_1 = \cdots = R^{(q)}_{2n_q}(\lambda^2) u^{(q)}_1,$$
$$\sum_{j=1}^{q} R^{(j)}_{2n_j-1}(\lambda^2) u^{(j)}_1 = 0.$$

Therefore, the spectrum of problem (2.2)–(2.4) coincides with the set of zeros of the polynomial

$$\phi(\lambda^2) = \sum_{j=1}^{q} R^{(j)}_{2n_j-1}(\lambda^2) \prod_{k=1,\ k\neq j}^{q} R^{(k)}_{2n_k}(\lambda^2). \tag{2.6}$$

It should be mentioned that according to [4] the fractions $\frac{R^{(j)}_{2n_j}(z)}{R^{(j)}_{2n_j-1}(z)}$ and can be expanded into continued fractions,

$$\frac{R^{(j)}_{2n_j}(z)}{R^{(j)}_{2n_j-1}(z)} = l^{(j)}_{n_j} + \cfrac{1}{-m^{(j)}_{n_j} z + \cfrac{1}{l^{(j)}_{n_j-1} + \cfrac{1}{-m^{(j)}_{n_j-1} z + \cdots + \cfrac{1}{l^{(j)}_1 + \cfrac{1}{-m^{(j)}_1 z + \cfrac{1}{l^{(j)}_0}}}}}}. \tag{2.7}$$

Definition 2.1. The function $\omega(\lambda)$ is said to be a Nevanlinna function (or an R-function in terms of [20]) if the following conditions are satisfied:
1) it is analytic in the half-planes $\mathrm{Im}\lambda > 0$ and $\mathrm{Im}\lambda < 0$;
2) $\omega(\overline{\lambda}) = \overline{\omega(\lambda)}$ ($\mathrm{Im}\lambda \neq 0$);
3) $\mathrm{Im}\lambda \ \mathrm{Im}\ \omega(\lambda) \geq 0$ for $\mathrm{Im}\lambda \neq 0$.

Definition 2.2. (see [20]). The Nevanlinna function $\omega(\lambda)$ is said to be an S-function if $\omega(\lambda) > 0$ for $\lambda < 0$.

Definition 2.3. An S-function $\omega(\lambda)$ is said to be an S_0-function if 0 is not a pole of $\omega(\lambda)$.

Theorem 2.1. *After cancellation of common factors (if any) in the numerator and the denominator, the function*

$$\frac{\prod_{j=1}^{q} R_{2n_j}^{(j)}(z)}{\phi(z)}$$

becomes an S_0-function.

Proof. Let us represent the ratio as follows:

$$\frac{\prod_{j=1}^{q} R_{2n_j}^{(j)}(z)}{\phi(z)} = \left(\sum_{j=1}^{q} \frac{R_{2n_j-1}^{(j)}(z)}{R_{2n_j}^{(j)}(z)}\right)^{-1}.$$

Since $\frac{R_{2n_j}^{(j)}(z)}{R_{2n_j-1}^{(j)}(z)}$ are Nevanlinna functions (see [4]), the functions $-\frac{R_{2n_j-1}^{(j)}(z)}{R_{2n_j}^{(j)}(z)}$ are also Nevanlinna as well as the functions $-\sum_{j=1}^{q}\left(\frac{R_{2n_j-1}^{(j)}(z)}{R_{2n_j}^{(j)}(z)}\right)$ and $\left(\sum_{j=1}^{q}\frac{R_{2n_j-1}^{(j)}(z)}{R_{2n_j}^{(j)}(z)}\right)^{-1}$. Since S_0-functions $\frac{R_{2n_j}^{(j)}(z)}{R_{2n_j-1}^{(j)}(z)}$ are positive for $z \in (-\infty, 0]$, we have that $\left(\sum_{j=1}^{q}\frac{R_{2n_j-1}(z)}{R_{2n_j}(z)}\right)^{-1}$ is also positive for $z \in (-\infty, 0]$. The theorem is proved. □

