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ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR PERIODIC POTENTIALS

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Dedicated to M. L. Gorbachuk on the occasion of his 70th birthday.

Abstract. We study the one-dimensional Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad u \in \mathrm{Dom}\,(S(q))\,,$$

with 1-periodic real-valued singular potentials $q(x) \in H_{\mathrm{per}}^{-1}(\mathbb{R},\mathbb{R})$ on the Hilbert space $L_2\,(\mathbb{R})$. We show equivalence of five basic definitions of the operators $S(q)$ and prove that they are self-adjoint. A new proof of continuity of the spectrum of the operators $S(q)$ is found. Endpoints of spectrum gaps are precisely described.

1. Introduction

On the complex Hilbert space $L_2(\mathbb{R})$, we consider the one-dimensional Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad u \in \mathrm{Dom}\ (S(q))\,, \tag{1}$$

with real-valued 1-periodic distribution potentials $q(x)$, the so-called the Hill-Schrödinger operators.

Assuming that

$$q(x) = \sum_{k\in 2\mathbb{Z}} \widehat{q}(k)e^{ik\pi x} \in H_{\mathrm{per}}^{-1}(\mathbb{R},\mathbb{R}), \tag{2}$$

that is,

$$\sum_{k\in 2\mathbb{Z}} (1+|k|)^{-2}|\widehat{q}(k)|^2 < \infty \quad \text{and} \quad \widehat{q}(k) = \overline{\widehat{q}(-k)} \quad \forall k \in 2\mathbb{Z},$$

the Hill-Schrödinger operators $S(q)$ can be well defined on the Hilbert space $L_2(\mathbb{R})$ in the following different ways:

(1) as minimal/maximal quasi-differential operators $S_{\min}(q)/S_{\max}(q)$;
(2) as Friedrichs extensions $S_F(q)$ of quasi-differential operators $S_{\min}(q)$;
(3) as form-sum operators $S_{\mathrm{form}}(q)$;
(4) as the limit $S_{\lim}(q)$ of sequences of the Hill-Schrödinger operators with smooth periodic potentials in the norm resolvent sense.

Hryniv and Mykytyuk [8], Djakov and Mityagin [5] studied the Friedrichs extensions $S_F(q)$, but Korotyaev [10] treated the form-sum operators $S_{\mathrm{form}}(q)$. We propose to join together these results showing an equivalence of all definitions.

More precisely, we will prove the following statements.

Theorem A. (Theorem 14). *The Hill-Schrödinger quasi-differential operators* $S_{\max}(q)$ *with distributional potentials* $q(x) \in H_{\mathrm{per}}^{-1}(\mathbb{R},\mathbb{R})$ *are self-adjoint.*

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Theorem B. (Corollary 15, Corollary 16, Theorem 18). *The quasi-differential operators $S_{\min}(q)$ and $S_{\max}(q)$, the Friedrichs extensions $S_F(q)$, the form-sum operators $S_{\rm form}(q)$, and the operators $S_{\lim}(q)$ coincide.*

In the paper [8, Theorem 3.5] the authors tried to show that the operators $S_{\max}(q)$ and $S_F(q)$ coincide. But the proof of this assertion was erroneous. Our proofs of Theorem A and Theorem B are based on a different idea (see Lemma 5).

The equality $S(q) = S_{\lim}(q)$, together with the classical Birkhoff-Lyapunov theorem, allow to prove the following statement.

Theorem C. (Theorem 19). *The Hill-Schrödinger operators $S(q)$ with distributional potentials $q(x) \in H^{-1}_{\rm per}(\mathbb{R},\mathbb{R})$ have continuous spectra with the band and the gap structures being such that the endpoints $\{\lambda_0(q), \lambda_k^{\pm}(q)\}_{k=1}^{\infty}$ of the spectrum gaps satisfy the inequalities*

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \cdots$$

Moreover, endpoints of the spectrum gaps for even (odd) numbers $k \in \mathbb{Z}_+$ are periodic (semiperiodic) eigenvalues of the following problem on the interval $[0, 1]$:

$$S_{\pm}(q)u = -u'' + q(x)u = \lambda u, \quad u \in \mathrm{Dom}\,(S_{\pm}(q))\,.$$

It is interesting to remark that the last assertion is nontrivial, and for more singular δ'-interactions, that is if

$$q(x) = \sum_{k\in\mathbb{Z}} \beta\, \delta'(x-k) \notin H^{-1}_{\rm per}(\mathbb{R}), \quad \beta < 0,$$

it could still occur that endpoints of the spectrum gaps for even (odd) numbers $k \in \mathbb{Z}_+$ are *semiperiodic (periodic)* eigenvalues of the problem on the interval $[0, 1]$, see [2, Theorem III.3.6].

In the closely related paper of Hryniv and Mykytyuk [8], the authors have established that spectra of the operators $S(q)$ are absolutely continuous.

2. Preliminaries

2.1. **Sobolev spaces.** Let us denote by $\mathfrak{D}'_1(\mathbb{R})$ the Schwartz space of 1-periodic distributions defined on the whole real axis $\mathbb{R}$ (see [24]). To have a detailed characterization of 1-periodic distributions, we will use Sobolev spaces.

Consider the Sobolev spaces $H^s_{\rm per}(\mathbb{R})$, $s \in \mathbb{R}$, of 1-periodic functions (distributions) defined by means of their Fourier coefficients,

$$H^s_{\rm per}(\mathbb{R}) := \left\{ f = \sum_{k\in 2\mathbb{Z}} \widehat{f}(k) e^{ik\pi x} \;\Big|\; \| f \|_{H^s_{\rm per}(\mathbb{R})} < \infty \right\},$$

$$\| f \|_{H^s_{\rm per}(\mathbb{R})} := \left(\sum_{k\in 2\mathbb{Z}} \langle k\rangle^{2s} |\widehat{f}(k)|^2 \right)^{1/2}, \quad \langle k\rangle := 1 + |k|,$$

$$\widehat{f}(k) := \langle f, e^{ik\pi x}\rangle_{L_{2,\rm per}(\mathbb{R})}, \quad k \in 2\mathbb{Z},$$

$$2\mathbb{Z} := \{ k \in \mathbb{Z} \mid k \equiv 0 \ (\mathrm{mod}\, 2) \}\,.$$

The sesquilinear form $\langle\cdot,\cdot\rangle_{L_{2,\rm per}(\mathbb{R})}$ pairs the dual, respectively $L_{2,\rm per}(\mathbb{R})$, spaces $H^s_{\rm per}(\mathbb{R})$ and $H^{-s}_{\rm per}(\mathbb{R})$, and is an extension by continuity of the $L_{2,\rm per}(\mathbb{R})$-inner product [3, 7],

$$\langle f, g\rangle_{L_{2,\rm per}(\mathbb{R})} := \int_0^1 f(x)\overline{g(x)}\, dx \quad \forall f, g \in L_{2,\rm per}(\mathbb{R}).$$

It should be noted that

$$H^0_{\rm per}(\mathbb{R}) = L_{2,\rm per}(\mathbb{R}),$$

and we denote by $\mathfrak{D}'_1(\mathbb{R},\mathbb{R})$ and $H^s_{\mathrm{per}}(\mathbb{R},\mathbb{R})$, $s\in\mathbb{R}$, the *real-valued* 1-periodic distributions from the correspondent spaces,

$$\mathfrak{D}'_1(\mathbb{R},\mathbb{R}) := \{f(x)\in\mathfrak{D}'_1(\mathbb{R})\,|\,\mathrm{Im}f(x)=0\},$$
$$H^s_{\mathrm{per}}(\mathbb{R},\mathbb{R}) := \left\{f(x)\in H^s_{\mathrm{per}}(\mathbb{R})\,|\,\mathrm{Im}f(x)=0\right\}.$$

Note that $\mathrm{Im}f(x)=0$ for a 1-periodic distribution $f(x)\in\mathfrak{D}'_1(\mathbb{R})$ means that

$$\widehat{f}(2k)=\overline{\widehat{f}(-2k)}\quad\forall k\in\mathbb{Z}.$$

2.2. **Quasi-differential equations.** The differential expressions in the right-hand of (1), by introducing quasi-derivatives

$$u^{[1]}(x) := u'(x)-Q(x)u(x),$$

can be re-written as quasi-differential expressions [22, 23],

$$l_Q[u] := -(u'-Qu)'-Q(u'-Qu)-Q^2u,$$

which are well defined if $u,u^{[1]}\in W^1_{1,\mathrm{loc}}(\mathbb{R})$ [19].

