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AN EXACT INNER STRUCTURE OF THE BLOCK JACOBI-TYPE UNITARY MATRICES CONNECTED WITH THE CORRESPONDING DIRECT AND INVERSE SPECTRAL PROBLEMS

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Dedicated to Myroslav Lvovych Gorbachuk on the occasion of his 70th birthday.

Abstract. We discuss a problem posed by M. J. Cantero, L. Moral, and L. Velázquez about representing an arbitrary unitary operator with a CMV-matrix. We consider this problem from the point of view of a one-to-one correspondence between a non-finite unitary operator and an infinite (five-diagonal) block three-diagonal Jacobi-type matrix in the form of the corresponding direct and inverse spectral problems for the trigonometric moment problem. Since the earlier obtained block three-diagonal Jacobi-type unitary matrix has not been fully described, we continue this investigations in the present article. In particular, we show that this exact inner structure coincides with an earlier obtained CMV-matrix.

1. Introduction

At the beginning let us make a few remarks about the previous paper [2].

Paper [2] deals with direct and inverse spectral problems for block three-diagonal Jacobi-type unitary matrices connected with trigonometric moment problem and corresponding to orthogonal polynomials on the unit circle. A solution of the inverse problem was presented using Theorem 1 [2]. Namely, a unitary operator (with a cyclic vector) defined by its spectral measure on the unit circle was represented by a block three-diagonal Jacobi-type unitary matrix, elements of which can be represented in an orthonormal basis of polynomials on the unit circle. The direct problem was presented in Theorem 2 [2]. Namely, for a given block three-diagonal Jacobi-type unitary matrix with a cyclic vector, we have recovered thew spectral measure (on the unite circle) in the sense of its Fourier coefficients that are a generalized eigenvectors of the corresponding unitary operator in its eigenfunction expansion.

The direct and inverse spectral problems (together) establish, obviously, a one-to-one correspondence between any non-finite unitary operator and an infinite (five-diagonal) block three-diagonal Jacobi-type matrices under the condition that the unitary operator has a cyclic vector.

Note that the five-diagonal matrices have appeared, in general, in [4, 5] and later were called CMV-matrices [8, 9, 10]. However, the question about the possibility to represent an arbitrary unitary operator in the CMV form was posed in [5]. A solution to a similar question was given for the first time in a talk of Yu. M. Berezansky at the International Conference on Difference Equations, Special Functions and Applications, July 25–30, 2005, Munich, Germany, and later realized in [2]. Namely, this question

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is about a one-to-one correspondence between any non-finite unitary operator and an infinite (five-diagonal) block three-diagonal Jacobi-type matrices with a cyclic vector.

But the inner structure of such a block Jacobi-type unitary matrix with special forms of its elements was not fully investigated in [2]. The form of this matrix coincides with the form formulated in [5] with a sketch of the proof. The present paper deals with a study of block Jacobi-type unitary matrix and, in fact, contains a detailed proof of the corresponding part of Theorem 3.2 in [5].

Note that a similar representation for a finite matrix of a unitary operator in a truncated CMV-matrix has been obtained in [7]. Let us also remark that an indirect answer to this problem can be found in [9, 10], where each unitary operator corresponds to Schur parameters and Schur parameters are equal to the Verblunsky coefficients related to the spectral measure of the unitary operator (Geronimus' Theorem) and finally the Verblunsky coefficients correspond to any CMV-matrix. Numerous properties of CMV-matrices are also presented in [1, 6–12] and in the cited works.

The exact inner structure of the block Jacobi-type normal matrix (given and investigated in [3]), an analogous theory of either Verblunsky coefficients or Schur parameters corresponding to the spectral measure of the normal operator connected with a given normal matrix are important open problems.

2. Preliminary results

Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$ denote the unit circle in the complex plane $\mathbb{C}$ and $d\rho(z) = d\rho(\theta)$ be a Borel probability measure on $\mathbb{T}$. Denote by $L^2 = L^2(\mathbb{T}, d\rho(\theta))$ the space of square integrable complex-valued functions defined on $\mathbb{T}$. Suppose that the support of this measure is an infinite set and, therefore, the functions $[0, 2\pi) \ni \theta \longmapsto e^{li\theta}$, $l \in \mathbb{Z}$, are linearly independent in L^2.