Let us denote by $n = \sum_{j=1}^{q} n_j$ and by $\{\lambda_k\}$ ($k = \pm 1, \pm 2, \ldots, \pm n$, $\lambda_{-k} = -\lambda_k$ and $\lambda_k \geq \lambda_{k'}$ for $k > k'$) the eigenvalues of problem (2.2)–(2.5). If we clamp our string at the point of joining of its parts then we obtain q problems on the edges with Dirichlet boundary conditions at both ends,

$$\frac{u_k^{(j)} - u_{k+1}^{(j)}}{l_k^{(j)}} + \frac{u_k^{(j)} - u_{k-1}^{(j)}}{l_{k-1}^{(j)}} - m_k^{(j)}\lambda^2 u_k^{(j)} = 0 \quad (k = 1, 2, \ldots, n_j), \tag{2.8}$$

$$u_{n_j+1}^{(j)} = 0, \tag{2.9}$$

$$u_0^{(j)} = 0. \tag{2.10}$$

Let us denote by $\{\nu_k^{(j)}\}$ ($k = \pm 1, \pm 2, \ldots, \pm n_j$), where $\nu_{-k}^{(j)} = -\nu_k^{(j)}$ and $\nu_k^{(j)} > \nu_{k'}^{(j)}$ for $k > k'$, $j = 1, 2, \ldots, q$, the spectra of problems (2.8)–(2.10) , respectively. By $\{\zeta_k\}_{-n}^{n}$ we denote the union $\{\zeta_k\}_{-n,k\neq 0}^{n} = \cup_{j=1}^{q}\{\nu_k^{(j)}\}_{j=-n_j,k\neq 0}^{n_j}$.

Theorem 2.2. *The sequences $\{\lambda_k\}_{-n,k\neq 0}^{n}$ and $\{\zeta_k\}_{-n,k\neq 0}^{n}$ interlace as follows:*

1) $\zeta_{-n} \leq \lambda_{-n} \leq \zeta_{-n+1} \leq \cdots \leq \zeta_{-1} < \lambda_{-1} < 0 < \lambda_1 < \zeta_1 \leq \lambda_2 \leq \cdots \leq \zeta_n$;
2) $\zeta_{k-1} = \lambda_k$ *if and only if* $\lambda_k = \zeta_k$;
3) *the multiplicity of* ζ_k *does not exceed* q.

Proof. By Lemma 5.1 in [20], Theorem 2.1 implies that

$$\frac{\lambda \prod_{j=1}^{q} R_{2n_j}^{(j)}(\lambda^2)}{\phi(\lambda^2)}$$

becomes a Nevanlinna function possibly after cancellation of equal factors in the numerator and the denominator. Suppose $\lambda_k = \nu_p^{(j)}$ for some k, p and j, then (2.6) implies

$$R_{2n_j-1}^{(j)}(\nu_p^{(j)2}) \prod_{k=1,k\neq j}^{q} R_{2n_k}^{(k)}(\nu_p^{(j)2}) = 0. \tag{2.11}$$

It is known (see [4]) that $R_{2n_j}^{(j)}(\nu_p^{(j)2}) = 0$ implies $R_{2n_j-1}^{(j)}(\nu_p^{(j)2}) \neq 0$ which, together with (2.11), yields

$$R_{2n_r}^{(r)}(\nu_p^{(j)2}) = 0$$

for some $r \neq j$.

This means $\nu_p^{(j)} \in \{\nu_k^{(r)}\}_{k=-n_r,k\neq 0}^{n_r}$. Assertions 1 and 2 are proved. The multiplicity of a zero of $\prod_{j=1}^{q} R_{2n_j}^{(j)}(\lambda^2)$ cannot exceed q because each factor in the product has only simple zeros. Theorem 2.2 is proved. $\square$

3. Inverse three spectral problem

Here we will consider the problem of recovering the sets $\{m_k^{(j)}\}_{k=1}^{n_j}$, $\{l_k^{(j)}\}_{k=0}^{n_j}$ $(j = 1, 2, \ldots, q)$ using the spectra $\{\lambda_k\}_{k=-n,k\neq 0}^{n}$, $\{\nu_k^{(j)}\}_{k=-n_j,k\neq 0}^{n_j}$ $(j = 1, 2, \ldots, q)$ and the lengths l_j of the strings. Here $n = \sum_{k=1}^{q} n_j$.