Proposition 1. (Existence and Uniqueness Theorem). *Let $\lambda\in\mathbb{C}$ and $f(x)\in L_{1,\mathrm{loc}}(\mathbb{R})$. Then, for any complex numbers c_0, $c_1\in\mathbb{C}$ and arbitrary $x_0\in\mathbb{R}$, the quasi-differential equation*

$$l_Q[u]=\lambda u+f,\quad \lambda\in\mathbb{C},\quad f\in L_{1,\mathrm{loc}}(\mathbb{R}), \tag{3}$$

has one and only one solution $u\in W^1_{1,\mathrm{loc}}(R)$ satisfying the initial conditions

$$u(x)\,|_{x=x_0}=c_0,\quad u^{[1]}(x)\,|_{x=x_0}=c_1.$$

For the quasi-differential equation (3) there is a relating normal 2-dimensional system of the first order differential equations with locally integrable coefficients,

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}'=\begin{pmatrix}Q & 1\\ -\lambda-Q^2 & -Q\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}0\\-f\end{pmatrix},$$

where $u_1(x):=u(x)$, $u_2(x):=u^{[1]}(x)$.

Then Proposition 1 follows from [19, Theorem 1, §16], also see [1].

Lemma 2. (Lagrange formula). *Let $u(x)$ and $v(x)$ be functions such that the quasi-differential expressions $l_Q[\cdot]$ are well defined. Then the following Lagrange formula holds:*

$$l_Q[u]\overline{v}-ul_Q[\overline{v}]=\frac{d}{dx}[u,v]_x,$$

where the sesquilinear forms $[u,v]_x$ are defined by

$$[u,v]_x := u(x)\overline{(v'(x)-Q(x)v(x))}-(u'(x)-Q(x)u(x))\,\overline{v(x)}.$$

Proof. It follows at once that $u(x)$ and $v(x)$ are such that

$$u,u'-Qu\in W^1_{1,\mathrm{loc}}(\mathbb{R})\quad\text{and}\quad v,v'-Qv\in W^1_{1,\mathrm{loc}}(\mathbb{R}).$$

Then we have

$$\begin{aligned}\frac{d}{dx}[u,v]_x &\equiv \frac{d}{dx}\left(u\overline{(v'-Qv)}-(u'-Qu)\,\overline{v}\right)\\ &= u'\overline{(v'-Qv)}+u\overline{(v'-Qv)'}-(u'-Qu)'\,\overline{v}-(u'-Qu)\,\overline{v'}\\ &= l_Q[u]\overline{v}-ul_Q[\overline{v}]+Qu'\overline{v}-Qu\overline{v'}+u'\overline{(v'-Qv)}-(u'-Qu)\,\overline{v'}\\ &= l_Q[u]\overline{v}-ul_Q[\overline{v}],\end{aligned}$$

since it follows from the assumptions that

$$u'\overline{v'},\ Q^2u\overline{v},\ Qu'\overline{v},\ Qu\overline{v'} \in L_{1,\mathrm{loc}}(\mathbb{R}).$$

The proof is complete. □

Integrating both sides of the Lagrange formula over the compact interval $[\alpha,\beta] \Subset \mathbb{R}$ we obtain the Lagrange identity in an integral form,

$$\int_\alpha^\beta l_Q[u]\overline{v}\,dx - \int_\alpha^\beta u l_Q[\overline{v}]\,dx = [u,v]_\alpha^\beta, \tag{4}$$

where

$$[u,v]_\alpha^\beta := [u,v]_\beta - [u,v]_\alpha.$$

2.3. **Quasi-differential operators on a finite interval.** Here, following Savchuk and Shkalikov [22], we give a brief review of results related to Sturm-Liouville operators with distribution potentials defined on a finite interval.

On the Hilbert space $L_2(0,1)$, we consider the Sturm-Liouville operators

$$L(q)u := -u'' + q(x)u, \quad u \in \operatorname{Dom}\left(L(q)\right),$$

with real-valued distribution potentials $q(x) \in H^{-1}\left([0,1],\mathbb{R}\right)$, i.e.,

$$Q(x) = \int q(\xi)\,d\xi \in L_2\left((0,1),\mathbb{R}\right).$$

Set

$$\begin{aligned} L_{\max}(q)u &:= l_Q[u], \\ \operatorname{Dom}(L_{\max}(q)) &:= \left\{u \in L_2(0,1) \mid u, u' - Qu \in W_1^1[0,1],\ l_Q[u] \in L_2(0,1)\right\}, \end{aligned}$$

and

$$\begin{aligned} \dot{L}_{\min}(q)u &:= l_Q[u], \\ \operatorname{Dom}(\dot{L}_{\min}(q)) &:= \left\{u \in \operatorname{Dom}(L_{\max}(q)) \mid \operatorname{supp} u \Subset [0,1]\right\}. \end{aligned}$$

We also consider the operators

$$\begin{aligned} L_{\min}(q)u &:= l_Q[u], \\ \operatorname{Dom}(L_{\min}(q)) &:= \left\{u \in \operatorname{Dom}(L_{\max}(q)) \mid u^{[j]}(0) = u^{[j]}(1) = 0,\ j = 0,1\right\}. \end{aligned}$$

Proposition 3. ([22]). *Suppose that $q(x) \in H^{-1}\left([0,1],\mathbb{R}\right)$. Then the following statements are true:*

(I) *The operators $L_{\min}(q)$ are densely defined on the Hilbert space $L_2(0,1)$.*

(II) *The operators $L_{\min}(q)$ and $L_{\max}(q)$ are mutually adjoint,*

$$L^*_{\min}(q) = L_{\max}(q), \qquad L^*_{\max}(q) = L_{\min}(q).$$

In particular, the operators $L_{\min}(q)$ and $L_{\max}(q)$ are closed.

In Statement 4, which is proved in Appendix A.1, we establish relationships between the operators $\dot{L}_{\min}(q)$ and $L_{\min}(q)$.

Statement 4. *The operators $L_{\min}(q)$ are closures of the operators $\dot{L}_{\min}(q)$,*

$$L_{\min}(q) = (\dot{L}_{\min}(q))^\sim = \dot{L}^{**}_{\min}(q).$$

3. Main results

3.1. **A principal lemma.** The following operator-theory result is an essential part of our approach. In this section, we will give two important applications.

Lemma 5. *Let A be a linear operator that is densely defined and closed on a complex Banach space X, and let B be a linear operator bounded on X such that*

(a) $BA \subset AB$ *(A and B commute);*
(b) $\sigma_p(B) = \varnothing$ *(the point spectrum $\sigma_p(B)$ of the operator B is empty).*

Then the operator A has no eigenvalues of finite multiplicity.

Proof. Suppose that the operator A has an eigenvalue $\lambda \in \sigma_p(A)$ of finite multiplicity, and let G_λ be the corresponding eigenspace.