Consider a sequence of functions,

$$1;\quad e^{i\theta}, e^{-i\theta};\quad e^{2i\theta}, e^{-2i\theta};\quad \ldots;\quad e^{ni\theta}, e^{-ni\theta};\quad \ldots, \tag{1}$$

and start the orthogonalization via the Schmidt procedure. As a result we obtain an orthonormal system of polynomials (each one is a polynomial in $e^{i\theta}$ and $e^{-i\theta}$) which we denote as follows:

$$P_0(\theta) \equiv 1; P_{1;1}(\theta), P_{1;2}(\theta); P_{2;1}(\theta), P_{2;2}(\theta); \cdots ; P_{n;1}(\theta), P_{n;2}(\theta); \cdots . \tag{2}$$

Let us consider the Hilbert space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathbb{C}^2. \tag{3}$$

Each vector $f \in \mathbf{l}_2$ has the form $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$ and, consequently, $\forall f, g \in \mathbf{l}_2$,

$$\|f\|^2_{\mathbf{l}_2} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} < \infty, \quad (f, g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n}.$$

For $n \in \mathbb{N}$, denote by $(f_{n;1}, f_{n;2})$ the coordinates of the vector $f_n \in \mathcal{H}$ in the space $\mathbb{C}^2$ with respect to the standard orthonormal basis $\{e_{n;1}, e_{n;2}\}$ and, hence, we have $f_n = (f_{n;1}, f_{n;2})$.

Using the orthonormal system (2) one can define a mapping from $\mathbf{l}_2$ into L^2. Putting $\forall\theta \in [0, 2\pi)$, $P_n(\theta) = (P_{n;1}(\theta), P_{n;2}(\theta)) \in \mathcal{H}_n$, we have

$$\mathbf{l}_2 \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(\theta) = \sum_{n=0}^{\infty} (f_n, P_n(\theta))_{\mathcal{H}_n} \in L^2, \quad Ff = \hat{f}. \tag{4}$$

Since, for $n \in \mathbb{N}$ $(f_n, P_n(\theta))_{\mathcal{H}_n} = f_{n;1}\overline{P_{n;1}(\theta)} + f_{n;2}\overline{P_{n;2}(\theta)}$ and $\|f\|^2_{\mathbf{l}_2} = \|(f_0, f_{1;1}, f_{1;2}, f_{2,1}, \ldots)\|^2_{l_2}$, we see that (4) is a mapping from the usual space l_2 into L^2 defined by the orthonormal system (2) and, hence, this mapping is isometric. The image of $\mathbf{l}_2$ under the mapping (4) coincides with the space L^2.

Let A be a bounded linear operator defined on the space $\mathbf{l}_2$. It is possible to construct an operator-valued matrix $A = (a_{j,k})_{j,k=0}^{\infty}$ so that $\forall f, g \in \mathbf{l}_2$ we have

$$(Af)_j = \sum_{k=0}^{\infty} a_{j,k} f_k, \; j \in \mathbb{N}_0, \quad (Af, g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} (a_{j,k} f_k, g_j)_{\mathcal{H}_j}, \tag{5}$$

where, for each $j, k \in \mathbb{N}$, $a_{j,k}$ is an operator $\mathcal{H}_k \longrightarrow \mathcal{H}_j$ that has the matrix representation $(a_{j,k;\alpha,\beta})_{\alpha,\beta=1}^{2}$, so that

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{l_2}. \tag{6}$$

Let us consider the image $\hat{A} = FAF^{-1} : L^2 \longrightarrow L^2$ of the above operator A, given by the mapping (4). Its matrix is equal to the usual matrix of the operator A in the basis (2), $l_2 \longrightarrow l_2$ in the corresponding basis $(e_0,\, e_{1;1},\, e_{1;2},\, e_{2;1},\, e_{2;2},\, \ldots)$. Using (6) and the above mentioned procedure, we get the operator matrix $(a_{j,k})_{j,k=0}^{\infty}$ for $A : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$. *By the definition this matrix is also the operator matrix of* $\hat{A} : L^2 \longrightarrow L^2$. The next theorem is related, in general, to the inverse spectral problem [2].

Theorem 1. (Related to the inverse spectral problem.) *The unitary operator $\hat{A}$ of multiplication by $e^{i\theta}$ in the space L^2 in the orthonormal basis (2) of polynomials has the form of a three-diagonal block Jacobi type unitary matrix, $J = (a_{j,k})_{j,k=0}^{\infty}$, that acts on the space (3).*