Theorem 3.1. *Let $l_j > 0$ $(j = 1, 2, \ldots, q)$ be given. Let the sequences of real numbers $\{\lambda_k\}_{k=-n,\ k\neq 0}^{n}$, $\{\nu_k^{(j)}\}_{k=-n_j,\ k\neq 0}^{n_j}$ $(j = 1, 2, \ldots, q,\ n = \sum_{j=1}^{q} n_j)$ satisfy the conditions*

1. $\lambda_{-k} = -\lambda_k$ for each k; $\lambda_k < \lambda_{k'}$ if $k < k'$, $\nu_{-k}^{(j)} = -\nu_k^{(j)}$ for each k and each j; and, for each $j = 1, 2, \ldots, n_j$, $\nu_k^{(j)} < \nu_{k'}^{(j)}$ if $k < k'$.

2. $\{\lambda_k\}_{k=-n,\ k\neq 0}^{n} \cap \{\nu_k^{(j)}\}_{k=-n_j,\ k\neq 0}^{n_j} = \emptyset$ for $j = 1, 2, \ldots, q$, and $\{\nu_k^{(j)}\}_{k=-n_j,\ k\neq 0}^{n_j} \cap \{\nu_k^{(s)}\}_{k=-n_s,\ k\neq 0}^{n_s} = \emptyset$ for $j, s \in \{1, 2, \ldots, q\}$ and $j \neq s$.

3. Elements of the set $\{\zeta_k\}_{k=-n}^{n} =^{def} \{0\} \cup_{j=1}^{q} \{\nu_k^{(j)}\}_{k=-n_j,\ k\neq 0}^{n_j}$ are indexed in such a way that $\zeta_{-k} = -\zeta_k$ for each k; $\zeta_k < \zeta_{k'}$ if $k < k'$ interlace with elements of $\{\lambda_k\}_{k=-n),\ k\neq 0}^{n}$,

$$\zeta_{-n} < \lambda_{-n} < \zeta_{-n+1} < \cdots < \lambda_{-1} < 0 < \lambda_1 < \zeta_1 < \cdots < \zeta_n. \tag{3.1}$$

Then there exist a unique collection of sets $\{m_k^{(j)}\}_{k=1}^{n_j}$, $(j = 1, 2, \ldots, q)$, $\{l_k^{(j)}\}_{k=0}^{n_j}$ $(j = 0, 1, 2, \ldots, q)$ such that $\sum_{k=0}^{n_j} l_k^{(j)} = l_j$, which generate problems (2.2)–(2.5) and (2.8)–(2.10) with the spectra $\{\lambda_k\}_{k=-n,\ k\neq 0}^{n}$, $\{\nu_k^{(j)}\}_{k=-n_j,\ k\neq 0}^{n_j}$, respectively.

Proof. Let us construct the polynomials

$$Q_0(\lambda^2) = \sum_{j=1}^{q} l_j^{-1} \prod_{j=1}^{q} l_j \prod_{k=1}^{n} \left(1 - \frac{\lambda^2}{\lambda_k^2}\right), \tag{3.2}$$

$$Q_j(\lambda^2) = l_j \prod_{k=1}^{n_j} \left(1 - \frac{\lambda^2}{\nu_k^{(j)2}}\right), \tag{3.3}$$

and the Lagrange interpolating polynomial

$$P_j(\lambda^2) = \sum_{k=1}^{n_j} \frac{\lambda^2 Q_0(\nu_k^{(j)2})}{\nu_k^{(j)2} \prod_{s=1,\ s\neq j}^{q} Q_s(\nu_k^{(j)2})} \prod_{p=1,\ p\neq k}^{n_j} \frac{(\lambda^2 - \nu_p^{(j)2})}{(\nu_k^{(j)2} - \nu_p^{(j)2})} + \prod_{k=1}^{n_j} \frac{\nu_k^{(j)2} - \lambda^2}{\nu_k^{(j)2}}. \tag{3.4}$$