Further, let f be an eigenvector of the operator A,

$$Af = \lambda f, \quad f \in G_\lambda.$$

Then

$$A(Bf) = B(Af) = \lambda(Bf), \quad f \in G_\lambda,$$

whence we conclude that

$$BG_\lambda \subset G_\lambda.$$

The assumption $\dim(G_\lambda) \in \mathbb{N}$ implies that the point spectrum $\sigma_p(B)$ of the operator B is not empty. This contradicts condition (b).

The proof is complete. □

Remark 6. The condition (b) is satisfied if $X = L_p(\mathbb{R}, \mathbb{C})$, $1 \leq p < \infty$, and B is a shift operator,

$$B : y(x) \mapsto y(x+T), \quad T > 0.$$

Indeed, the operator B is unitary on the space $X = L_p(\mathbb{R}, \mathbb{C})$. Therefore,

$$\sigma_p(B) \subset \sigma(B) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\},$$

and the identity

$$By(x) = \lambda y(x) = y(x+T), \quad y(x) \not\equiv 0, \quad |\lambda| = 1,$$

implies that the function $|y(x)|$ is T-periodic. Then $y(x) \notin L_p(\mathbb{R}, \mathbb{C})$, and we conclude that $\sigma_p(B) = \varnothing$.

Condition (a) means in this case that the operator A is T-periodic on the line.

3.2. **Self-adjointness of the Hill-Schrödinger operators with distribution potentials.** If assumption (2) is true, then the distribution potentials $q(x)$ can be represented as

$$q(x) = C + Q'(x)$$

with

$$C = \widehat{q}(0)$$

and

$$Q(x) = \sum_{k \in 2\mathbb{Z} \setminus \{0\}} \frac{1}{ik\pi} \widehat{q}(2k) e^{ik\pi x} \in L_{2,\mathrm{per}}(\mathbb{R}, \mathbb{R})$$

such that

$$\langle q, \varphi \rangle = -\langle Q, \varphi' \rangle \quad \forall \varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}),$$

see [5, Proposition 1], [24]. Here, by $\langle f, \cdot \rangle$, $f \in \mathfrak{D}'(\mathbb{R})$, we denote sesquilinear functionals on the space $C^\infty_{\mathrm{comp}}(\mathbb{R})$.

Remark 7. Without loss of generality, everywhere in the sequel we will assume that

$$\widehat{q}(0) = 0.$$

Then, the Hill-Schrödinger operators can be well defined on the Hilbert space $L_2(\mathbb{R})$ as quasi-differential operators [22, 23] by means of the quasi-expressions

$$l_Q[u] = -(u' - Qu)' - Q(u' - Qu) - Q^2 u.$$

Set

$$S_{\max}(q)u := l_Q[u],$$
$$\mathrm{Dom}(S_{\max}(q)) := \left\{ u \in L_2(\mathbb{R}) \mid u, u' - Qu \in W^1_{1,\mathrm{loc}}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R}) \right\},$$

and

$$\dot{S}_{\min}(q)u := l_Q[u],$$
$$\mathrm{Dom}(\dot{S}_{\min}(q)) := \{ u \in \mathrm{Dom}(S_{\max}(q)) \mid \operatorname{supp} u \Subset \mathbb{R} \}.$$

It is obvious that the operators $S_{\max}(q)$ are defined on maximal linear manifolds where the quasi-expressions $l_Q[\cdot]$ are well defined.

Proposition 8. *Let $q(x) \in H^{-1}_{\mathrm{per}}(\mathbb{R}, \mathbb{R})$. Then the following statements hold true.*

(I) *The operators $\dot{S}_{\min}(q)$ are symmetric and lower semibounded on the Hilbert space $L_2(\mathbb{R})$. In particular, they are closable.*

(II) *The closures $S_{\min}(q)$ of the operators $\dot{S}_{\min}(q)$, $S_{\min}(q) := (\dot{S}_{\min}(q))^\sim$, are symmetric, lower semibounded operators on the Hilbert space $L_2(\mathbb{R})$ with deficiency indices of the form (m, m) where $0 \le m \le 2$. The operators $S_{\max}(q)$ are adjoint to the operators $S_{\min}(q)$,*

$$S^*_{\min}(q) = S_{\max}(q).$$

In particular, $S_{\max}(q)$ are closed operators on the Hilbert space $L_2(\mathbb{R})$, and

$$S^*_{\max}(q) = S_{\min}(q).$$

(III) *Domains $\mathrm{Dom}(S_{\min}(q))$ of the operators $S_{\min}(q)$ consist of those and only those functions $u \in \mathrm{Dom}(S_{\max}(q))$ which satisfy the conditions*

$$[u, v]_{+\infty} - [u, v]_{-\infty} = 0 \quad \forall v \in \mathrm{Dom}(S_{\max}(q)),$$

where the limits

$$[u, v]_{+\infty} := \lim_{x \to +\infty} [u, v]_x \quad \text{and} \quad [u, v]_{-\infty} := \lim_{x \to -\infty} [u, v]_x$$

are well defined and exist.

Proposition 8, which describes properties of the operators $\dot{S}_{\min}(q)$ and $S_{\max}(q)$, is proved in Appendix A.2 by using methods of the theory of linear quasi-differential operators.

In Proposition 10 we define Friedrichs extensions of the minimal operators $S_{\min}(q)$. But for convenience we first recall some related facts and prove useful Lemma 9.

Let H be a Hilbert space, and $\dot{A}$ be a densely defined, lower semibounded linear operator on H. Hence, $\dot{A}$ is a closable, symmetric operator. Define by A its closure, $A := (\dot{A})^\sim$.

Set

$$\dot{t}[u, v] := (\dot{A}u, v), \quad \mathrm{Dom}(\dot{t}) := \mathrm{Dom}(\dot{A}).$$

As known [9], the sesquilinear form $\dot{t}[u, v]$ is closable, lower semibounded and symmetric on the Hilbert space H. Let $t[u, v]$ be its closure, $t := (\dot{t})^\sim$.

For the operator $\dot{A}$ there is a uniquely defined its Friedrichs extension A_F [9],

$$t[u, v] = (A_F u, v), \quad u \in \mathrm{Dom}(A_F) \subset \mathrm{Dom}(t), \quad v \in \mathrm{Dom}(t).$$

Due to the First Representation Theorem [9], the operator A_F is lower semibounded and self-adjoint. In Lemma 9 we describe its domain, but at first note that the following inclusions take place:

$$\dot{A} \subset A \subset A_F \subset A^*.$$

Lemma 9. *Let A_F be a Friedrichs extension of a densely defined, lower semibounded operator $\dot{A}$ on a Hilbert space H, and let $t[u,v]$ be the densely defined, closed, symmetric, and bounded from below sesquilinear form on H constructed from the operator $\dot{A}$. Then*

$$\mathrm{Dom}(A_F) = \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*).$$

Proof. It is obvious that

$$\mathrm{Dom}(A_F) \subset \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*).$$

Let us prove the inverse inclusion.