The norms of all operators $a_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$ are uniformly bounded with respect to $j, k \in \mathbb{N}_0$. This matrix has the form

$$J = \begin{pmatrix} b_0 & c_0 & 0 & 0 & 0 & \ldots \\ a_0 & b_1 & c_1 & 0 & 0 & \ldots \\ 0 & a_1 & b_2 & c_2 & 0 & \ldots \\ 0 & 0 & a_2 & b_3 & c_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \begin{array}{l} c_n := a_{n,n+1}, \\ b_n := a_{n,n}, \\ a_n := a_{n+1,n}, \quad n \in \mathbb{N}_0, \end{array} \tag{7}$$

and, more precisely,

$$J = \left(\begin{array}{c|cc|cc|cc|cc|c}
*b_0 & * \;\; c_0 & + & & & & & & & \ldots \\ \hline
+ & * & * & 0 & 0 & & & & & \\
a_0 & b_1 & & c_1 & & & & 0 & & \ldots \\
0 & * & * & * & + & & & & & \\ \hline
 & + & * & * & * & 0 & 0 & & & \\
 & a_1 & & b_2 & & c_2 & & & & \ldots \\
 & 0 & 0 & * & * & * & + & & & \\ \hline
 & & & + & * & * & * & 0 & 0 & \\
 & 0 & & a_2 & & b_3 & & c_3 & & \ldots \\
 & & & 0 & 0 & * & * & * & + & \\ \hline
\vdots & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}\right). \tag{8}$$

In (8), $b_0 = b_{0;2,2}$ is a (1×1)-matrix (i.e., a scalar matrix), a_0 is a (2×1)-matrix: $a_0 = (a_{0;\alpha,2})_{\alpha=1}^{2}$, c_0 is a (1×2)-matrix: $c_0 = (c_{0;2,\beta})_{\beta=1}^{2}$; for $j \in \mathbb{N}$, the elements $a_j = (a_{j;\alpha,\beta})_{\alpha,\beta=1}^{2}$, $b_j = (b_{j;\alpha,\beta})_{\alpha,\beta=1}^{2}$, $c_j = (c_{j;\alpha,\beta})_{\alpha,\beta=1}^{2}$ are (2×2)-matrices. In these matrices a_j, b_j and c_j, some elements are zero and others are positive, i.e.

$$\begin{array}{lll} a_{0;1,2} > 0, \quad a_{0;2,2} = 0; & & c_{0;2,2} > 0; \\ a_{n;2,1} = a_{n;2,2} = 0, \quad a_{n;1,1} > 0; & & c_{n;1,1} = c_{n;1,2} = 0, \quad c_{n;2,2} > 0; \quad n \in \mathbb{N}. \end{array} \tag{9}$$

The matrix (8), in the scalar form, is five-diagonal of the indicated structure.

The adjoint operator $(\hat{A})^*$ has the form of an analogous three-diagonal block Jacobi-type matrix J^+ in basis (2).

These matrices J, J^+ act as follows: $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$,

$$(10) \qquad \begin{aligned} (Jf)_n &= a_{n-1} f_{n-1} + b_n f_n + c_n f_{n+1}, \\ (J^+ f)_n &= c_{n-1}^* f_{n-1} + b_n^* f_n + a_n^* f_{n+1}, \quad n \in \mathbb{N}_0, \quad f_{-1} = 0 \end{aligned}$$

(here * denotes the usual adjoint of a matrix).

The next theorem contains a solution of the direct spectral problem corresponding to the inverse one mentioned in Theorem 1 in this article and in [2]. (This theorem combines Theorem 2 and Lemma 5 from [2].

Theorem 2. (The direct spectral problem.) *Consider the space (3) and the linear operator A which is defined on finite vectors $\mathbf{l}_{\mathrm{fin}}$ by a block three-diagonal Jacobi-type matrix J of the form (8) in terms of the first expression in (10). We suppose that all its coefficients a_n, b_n and c_n are uniformly bounded, some elements of these matrices are equal to zero or positive according to (9) and the extension of A by continuity is a unitary operator on this space.*

Let $\varphi(z) = (\varphi_n(z))_{n=0}^{\infty}$, $\varphi_n(z) \in \mathcal{H}_n$, $z \in \mathbb{C}$, be a fixed solution from $(\mathbf{l}_{\mathrm{fin}})'$ of the following system with the initial condition $\varphi_0(z) = \varphi_0 \in \mathbb{C}$,

$$(11) \qquad \begin{gathered} (J\varphi(z))_n = a_{n-1}\varphi_{n-1}(z) + b_n \varphi_n(z) + c_n \varphi_{n+1}(z) = z\varphi_n(z), \\ (J^+\varphi(z))_n = c_{n-1}^*\varphi_{n-1}(z) + b_n^* \varphi_n(z) + a_n^* \varphi_{n+1}(z) = \bar{z}\varphi_n(z), \\ n \in \mathbb{N}_0, \quad \varphi_{-1}(z) = 0. \end{gathered}$$