It is clear that

$$
\begin{aligned}
&P_j(0) = 1, \\
&P_j(\nu_k^{(j)2}) = \frac{Q_0(\nu_k^{(j)2})}{\prod_{s=1,\ s\neq j}^q Q_s(\nu_k^{(j)2})} \\
&\qquad = Q_0(\nu_k^{(j)2}) \left.\frac{dQ_j(z)}{dz}\right|_{z=(\nu_k^{(j)})^2} \left(\left.\frac{d\prod_{p=1}^q Q_j(z)}{dz}\right|_{z=\nu_k^{(j)2}} \right)^{-1}, \\
&\qquad\qquad k = 1,2,\dots,n_j.
\end{aligned}
\tag{3.5}
$$

From (3.5) and (3.1) we conclude that

$$P_j(0) > 0, \quad P_j(\nu_k^{(j)2})(-1)^k > 0, \quad k = 1,2,\dots,n_j.$$

Consequently, the zeros $\mu_k^{(j)2}$ of $P_j(z)$ are all positive and interlace with the zeros $\nu_k^{(j)2}$ of $Q_j(z)$

$$0 < \mu_1^{(j)2} < \nu_1^{(j)2} < \cdots < \mu_{n_1}^{(j)2} < \nu_{n_1}^{(j)2}.$$

Thus, due to the evident inequality $\frac{Q_j(0)}{P_j(0)} = l_j > 0$, we conclude that $\frac{Q_j(z)}{P_j(z)}$ is an S-function.

It is shown in [4] that the an S_0-function can be expanded into a continued fraction,

$$\frac{Q_j(z)}{P_j(z)} = l_{n_j}^{(j)} + \cfrac{1}{-m_{n_j}^{(j)} z + \cfrac{1}{l_{n_j-1}^{(j)} + \cfrac{1}{-m_{n_j-1}^{(j)} z + \dots + \cfrac{1}{l_1^{(j)} + \cfrac{1}{-m_1^{(j)} z + \cfrac{1}{l_0^{(j)}}}}}}}, \tag{3.6}$$

where $\{l_k^{(j)}\}_0^{n_j}$ and $\{m_k^{(j)}\}_1^{n_j}$ are sequences of positive numbers which we identify with the subintervals and masses we are looking for. From (3.3), (3.5) and (3.7) we obtain

$$l_j = \frac{Q_j(0)}{P_j(0)} = l_{n_j}^{(j)} + l_{n_j-1}^{(j)} + \cdots + l_1^{(j)} + l_0^{(j)}.$$

Comparing (3.6) with (2.7) we conclude that

$$R_{2n_j}^{(j)}(z) = T_j Q_j(z), \quad R_{2n_j-1}^{(j)}(z) = T_j P_j(z), \tag{3.7}$$

where T_j are nonzero constants and $R_{2n_j}^{(j)}(z)$, $R_{2n_j-1}^{(j)}(z)$ are the polynomials described in the previous section.

From (3.7) we conclude that the spectra of problems (2.8)–(2.9) generated by the obtained sets $\{l_k^{(j)}\}_{k=0}^{n_j}$, $\{m_k^{(j)}\}_{k=1}^{n_j}$ coincide with $\{\nu_k^{(j)}\}_{k=1}^{n_j}$. Let us show that the spectrum of problem (2.2)–(2.5) generated by the sets $\{l_k^{(j)}\}_{k=0}^{n_j}$, $\{m_k^{(j)}\}_{k=1}^{n_j}$ coincides with $\{\lambda_k\}_{-n,k\neq 0}^{n}$. As it was shown in Section 2, the spectrum is nothing but the set of zeros of the polynomial

$$\sum_{j=1}^q R_{2n_j-1}^{(j)}(\lambda^2) \prod_{s=1,\ s\neq j}^q R_{2n_s}^{(s)}(\lambda^2).$$