Let $u \in \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*)$, and $v \in \mathrm{Dom}(\dot{A}) \subset \mathrm{Dom}(A_F) \subset \mathrm{Dom}(t)$. Remark that $\mathrm{Dom}(\dot{A})$ is a core of the form $t[u,v]$ and that $\mathrm{Dom}(t) \cap \mathrm{Dom}(A^*)$ contains $\mathrm{Dom}(\dot{A})$. Then we have

$$(A^*u, v) = (u, \dot{A}v) = (u, A_F v) = \overline{(A_F v, u)} = \overline{t[v,u]} = t[u,v],$$

i.e.,

$$t[u,v] = (A^*u, v), \quad u \in \mathrm{Dom}(t) \cap \mathrm{Dom}(A^*), \quad v \in \mathrm{Dom}(\dot{A}).$$

Due to the First Representation Theorem [9] we get that $u \in \mathrm{Dom}(A_F)$, i.e.,

$$\mathrm{Dom}(t) \cap \mathrm{Dom}(A^*) \subset \mathrm{Dom}(A_F).$$

The proof is complete. □

Proposition 10. *Friedrichs extensions $S_F(q)$ of the operators $S_{\min}(q)$ are defined in the following way:*

$$S_F(q)u := l_Q[u],$$
$$\mathrm{Dom}(S_F(q)) := \left\{ u \in H^1(\mathbb{R}) \mid u' - Qu \in W^1_{1,\mathrm{loc}}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R}) \right\}.$$

Proof. Let us introduce the sesquilinear forms

$$\dot{t}[u,v] := (\dot{S}_{\min}(q)u, v), \quad \mathrm{Dom}(\dot{t}) := \mathrm{Dom}(\dot{S}_{\min}(q)).$$

As is well known [9], the sesquilinear forms $\dot{t}[u,v]$ are densely defined, closable, symmetric and bounded from below on the Hilbert space $L_2(\mathbb{R})$. Taking into account that $\mathrm{Dom}(\dot{S}_{\min}(q)) \subset H^1_{\mathrm{comp}}(\mathbb{R})$, the forms $\dot{t}[u,v]$ can be written as

$$\dot{t}[u,v] = (u', v') - (Qu, v') - (Qu', v), \quad \mathrm{Dom}(\dot{t}) \subset H^1_{\mathrm{comp}}(\mathbb{R}).$$

Set

$$\dot{t}_1[u,v] := (u', v') + (u, v), \qquad \mathrm{Dom}(\dot{t}_1) := \mathrm{Dom}(\dot{S}_{\min}(q)) \subset H^1_{\mathrm{comp}}(\mathbb{R}),$$
$$\dot{t}_2[u,v] := -(Qu, v') - (Qu', v) - (u, v), \qquad \mathrm{Dom}(\dot{t}_2) := \mathrm{Dom}(\dot{S}_{\min}(q)) \subset H^1_{\mathrm{comp}}(\mathbb{R}),$$

i.e.,

$$\dot{t} = \dot{t}_1 + \dot{t}_2.$$

It is well known that the form $\dot{t}_1[u,v]$ is closable, and its closure, $t_1[u,v]$, $t_1 := (\dot{t}_1)^\sim$, has the representation

$$t_1[u,v] = (u', v') + (u, v), \quad \mathrm{Dom}(t_1) = H^1(\mathbb{R}).$$

As was shown in [8], the forms $\dot{t}_2[u,v]$ are t_1-bounded with relative boundary 0. So, we finally obtain that the forms $\dot{t}[u,v]$, which are closures of $t[u,v]$, $t := (\dot{t})^\sim$, are defined as follows:

$$t[u,v] = (u', v') - (Qu, v') - (Qu', v), \quad \mathrm{Dom}(t) = H^1(\mathbb{R}).$$

And the sesquilinear forms $t[u, v]$ are densely defined, closed, symmetric, and lower semi-bounded on the Hilbert space $L_2(\mathbb{R})$.

Further, since

$$S^*_{\min}(q)u = l_Q[u],$$
$$\mathrm{Dom}(S^*_{\min}(q)) = \left\{u \in L_2(\mathbb{R}) \mid u, u' - Qu \in W^1_{1,\mathrm{loc}}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R})\right\},$$

applying Lemma 9 we get the needed representations for Friedrichs extensions of the operators $\dot{S}_{\min}(q)$.

The proof is complete. □

Statement 11. *The following inclusions take place:*

$$\dot{S}_{\min}(q) \subset S_{\min}(q) \subset S_F(q) \subset S_{\max}(q)$$

and

$$\mathrm{Dom}(\dot{S}_{\min}(q)) \subset H^1_{\mathrm{comp}}(\mathbb{R}),$$
$$\mathrm{Dom}(S_{\min}(q)) \subset H^1(\mathbb{R}), \quad \mathrm{Dom}(S_F(q)) \subset H^1(\mathbb{R}),$$
$$\mathrm{Dom}(S_{\max}(q)) \subset L_2(\mathbb{R}) \cap H^1_{loc}(\mathbb{R}).$$

Statement 11 immediately follows from the corresponding definitions and not very complicated computations.

Now, our aim is to prove that the maximal quasi-differential operators $S_{\max}(q)$ are self-adjoint.

Proposition 12. *Let $q(x) \in H^{-1}_{\mathrm{per}}(\mathbb{R}, \mathbb{R})$. The following statements are equivalent.*

(a) *The operators $S_{\max}(q)$ are self-adjoint.*
(b) $\mathrm{Dom}(S_{\max}(q)) \subset H^1(\mathbb{R})$.
(c) $u' - Qu \in L_2(\mathbb{R}) \cap W^1_{1,\mathrm{loc}}(\mathbb{R}) \quad \forall u \in \mathrm{Dom}(S_{\max}(q))$.

Proof. (a) Let $S_{\max}(q)$ be self-adjoint. Then it follows from Proposition 8.II and Statement 11 that

$$S_{\min}(q) = S_F(q) = S_{\max}(q),$$
$$\mathrm{Dom}(S_{\min}(q)) = \mathrm{Dom}(S_F(q)) = \mathrm{Dom}(S_{\max}(q)) \subset H^1(\mathbb{R}),$$

and (b) is true.

Further, under the assumptions $Q \in L_{2,\mathrm{per}}(\mathbb{R})$ and $u \in H^1(\mathbb{R})$ we get that $Qu \in L_2(\mathbb{R})$ [8], which yields (c).

(b) Let us now assume that $\mathrm{Dom}(S_{\max}(q)) \subset H^1(\mathbb{R})$. As above, we get $Qu \in L_2(\mathbb{R})$, and, as a consequence, (c) follows. Then statement (a) follows from the Lagrange identity (4), taking into account that

$$[u, v]_{+\infty} = 0 \quad \text{and} \quad [u, v]_{-\infty} = 0$$

for $u, v \in L_2(\mathbb{R})$ and $u' - Qu, v' - Qv \in L_2(\mathbb{R}) \cap W^1_{1,\mathrm{loc}}(\mathbb{R})$.

(c) Assume that $u' - Qu \in L_2(\mathbb{R}) \cap W^1_{1,\mathrm{loc}}(\mathbb{R}) \quad \forall u \in \mathrm{Dom}(S_{\max}(q))$. Then applying the Lagrange identity (4) as above we get (a) and, as a consequence, (b).

The proof is complete. □

Hryniv and Mykytyuk [8] studied operators associated via the First Representation Theorem [9] to the sesquilinear forms

$$t[u, v] = (u', v') - (Qu, v') - (Qu', v), \quad \mathrm{Dom}(t) = H^1(\mathbb{R}),$$

that is, they have actually studied the Friedrichs extensions $S_F(q)$.

Djakov and Mityagin [5] have also treated the Friedrichs extensions $S_F(q)$ *a priori* considering the operators on the domains

$$\mathrm{Dom}(S_F(q)) = \left\{u \in H^1(\mathbb{R}) \mid u' - Qu \in W^1_{1,\mathrm{loc}}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R})\right\},$$

see Proposition 10 and Proposition 12.

So, due to Proposition 8.II, we have

$$S_{\max}(q) \supset S^*_{\max}(q),$$

and, therefore, it remains to show that the operators $S_{\max}(q)$,

$$S_{\max}(q) \subset S^*_{\max}(q)$$

are symmetric. We do it by applying Lemma 5.

