Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 2, pp. 142–158

DOUGLIS-NIRENBERG ELLIPTIC SYSTEMS IN THE REFINED SCALE OF SPACES ON A CLOSED MANIFOLD

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Dedicated to Professor Gorbachuk on the occasion of his 70th birthday.

Abstract. Douglis-Nirenberg elliptic systems of linear pseudodifferential equations are studied on a smooth closed manifold. We prove that the operator generated by the system is a Fredholm one on the refined two-sided scale of the functional Hilbert spaces. Elements of this scale are the special isotropic spaces of Hörmander–Volevich–Paneah. The refined smoothness of a solution of the system is studied. The elliptic systems with a parameter are investigated as well.

0. Introduction

In this paper we consider Douglis-Nirenberg elliptic systems [1] of linear pseudodifferential equations on a smooth closed manifold. L. Hörmander [2, Sec. 1.0] proved a priori estimates for solutions of these systems in appropriate pairs of Sobolev spaces. These estimates are equivalent to the fact that the linear operator A corresponding to the elliptic system is bounded, Fredholm, and establishes a complete collection of topological isomorphisms in the two-sided scale of Sobolev spaces [3, 4]. This fact has important applications in the theory of elliptic boundary-value problems [5, 3], in the index theory for elliptic operators [6], in the spectral theory [7, 8, 9] and others (see survey [4]).

In contrast to the papers cited above, we investigate the operator A on a Hilbert scale of special isotropic Hörmander–Volevich–Paneah spaces [10–13],

$$H^{s,\varphi} := H_2^{\langle\cdot\rangle^s\,\varphi(\langle\cdot\rangle)}, \quad \langle\xi\rangle := \bigl(1+|\xi|^2\bigr)^{1/2}. \tag{0.1}$$

Here $s\in\mathbb{R}$ and φ is a functional parameter slowly varying at $+\infty$ in Karamata's sense. In particular, every standard function

$$\varphi(t) = (\log t)^{r_1}(\log\log t)^{r_2}\dots(\log\dots\log t)^{r_n}, \quad \{r_1,r_2,\dots,r_n\}\subset\mathbb{R}, \quad n\in\mathbb{N},$$

is admissible. This scale was introduced and investigated by the authors in [14, 15]. It contains the Sobolev scale $\{H^s\}\equiv\{H^{s,1}\}$ and is attached to it with the number parameter s, and is considerably finer.

The spaces of form (0.1) naturally arise in different spectral problems including convergence of spectral expansions of self-adjoint elliptic operators almost everywhere in the norm of the spaces L_p with $p>2$ or C (see survey [16]); spectral asymptotics of general self-adjoint elliptic operators in a bounded domain, the Weyl formula, a sharp estimate of the remainder in it (see [17, 18]) and others. We also think that they can be useful in other "fine" questions. Due to their interpolation properties, the spaces $H^{s,\varphi}$ occupy a special place among spaces that define generalized smoothness and which are actively

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2000 *Mathematics Subject Classification.* Primary 35J45; Secondary 46E35.
Key words and phrases. Douglis-Nirenberg elliptic system, pseudodifferential operator, regularly varying function, scale of spaces, the Hörmander spaces, the Fredholm property, elliptic system with a parameter.

investigated and used today (see survey [19], recent articles [20, 21] and the bibliography given there).

The main result of this paper is a theorem on a collection of topological isomorphisms established by the operator A on refined scale (0.1). The refined local smoothness of a solution of the elliptic system is obtained as a significant application. The elliptic systems with a parameter are investigated in the refined scale as well. We also give some auxiliary results which may be of interest by themselves.

The case of scalar differential operators was investigated earlier in [15, 22–26].

1. The statement of the problem and the main result

Let Γ be a closed (compact and without boundary) infinitely smooth manifold of dimension $n \geq 1$. We suppose that a certain C^∞-density dx is defined on Γ. By $\mathcal{D}'(\Gamma)$ we denote the linear topological space of all distributions on Γ, i.e., $\mathcal{D}'(\Gamma)$ is the space antidual to the space $C^\infty(\Gamma)$ with respect to the sesquilinear form

$$(h,\omega)_\Gamma := \int_\Gamma h(x)\,\overline{\omega(x)}\,dx.$$

This form can be extended by continuity to the form $(h,\omega)_\Gamma$ of $h \in \mathcal{D}'(\Gamma)$ and $\omega \in C^\infty(\Gamma)$ which is equal to the value of the distribution h at the test function ω.

According to [4, Sec. 2.1] we denote by $\Psi^m_{\mathrm{ph}}(\Gamma)$ the class of polyhomogeneous (in other words, classical) pseudodifferential operators (PsDOs) of the order m, defined on the manifold Γ. Note that for a PsDO from the class $\Psi^m_{\mathrm{ph}}(\Gamma)$, the principal symbol is defined on the cotangent bundle $T^*\Gamma \setminus 0$ (here 0 is the zero-section). This principal symbol is an infinitely smooth and complex-valued function which is positively homogeneous of the degree m with respect to ξ in every section $T^*_x\Gamma \setminus \{0\}$, $x \in \Gamma$. We assume that the principal symbol can be equal to zero identically. Then $\Psi^m_{\mathrm{ph}}(\Gamma) \subset \Psi^r_{\mathrm{ph}}(\Gamma)$ for $m < r$. A linear differential operator of the order $\leq m$ on the manifold Γ with infinitely smooth coefficients is a particular and important case of a PsDO from the class $\Psi^m_{\mathrm{ph}}(\Gamma)$ with $m \geq 1$. Note that the PsDOs under consideration are linear and continuous in both topological spaces, $C^\infty(\Gamma)$ and $\mathcal{D}'(\Gamma)$.

We consider the system of linear equations

$$\sum_{k=1}^{p} A_{j,k}\,u_k = f_j \quad \text{on} \quad \Gamma, \quad j = 1,\dots,p. \tag{1.1}$$

Here $p \in \mathbb{N}$ and $A_{j,k}$, where $j,k = 1,\dots,p$, is a scalar classical PsDO defined on the manifold Γ. Equations (1.1) are understood in the sense of the distribution theory.

Further we suppose system (1.1) to be *elliptic* in Douglis-Nirenberg's sense [4, Sec. 3.2], i.e., the following two conditions are fulfilled:

a) there are sets of real numbers $\{l_1,\dots,l_p\}$ and $\{m_1,\dots,m_p\}$ such that $A_{j,k} \in \Psi^{l_j+m_k}_{\mathrm{ph}}(\Gamma)$ for all indices $j,k = 1,\dots,p$;

b) for each point $x \in \Gamma$ and covector $\xi \in T^*_x\Gamma\setminus\{0\}$, the inequality $\det\big(\,a^{(0)}_{j,k}(x,\xi)\,\big)^{p}_{j,k=1} \neq 0$ holds; here $a^{(0)}_{j,k}(x,\xi)$ is the principal symbol of the PsDO $A_{j,k}$.

Let us write down the system of equations (1.1) in a matrix form, $Au = f$ on Γ. Here $A := \big(A_{j,k}\big)$ is a square matrix of the order p, and $u = \mathrm{col}\,(u_1,\dots,u_p)$, $f = \mathrm{col}\,(f_1,\dots,f_p)$ are function columns. We study the mapping $u \mapsto Au$ on the refined scale of spaces over the manifold Γ. This scale consists of the Hilbert spaces $H^{s,\varphi}(\Gamma)$, where the number parameter s is arbitrary real and the function parameter φ runs over a certain class $\mathcal{M}$ of functions slowly varying in the Karamata sense at $+\infty$. The definition of the refined scale is given in Sec. 2. Now we only note that this scale contains the Hilbert scale of the Sobolev spaces $H^s(\Gamma) = H^{s,1}(\Gamma)$ and is much finer than the Sobolev scale.