Using (3.7) we obtain

$$\sum_{j=1}^q R_{2n_j-1}^{(j)}(z) \prod_{s=1,\ s\neq j}^q R_{2n_s}^{(s)}(z). = \prod_{j=1}^q T_j \hat{Q}_0(z),$$

where we use the notation

$$\hat{Q}_0(z) = \sum_{j=1}^{q} P_j(z) \prod_{s=1,\ s\neq j}^{q} Q_s(z).$$

Let us compare the polynomial $Q_0(z)$ with the polynomial $\hat{Q}_0(z)$. It follows from (3.2)–(3.5) that

$$Q_0(0) = \sum_{j=1}^{q} l_j^{-1} \prod_{j=1}^{q} l_j = \hat{Q}_0(0),$$

$$Q_0(\nu_k^{(j)2}) = \prod_{s=1,\ s\neq j}^{q} Q_s(\nu_k^{(j)2}) P_j(\nu_k^{(j)2}) = \hat{Q}_0(\nu_k^{(j)2}), \quad k = 1, 2, \ldots, n_j, \quad j = 1, 2, \ldots, q,$$

and, consequently, $\hat{Q}_0(z) \equiv Q_0(z)$. Hence, $\phi(z) = \prod_{j=1}^{q} T_j Q_0(z)$ and the set of zeros of $\phi(\lambda^2)$ is nothing but $\{\lambda_k\}_{k=-n,k\neq 0}^{n}$.

Let us prove now uniqueness of the solution of the inverse problem. Suppose there exist two collections of sets $\{\{m_k^{(j)}\}_{k=1}^{n_j}, \{l_k^{(j)}\}_{k=0}^{n_j}, j = 1, 2, \ldots, q\}$ and $\{\{\tilde{m}_k^{(j)}\}_{k=1}^{n_j}, \{\tilde{l}_k^{(j)}\}_{k=0}^{n_j}, j = 1, 2, \ldots, q\}$ which satisfy $\sum_{k=0}^{n_j} l_k^{(j)} = \sum_{k=0}^{n_j} \tilde{l}_k^{(j)} = l_j$ for $j = 1, 2, \ldots, q$ and which generate problems (2.8)–(2.10) with the same spectra, $\{\nu_k^{(j)}\}_{k=-n_j,k\neq 0}^{n_j} = \{\tilde{\nu}_k^{(j)}\}_{k=-n_j,k\neq 0}^{n_j}$ ($j = 1, 2, \ldots, q$) and problems (2.2)–(2.5) with the same spectra $\{\lambda_k\}_{k=-n,k\neq 0}^{n} = \{\tilde{\lambda}_k\}_{k=-n,k\neq 0}^{n}$ ($n = \sum_1^q n_j$). Then solving the corresponding direct problems (2.8)–(2.10) we find polynomials $R_{2n_j}^{(j)}(z)$ and $\tilde{R}_{2n_j}^{(j)}(z)$ the sets of zeros of which coincide. This means that

$$\tilde{R}_{2n_j}^{(j)}(z) = T_j R_{2n_j}^{(j)}(z), \quad j = 1, 2, \ldots, q, \tag{3.8}$$

where T_j are nonzero constants. Solving problem (2.2)–(2.5) for both collections $\{\{m_k^{(j)}\}_{k=1}^{n_j}, \{l_k^{(j)}\}_{k=0}^{n_j}, j = 1, 2, \ldots, q\}$ and $\{\{\tilde{m}_k^{(j)}\}_{k=1}^{n_j}, \{\tilde{l}_k^{(j)}\}_{k=0}^{n_j}, j = 1, 2, \ldots, q\}$ we obtain the corresponding characteristic polynomials $\phi(z)$ and $\tilde{\phi}(z)$ which have the same set of zeros $\{\lambda_k\}$. Therefore,