Let us consider the following shift operator on the Hilbert space $L_2(\mathbb{R})$:

$$(Uf)(x) := f(x+1), \quad \mathrm{Dom}(U) := L_2(\mathbb{R}).$$

Then $\sigma_p(U) = \varnothing$.

Further, let $f \in \mathrm{Dom}(S_{\max}(q))$. It is obvious that $Uf \in \mathrm{Dom}(S_{\max}(q))$ too, and it is also true that

$$U(S_{\max}(q)f) = Ul_Q[f(x)] = l_Q[f(x+1)] = l_Q[(Uf)(x)] = S_{\max}(q)(Uf),$$

i.e., the operators $S_{\max}(q)$ and U commute.

Taking into account that $S_{\max}(q)$ are the second order quasi-differential operators, i.e., their possible eigenvalues cannot have multiplicities more than two, and applying Lemma 5 to the operators $S_{\max}(q)$ and U we obtain the following proposition.

Proposition 13. *The point spectra $\sigma_p(S_{\max}(q))$ of the quasi-differential operators $S_{\max}(q)$ are empty.*

Theorem 14. *The quasi-differential operators $S_{\max}(q)$ are self-adjoint.*

Proof. It follows from Proposition 8.II and Proposition 13 that the minimal symmetric operators $S_{\min}(q)$ have deficiency indices of the form $(0,0)$, i.e., they are self-adjoint. Due to Proposition 8.II, this implies that the operators $S_{\max}(q)$ are also self-adjoint.

The proof is complete. □

Corollary 15. *The minimal operators $S_{\min}(q)$, the Friedrichs extensions $S_F(q)$, and the maximal operators $S_{\max}(q)$ coincide. In particular, they are self-adjoint and lower semibounded.*

Corollary 16. *Let $q(x) \in H^{-1}_{\mathrm{per}}(\mathbb{R},\mathbb{R})$, and $q_n(x) \in H^{-1}_{\mathrm{per}}(\mathbb{R},\mathbb{R})$, $n \in \mathbb{N}$, be such that*

$$q_n(x) \overset{H^{-1}_{\mathrm{per}}(\mathbb{R})}{\longrightarrow} q(x) \quad \text{as} \quad n \to \infty.$$

Then the Hill-Schrödinger operators $S(q_n)$, $n \in \mathbb{N}$, converge to the operators $S(q)$ in the norm resolvent sense,

$$\left\|(S(q_n) - \lambda I)^{-1} - (S(q) - \lambda I)^{-1}\right\| \to 0 \quad \text{as} \quad n \to \infty,$$

for any λ belonging to the resolvent sets of $S(q)$ and $S(q_n)$, $n \in \mathbb{N}$.

Proof. The proof immediately follows from [8, Theorem 4.1] and Corollary 15. □

In particular, the Hill-Schrödinger operators $S(q)$ with distribution potentials $q(x) \in H^{-1}_{\mathrm{per}}(\mathbb{R},\mathbb{R})$ are the limits $S_{\lim}(q)$ of a sequence of operators $S(q_n)$, $n \in \mathbb{N}$, with smooth potentials $q_n(x) \in L_{2,\mathrm{per}}(\mathbb{R},\mathbb{R})$. For instance, taking

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(2k)\, e^{i\,2k\pi x} \in H^{-1}_{\mathrm{per}}(\mathbb{R},\mathbb{R})$$

one can choose

$$q_n(x) := \sum_{|k|\leq n} \widehat{q}(2k)\, e^{i\,2k\pi x} \in C^{\infty}_{\mathrm{per}}(\mathbb{R},\mathbb{R}), \quad n\in\mathbb{N}.$$

Now, we are going to define the Hill-Schrödinger operators with distribution potentials as form-sum operators [10]. We will show that this definition coincides with the definitions given above.

Let us consider the following sesquilinear forms on the Hilbert space $L_2(\mathbb{R})$:

$$\tau[u,v] := \left\langle -\frac{d^2}{dx^2}u, v\right\rangle_{L_2(\mathbb{R})} + \langle q(x)u, v\rangle_{L_2(\mathbb{R})}, \quad \mathrm{Dom}(\tau) = H^1(\mathbb{R}),$$

generated by the one-dimensional Schrödinger operators with $q(x)\in H^{-1}_{\mathrm{per}}(\mathbb{R},\mathbb{R})$.

Here, $\langle\cdot,\cdot\rangle_{L_2(\mathbb{R})}$ denotes the sesquilinear form on the space $L_2(\mathbb{R})$, the spaces $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$ for $s\in\mathbb{R}$, respectively, which is a (sesquilinear) continuous extension of the inner product in $L_2(\mathbb{R})$ [3, 7],

$$\langle f, g\rangle_{L_2(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx \quad \forall f, g\in L_2(\mathbb{R}).$$

As is known [10], the sesquilinear forms $\tau[u,v]$ are densely defined, closed, bounded from below, and are defined on the Hilbert space $L_2(\mathbb{R})$. Due to the First Representation Theorem [9], there are associated operators $S_{\mathrm{form}}(q)$ that are uniquely defined on the Hilbert space $L_2(\mathbb{R})$, self-adjoint, lower semibounded, and such that

i) $\mathrm{Dom}\,(S_{\mathrm{form}}(q)) \subset \mathrm{Dom}\,(\tau)$ and

$$\tau[u,v] = (S_{\mathrm{form}}(q)u, v) \quad \forall u\in \mathrm{Dom}\,(S_{\mathrm{form}}(q)), \quad \forall v\in\mathrm{Dom}\,(\tau);$$

ii) $\mathrm{Dom}\,(S_{\mathrm{form}}(q))$ are cores of the forms $\tau[u,v]$;

iii) if $u\in\mathrm{Dom}\,(\tau)$, $w\in L_2(\mathbb{R})$, and

$$\tau[u,v] = (w,v)$$

holds for every v in cores of the forms $\tau[u,v]$, then $u\in\mathrm{Dom}\,(S_{\mathrm{form}}(q))$ and

$$S_{\mathrm{form}}(q)u = w.$$

The operators $S_{\mathrm{form}}(q)$ are called form-sum operators associated with the forms $\tau[u,v]$, and denoted by

$$S_{\mathrm{form}}(q) := -\frac{d^2}{dx^2} \dotplus q(x).$$

It will also be convenient to use the notations

$$\tau_{S_{\mathrm{form}}(q)}[u,v] \equiv \tau[u,v].$$

Proposition 17. ([10]). *The Hill-Schrödinger operators with distribution potentials from the negative Sobolev space $H^{-1}_{\mathrm{per}}(\mathbb{R},\mathbb{R})$ are well defined on the Hilbert space $L_2(\mathbb{R})$ as self-adjoint, lower semibounded form-sum operators $S_{\mathrm{form}}(q)$,*

$$S_{\mathrm{form}}(q) = -\frac{d^2}{dx^2} \dotplus q(x),$$

associated with the sesquilinear forms

$$\tau_{S_{\mathrm{form}}(q)}[u,v] = \left\langle -\frac{d^2}{dx^2}u, v\right\rangle_{L_2(\mathbb{R})} + \langle q(x)u, v\rangle_{L_2(\mathbb{R})}, \quad \mathrm{Dom}(\tau) = H^1(\mathbb{R}),$$

acting on the dense domains

$$\mathrm{Dom}\,(S_{\mathrm{form}}(q)) := \left\{ u\in H^1(\mathbb{R}) \,\middle|\, -\frac{d^2}{dx^2}u + q(x)u \in L_2(\mathbb{R}) \right\}$$

as

$$S_{\text{form}}(q)u := -\frac{d^2}{dx^2}u + q(x)u \in L_2(\mathbb{R}), \quad u \in \operatorname{Dom}\left(S_{\text{form}}(q)\right).$$