Then this solution exist $\forall \varphi_0$ and has the form: $\forall n \in \mathbb{N}$

$$(12) \qquad \varphi_n(z) = Q_n(z)\varphi_0 = (Q_{n;1}, Q_{n;2})\varphi_0,$$

where $Q_{n;1}$ and $Q_{n;2}$ are polynomials in z and $\bar{z}$ and these polynomials have the form

$$(13) \qquad Q_{n;1}(z) = l_{n;1}\bar{z}^n + q_{n;1}(z), \quad Q_{n;2}(z) = l_{n;2} z^n + q_{n;2}(z).$$

Here $l_{n;1} > 0$, $l_{n;2} > 0$ and $q_{n;1}(z)$, $q_{n;2}(z)$ are linear combinations of $z^j \bar{z}^k$ for $0 \le j+k \le n-1$; $Q_0(z) = 1$, $z \in \mathbb{C}$.

Then the eigenfunction expansion of the operator A has the following description. Namely, the Fourier transform has the form

$$(14) \qquad \begin{aligned} \mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(z) &= \sum_{n=0}^{\infty} Q_n^*(z) f_n \\ &= f_0 + \sum_{n=1}^{\infty} (\overline{Q_{n;1}(z)} f_{n;1} + \overline{Q_{n;2}(z)} f_{n;2}) \in L^2(\mathbb{T}, d\rho(z)). \end{aligned}$$

Here $Q_n^*(z) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ is the adjoint to the operator $Q_n(z) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$, $d\rho(z)$ is the probability spectral measure of A.

The Parseval equality has the following form: $\forall f, g \in \mathbf{l}_{\mathrm{fin}}$

$$(15) \qquad (f,g)_{\mathbf{l}_2} = \int_{\mathbb{T}} \hat{f}(z)\overline{\hat{g}(z)}\, d\rho(z), \quad (Jf,g)_{\mathbf{l}_2} = \int_{\mathbb{T}} z\hat{f}(z)\overline{\hat{g}(z)}\, d\rho(z).$$

Identities (14) and (15) are extended by continuity to $\forall f, g \in \mathbf{l}_2$ making the operator (14) unitary, which maps $\mathbf{l}_2$ onto the whole $L^2(\mathbb{T}, d\rho(z))$.

The polynomials $\overline{Q_{n;\alpha}(z)}$*,* $n \in \mathbb{N}$*,* $\alpha = 1, 2$*, and* $Q_{0,\alpha}(z) = 1$*, form an orthonormal system in* $L^2(\mathbb{T}, d\rho(z))$*. The orthogonality in the sense*

$$\int_{\mathbb{T}} \overline{Q^*_{k;\beta}(z)} Q_{j;\alpha} d\rho(z) = \delta_{j,k}\delta_{\alpha,\beta} \quad (Q_{0;\alpha} = Q_0(z)).$$

This system is total in $L^2(\mathbb{T}, d\rho(z))$*.*

3. The inner structure of the unitary matrix (8)

We will find a condition that would guarantee the unitary for the matrix J of the type (8). The formal adjoint matrix J^+ has the form

$$J^+ = \begin{pmatrix} b_0^* & a_0^* & 0 & 0 & \cdots \\ c_0^* & b_1^* & a_1^* & 0 & \cdots \\ 0 & c_1^* & b_2^* & a_2^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \begin{array}{lll} c_n^* & : & \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ b_n^* & : & \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ a_n^* & : & \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{16}$$

Multiplying matrices (7) and (16) we get

$$JJ^+ = \begin{pmatrix} b_0b_0^* + c_0c_0^* & b_0a_0^* + c_0b_1^* & c_0a_1^* & 0 & 0 & \cdots \\ a_0b_0^* + b_1c_0^* & a_0a_0^* + b_1b_1^* + c_1c_1^* & b_1a_1^* + c_1b_2^* & c_1a_2^* & 0 & \cdots \\ a_1c_0^* & a_1b_1^* + b_2c_1^* & a_1a_1^* + b_2b_2^* + c_2c_2^* & b_2a_2^* + c_2b_3^* & c_2a_3^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{17}$$

The expression for J^+J is analogous to (17) if a_n, b_n and c_n are replaced with c_n^*, b_n^* and a_n^*, respectively, and vice versa.

Comparing these expressions for JJ^+ and J^+J with the identity I we conclude that the equality $JJ^+ = J^+J = I$ is equivalent to fulfillment of the following equalities:

$$\begin{array}{l} b_0b_0^* + c_0c_0^* = b_0^*b_0 + a_0^*a_0 = 1, \\ c_na_{n+1}^* = a_n^*c_{n+1} = 0, \\ b_na_n^* + c_nb_{n+1}^* = b_n^*c_n + a_n^*b_{n+1} = 0, \\ a_na_n^* + b_{n+1}b_{n+1}^* + c_{n+1}c_{n+1}^* = c_n^*c_n + b_{n+1}^*b_{n+1} + a_{n+1}^*a_{n+1} = I, \quad n \in \mathbb{N}_0, \end{array} \tag{18}$$

where I is the identity operator in the corresponding space. In [3] we obtained equalities of the type (18) in any general case, where J is a bounded normal operator i.e. satisfies the relation $JJ^+ = J^+J$.