$$\tilde{\phi}(z) = C\phi(z), \tag{3.9}$$

where C is a nonzero constant. Using (2.6) we obtain

$$\phi(\nu_k^{(j)2}) = R_{2n_j-1}^{(j)}(\nu_k^{(j)2}) \prod_{s=1,\ s\neq j}^{q} R_{2n_s}^{(s)}(\nu_k^{(j)}) \tag{3.10}$$

and

$$\tilde{\phi}(\nu_k^{(j)2}) = \tilde{R}_{2n_j-1}^{(j)}(\nu_k^{(j)2}) \prod_{s=1,\ s\neq j}^{q} \tilde{R}_{2n_s}^{(s)}(\nu_k^{(j)}). \tag{3.11}$$

Using (3.7), (3.10), (3.11) we obtain

$$C = \frac{\tilde{R}_{2n_j-1}^{(j)}(\nu_k^{(j)2})}{R_{2n_j-1}^{(j)}(\nu_k^{(j)2})} \prod_{s=1,\ s\neq j}^{q} T_s. \tag{3.12}$$

Using (2.7) taking into account the equation $\tilde{l}_j = l_j$ we obtain

$$\frac{\tilde{R}_{2n_j}^{(j)}(0)}{\tilde{R}_{2n_j-1}^{(j)}(0)} = \tilde{l}_j = l_j = \frac{R_{2n_j}^{(j)}(0)}{R_{2n_j-1}^{(j)}(0)}. \tag{3.13}$$

Now (3.8) and (3.13) imply

$$\frac{\tilde{R}^{(j)}_{2n_j-1}(0)}{R^{(j)}_{2n_j-1}(0)} = \frac{\tilde{R}^{(j)}_{2n_j}(0)}{R^{(j)}_{2n_j}(0)} = T_j. \tag{3.14}$$

Using (3.9)–(3.12) and (3.14) we obtain

$$C = T_1 T_2 \dots T_q. \tag{3.15}$$

Substituting (3.15) into (3.12) we obtain

$$\tilde{R}^{(j)}_{2n_j-1}(\nu_k^{(j)2}) - T_j R^{(j)}_{2n_j-1}(\nu_k^{(j)2}) = 0, \quad k = 1, 2, \dots, n_j, \quad j = 1, 2, \dots, q. \tag{3.16}$$

According to (3.13) and (3.16) the polynomial $\tilde{R}^{(j)}_{2n_j-1}(z) - T_j R^{(j)}_{2n_j-1}(z)$ of degree not greater than n_j vanishes at $n_j + 1$ points and is identically 0. This means that

$$\tilde{R}^{(j)}_{2n_j-1}(z) = T_j R^{(j)}_{2n_j-1}(z).$$

Together with (3.8), this implies

$$\frac{\tilde{R}^{(j)}_{2n_j}(z)}{\tilde{R}^{(j)}_{2n_j-1}(z)} = \frac{R^{(j)}_{2n_j}(z)}{R^{(j)}_{2n_j-1}(z)}.$$

The left-hand side and the right-hand side possess the same decomposition into continued fractions (2.7). Theorem 3.1 is proved. □

Remark 3.1. If Condition 2 of Theorem 3.1 is violated but Condition 2 of Theorem 2.2 is satisfied instead, then the sets of positive numbers $\{m_k^{(j)}\}_{k=1}^{n_j}$, $\{l_k^{(j)}\}_{k=0}^{n_j}$ exist such that $\sum_{k=0}^{n_j} l_k^{(j)} = l_j$ and the sets generate problems (2.2)–(2.5), (2.8)–(2.10) with the spectra $\{\lambda_k\}_{k=-n,\ k\neq 0}^{n}$, $\{\nu_k^{(j)}\}_{k=-n_j,\ k\neq 0}^{n_j}$ respectively, but the collection of sets $\{\ \{m_k^{(j)}\}_{k=1}^{n_j}$, $\{l_k^{(j)}\}_{k=0}^{n_j}\ \}$ is not unique.