Let us formulate the main result of the paper. Denote by A^+ the matrix PsDO formally adjoint to the operator A with respect to the density dx. The ellipticity of the system $Au = f$ is equivalent to the ellipticity of the adjoint system $A^+v = g$ (in Douglis-Nirenberg's sense). We set

$$N := \{\, u \in (C^\infty(\Gamma))^p : \, Au = 0 \text{ on } \Gamma \,\},$$
$$N^+ := \{\, v \in (C^\infty(\Gamma))^p : \, A^+v = 0 \text{ on } \Gamma \,\}. \tag{1.2}$$

Since the systems $Au = f$ and $A^+v = g$ are elliptic, the spaces N and N^+ are finite-dimensional [4, Theorem 3.2.1].

Theorem 1.1. *Let us assume that the spaces N and N^+ are trivial. Then, for arbitrary parameters $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$, the following topological isomorphism holds true:*

$$A : \prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma) \leftrightarrow \prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma). \tag{1.3}$$

A more general statement is given in Sec. 4. We can see that the matrix PsDO A leaves the parameter φ invariant. Theorem 1.1 refines the known result [4, Theorem 3.2.1] concerning properties of elliptic system (1.1) in a Sobolev scale (the case where $\varphi \equiv 1$). This theorem permits us to investigate the local smoothness of a solution of the system in the refined scale.

2. The refined scale of spaces

The refined scale was introduced and studied in [15, 26]. We formulate (for reader's convenience) the definition and some properties of this scale.

We denote by $\mathcal{M}$ the set of all Borel measurable functions $\varphi : [1, +\infty) \to (0, +\infty)$ such that:

a) the functions φ and $1/\varphi$ are bounded on every closed interval $[1, b]$, where $1 < b < +\infty$;

b) φ is a slowly varying function at $+\infty$ in the Karamata sense, i.e. [27, Sec. 1.1]

$$\lim_{t\to+\infty} \varphi(\lambda\, t)/\varphi(t) = 1 \quad \text{for each} \quad \lambda > 0.$$

Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. We denote by $H^{s,\varphi}(\mathbb{R}^n)$ the space of all tempered distributions w in the Euclidean space $\mathbb{R}^n$ such that the Fourier transform $\widehat{w}$ of the distribution w is a function locally Lebesgue integrable in $\mathbb{R}^n$ and satisfies the condition

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2s}\, \varphi^2(\langle\xi\rangle)\, |\widehat{w}(\xi)|^2\, d\xi < \infty.$$

Here $\langle\xi\rangle = (1+\xi_1^2+\cdots+\xi_n^2)^{1/2}$ is the smoothed modulus of a vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. We define the inner product in the space $\mathrm{H}^{s,\varphi}(\mathbb{R}^n)$ by the formula

$$\big(w_1, w_2\big)_{\mathrm{H}^{s,\varphi}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \varphi^2(\langle\xi\rangle)\, \widehat{w_1}(\xi)\, \overline{\widehat{w_2}(\xi)}\, d\xi.$$

This inner product induces a norm in $\mathrm{H}^{s,\varphi}(\mathbb{R}^n)$ in the usual way.

The space $H^{s,\varphi}(\mathbb{R}^n)$ is a special isotropic Hilbert case of the spaces introduced by L. Hörmander [10, Sec. 2.2], [11, Sec. 10.1] and L. R. Volevich, B. P. Paneah [12, Sec. 2], [13, Sec. 1.4.2]. In the simplest case where $\varphi(\cdot) \equiv 1$, the space $H^{s,\varphi}(\mathbb{R}^n)$ coincides with the Sobolev space $H^s(\mathbb{R}^n)$. Inclusions

$$\bigcup_{\varepsilon>0} H^{s+\varepsilon}(\mathbb{R}^n) =: H^{s+}(\mathbb{R}^n) \subset H^{s,\varphi}(\mathbb{R}^n) \subset H^{s-}(\mathbb{R}^n) := \bigcap_{\varepsilon>0} H^{s-\varepsilon}(\mathbb{R}^n)$$

imply that, in the collection of spaces

$$\{H^{s,\varphi}(\mathbb{R}^n) : s \in \mathbb{R}, \varphi \in \mathcal{M}\}, \tag{2.1}$$

the functional parameter φ defines an additional (subpower) smoothness with respect to the basic (power) s-smoothness. In other words, φ *refines* the power smoothness s. Therefore the collection of spaces (2.1) is naturally called a *refined* scale over $\mathbb{R}^n$ (with respect to the Sobolev scale).

The refined scale over the manifold Γ is constructed from scale (2.1) in the usual way. Let us choose a finite atlas from the C^∞-structure on Γ, consisting of local charts $\alpha_j : \mathbb{R}^n \leftrightarrow U_j$, $j = 1, \ldots, r$. Here the open sets U_j form a finite covering of the manifold Γ. Let functions $\chi_j \in C^\infty(\Gamma)$, $j = 1, \ldots, r$, form a partition of unity on Γ satisfying the condition $\operatorname{supp}\chi_j \subset U_j$.

We set

$$H^{s,\varphi}(\Gamma) := \{h \in \mathcal{D}'(\Gamma) : \ (\chi_j h) \circ \alpha_j \in H^{s,\varphi}(\mathbb{R}^n) \ \ \forall\, j = 1, \ldots, r\}\,.$$

Here $(\chi_j h) \circ \alpha_j$ is a representation of the distribution $\chi_j h$ in the local chart α_j. The inner product in the space $H^{s,\varphi}(\Gamma)$ is defined by the formula

$$(h_1, h_2)_{s,\varphi} := \sum_{j=1}^{r} ((\chi_j h_1) \circ \alpha_j, (\chi_j h_2) \circ \alpha_j)_{H^{s,\varphi}(\mathbb{R}^n)}$$

and induces the norm $\|h\|_{s,\varphi} := \bigl(h, h\bigr)_{s,\varphi}^{1/2}$. In the Sobolev case where $\varphi \equiv 1$ we omit the index φ in the notations.

The Hilbert space $H^{s,\varphi}(\Gamma)$ is separable, continuously embedded into the space $\mathcal{D}'(\Gamma)$, and independent (up to equivalent norms) of the choice of the atlas and the partition of unity. The collection of function spaces

$$\{H^{s,\varphi}(\Gamma) : s \in \mathbb{R},\ \varphi \in \mathcal{M}\}$$

is called the refined scale over the manifold Γ. We note the following properties of this scale [15, Theorem 3.6].

Proposition 2.1. *Let $s \in \mathbb{R}$ and $\varphi, \varphi_1 \in \mathcal{M}$. The following assertions are true.*

(i) *The set $C^\infty(\Gamma)$ is dense in the space $H^{s,\varphi}(\Gamma)$.*

(ii) *For each $\varepsilon > 0$ the following embeddings are compact and dense:*

$$H^{s+\varepsilon}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma) \hookrightarrow H^{s-\varepsilon}(\Gamma) \quad \textit{and} \quad H^{s+\varepsilon,\varphi_1}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma).$$

(iii) *Suppose that the function φ/φ_1 is bounded in a neighborhood of $+\infty$. Then the embedding $H^{s,\varphi_1}(\Gamma) \hookrightarrow H^{s,\varphi}(\Gamma)$ is continuous and dense. It is compact if $\varphi(t)/\varphi_1(t) \to 0$ as $t \to +\infty$.*

(iv) *For every fixed integer $\rho \geq 0$ the inequality*

$$\int_1^{+\infty} \frac{dt}{t\,\varphi^2(t)} < \infty \tag{2.2}$$

is equivalent to existence of the embedding $H^{\rho+n/2,\,\varphi}(\Gamma) \hookrightarrow C^{\,\rho}(\Gamma)$. This embedding is compact.