Theorem 18. *The quasi-differential operators $S(q)$ and the form-sum operators $S_{\text{form}}(q)$ coincide.*

Proof. Let $u \in \operatorname{Dom}(S(q))$. Recall that

$$\operatorname{Dom}(S(q)) = \left\{u \in H^1(\mathbb{R}) \mid u' - Qu \in W^1_{1,\text{loc}}(\mathbb{R}),\, l_Q[u] \in L_2(\mathbb{R})\right\},$$

so that

$$\operatorname{Dom}(S(q)) \subset \operatorname{Dom}(\tau_{S_{\text{form}}(q)}) = H^1(\mathbb{R}).$$

Then we have

$$\begin{aligned}\tau_{S_{\text{form}}(q)}[u,v] &= \langle -u'', v\rangle_{L_2(\mathbb{R})} + \langle q(x)u, v\rangle_{L_2(\mathbb{R})} = \langle u', v'\rangle_{L_2(\mathbb{R})} - \langle Q(x), \overline{u'}v + \overline{u}v'\rangle_{L_2(\mathbb{R})} \\ &= (u', v') - (Qu, v') - (Qu', v) = (l_Q[u], v) \quad \forall v \in C^\infty_{\text{comp}}(\mathbb{R}).\end{aligned}$$

And, due to the First Representation Theorem [9], we conclude that

$$u \in \operatorname{Dom}(S_{\text{form}}(q)) \quad \text{and} \quad S_{\text{form}}(q)u = l_Q[u],$$

i.e.,

$$S(q) \subset S_{\text{form}}(q).$$

Taking into account that the operators $S(q)$ and $S_{\text{form}}(q)$ are self-adjoint, the latter also gives the inverse inclusions

$$S(q) \supset S_{\text{form}}(q).$$

The proof is complete. □

3.3. **Spectra of the Hill-Schrödinger operators with distribution potentials.** In this section, we will establish characteristic properties of the structure of the spectrum of the Hill-Schrödinger operators $S(q)$ with distribution potentials $q(x) \in H^{-1}_{\text{per}}(\mathbb{R},\mathbb{R})$. Using a limit process in the generalized sense applied to the Hill-Schrödinger operators $S(q_n)$, $n \in \mathbb{N}$, with smooth potentials $q_n(x) \in L_{2,\text{per}}(\mathbb{R},\mathbb{R})$ (see Corollary 16) we show that the Hill-Schrödinger operators $S(q)$, with the distributions $q(x) \in H^{-1}_{\text{per}}(\mathbb{R},\mathbb{R})$ as potentials, have continuous spectra with a band and gap structure.

For different approaches, see [8, 10, 5].

At first, let us recall well known results related to the classical case of $L_{2,\text{per}}(\mathbb{R},\mathbb{R})$-potentials $q(x)$,

$$q(x) \in L_{2,\text{per}}(\mathbb{R},\mathbb{R}), \tag{5}$$

see, for an example, [6, 21]. Under assumption (5), the Hill-Schrödinger operators $S(q)$ are lower semibounded and self-adjoint on the Hilbert space $L_2(\mathbb{R})$; they have absolutely continuous spectra with a band and gap structure.

Spectra of the Hill-Schrödinger operators are well defined by locating the spectrum gap endpoints. It is known that for the endpoints $\{\lambda_0(q), \lambda^\pm_k(q)\}_{k=1}^\infty$ of the spectrum gaps, we have the following inequalities:

$$-\infty < \lambda_0(q) < \lambda^-_1(q) \le \lambda^+_1(q) < \lambda^-_2(q) \le \lambda^+_2(q) < \cdots \tag{6}$$

The spectrum bands (or stability zones),

$$\mathcal{B}_0(q) := [\lambda_0(q), \lambda^-_1(q)], \quad \mathcal{B}_k(q) := [\lambda^+_k(q), \lambda^-_{k+1}(q)], \quad k \in \mathbb{N},$$

are characterized as a set of real $\lambda \in \mathbb{R}$ for which all solutions of the equation

$$S(q)u = \lambda u \tag{7}$$

are bounded. On the other hand, spectrum gaps (or instability zones),

$$\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \quad \mathcal{G}_k(q) := (\lambda^-_k(q), \lambda^+_k(q)), \quad k \in \mathbb{N},$$

make a set of real $\lambda \in \mathbb{R}$ for which any nontrivial solution of the equation (7) is unbounded.

As follows from (6), it could happen that

$$\lambda_k^-(q) = \lambda_k^+(q)$$

for some $k \in \mathbb{N}$. In such a case, we say that the corresponding spectrum gap $\mathcal{G}_k(q)$ is *collapsed* or *closed*. Note that this cannot happen for spectrum bands.

Further, it could happen that the endpoints of spectrum gaps for even numbers $k \in \mathbb{Z}_+$ are periodic eigenvalues of the problem on the interval $[0,1]$,

$$S_+(q)u := -u'' + q(x)u = \lambda u, \quad u \in \operatorname{Dom}(S_+(q)),$$

and the endpoints of spectrum gaps for odd numbers $k \in \mathbb{N}$ are semiperiodic eigenvalues of the problem on the interval $[0,1]$,

$$S_-(q)u := -u'' + q(x)u = \lambda u, \quad u \in \operatorname{Dom}(S_-(q)).$$

Under the assumption (5), domains of the operators $S_+(q)$ and $S_-(q)$ have the form

$$\operatorname{Dom}(S_\pm(q)) = \left\{ u \in H^2[0,1] \,\middle|\, u^{(j)}(0) = \pm u^{(j)}(1),\ j = 0,1 \right\}.$$

Now, applying the limit process in the generalized sense (see Corollary 16) to the Hill-Schrödinger operators $S(q_n)$, $n \in \mathbb{N}$, with $L_{2,\mathrm{per}}(\mathbb{R},\mathbb{R})$-potentials $q_n(x)$ we establish the following statement.

Theorem 19. *Suppose that $q(x) \in H^{-1}_{\mathrm{per}}(\mathbb{R},\mathbb{R})$. Then the Hill-Schrödinger operators $S(q)$ have continuous spectra with a band and gap structure such that the endpoints $\{\lambda_0(q), \lambda_k^\pm(q)\}_{k=1}^\infty$ of the spectrum gaps satisfy the inequalities*

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \leq \lambda_1^+(q) < \lambda_2^-(q) \leq \lambda_2^+(q) < \cdots$$

Moreover, the endpoints of the spectrum gaps for even (odd) numbers $k \in \mathbb{Z}_+$ are periodic (semiperiodic) eigenvalues of the problem on the interval $[0,1]$,

$$S_\pm(q)u = -u'' + q(x)u = \lambda u, \quad u \in \operatorname{Dom}(S_\pm(q)).$$

Remark 20. The operators $S_+(q)$ and $S_-(q)$ are well defined on the Hilbert space $L_2(0,1)$ as lower semi-bounded, self-adjoint form-sum operators,

$$S_\pm(q) = \left(-\frac{d^2}{dx^2}\right)_\pm \dotplus q(x).$$

They also can be well defined in alternative equivalent ways, — as quasi-differential operators or as limits, in the norm resolvent sense, of a sequence of operators with smooth potentials.

In the papers [13, 14, 15], the authors meticulously treated the form-sum operators

$$S_\pm(V) = \left((-1)^m \frac{d^{2m}}{dx^{2m}}\right)_\pm \dotplus V(x), \quad V(x) \in H^{-m}_{\mathrm{per}}[0,1], \quad m \in \mathbb{N},$$

defined on $L_2(0,1)$.

In [18, 11, 12], the authors studied two terms differential operators of even order defined in the *negative* Sobolev spaces.