Remark 3.2. Similar to Theorem 3.1, the result is true if we impose Neumann or Robin conditions at pendant ends instead of the Dirichlet conditions.

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References

1. H. Dym, H. P. McKean, *Gaussian Processes, Function Theory and the Inverse Spectral Problem*, Academic Press, New York—San Francisco—London, 1976.
2. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, 1953.
3. B. M. Levitan, M. G. Gasymov, *Determination of a differential equation by two of its spectra*, Uspekhi Mat. Nauk **19** (1964), no. 2(116), 3–63. (Russian)
4. F. R. Gantmakher, M. G. Krein, *Oscillating Matrices and Kernels and Small Vibrations of Mechanical Systems*, GITTL, Moscow—Leningrad, 1950. (Russian); German transl.: Akademie Verlag, Berlin, 1960.
5. M. G. Krein, *Some new problems in the theory of oscillations of Sturm systems*, Prikladnaya matematika i mekhanika **16** (1952), no. 5, 555–568. (Russian)
6. P. F. Kurchanov, A. D. Myshkis, A. M. Filimonov, *Train vibrations and Kronecker's theorem*, Prikladnaya matematika i mekhanika **55** (1991), no. 6, 989–995. (Russian)
7. A. M. Filimonov, P. F. Kurchanov, A. D. Myshkis, *Some unexpected results in the classical problem of vibrations of the string with n beads when n is large*, C.R. Acad. Sci. Paris, Ser. I. Math. **313** (1991), 961–965.
8. A. F. Filimonov, A. D. Myshkis, *On properties of large wave effect in classical problem bead string vibration*, J. Difference Equations and Applications **10** (2004), no. 13–15, 1171–1175.
9. M. R. Wohlers, *Lumped and Distributed Passive Networks*, Academic Press, New York, 1969.

10. V. N. Pivovarchik, *An inverse Sturm-Liouville problem by three spectra*, Integral Equations and Operator Theory **34** (1999), no. 2, 234–243.
11. F. Gesztesy and B. Simon, *On the determination of a potential from three spectra*, Advances in Mathematical Sciences, V. Buslaev and M. Solomyak, eds., Amer. Math. Soc. Transl., Ser. 2, Vol. 189, 1999, pp. 85–92.
12. F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential,* II. *The case of discrete spectrum*, Trans. Amer. Math. Soc. **352** (1999), no. 6, 2765–2787.
13. R. O. Hryniv, Ya. V. Mykytyuk, *Inverse spectral problems for Sturm-Liouville operators with singular potentials. Part* III*: Reconstruction by three spectra*, J. Math. Anal. Appl. **248** (2003), 626–646.
14. S. Albeverio, R. O. Hryniv and Ya. V. Mykytyuk, *Inverse problems for coupled oscillating systems: reconstruction from three spectra*, Methods Funct. Anal. Topology **13** (2007), no. 2, 110–123.
15. Johanna Michor, Gerald Teschl, *Reconstructing Jacobi matrices by three spectra*, Oper. Theory Adv. Appl., **154**, Birkhäuser, Basel, 2004, 151–154.
16. V. Pivovarchik, *Inverse problem for the Sturm-Liouville equation on a simple graph*, SIAM J. Math. Anal. **32** (2000), no. 4, 801–819.
17. V. Pivovarchik, *Inverse problem for the Sturm-Liouville equation on a star-shaped graph*, Math. Nachr. **280** (2007), 1595–1619.
18. O. Boyko, V. Pivovarchik, *Inverse three spectral problem for a Stieltjes string and inverse problem with one dimensional damping*, Inverse Problems **24** (2008), no. 1, 015019.
19. T.-L. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse **8** (1894), 1–122; **9** (1895), 1–47.
20. I. S. Kac, M. G. Krein, *On the Spectral Function of the String*, Amer. Math. Soc. Transl., Ser. 2, Vol. 103, 1974, 1–18.
21. B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Transl. Math. Monographs, Vol. 5, Amer. Math. Soc., Providence, R. I., 1980.

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