(v) *The spaces $H^{s,\varphi}(\Gamma)$ and $H^{-s,1/\varphi}(\Gamma)$ are mutually dual (up to equivalent norms) with respect to the sesquilinear form $(\cdot\,,\,\cdot)_\Gamma$.*

In connection with assertion (v) we note that $\varphi \in \mathcal{M} \Leftrightarrow 1/\varphi \in \mathcal{M}$. Hence the space $H^{-s,\,1/\varphi}(\Gamma)$ is well defined.

At the end of this section, let us give the following alternative (and equivalent) description of the refined scale over a closed manifold Γ [26, Theorem 3.8].

Let a Riemannian structure on the manifold Γ, which defines the density dx, be given (this is always possible), and let $\triangle_\Gamma$ be the Beltrami-Laplace operator on Γ. We define the function

$$\varphi_s(t) := t^{s/2}\varphi(t^{1/2}) \quad \text{for} \quad t \geq 1 \quad \text{and} \quad \varphi_s(t) := \varphi(1) \quad \text{for} \quad 0 < t < 1.$$

We consider the operator $\varphi_s(1-\triangle_\Gamma)$ on the space $L_2(\Gamma) = L_2(\Gamma, dx)$ as a Borel function of the self-adjoint positive operator $1-\triangle_\Gamma$.

Proposition 2.2. *For arbitrary $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$, the space $H^{s,\varphi}(\Gamma)$ coincides with the completion of the set of functions $u \in C^\infty(\Gamma)$ with respect to the norm $\|\varphi_s(1-\triangle_\Gamma)\, u\|_{L_2(\Gamma)}$ which is equivalent to the norm $\|u\|_{s,\varphi}$.*

3. The interpolation with a function parameter

An interpolation, with a function parameter, of couples of Hilbert spaces is a natural generalization of the classical Lions-Krein interpolation method to the case where a more general function then the power function is used as an interpolation parameter [28–31, 14, 26]. Now we give a definition of this interpolation. For our purpose, it is sufficient to restrict ourselves to the case of *separable* Hilbert spaces.

An ordered couple $[X_0, X_1]$ of complex Hilbert spaces X_0 and X_1 is called *admissible* if the spaces X_0 and X_1 are separable and there is a continuous dense embedding $X_1 \hookrightarrow X_0$. If, in addition, $\|\,u\,\|_{X_0} \leq \|\,u\,\|_{X_1}$ for each $u \in X_1$, then the admissible couple $[X_0, X_1]$ is called *normal*. Note that we can transform every admissible couple $[X_0, X_1]$ into the normal one by replacing the norm $\|\,u\,\|_{X_1}$ with the equivalent norm $c\,\|\,u\,\|_{X_1}$ where c is a sufficiently large positive number.

Let an admissible couple $X = [X_0, X_1]$ of Hilbert spaces be given. It is known [32, Ch. 1, Sec. 2.1] that for this couple X there exists an isometric isomorphism $J : X_1 \leftrightarrow X_0$ such that J is a self-adjoint positive operator on the space X_0 with the domain X_1. The operator J is called a *generating* one for the couple X. This operator is uniquely determined by the couple X.

We denote by $\mathcal{B}$ the set of all positive functions defined and Borel measurable in the open positive semiaxis $(0, +\infty)$. Let $\psi \in \mathcal{B}$. Since $\operatorname{Spec} J \subset (0, +\infty)$, generally the unbounded operator $\psi(J)$ is defined on the space X_0 as a function of J. The domain of the operator $\psi(J)$ is a linear manifold which is dense in X_0. We denote by $[X_0, X_1]_\psi$ or simply by X_ψ the domain of the operator $\psi(J)$ endowed with the graph inner product and the corresponding norm,

$$(u, v)_{X_\psi} := (u, v)_{X_0} + (\psi(J)u, \psi(J)v)_{X_0}, \quad \|\,u\,\|_{X_\psi} = (u, u)_{X_\psi}^{1/2}.$$

This makes X_ψ a separable Hilbert space.

A function ψ is called an *interpolation parameter* if the following condition is satisfied for all admissible couples $X = [X_0, X_1]$, $Y = [Y_0, Y_1]$ of Hilbert spaces and an arbitrary linear mapping T given on X_0: if the restriction of the mapping T to the space X_j is a bounded operator $T : X_j \to Y_j$ for each $j = 0,\,1$, then the restriction of the mapping T to the space X_ψ is also a bounded operator $T : X_\psi \to Y_\psi$.

In other words, ψ is an interpolation parameter if and only if the mapping $X \mapsto X_\psi$ is an interpolation functor given on the category of all admissible couples X of Hilbert spaces. In the case where ψ is an interpolation parameter, we say that the space X_ψ is obtained by the interpolation with the function parameter ψ of the admissible couple X.

The classical result of J.-L. Lions and S. G. Krein in the interpolation theory consists in the fact that the power function $\psi(t) = t^\theta$ with the index $\theta \in (0, 1)$ is an interpolation parameter (see e.g. [32, Ch. 1, Sec. 5.1]). In this case the index θ plays the role of numerical interpolation parameter. However the class of interpolation parameters is not exhausted by power functions. We use the following functions as interpolation parameters (see [14, Theorem 2.1], [24, Lemma 7.1]).

Proposition 3.1. *Let a function $\psi \in \mathcal{B}$ be such that:*

a) *ψ is bounded on every closed interval $[a; b]$ where $0 < a < b < +\infty$;*
b) *ψ is a function regularly varying in Karamata's sense at $+\infty$ with the index θ where $0 < \theta < 1$, i.e.* [27, Sec. 1.1]

$$\lim_{t \to +\infty} \psi(\lambda t)/\psi(t) = \lambda^{\theta} \quad \text{for each} \quad \lambda > 0.$$

Then ψ is an interpolation parameter. Moreover, there is a number $c_\psi > 0$ such that

$$\| T \|_{X_\psi \to Y_\psi} \leq c_\psi \max \left\{ \| T \|_{X_j \to Y_j} : j = 0, 1 \right\}.$$

Here $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ are arbitrary admissible couples of Hilbert spaces and T is an arbitrary linear mapping on X_0 such that the operators $T : X_j \to Y_j$ are bounded for $j = 0, 1$. The number $c_\psi > 0$ does not depend on T and on the couples X, Y if these couples are normal.

The interpolation with a function parameter establishes a closed connection between the classical Sobolev scale and the refined scale. Namely, we have the following proposition [15, Theorem 3.5].

Proposition 3.2. *Let a function $\varphi \in \mathcal{M}$ and positive numbers ε, δ be given. We set $\psi(t) := t^{\varepsilon/(\varepsilon+\delta)} \varphi(t^{1/(\varepsilon+\delta)})$ for $t \geq 1$ and $\psi(t) := \varphi(1)$ for $0 < t < 1$. Then*

(i) *The function $\psi \in \mathcal{B}$ satisfies the all conditions of Proposition* 3.1 *with $\theta = \varepsilon/(\varepsilon + \delta)$ and, therefore, ψ is an interpolation parameter.*
(ii) *For an arbitrary $s \in \mathbb{R}$, we have*

$$\left[H^{s-\varepsilon}(\Gamma),\, H^{s+\delta}(\Gamma) \right]_\psi = H^{s,\varphi}(\Gamma) \quad \text{with the equivalence of norms.}$$

This implies the following fact used below repeatedly.