Note that the necessary equalities

$$a_nb_n^* + b_{n+1}c_n^* = c_n^*b_n + b_{n+1}^*a_n = 0, \quad a_{n+1}c_n^* = c_{n+1}^*a_n = 0, \quad n \in \mathbb{N}_0,$$

follow from the third and the second equalities in (18) by taking $*$ the adjoints.

Taking the initial matrices a_0, b_0, c_0 and finding from (18) step by step a_1, b_1, c_1; a_2, b_2, c_2; ... etc. (in a non-unique manner) we can construct some unitary matrix J. But for such a matrix, Theorem 2 in general is not valid, because it is necessary to find these matrices in such way that a_n and c_n must be of the form (8). Only in this case the functions (1) are linearly independent and Theorem 2 is applicable.

For an analysis we write (18) in the matrix form

$$b_0b_0^* + (c_{0;2,1}\ c_{0;2,2}) \begin{pmatrix} \bar{c}_{0;2,1} \\ \bar{c}_{0;2,2} \end{pmatrix} = b_0^*b_0 + (a_{0;1,1}\ 0) \begin{pmatrix} \bar{a}_{0;1,1} \\ 0 \end{pmatrix}, \tag{19}$$

and for $n \in \mathbb{N}$

$$\begin{pmatrix} 0 & 0 \\ c_{n;2,1} & c_{n;2,2} \end{pmatrix} \begin{pmatrix} \bar{a}_{n+1;1,1} & 0 \\ \bar{a}_{n+1;2,1} & 0 \end{pmatrix} = \begin{pmatrix} \bar{a}_{n;1,1} & 0 \\ \bar{a}_{n;2,1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c_{n+1;2,1} & c_{n+1;2,2} \end{pmatrix}, \tag{20}$$

$$(21)\quad \begin{pmatrix} b_{n;1,1} & b_{n;1,2} \\ b_{n;2,1} & b_{n;2,2} \end{pmatrix} \begin{pmatrix} \bar{a}_{n;1,1} & 0 \\ \bar{a}_{n;1,2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_{n;2,1} & c_{n;2,2} \end{pmatrix} \begin{pmatrix} \bar{b}_{n+1;1,1} & \bar{b}_{n+1;2,1} \\ \bar{b}_{n+1;1,2} & \bar{b}_{n+1;2,2} \end{pmatrix} = 0,$$

$$(22)\quad \begin{pmatrix} \bar{b}_{n;1,1} & \bar{b}_{n;2,1} \\ \bar{b}_{n;1,2} & \bar{b}_{n;2,2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c_{n;2,1} & c_{n;2,2} \end{pmatrix} + \begin{pmatrix} \bar{a}_{n;1,1} & 0 \\ \bar{a}_{n;1,2} & 0 \end{pmatrix} \begin{pmatrix} b_{n+1;1,1} & b_{n+1;1,2} \\ b_{n+1;2,1} & b_{n+1;2,2} \end{pmatrix} = 0,$$

(23)

$$\begin{pmatrix} a_{n;1,1} & a_{n;1,2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_{n;1,1} & 0 \\ \bar{a}_{n;1,2} & 0 \end{pmatrix} + \begin{pmatrix} b_{n+1;1,1} & b_{n+1;1,2} \\ b_{n+1;2,1} & b_{n+1;2,2} \end{pmatrix} \begin{pmatrix} \bar{b}_{n+1;1,1} & \bar{b}_{n+1;2,1} \\ \bar{b}_{n+1;1,2} & \bar{b}_{n+1;2,2} \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 \\ c_{n+1;2,1} & c_{n+1;2,2} \end{pmatrix} \begin{pmatrix} 0 & \bar{c}_{n+1;2,1} \\ 0 & \bar{c}_{n+1;2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(24)