Proof. Let $\{q_n(x)\}_{n\in\mathbb{N}}$ be a sequence of real-valued trigonometric polynomials, which converges to the singular potential $q(x)$ in the norm of the space $H^{-1}_{\mathrm{per}}(\mathbb{R})$. With this sequence one can associate a sequence of self-adjoint operators $\{S_\pm(q_n)\}_{n\in\mathbb{N}}$ defined in $L_2(0,1)$, and a sequence of Hill operators $\{S(q_n)\}_{n\in\mathbb{N}}$ defined on $L_2(\mathbb{R})$. As was proved by the authors in [13, 15], the sequences $\{S_\pm(q_n)\}_{n\in\mathbb{N}}$ converge to the operators $S_\pm(q)$ in the norm resolvent sense. Hence, eigenvalues of these operators $\{S_\pm(q_n)\}_{n\in\mathbb{N}}$ converge to the corresponding eigenvalues of the limit operators $S_\pm(q)$ [20, Theorem

VIII.23 and Theorem VIII.24] (also see [9]). Further, as is well known [4, 6], for the operators $\{S_{\pm}(q_n)\}_{n\in\mathbb{N}}$, the assertion of the theorem is true, i.e.,

$$-\infty < \lambda_0(q_n) < \lambda_1^-(q_n) \leq \lambda_1^+(q_n) < \lambda_2^-(q_n) \leq \lambda_2^+(q_n) < \cdots \tag{8}$$

Moreover, as we have already proved (see Corollary 16), the sequence $\{S(q_n)\}_{n\in\mathbb{N}}$ converges to the operator $S(q)$ in the norm resolvent sense. Therefore, from (8) we get

$$-\infty < \lambda_0(q) \leq \lambda_1^-(q) \leq \lambda_1^+(q) \leq \lambda_2^-(q) \leq \lambda_2^+(q) \leq \cdots ,$$

where $\lambda_0(q)$, $\lambda_{2k}^{\pm}(q) \in \sigma(S_+(q))$ and $\lambda_{2k-1}^{\pm}(q) \in \sigma(S_-(q))$, $k \in \mathbb{N}$.

Now it remains to show that the strict inequalities

$$\lambda_k^+(q_n) < \lambda_{k+1}^-(q_n), \quad k \in \mathbb{Z}_+,$$

can not become equalities. Indeed, suppose the contrary. Then, one of the spectrum zones of the operator $S(q)$ degenerates into a point,

$$\lambda_{k_0}^+(q) = \lambda_{k_0+1}^-(q), \quad k_0 \in \mathbb{Z}_+.$$

Since it is an isolated point of the spectrum of the operator $S(q)$, it cannot belong to the continuous spectrum $\sigma_c(S(q))$. On the other hand, it cannot belong to the point spectrum of the operator $S(q)$, since $\sigma_p(S(q)) = \varnothing$. The obtained contradiction proves the inequalities in theorem.

The proof is complete. □

4. Concluding remarks

It follows from the direct integral decomposition of the Hill-Schrödinger operators $S(q)$ [8] and [21, Theorem XIII.86] that $\sigma_{sc}(S(q)) = \varnothing$. Therefore, the continuity of spectra of the operators $S(q)$, which was proved in this paper, shows that they are absolutely continuous [17].

From Theorem C and the results of the authors in [13], one obtains a series of theorems establishing relationships between the lengths of the spectrum gaps and smoothness of the distribution potentials $q(x) \in H_{\mathrm{per}}^{-s}(\mathbb{R},\mathbb{R})$, $s \geq -1$, of the Hill-Schrödinger operators $S(q)$ [16].

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Appendix: Some proofs

A.1. **Proof of Statement 4.** At first note that the relations

$$\dot{L}_{\min}(q) \subset L_{\min}(q)$$

give

$$(\dot{L}_{\min}(q))^{\sim} \subset L_{\min}(q),$$

see Proposition 3.III. Therefore, it suffices to show the inverse inclusions,

$$(\dot{L}_{\min}(q))^{\sim} \supset L_{\min}(q).$$

Let $\Delta = [\alpha,\beta]$ denote a fixed, closed interval that completely lies in the interval $[0,1]$, and let

$$\mathfrak{H}_\Delta := L_2(\alpha,\beta).$$

On the Hilbert space $\mathfrak{H}_\Delta$, consider the operators $L_{\min,\Delta}(q)$ and $L_{\max,\Delta}(q)$ generated by $l_Q[\cdot]$ on the interval Δ, which are are mutually adjoint due to Proposition 3.III,

$$L_{\min,\Delta}^*(q) = L_{\max,\Delta}(q), \quad L_{\max,\Delta}^*(q) = L_{\min,\Delta}(q).$$

On the other hand the Hilbert space $\mathfrak{H}_\Delta$ can be well embedded into the space $\mathfrak{H} := L_2(0,1)$ assuming that the function $u \in \mathfrak{H}_\Delta$ equals zero on the interval Δ. Thus, the domains $\mathrm{Dom}(L_{\min,\Delta}(q))$ of the operators $L_{\min,\Delta}(q)$ become a part of the domains $\mathrm{Dom}(L_{\max}(q))$ of the operators $L_{\max}(q)$, since continuity of the quasi-derivatives $u^{[j]}(x)$, $j = 0, 1$, of the function $u \in \mathrm{Dom}(L_{\min,\Delta}(q))$ is preserved when extending the function over the interval Δ. Moreover, extended in such a way, the function $u \in \mathrm{Dom}(L_{\min,\Delta}(q))$ then belongs to $\mathrm{Dom}(\dot{L}_{\min}(q))$. Therefore, if $v \in \mathrm{Dom}(\dot{L}^*_{\min}(q))$, then we have

$$\left(\dot{L}^*_{\min}(q)v, u\right) = \left(v, \dot{L}_{\min}(q)u\right) \quad \forall u \in \mathrm{Dom}(L_{\min,\Delta}(q)). \tag{9}$$

Since $u(x) = 0$ on the interval Δ, the scalar product in (9) is the $\mathfrak{H}_\Delta$-inner product. Denoting these scalar products with the index Δ we can rewrite (9) as follows:

$$\left((\dot{L}^*_{\min}(q)v)_\Delta, u\right)_\Delta = (v_\Delta, L_{\min,\Delta}(q)u)_\Delta \quad \forall u \in \mathrm{Dom}(L_{\min,\Delta}(q)).$$

Here, $(\dot{L}^*_{\min}(q)v)_\Delta$, v_Δ denote the functions $\dot{L}^*_{\min}(q)v$ and v considered only in the interval Δ. So, from the latter we obtain

$$v_\Delta \in \mathrm{Dom}(L^*_{\min,\Delta}(q)) = \mathrm{Dom}(L_{\max,\Delta}(q))$$

and

$$(\dot{L}^*_{\min}(q)v)_\Delta = L^*_{\min,\Delta}(q)v_\Delta = L_{\max,\Delta}(q)v_\Delta = (l_Q[v])_\Delta\,.$$

Since these relations hold for any interval $\Delta \subset [0, 1]$, we conclude that

$$v \in \mathrm{Dom}(L_{\max}(q)) \quad \text{and} \quad \dot{L}^*_{\min}(q)v = l_Q[v] = L_{\max}(q)v.$$

Thus, we have proved that

$$\dot{L}^*_{\min}(q) \subset L_{\max}(q),$$

i.e.,

$$\dot{L}^{**}_{\min}(q) \supset L^*_{\max}(q) = L_{\min}(q),$$

which implies the required inclusions

$$(\dot{L}_{\min}(q))^\sim \supset L_{\min}(q).$$

The proof is complete. □

A.2. **Proof of Proposition 8.** (I) At first note that

$$\mathrm{Dom}(\dot{S}_{\min}(q)) \subset H^1_{\mathrm{comp}}(\mathbb{R}). \tag{10}$$