Lemma 3.1. *Let T be a PsDO from the class $\Psi^r_{\mathrm{ph}}(\Gamma)$ where $r \in \mathbb{R}$. Then the restriction of the mapping $u \mapsto Tu$, $u \in \mathcal{D}'(\Gamma)$, to the space $H^{s,\varphi}(\Gamma)$ is a bounded operator*

$$T : H^{s,\varphi}(\Gamma) \to H^{s-r,\varphi}(\Gamma) \quad \text{for every} \quad s \in \mathbb{R}, \;\; \varphi \in \mathcal{M}. \tag{3.1}$$

Proof. In the case where $\varphi \equiv 1$ this result is known [4, Theorem 2.1.2]. Let us choose arbitrary parameters $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. We consider the linear bounded operators

$$T : H^{s\mp 1}(\Gamma) \to H^{s\mp 1-r}(\Gamma).$$

Let us use the interpolation with the function parameter ψ from Proposition 3.2 where $\varepsilon = \delta = 1$. We obtain the bounded operator

$$T : \left[H^{s-1}(\Gamma),\, H^{s+1}(\Gamma) \right]_\psi \to \left[H^{s-1-r}(\Gamma),\, H^{s+1-r}(\Gamma) \right]_\psi.$$

Proposition 3.2 (ii) implies boundedness of the operator (3.1). □

We also need the following two propositions on interpolation of Fredholm operators and direct products of spaces [33, Sec. 5]. We recall that a linear bounded operator $T : X \to Y$ is called *Fredholm* if its kernel is finite-dimensional and its range $T(X)$ is closed in the space Y and has finite codimension. Here X, Y are Banach spaces. A Fredholm operator T has finite *index* defined by $\operatorname{ind} T := \dim \ker T - \dim(Y/\,T(X))$.

Proposition 3.3. *Let two admissible couples of Hilbert spaces $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ be given. Moreover, let a linear mapping T on X_0 be such that there exist bounded Fredholm operators $T : X_j \to Y_j$ for $j = 0, 1$ which have a common kernel $\mathcal{N}$ and the same index κ. Then, for an arbitrary interpolation parameter $\psi \in \mathcal{B}$, the bounded operator $T : X_\psi \to Y_\psi$ is Fredholm and has the kernel $\mathcal{N}$, the range $Y_\psi \cap T(X_0)$, and the same index κ.*

Proposition 3.4. *Let* $[X_0^{(k)}, X_1^{(k)}]$, $k = 1, \dots, p$, *be a finite collection of admissible couples of Hilbert spaces. Then, for every function,* $\psi \in \mathcal{B}$ *we have*

$$\Big[\prod_{k=1}^{p} X_0^{(k)},\, \prod_{k=1}^{p} X_1^{(k)}\Big]_\psi = \prod_{k=1}^{p}\Big[X_0^{(k)},\, X_1^{(k)}\Big]_\psi \quad \textit{with equality of norms.}$$

4. The elliptic system in the refined scale

Let us return to the elliptic system (1.1) which is written in the matrix form $Au = f$. We study properties of the matrix PsDO A in the refined scale of spaces. First note that by Lemma 3.1 the condition $A_{j,k} \in \Psi_{\mathrm{ph}}^{l_j+m_k}(\Gamma)$ implies boundedness of the operator

$$(4.1) \qquad A : \prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma) \to \prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma) \quad \text{for each } \ s \in \mathbb{R},\ \varphi \in \mathcal{M}.$$

We assume that the finite-dimensional spaces N and N^+ given by formula (1.2) are, generally, not trivial. We have the following assertion containing Theorem 1.1 as a particular case.

Theorem 4.1. *For arbitrary parameters* $s \in \mathbb{R}$ *and* $\varphi \in \mathcal{M}$, *the bounded operator* (4.1) *is Fredholm. Its kernel coincides with the space* N, *whereas the range consists of all vector-valued functions*

$$(4.2) \qquad f = \mathrm{col}\,(f_1, \dots, f_p) \in \prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma)$$

such that

$$(4.3) \qquad \sum_{j=1}^{p} (f_j, w_j)_\Gamma = 0 \quad \textit{for each} \quad w = (w_1, \dots, w_p) \in N^+.$$

The index of operator (4.1) *is equal to* $\dim N - \dim N^+$ *and independent of* s, φ.

Proof. In the case where $\varphi \equiv 1$ (the Sobolev scale) this theorem is known [4, Theorem 3.2.1]. From this, the general case $\varphi \in \mathcal{M}$ is deduced by using interpolation with a function parameter in the following way. Let $s \in \mathbb{R}$ and consider the bounded Fredholm operators

$$(4.4) \qquad A : \prod_{k=1}^{p} H^{s\mp1+m_k}(\Gamma) \to \prod_{j=1}^{p} H^{s\mp1-l_j}(\Gamma)$$

which have the common kernel N, the common index $\kappa := \dim N - \dim N^+$ and the closed ranges

$$(4.5) \quad A\Big(\prod_{k=1}^{p} H^{s\mp1+m_k}(\Gamma)\Big) = \Big\{ f = \mathrm{col}\,(f_1, \dots, f_p) \in \prod_{j=1}^{p} H^{s\mp1-l_j}(\Gamma) : \text{(4.3) is true}\Big\}.$$

Let ψ be an interpolation parameter from Proposition 3.2 in which $\varepsilon = \delta = 1$. We apply the interpolation with the parameter ψ to operators (4.4). We get the bounded operator

$$A : \Big[\prod_{k=1}^{p} H^{s-1+m_k}(\Gamma), \prod_{k=1}^{p} H^{s+1+m_k}(\Gamma)\Big]_\psi \to \Big[\prod_{j=1}^{p} H^{s-1-l_j}(\Gamma), \prod_{j=1}^{p} H^{s+1-l_j}(\Gamma)\Big]_\psi$$

that coincides with operator (4.1) by virtue of Propositions 3.2 and 3.4. Hence according to Proposition 3.3, operator (4.1) is Fredholm and has the kernel N and index $\kappa =$

$\dim N - \dim N^{+}$. Moreover, the range of this operator is equal to

$$\Big(\prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma)\Big) \cap A\Big(\prod_{k=1}^{p} H^{s-1+m_k}(\Gamma)\Big).$$

From this it follows with a use of (4.5) that the range is as needed. □

According to this theorem, N^{+} is a deficiency subspace for operator (4.1). Note that the operator

$$A^{+}: \prod_{j=1}^{p} H^{-s+l_j,1/\varphi}(\Gamma) \to \prod_{k=1}^{p} H^{-s-m_k,1/\varphi}(\Gamma) \tag{4.6}$$

is adjoint to operator (4.1) in view of Proposition 2.1 (v). Since the adjoint system $A^{+}v = g$ is elliptic, bounded operator (4.6) is Fredholm and has the kernel N^{+} and the deficiency subspace N, in virtue of Theorem 4.1. We also note that, in the scalar case $p = 1$, the indices of operators (4.1) and (4.6) are equal to 0 provided that $\dim \Gamma \geq 2$ (see [6], [4, Sec. 2.3 f]).

If the spaces N and N^{+} are trivial, then operator (4.1) coincides with topological isomorphism (1.3). This follows from Theorem 4.1 and the Banach theorem on inverse operator. In general, it is convenient to define the appropriate isomorphism with the help of the following projections.

Let us represent the spaces in formula (4.1) as the following direct sums of closed spaces:

$$\prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma) = N \dotplus \Big\{u \in \prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma) : \sum_{k=1}^{p} (u_k, v_k)_{\Gamma} = 0 \ \ \forall \ \ v \in N\Big\},$$

$$\prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma) = N^{+} \dotplus \Big\{f \in \prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma) : \sum_{j=1}^{p} (f_j, w_j)_{\Gamma} = 0 \ \ \forall \ \ w \in N^{+}\Big\}.$$

As above, we write $u = \mathrm{col}\,(u_1, \ldots, u_p)$, $f = \mathrm{col}\,(f_1, \ldots, f_p)$, and also let $v = (v_1, \ldots, v_p)$, $w = (w_1, \ldots, w_p)$. These decompositions into direct sums exist because the spaces N and N^{+} are finite-dimensional. We denote by P and P^{+}, respectively, the oblique projections of the spaces

$$\prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma) \quad \text{and} \quad \prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma)$$

onto the second terms in the sums. These projections do not depend on s and φ.