$$\begin{pmatrix} 0 & \bar{c}_{n;2,1} \\ 0 & \bar{c}_{n;2,2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c_{n;2,1} & c_{n;2,2} \end{pmatrix} + \begin{pmatrix} \bar{b}_{n+1;1,1} & \bar{b}_{n+1;2,1} \\ \bar{b}_{n+1;1,2} & \bar{b}_{n+1;2,2} \end{pmatrix} \begin{pmatrix} b_{n+1;1,1} & b_{n+1;1,2} \\ b_{n+1;2,1} & b_{n+1;2,2} \end{pmatrix}$$
$$+ \begin{pmatrix} \bar{a}_{n+1;1,1} & 0 \\ \bar{a}_{n+1;1,2} & 0 \end{pmatrix} \begin{pmatrix} a_{n+1;1,1} & a_{n+1;1,2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we suppose for a convenience that b_0, a_0 and c_0 are of the form

$$(25)\qquad b_0 = \begin{pmatrix} 0 & 0 \\ 0 & b_{0;2,2} \end{pmatrix}, \quad a_0 = \begin{pmatrix} 0 & a_{0;1,2} \\ 0 & 0 \end{pmatrix}, \quad c_0 = \begin{pmatrix} 0 & 0 \\ c_{0;2,1} & c_{0;2,2} \end{pmatrix},$$

than we can write equalities (20–24) for $n \in \mathbb{N}_0$.

From (19) we obtain

$$(26)\qquad |c_{0;2,1}|^2 + |c_{0;2,2}|^2 + |b_{0;2,2}|^2 = |a_{0;1,1}|^2 + |b_{0;2,2}|^2 = 1.$$

Taking into account (25) for equalities (20–24) we recover from (20):

$$(27)\qquad c_{n;2,1}\bar{a}_{n+1;1,1} + c_{n;2,2}\bar{a}_{n+1;1,2} = 0;$$

from (21):

$$(28)\qquad \begin{aligned} b_{n;1,1}\bar{a}_{n;1,1} + b_{n;1,2}\bar{a}_{n;1,2} &= 0, \\ b_{n;2,1}\bar{a}_{n;1,1} + b_{n;2,2}\bar{a}_{n;1,2} + c_{n;2,1}\bar{b}_{n+1;1,1} + c_{n;2,2}\bar{b}_{n+1;1,2} &= 0, \\ c_{n;2,1}\bar{b}_{n+1;2,1} + c_{n;2,2}\bar{b}_{n+1;2,2} &= 0; \end{aligned}$$

from (22):

$$(29)\qquad \begin{aligned} \bar{b}_{n;2,1}c_{n;2,1} + \bar{a}_{n;1,1}b_{n+1;1,1} &= 0, \\ \bar{b}_{n;2,1}c_{n;2,2} + \bar{a}_{n;1,1}b_{n+1;1,2} &= 0, \\ \bar{b}_{n;2,2}c_{n;2,1} + \bar{a}_{n;1,2}b_{n+1;1,1} &= 0, \\ \bar{b}_{n;2,2}c_{n;2,2} + \bar{a}_{n;1,2}b_{n+1;1,2} &= 0; \end{aligned}$$

from (23):

$$(30)\qquad \begin{aligned} |a_{n;1,1}|^2 + |a_{n;1,2}|^2 + |b_{n+1;1,1}|^2 + |b_{n+1;1,2}|^2 &= 1, \\ b_{n+1;1,1}\bar{b}_{n+1;2,1} + b_{n+1;1,2}\bar{b}_{n+1;2,2} &= 0, \\ b_{n+1;2,1}\bar{b}_{n+1;1,1} + b_{n+1;2,2}\bar{b}_{n+1;1,2} &= 0, \\ |b_{n+1;2,1}|^2 + |b_{n+1;2,2}|^2 + |c_{n+1;2,1}|^2 + |c_{n+1;2,2}|^2 &= 1; \end{aligned}$$

from (24):

$$
\begin{aligned}
|c_{n;2,1}|^2 + |b_{n+1;1,1}|^2 + |b_{n+1;2,1}|^2 + |a_{n+1;1,1}|^2 &= 1,\\
\bar{c}_{n;2,1}c_{n;2,2} + \bar{b}_{n+1;1,1}b_{n+1;1,2} + \bar{b}_{n+1;2,1}b_{n+1;2,2} + \bar{a}_{n+1;1,1}a_{n+1;1,2} &= 0,\\
\bar{c}_{n;2,2}c_{n;2,1} + \bar{b}_{n+1;1,2}b_{n+1;1,1} + \bar{b}_{n+1;2,2}b_{n+1;2,1} + \bar{a}_{n+1;1,2}a_{n+1;1,1} &= 0,\\
|c_{n;2,2}|^2 + |b_{n+1;1,2}|^2 + |b_{n+1;2,2}|^2 + |a_{n+1;1,2}|^2 &= 1.
\end{aligned} \tag{31}
$$

We remark that the second equalities of (30) and (31) are adjoint to third equalities correspondingly.