Let $u \in \mathrm{Dom}(\dot{S}_{\min}(q))$. Then we have

$$(\dot{S}_{\min}(q)u, u) = (l_Q[u], u) = (u', u') - (Qu, u') - (Qu', u),$$

taking into account that, due to the (10),

$$|u'|^2,\ Quu' \in L_{1,\mathrm{comp}}(\mathbb{R}).$$

Now, we estimate (Qu, u') and (Qu', u) as in [8],

$$|(Qu, u')| \leq \|Q\|_{L_{2,\mathrm{per}}(\mathbb{R})} \left(\varepsilon \|u'\|_{L_2(\mathbb{R})} + b(\varepsilon^{-1})\|u\|_{L_2(\mathbb{R})}\right), \quad \varepsilon \in (0, 1], \quad b \geq 0,$$

which yields

$$(\dot{S}_{\min}(q)u, u) \geq -\gamma(\varepsilon^{-1})\|u\|_{L_2(\mathbb{R})} \quad \forall u \in \mathrm{Dom}(\dot{S}_{\min}(q)), \quad \gamma \geq 0.$$

We can conclude that $\dot{S}_{\min}(q)$ are Hermitian operators, lower semibounded on $L_2(\mathbb{R})$.

Now, let us show that $\mathrm{Dom}(\dot{S}_{\min}(q))$ are dense in the Hilbert space $L_2(\mathbb{R})$.

Obviously, it is sufficient to prove that any element $h \in \mathfrak{H}$, $\mathfrak{H} := L_2(\mathbb{R})$, which is orthogonal to $\mathrm{Dom}(\dot{S}_{\min}(q))$ is equal to zero. Suppose that $h(x)$ is such a function,

$$h(x) \perp \mathrm{Dom}(\dot{S}_{\min}(q)),$$

and let $\Delta = [\alpha, \beta]$ be a fixed, closed interval compactly lying in the real axis $\mathbb{R}$ ($\Delta \Subset \mathbb{R}$). Any element $u \in \mathrm{Dom}(S_{\min,\Delta}(q))$ can be viewed as an element of $\mathrm{Dom}(\dot{S}_{\min}(q))$

(for the notations see the proof of Statement 4), consequently, $h(x)$ is orthogonal to $\mathrm{Dom}(S_{\min,\Delta}(q))$. Due to Proposition 3.II, $\mathrm{Dom}(S_{\min,\Delta}(q))$ is dense in $\mathfrak{H}_\Delta = L_2(\alpha,\beta)$, hence the function $h(x)$ considered in the interval Δ has to be equal to zero almost everywhere in Δ.

Since the interval $\Delta \Subset \mathbb{R}$ was arbitrary, we conclude that $h(x) = 0$ almost everywhere on $\mathbb{R}$.

So, statement (I) of Proposition 8 has been proved completely.

(II) It is obvious that the operators $S_{\min}(q)$ are symmetric, lower semibounded on the Hilbert space $L_2(\mathbb{R})$.

Let us show that the operators $S_{\min}(q)$ and $S_{\max}(q)$ are adjoint to each other. Since $(\dot{S}_{\min}(q))^{\sim} = S_{\min}(q)$, we have $\dot{S}^*_{\min}(q) = S^*_{\min}(q)$, and it suffices to show that

$$\dot{S}^*_{\min}(q) = S_{\max}(q).$$

Applying the Lagrange identity (4), we have

$$(S_{\max}(q)u, v) = (u, \dot{S}_{\min}(q)v) \quad \forall u \in \mathrm{Dom}(S_{\max}(q)), \quad \forall v \in \mathrm{Dom}(\dot{S}_{\min}(q)),$$

which implies that

$$S_{\max}(q) \subset \dot{S}^*_{\min}(q).$$

So, it remains to prove the inverse inclusions,

$$S_{\max}(q) \supset \dot{S}^*_{\min}(q).$$

We do it in a similar manner as in the proof of Statement 4.

Let $v(x)$ be an arbitrary element in the domains $\mathrm{Dom}(\dot{S}^*_{\min}(q))$ of the operators $\dot{S}^*_{\min}(q)$, and let $\Delta = [\alpha,\beta]$ be a fixed, compact interval ($\Delta \Subset \mathbb{R}$). As in the proof of Statement 4, we obtain

$$\Big((\dot{S}^*_{\min}(q)v)_\Delta, u\Big)_\Delta = (v_\Delta, S_{\min,\Delta}(q)u)_\Delta \quad \forall u \in \mathrm{Dom}(S_{\min,\Delta}(q)).$$

So, one can conclude that

$$v_\Delta \in \mathrm{Dom}(S_{\max,\Delta}(q))$$

and

$$(\dot{S}^*_{\min}(q)v)_\Delta = S^*_{\min,\Delta}(q)v_\Delta = S_{\max,\Delta}(q)v_\Delta = (l_Q[v])_\Delta\,.$$

Taking into account that the interval $\Delta \subset \mathbb{R}$ is arbitrarily chosen, we finally get that

$$v \in \mathrm{Dom}(S_{\max}(q)) \quad \text{and} \quad \dot{S}^*_{\min}(q)v = l_Q[v] = S_{\max}(q)v,$$

so that the required inclusions hold,

$$S_{\max}(q) \supset \dot{S}^*_{\min}(q).$$

Further, let us find the deficiency index of the operators $S_{\min}(q)$. At first it is necessary to note that, since the operators $S_{\min}(q)$ are lower semibounded, their deficiency indices are equal.

Let $\lambda \in \mathbb{C}$, $\mathrm{Im}\,\lambda \neq 0$. Then the deficiency indices of the operators $S_{\min}(q)$, which will be denoted by m, are equal to the number of linearly independent solutions of the equation

$$S^*_{\min}(q)u = \lambda u,$$

i.e., of the equation (Proposition 8.II)

$$S_{\max}(q)u = \lambda u.$$

In other words the deficiency index is a maximal number of linear independent solutions of the equation

$$l_Q[u] = \lambda u$$

in the Hilbert space $L_2(\mathbb{R})$. Since the total number of linearly independent solutions of this equation is 2, we conclude that

$$0 \leq m \leq 2.$$

Assertion (II) is proved.

(III) Let $u, v \in \mathrm{Dom}(S_{\max}(q))$. Then applying the Lagrange identity (4) we conclude that the following limits exist:

$$[u, v]_{+\infty} := \lim_{x\to+\infty}[u, v]_x \quad \text{and} \quad [u, v]_{-\infty} := \lim_{x\to-\infty}[u, v]_x,$$

and, as a consequence, the Lagrange identity (4) becomes

$$(l_Q[u], v) - (u, l_Q[v]) = [u, v]_{-\infty}^{+\infty} \quad \forall u, v \in \mathrm{Dom}(S_{\max}(q)). \tag{11}$$

Now, due to Proposition 8.II, we have

$$S_{\min}(q) = S^*_{\max}(q).$$

Therefore, the domains $\mathrm{Dom}(S_{\min}(q))$ consist of only the functions $u \in \mathrm{Dom}(S_{\max}(q))$ that satisfy the identities

$$(u, S_{\max}(q)v) = (S_{\max}(q)u, v) \quad \forall v \in \mathrm{Dom}(S_{\max}(q))$$

and only of them. Together with the Lagrange identity (11), the latter implies the required assertion, i.e.,

$$u \in \mathrm{Dom}(S_{\min}(q)) \Leftrightarrow [u, v]_{+\infty} - [u, v]_{-\infty} = 0, \quad u \in \mathrm{Dom}(S_{\max}(q)) \quad \forall v \in \mathrm{Dom}(S_{\max}(q)).$$

Proposition 8 is proved. □

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