Theorem 4.2. *For arbitrary parameters $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$, the restriction of operator (4.1) to the subspace $P\big(\prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma)\big)$ is the topological isomorphism*

$$A : P\Big(\prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma)\Big) \leftrightarrow P^{+}\Big(\prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma)\Big). \tag{4.7}$$

Proof. According to Theorem 4.1, N is the kernel and $P^{+}\big(\prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma)\big)$ is the range of operator (4.1). Therefore operator (4.7) is a bijection. Moreover, this operator is bounded. Thus it is a topological isomorphism by virtue of the Banach theorem on inverse operator. □

Theorem 4.2 implies the following a priory estimate of a solution of elliptic system (1.1).

Theorem 4.3. *Let $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$. Assume that the vector-valued function*

$$u = \mathrm{col}\,(u_1, \dots, u_p) \in \prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma) \tag{4.8}$$

is a solution of the equation $Au = f$ on Γ, where f in the right-hand side satisfies condition (4.2)*. Then for the chosen parameters s, φ and an arbitrary number $\sigma > 0$, there exists a number $c > 0$ independent of u, f such that*

$$\sum_{k=1}^{p} \|\, u_k \,\|_{s+m_k,\varphi} \leq c \Big(\sum_{j=1}^{p} \|\, f_j \,\|_{s-l_j,\varphi} + \sum_{k=1}^{p} \|\, u_k \,\|_{s-\sigma+m_k} \Big). \tag{4.9}$$

Proof. We denote for brevity by $\|\cdot\|'_{s,\varphi}$, $\|\cdot\|''_{s,\varphi}$ and $\|\cdot\|'_{s-\sigma}$, respectively, the (non-Hilbert) norms in the spaces

$$\prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma), \quad \prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma), \quad \text{and} \quad \prod_{k=1}^{p} H^{s-\sigma+m_k}(\Gamma)$$

used in (4.9). Since N is a finite-dimensional subspace of these spaces, the norms mentioned above are equivalent on N. In particular, we have for a vector-valued function $u - Pu \in N$ the inequality

$$\|\, u - Pu \,\|'_{s,\varphi} \leq c_1 \,\|\, u - Pu \,\|'_{s-\sigma}$$

with a constant $c_1 > 0$ independent of u. Therefore we can write the following:

$$\begin{aligned} \|\, u \,\|'_{s,\varphi} &\leq \|\, u - Pu \,\|'_{s,\varphi} + \|\, Pu \,\|'_{s,\varphi} \\ &\leq c_1 \,\|\, u - Pu \,\|'_{s-\sigma} + \|\, Pu \,\|'_{s,\varphi} \leq c_1\, c_2 \,\|\, u \,\|'_{s-\sigma} + \|\, Pu \,\|'_{s,\varphi}. \end{aligned}$$

Here c_2 is the norm of the projection $1 - P$ in the space $\prod_{k=1}^{p} H^{s-\sigma+m_k}(\Gamma)$. Thus

$$\|\, u \,\|'_{s,\varphi} \leq \|\, Pu \,\|'_{s,\varphi} + c_1\, c_2 \,\|\, u \,\|'_{s-\sigma}. \tag{4.10}$$

Now let us use the condition $Au = f$. Since N is the kernel of operator (4.1) and $u - Pu \in N$, we have $APu = f$. Therefore, Pu is the preimage of the vector-valued function f under topological isomorphism (4.7). Hence,

$$\|\, Pu \,\|'_{s,\varphi} \leq c_3 \,\|\, f \,\|''_{s,\varphi},$$

where c_3 is the norm of the inverse operator to (4.7). The last inequality and (4.10) yield estimate (4.9). □

Now note the following: if $N = \{0\}$, i.e., the equation $Au = f$ has at the most one solution, then the term $\sum_{k=1}^{p} \|\, u_k \,\|_{s-\sigma+m_k}$ in the right-hand side of estimate (4.9) is absent. Generally (if $N \neq \{0\}$), this term can be made arbitrarily small for every fixed vector-valued function u by choosing the parameter σ large enough.

5. Local refined smoothness of the elliptic system solution

Let us pose the following question. Assume that the right-hand side of the elliptic equation has a certain local smoothness in the refined scale in a given open subset Γ_0 of the manifold Γ. What can we say about the local smoothness of a solution u of the equation? The answer to this question will be given below. First, let us consider the case where $\Gamma_0 = \Gamma$.

Theorem 5.1. *Suppose that the vector-valued function* $u \in (\mathcal{D}'(\Gamma))^p$ *is a solution of the equation* $Au = f$ *on the manifold* Γ *where*

$$f_j \in H^{s-l_j,\varphi}(\Gamma) \quad \text{for all} \quad j = 1, \ldots, p \tag{5.1}$$

and some parameters $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$. *Then*

$$u_k \in H^{s+m_k,\varphi}(\Gamma) \quad \text{for all} \quad k = 1, \ldots, p. \tag{5.2}$$

Proof. Since the manifold Γ is compact, the space $\mathcal{D}'(\Gamma)$ is the union of the Sobolev spaces $H^{\sigma}(\Gamma)$, $\sigma \in \mathbb{R}$. Hence for the vector-valued function $u \in (\mathcal{D}'(\Gamma))^p$, there exists a number $\sigma < s$ such that $u \in \prod_{k=1}^{p} H^{\sigma+m_k}(\Gamma)$. By virtue of Theorem 4.1 we have the equality

$$\Big(\prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma)\Big) \cap A\Big(\prod_{k=1}^{p} H^{\sigma+m_k}(\Gamma)\Big) = A\Big(\prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma)\Big).$$

Therefore it follows from condition (5.1) that

$$f = Au \in A\Big(\prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma)\Big).$$

Thus the equality $Av = f$ holds true on Γ for some vector-valued function

$$v \in \prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma)$$

as well as $Au = f$. Hence, $A(u-v) = 0$ on Γ and by Theorem 4.1 we have $w := u-v \in N$. However,

$$N \subset (C^{\infty}(\Gamma))^p \subset \prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma).$$

Therefore,

$$u = v + w \in \prod_{k=1}^{p} H^{s+m_k,\varphi}(\Gamma),$$

i.e. property (5.2) is verified. □

Now we consider the general case where Γ_0 is an arbitrary nonempty subset of the manifold Γ. For $\sigma \in \mathbb{R}$, $\varphi \in \mathcal{M}$, we denote

$$H^{\sigma,\varphi}_{\mathrm{loc}}(\Gamma_0) := \{h \in \mathcal{D}'(\Gamma) : \chi h \in H^{\sigma,\varphi}(\Gamma) \ \forall \ \chi \in C^{\infty}(\Gamma), \ \operatorname{supp}\chi \subset \Gamma_0\}.$$

Theorem 5.2. *Suppose that a vector-valued function* $u \in (\mathcal{D}'(\Gamma))^p$ *is a solution of the equation* $Au = f$ *on the set* Γ_0, *where*

$$f_j \in H^{s-l_j,\varphi}_{\mathrm{loc}}(\Gamma_0) \quad \text{for all} \quad j = 1, \ldots, p \tag{5.3}$$

and some parameters $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$. *Then*

$$u_k \in H^{s+m_k,\varphi}_{\mathrm{loc}}(\Gamma_0) \quad \text{for all} \quad k = 1, \ldots, p. \tag{5.4}$$

Proof. We will show that condition (5.3) implies that smoothness of a solution u increases, that is, for every number $r \geq 1$

$$u \in \prod_{k=1}^{p} H^{s-r+m_k,\varphi}_{\mathrm{loc}}(\Gamma_0) \ \Rightarrow \ u \in \prod_{k=1}^{p} H^{s-r+1+m_k,\varphi}_{\mathrm{loc}}(\Gamma_0). \tag{5.5}$$