Denote h_k and v_k, $k = 0, 1, 2, \ldots$, rows and columns of the matrix (8). Each h_k and v_k we understand as a finite vector in l_2. Such vector contains no more than fore nonzero elements. Put $(h_i, \bar{h}_j) = (h_i, \bar{h}_j)_{l_2}$, $(v_i, \bar{v}_j) = (v_i, \bar{v}_j)_{l_2}$ as usual scalar product in l_2, ($\bar{h}_j$, ($\bar{v}_j$) is a vector with adjoint elements).

Lemma 1. *The equalities (26–31) are equivalent the following equalities:*

$$(v_m, \bar{v}_n) = 0, \quad m \neq n, \quad (v_m, \bar{v}_m) = \|v_m\|^2 = 1,$$

$$(h_m, \bar{h}_n) = 0, \quad m \neq n, \quad (h_m, \bar{h}_m) = \|h_m\|^2 = 1, \quad m, n \in \mathbb{N}_0.$$

Proof. Proof follows immediately from the direct observation of (26–31) using (8). Namely, the equalities (26–31) are equivalent to the following (32–37) correspondingly:

$$\|v_0\| = \|h_0\| = 1; \tag{32}$$

$$(h_{2n+1}, \bar{h}_{2(n+2)}) = 0; \tag{33}$$

$$
\begin{aligned}
(h_{2n}, \bar{h}_{2(n+1)}) &= 0,\\
(h_{2n+1}, \bar{h}_{2(n+1)}) &= 0,\\
(h_{2n+1}, \bar{h}_{2(n+1)+1}) &= 0;
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
(\bar{v}_{2n}, v_{2(n+1)}) &= 0,\\
(\bar{v}_{2n}, v_{2(n+1)+1}) &= 0,\\
(\bar{v}_{2n+1}, v_{2(n+1)}) &= 0,\\
(\bar{v}_{2n+1}, v_{2(n+1)+1}) &= 0;
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\|h_{2(n+1)}\| &= 1,\\
(h_{2(n+1)}, \bar{h}_{2(n+1)+1}) &= 0,\\
(\bar{h}_{2(n+1)+1}, h_{2(n+1)}) &= 0,\\
\|h_{2(n+1)+1}\| &= 1;
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\|v_{2(n+1)}\| &= 1,\\
(\bar{v}_{2(n+1)}, v_{2(n+1)+1}) &= 0,\\
(v_{2(n+1)+1}, \bar{v}_{2(n+1)}) &= 0,\\
\|v_{2(n+1)+1}\| &= 1.
\end{aligned} \tag{37}
$$

□

Now, using Lemma 1 we will obtain the inner structure of J. In the beginning we put $b_{0;2,2} := \bar{\alpha}_0 \in \mathbb{C}$. Since $(v_0, \bar{v}_0) = \|v_0\|^2 = 1$ i.e. (32), we conclude that $|\alpha_0| \leq 1$ and $a_{0;1,2} := \rho_0 = \sqrt{1 - |\alpha|^2}$. In the "zero" step we have

$$
J = \left(\begin{array}{c|ccc|ccccc}
\bar{\alpha}_0 & * & c_0 & + & & & & & \cdots \\
\hline
\rho_0 & * & & * & 0 & 0 & & & \\
 & & b_1 & & & c_1 & & 0 & \cdots \\
0 & * & & * & * & + & & & \\
\hline
\vdots & & \vdots & & & \vdots & \vdots & \vdots & \ddots
\end{array} \right). \tag{38}
$$

Then we use $(v_0, \bar{v}_1) = 0$ and $(v_0, \bar{v}_2) = 0$, Lemma 1, i.e. the first and the second equalities in (35) with $n = 0$. Put

$$
c_{0;2,1} := \bar{\alpha}_1 \rho_0, \quad c_{0;2,2} := \rho_1 \rho_0, \quad b_{1;1,1} := -\bar{\alpha}_1 \alpha_0, \quad b_{1;1,2} := -\rho_1 \alpha_0,
$$

where $\alpha_1 \in \mathbb{C}$ and $\rho_1 \in \mathbb{R}$ are some proportion coefficients. Since $(h_0, \bar{h}_0) = 1$, i.e. (32),

$$
|\alpha_0|^2 + |\bar{\alpha}_1 \rho_0|^2 + |\rho_1 \rho_0|^2 = 1,
$$

and taking into account $|\alpha_0|^2 + |\rho_0|^2 = 1$, we recover $|\alpha_1|^2 + |\rho_1|^2 = 1$.