Let us chose arbitrary functions $\chi, \eta \in C^\infty(\Gamma)$ such that $\operatorname{supp}\chi$, $\operatorname{supp}\eta \subset \Gamma_0$ and $\eta = 1$ in a neighborhood of $\operatorname{supp}\chi$. Rearranging the matrix PsDO A and the operator of multiplication by the function χ we can write the following:

$$\begin{aligned} A\chi u = A\chi\eta u &= \chi\, A\eta u + A'\eta u = \chi\, Au + \chi\, A(\eta-1)u + A'\eta u \\ &= \chi f + \chi\, A(\eta-1)u + A'\eta u \quad \text{on} \quad \Gamma. \end{aligned} \tag{5.6}$$

Here the matrix PsDO $A' = (\, A'_{j,k}\,)_{j,k=1}^{p}$ is the commutator of these operators. Since $A'_{j,k} \in \Psi_{\mathrm{ph}}^{l_j+m_k-1}(\Gamma)$, there exists (by Lemma 3.1) the bounded operator

$$A' : \prod_{k=1}^{p} H^{s-r+m_k,\varphi}(\Gamma) \,\to\, \prod_{j=1}^{p} H^{s-r+1-l_j,\varphi}(\Gamma).$$

Therefore,

$$u \in \prod_{k=1}^{p} H_{\mathrm{loc}}^{s-r+m_k,\varphi}(\Gamma_0) \;\Rightarrow\; A'\eta u \in \prod_{j=1}^{p} H^{s-r+1-l_j,\varphi}(\Gamma). \tag{5.7}$$

Further, according to condition (5.3) and in view of Proposition 2.1 (ii) we have

$$\chi f \in \prod_{j=1}^{p} H^{s-l_j,\varphi}(\Gamma) \hookrightarrow \prod_{j=1}^{p} H^{s-r+1-l_j,\varphi}(\Gamma). \tag{5.8}$$

In addition, since the supports of the functions χ and $\eta - 1$ are disjoint, we get

$$\chi\, A(\eta-1)u \in \bigl(C^\infty(\Gamma)\bigr)^p. \tag{5.9}$$

It follows from (5.6)–(5.9) that

$$u \in \prod_{k=1}^{p} H_{\mathrm{loc}}^{s-r+m_k,\varphi}(\Gamma_0) \;\Rightarrow\; A\chi u \in \prod_{j=1}^{p} H^{s-r+1-l_j,\varphi}(\Gamma).$$

Moreover, according to Theorem 5.1 we have

$$A\chi u \in \prod_{j=1}^{p} H^{s-r+1-l_j,\varphi}(\Gamma) \;\Rightarrow\; \chi u \in \prod_{k=1}^{p} H^{s-r+1+m_k,\varphi}(\Gamma).$$

Therefore implication (5.5) holds true because the function $\chi \in C^\infty(\Gamma)$ satisfying the condition $\operatorname{supp}\chi \subset \Gamma_0$ is chosen arbitrarily.

Now it is easy to deduce property (5.4) with the help of implication (5.5). As has been noted in the proof of Theorem 5.1, there is a sufficiently large integer r_0 such that

$$u \in \prod_{k=1}^{p} H^{s-r_0+1+m_k}(\Gamma) \subset \prod_{k=1}^{p} H_{\mathrm{loc}}^{s-r_0+m_k,\varphi}(\Gamma_0).$$

From this by applying implication (5.5) for $r = r_0, r_0 - 1, \ldots, 1$ successively, we deduce property (5.4),

$$\begin{aligned} &u \in \prod_{k=1}^{p} H_{\mathrm{loc}}^{s-r_0+m_k,\varphi}(\Gamma_0) \\ &\quad\Rightarrow\; u \in \prod_{k=1}^{p} H_{\mathrm{loc}}^{s-r_0+1+m_k,\varphi}(\Gamma_0) \;\Rightarrow\; \ldots \;\Rightarrow\; u \in \prod_{k=1}^{p} H_{\mathrm{loc}}^{s+m_k,\varphi}(\Gamma_0). \end{aligned}$$

Theorem is proved. □

This theorem specifies, with regard to the refined scale of the spaces $H^{s,\varphi}(\Gamma)$, the known propositions on the local increase of interior smoothness of a solution to an elliptic system in the Sobolev scale (see e.g. [1, 10, 34]). Note that the refined local smoothness φ of the right-hand side of the elliptic system is inherited by its solution.

Theorem 5.2 and Proposition 2.1 (iv) imply immediately the following sufficient condition for a chosen component u_k of the solution of system (1.1) to have continuous derivatives of a prescribed order.

Corollary 5.1. *Suppose that vector-valued functions $u, f \in \big(\mathcal{D}'(\Gamma)\big)^p$ satisfy the equation $Au = f$ on Γ_0. Let integers $\rho \geq 0$ and $k \in \{1, \ldots, p\}$ be given and a function $\varphi \in \mathcal{M}$ be such that inequality* (2.2) *holds true. Then*

$$\Big(\ f_j \in H^{\rho - m_k - l_j + n/2, \varphi}_{\mathrm{loc}}(\Gamma_0)\ \ \forall\ \ j = 1, \ldots, p\ \Big)\ \ \Rightarrow\ \ u_k \in C^{\rho}(\Gamma_0).$$

6. Elliptic system with a parameter

Elliptic operators with a parameter were studied by S. Agmon, L. Nirenberg [35], M. S. Agranovich, M. I. Vishik [36], A. N. Kozhevnikov [8] and their successors (see survey [4] and the literature cited therein). They found that a parameter-elliptic operator establishes a topological isomorphism in appropriate pairs of the Sobolev spaces for all values of the complex parameter large enough in modulus. Moreover, the norm of this operator admits a certain two-sided estimate with constants independent of the parameter. We specify this result for an elliptic system with a parameter with regard to the refined scale of spaces over a closed manifold. Note that scalar parameter-elliptic PsDOs were studied on the refined scale in [25], whereas elliptic boundary-value problems with a parameter (for a differential equation) were studied in [24].

We recall the definition of an elliptic system with a parameter given in survey [4, Sec. 4.3 e]. Let us fix arbitrarily numbers $p, q \in \mathbb{N}$, $m > 0$, and $m_1, \ldots, m_p \in \mathbb{R}$. We consider the matrix PsDO $A(\lambda)$ which depends on a complex parameter λ in the following way:

$$A(\lambda) := \sum_{r=0}^{q} \lambda^{q-r} A^{(r)}. \tag{6.1}$$

Here $A^{(r)} := \big(A^{(r)}_{j,k} \big)_{j,k=1}^{p}$ is a square matrix formed by scalar PsDOs $A^{(r)}_{j,k}$ from the class $\Psi^{mr+m_k-m_j}_{\mathrm{ph}}(\Gamma)$. Moreover, we set $A^{(0)} = -I$ where I is the identity matrix.

We consider the parameter-depending system of linear equations

$$A(\lambda)\, u = f \quad \text{on} \quad \Gamma. \tag{6.2}$$

Here, as above, $u = \mathrm{col}\,(u_1, \ldots, u_p)$, $f = \mathrm{col}\,(f_1, \ldots, f_p)$ are function columns components of which are distributions on the manifold Γ.

Let K be a fixed closed angle in the complex plain with the vertex at the origin (we do not exclude the case where K degenerates into a ray). Further we suppose that system (6.2) is *elliptic with a parameter* in the angle K, i.e., the following condition is fulfilled:

$$\det \sum_{r=0}^{q} \lambda^{q-r} a^{r,0}(x,\xi) \neq 0 \tag{6.3}$$
$$\text{for all} \quad x \in \Gamma, \quad \xi \in T^{*}_{x}\Gamma, \quad \lambda \in K \quad \text{such that} \quad (\xi, \lambda) \neq 0.$$

Here $a^{r,0}(x,\xi) := \big(a^{r,0}_{j,k}(x,\xi) \big)_{j,k=1}^{p}$ is a square matrix formed by the principal symbols $a^{r,0}_{j,k}(x,\xi)$ of PsDOs $A^{(r)}_{j,k}$. Furthermore, in the case where $r \geq 1$ we agree that the function $a^{r,0}_{j,k}(x,\xi)$ is equal to 0 at $\xi = 0$ (such an agreement is connected with the fact that the principal symbols are not initially defined at $\xi = 0$).

The ellipticity of system (6.2), with a parameter in the angle K, implies its Douglis-Nirenberg ellipticity for every fixed $\lambda \in \mathbb{C}$. Indeed, due to (6.1), the matrix $A(\lambda)$ is formed by the elements

$$\sum_{r=0}^{q} \lambda^{q-r} A_{j,k}^{(r)}, \quad j,k = 1,\dots,p, \tag{6.4}$$

which are scalar PsDOs from $\Psi_{\mathrm{ph}}^{l_j+m'_k}(\Gamma)$, where $l_j := -m_j$, $m'_k := mq + m_k$. The principal symbol of PsDO (6.4) is equal to $a_{j,k}^{q,0}(x,\xi)$ for every fixed λ. According to condition (6.3) with $\lambda = 0$ we have the following:

$$\det\big(a_{j,k}^{q,0}(x,\xi)\big)_{j,k=1}^{p} \neq 0 \quad \text{for all} \quad x \in \Gamma, \quad \xi \in T_x^*\Gamma \setminus \{0\}.$$

This means that system (6.2) is Douglis-Nirenberg elliptic for every $\lambda \in \mathbb{C}$.

Therefore Theorem 4.1 holds true for elliptic system (6.2) and, according to this theorem, the bounded operator

$$A(\lambda) : \prod_{k=1}^{p} H^{s+mq+m_k,\varphi}(\Gamma) \to \prod_{j=1}^{p} H^{s+m_j,\varphi}(\Gamma) \tag{6.5}$$

is Fredholm for arbitrary $\lambda \in \mathbb{C}$, $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$. Moreover, since system (6.2) is elliptic with the parameter in the angle K, this operator possesses the following additional properties.

Theorem 6.1.

(i) *There is a number $\lambda_0 > 0$ such that for every $\lambda \in K$ satisfying the condition $|\lambda| \geq \lambda_0$ we have the topological isomorphism*

$$A(\lambda) : \prod_{k=1}^{p} H^{s+mq+m_k,\varphi}(\Gamma) \leftrightarrow \prod_{j=1}^{p} H^{s+m_j,\varphi}(\Gamma) \quad \text{for all} \quad s \in \mathbb{R}, \quad \varphi \in \mathcal{M}. \tag{6.6}$$

(ii) *For each fixed parameters $s \in \mathbb{R}$, $\varphi \in \mathcal{M}$ there is a number $c \geq 1$ such that for every $\lambda \in K$, $|\lambda| \geq \lambda_0$, and arbitrary vector-valued functions*

$$\begin{aligned} u &= \mathrm{col}\,(u_1,\dots,u_p) \in \prod_{k=1}^{p} H^{s+mq+m_k,\varphi}(\Gamma), \\ f &= \mathrm{col}\,(f_1,\dots,f_p) \in \prod_{j=1}^{p} H^{s+m_j,\varphi}(\Gamma) \end{aligned} \tag{6.7}$$

satisfying equation (6.2), *we have the two-sided estimate*

$$\begin{aligned} c^{-1} \sum_{j=1}^{p} \| f_j \|_{s+m_j,\varphi} &\leq \sum_{k=1}^{p} \| u_k \|_{s+mq+m_k,\varphi} + |\lambda|^q \sum_{k=1}^{p} \| u_k \|_{s+m_k,\varphi} \\ &\leq c \sum_{j=1}^{p} \| f_j \|_{s+m_j,\varphi}. \end{aligned} \tag{6.8}$$

Here the number c does not depend on the parameter λ and the vector-valued functions u, f.

In the case where $\varphi \equiv 1$ (the Sobolev scale) this theorem is known [4, Sec. 4.3 e]. Note that the left-hand side inequality in the two-sided estimate (6.8) holds true without the assumption about ellipticity, with a parameter, of equation (6.2) (compare with [36, Proposition 2.1]).

We will prove separately assertions (i) and (ii) of Theorem 6.1. We deduce the general case of $\varphi \in \mathcal{M}$ from the Sobolev case of $\varphi \equiv 1$.

Proof of assertion (i). Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. Since system (6.2) is Douglis-Nirenberg elliptic for every $\lambda \in \mathbb{C}$, the bounded operator (6.5) has both a finite-dimensional kernel $N(\lambda)$ and a deficiency subspace $N^{+}(\lambda)$ which are independent of s and φ. This follows from Theorem 4.1. Next we use the mentioned above fact that Theorem 6.1 is true in the case where $\varphi \equiv 1$. So there is a number $\lambda_0 > 0$ such that for every $\lambda \in K$ satisfying the condition $|\lambda| \geq \lambda_0$, we have the topological isomorphism

$$A(\lambda) : \prod_{k=1}^{p} H^{s+mq+m_k,1}(\Gamma) \leftrightarrow \prod_{j=1}^{p} H^{s+m_j,1}(\Gamma).$$

Therefore for this λ, the spaces $N(\lambda)$ and $N^{+}(\lambda)$ are trivial, i.e., the linear bounded operator (6.5) is a bijection. This yields topological isomorphism (6.6) by the Banach theorem on inverse operator. Assertion (i) is proved. □

We will prove assertion (ii) with the help of the interpolation with a function parameter. For this purpose we need the following space.

Let a function $\varphi \in \mathcal{M}$ and numbers $\sigma \in \mathbb{R}$, $\rho > 0$, $\theta > 0$ be given. We denote by $H^{\sigma,\varphi}(\Gamma,\rho,\theta)$ the space $H^{\sigma,\varphi}(\Gamma)$ endowed with the norm which depends on the number parameters ρ and θ in the following way:

$$\|\, h \,\|_{H^{\sigma,\varphi}(\Gamma,\rho,\theta)} := \Big(\|\, h \,\|^2_{\sigma,\varphi} + \rho^2 \,\|\, h \,\|^2_{\sigma-\theta,\varphi} \Big)^{1/2}.$$

This definition is correct because of the continuous embedding $H^{\sigma,\varphi}(\Gamma) \hookrightarrow H^{\sigma-\theta,\varphi}(\Gamma)$. It also follows from this embedding that the norms in the spaces $H^{\sigma,\varphi}(\Gamma,\rho,\theta)$ and $H^{\sigma,\varphi}(\Gamma)$ are equivalent. The norm in the space $H^{\sigma,\varphi}(\Gamma,\rho,\theta)$ is induced by the inner product

$$\big(\, h_1, h_2 \,\big)_{H^{\sigma,\varphi}(\Gamma,\rho,\theta)} := \big(\, h_1, h_2 \,\big)_{\sigma,\varphi} + \rho^2 \,\big(\, h_1, h_2 \,\big)_{\sigma-\theta,\varphi}.$$

Therefore this is a Hilbert space. As above in the case where $\varphi \equiv 1$, we omit the index φ in the notations. Returning to assertion (ii) of Theorem 6.1, we note that

$$\begin{aligned} \|u_k\|_{H^{s+mq+m_k,\varphi}(\Gamma,|\lambda|^q,mq)} &\leq \|u_k\|_{s+mq+m_k,\varphi} + |\lambda|^q \,\|u_k\|_{s+m_k,\varphi} \\ &\leq \sqrt{2}\,\|u_k\|_{H^{s+mq+m_k,\varphi}(\Gamma,|\lambda|^q,mq)}. \end{aligned}$$

According to Proposition 3.2, the spaces

$$\big[\, H^{\sigma-\varepsilon}(\Gamma,\rho,\theta),\, H^{\sigma+\delta}(\Gamma,\rho,\theta) \,\big]_{\psi} \quad \text{and} \quad H^{\sigma,\varphi}(\Gamma,\rho,\theta)$$

are equal with equivalence of the norms in them. Here the numbers ε and δ are positive, whereas the function parameter ψ is the same as in Proposition 3.2. We can chose the constants in the estimates of these equivalent norms such that the constants do not depend on the parameter ρ. Namely the following holds true [25, Lemma 6.1].