After the "zero" and "first" steps we have

$$
J = \left(\begin{array}{c|cc|ccccc}
\bar{\alpha}_0 & \bar{\alpha}_1 \rho_0 & \rho_1 \rho_0 & & & & & \cdots \\
\hline
\rho_0 & -\bar{\alpha}_1 \alpha_0 & -\rho_1 \alpha_0 & 0 & 0 & & & \\
 & & & & c_1 & & 0 & \cdots \\
0 & * & * & * & + & & & \\
\hline
\vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots
\end{array} \right). \tag{39}
$$

By the way, we obtain $(h_1, \bar{h}_1) = 1$ and $(h_0, \bar{h}_1) = 0$, Lemma 1, i.e

$$
|\rho_0|^2 + |\bar{\alpha}_1 \alpha_0|^2 + |\rho_1 \alpha_0|^2 = 1,
$$

and

$$
(\bar{\alpha}_0)(\rho_0) + (\bar{\alpha}_1 \rho_0)(-\alpha_1 \rho_0) + (\rho_1 \rho_0)(-\rho_1 \bar{\alpha}_0) = 0.
$$

In the "second" step we use $(h_0, \bar{h}_2) = 0$, Lemma 1, i.e. the third equality (34) for $n = 0$ and $(h_0, \bar{h}_3) = 0$, Lemma 1, i.e. (33) for $n = 0$. (Obviously, we can use $(h_1, \bar{h}_2) = 0$ and $(h_1, \bar{h}_3) = 0$, i.e. the second equality in (36) and the first equality in (34)). Put

$$
b_{1;2,1} := \bar{\alpha}_2 \rho_1, \quad b_{1;2,2} := -\bar{\alpha}_2 \alpha_1, \quad a_{1;1,1} := \rho_2 \rho_1, \quad a_{1;1,2} := -\rho_2 \alpha_0,
$$

where $\alpha_2 \in \mathbb{C}$ and $\rho_2 \in \mathbb{R}$ some new proportion coefficient.

Using $(v_1, \bar{v}_1) = 1$, Lemma 1, and taking into account $|\alpha_0|^2 + |\rho_0|^2 = 1$, $|\alpha_1|^2 + |\rho_1|^2 = 1$, we conclude that $|\alpha_2|^2 + |\rho_2|^2 = 1$.

By the way, we obtain $(v_1, \bar{v}_2) = 0$, Lemma 1 i.e. the third equality in (37).

Hence after the "zero" (38), the "first" (39) and the "second" steps we have

$$
J = \left(\begin{array}{c|cc|cc|cccc}
\bar{\alpha}_0 & \bar{\alpha}_1 \rho_0 & \rho_1 \rho_0 & & & & & & \cdots \\
\hline
\rho_0 & -\bar{\alpha}_1 \alpha_0 & -\rho_1 \alpha_0 & 0 & 0 & & & & \\
 & & & & c_1 & & & 0 & \cdots \\
0 & \bar{\alpha}_2 \rho_1 & -\bar{\alpha}_2 \alpha_1 & * & + & & & & \\
\hline
 & \rho_2 \rho_1 & -\rho_2 \alpha_1 & * & * & 0 & 0 & & \\
 & & & & b_2 & & c_2 & & \cdots \\
 & 0 & 0 & * & * & * & + & & \\
\hline
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \ddots
\end{array} \right).
$$

Now we can prove obviously by induction the following proposition, i.e. the part of Theorem 3.2 [5].

Theorem 3. *The matrix J in (8) has a form*

(40)

$$J = \left(\begin{array}{c|cc|cc|cc|cc|c}
\bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & & & & & & & \dots \\
\rho_0 & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & & & 0 & & \dots \\
0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & & & & & \dots \\
 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & 0 & 0 & & & \dots \\
 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \bar{\alpha}_5\rho_4 & \rho_5\rho_4 & & & \dots \\
 & & & \rho_4\rho_3 & -\rho_4\alpha_3 & -\bar{\alpha}_5\alpha_4 & -\rho_5\alpha_4 & 0 & 0 & \dots \\
 & 0 & & 0 & 0 & \bar{\alpha}_6\rho_5 & -\bar{\alpha}_6\alpha_5 & \bar{\alpha}_7\rho_6 & \rho_7\rho_6 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right),$$

where $\alpha_k \in \mathbb{C}$, $k \in \mathbb{N}_0$ any coefficients with a condition $|\alpha_k| < 1$, and $\rho_k = \sqrt{1 - |\alpha_k|^2}$. These coefficients are named Verblunsky coefficients of the measure corresponding to the matrix J.

Conversely, each matrix of the form (40) is a unitary operator in l_2 as five-diagonal one and in $\mathbf{l}_2$ also as block tree-diagonal one.

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