Proposition 6.1. *Let $\sigma \in \mathbb{R}$, $\varphi \in \mathcal{M}$ and positive numbers θ, ε, δ be given. Then there is a number $c_0 \geq 1$ such that for arbitrary $\rho > 0$ and $h \in H^{\sigma,\varphi}(\Gamma)$ we have the following two-sided estimate of the norms:*

$$c_0^{-1}\,\|\, h \,\|_{H^{\sigma,\varphi}(\Gamma,\rho,\theta)} \leq \|\, h \,\|_{[\, H^{\sigma-\varepsilon}(\Gamma,\rho,\theta),\, H^{\sigma+\delta}(\Gamma,\rho,\theta) \,]_{\psi}} \leq c_0 \,\|\, h \,\|_{H^{\sigma,\varphi}(\Gamma,\rho,\theta)}.$$

Here ψ is the interpolation parameter from Proposition 3.2, whereas the number c_0 does not depend on ρ and h.

This proposition will play a decisive role in the proof of assertion (ii) of Theorem 6.1.

Proof of assertion (ii). Let $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. We recall that Theorem 6.1 holds true in the Sobolev case where $\varphi \equiv 1$. Therefore there is a number $\lambda_0 > 0$ such that for every $\lambda \in K$ satisfying the condition $|\lambda| \geq \lambda_0$ we have the topological isomorphisms

$$(6.9) \qquad A(\lambda):\ \prod_{k=1}^{p} H^{s\mp 1+mq+m_k}(\Gamma,|\lambda|^q,mq) \leftrightarrow \prod_{j=1}^{p} H^{s\mp 1+m_j}(\Gamma).$$

Moreover, the norm of operator (6.9) and the norm of the inverse operator are uniformly bounded with respect to the parameter λ. Let ψ be the interpolation parameter from Proposition 3.2 where we set $\varepsilon = \delta = 1$. Applying the interpolation with this parameter to (6.9) we get the topological isomorphism

$$(6.10) \qquad \begin{aligned} A(\lambda):\ &\Big[\prod_{k=1}^{p} H^{s-1+mq+m_k}(\Gamma,|\lambda|^q,mq),\ \prod_{k=1}^{p} H^{s+1+mq+m_k}(\Gamma,|\lambda|^q,mq)\Big]_{\psi} \\ &\leftrightarrow \Big[\prod_{j=1}^{p} H^{s-1+m_j}(\Gamma),\ \prod_{j=1}^{p} H^{s+1+m_j}(\Gamma)\Big]_{\psi}. \end{aligned}$$

In addition, by Proposition 3.1 the norm of operator (6.10) and the norm of the operator inverse to (6.10) are uniformly bounded with respect to the parameter λ. (Note that the couples of spaces written in formula (6.10) are normal.) We obtain from this, by Proposition 3.4, the following topological isomorphism

$$(6.11) \qquad \begin{aligned} A(\lambda):\ &\prod_{k=1}^{p} \big[H^{s-1+mq+m_k}(\Gamma,|\lambda|^q,mq),\ H^{s+1+mq+m_k}(\Gamma,|\lambda|^q,mq)\big]_{\psi} \\ &\leftrightarrow \prod_{j=1}^{p} \big[\,H^{s-1+m_j}(\Gamma),\ H^{s+1+m_j}(\Gamma)\big]_{\psi}. \end{aligned}$$

Here the norms of operators (6.10) and (6.11) are equal as well as the norms of the operators inverse to them. Now we need Proposition 6.1 where we set

$$\sigma := s + mq + m_k, \quad \rho := |\lambda|^q, \quad \theta := mq, \quad \varepsilon = \delta = 1$$

and Proposition 3.2. Using them we see that operator (6.11) gives the topological isomorphism

$$(6.12) \qquad A(\lambda):\ \prod_{k=1}^{p} H^{s+mq+m_k,\varphi}(\Gamma,|\lambda|^q,mq) \leftrightarrow \prod_{j=1}^{p} H^{s+m_j,\varphi}(\Gamma)$$

such that the norm of operator (6.12) and the norm of the inverse operator are uniformly bounded with respect to the parameter λ. This means that the two-sided estimate (6.8) with the constant c is independent of the parameter λ and vector-valued functions (6.7). Assertion (ii) is proved. □

Theorem 6.1 (i) implies the following proposition on the index of the operator corresponding to an elliptic system with a parameter (compare with [36, Sec. 6.4]).

Corollary 6.1. *Let system* (6.2) *be elliptic with a parameter on a certain closed ray* $K := \{\lambda \in \mathbb{C} : \arg\lambda = \mathrm{const}\}$. *Then operator* (6.5) *has the zero index for every* $\lambda \in \mathbb{C}$.

Proof. System (6.2) is Douglis-Nirenberg elliptic for every fixed $\lambda \in \mathbb{C}$. Therefore, by Theorem 5.1, operator (6.5) has a finite index independent of $s \in \mathbb{R}$ and $\varphi \in \mathcal{M}$. Moreover, this index does not depend on the parameter λ as well. Indeed, by virtue of

(6.1), the parameter λ influences only the lowest terms of the elements of the matrix PsDO $A(\lambda)$,

$$A(\lambda)-A(0)=\sum_{r=0}^{q-1}\lambda^{q-r}A^{(r)}=\Big(\sum_{r=0}^{q-1}\lambda^{q-r}A^{(r)}_{j,k}\Big)_{j,k=1}^{p},$$

where

$$\sum_{r=0}^{q-1}\lambda^{q-r}A^{(r)}_{j,k}\ \in\ \Psi_{\mathrm{ph}}^{m(q-1)+m_k-m_j}(\Gamma).$$

Therefore, in view of Lemma 3.1, we have the bounded operator

$$A(\lambda)-A(0):\ \prod_{k=1}^{p}H^{s+mq+m_k,\varphi}(\Gamma)\ \to\ \prod_{j=1}^{p}H^{s+m+m_j,\varphi}(\Gamma).$$

However Proposition 2.1 (i) and the condition $m>0$ imply compactness of the embedding of the space $H^{s+m+m_j,\varphi}(\Gamma)$ into the space $H^{s+m_j,\varphi}(\Gamma)$. Thus the operator

$$A(\lambda)-A(0):\ \prod_{k=1}^{p}H^{s+mq+m_k,\varphi}(\Gamma)\ \to\ \prod_{j=1}^{p}H^{s+m_j,\varphi}(\Gamma)$$

is compact. This implies that the operators $A(\lambda)$ and $A(0)$ have the same index, i.e., the index does not depend on the parameter λ (see [3, Corollary 19.1.8]). Further, according to Theorem 6.1 (i) we have topological isomorphism (6.6) for every $\lambda\in K$ large enough in modulus. Hence the index of the operator $A(\lambda)$ is equal to zero for $\lambda\in K$, $|\lambda|\gg 1$, so it is for all $\lambda\in\mathbb{C}$. Assertion (ii) is proved. □

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Received 28/01/2008