

## A STOCHASTIC INTEGRAL OF OPERATOR-VALUED FUNCTIONS

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*To Professor M. L. Gorbachuk on the occasion of his 70th birthday.*

ABSTRACT. In this note we define and study a Hilbert space-valued stochastic integral of operator-valued functions with respect to Hilbert space-valued measures. We show that this integral generalizes the classical Itô stochastic integral of adapted processes with respect to normal martingales and the Itô integral in a Fock space.

### 1. INTRODUCTION

Here and subsequently, we fix a real number  $T > 0$ . Let  $\mathcal{H}$  be a complex Hilbert space,  $M$  be a fixed vector from  $\mathcal{H}$  and  $[0, T] \ni t \mapsto E_t$  be a resolution of identity in  $\mathcal{H}$ . Consider the  $\mathcal{H}$ -valued function (*abstract martingale*)

$$[0, T] \ni t \mapsto M_t := E_t M \in \mathcal{H}.$$

In this paper we construct and study an integral

$$(1) \quad \int_{[0, T]} A(t) dM_t$$

for a certain class of operator-valued functions  $[0, T] \ni t \mapsto A(t)$  whose values are linear operators in the space  $\mathcal{H}$ . We define such an integral as an element of the Hilbert space  $\mathcal{H}$  and call it a *Hilbert space-valued stochastic integral* (or *H-stochastic integral*). By analogy with the classical integration theory we first define integral (1) for a certain class of simple operator-valued functions and then extend this definition to a wider class.

We illustrate our abstract constructions with a few examples. Thus, we show that the classical Itô stochastic integral is a particular case of the *H-stochastic integral*. Namely, let  $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$  be a space of square integrable functions on a complete probability space  $(\Omega, \mathcal{A}, P)$ ,  $\{\mathcal{A}_t\}_{t \in [0, T]}$  be a filtration satisfying the usual conditions and  $\{N_t\}_{t \in [0, T]}$  be a normal martingale on  $(\Omega, \mathcal{A}, P)$  with respect to  $\{\mathcal{A}_t\}_{t \in [0, T]}$ , i.e.,

$$\{N_t\}_{t \in [0, T]} \quad \text{and} \quad \{N_t^2 - t\}_{t \in [0, T]}$$

are martingales for  $\{\mathcal{A}_t\}_{t \in [0, T]}$ . It follows from the properties of martingales that

$$N_t = \mathbb{E}[N_T | \mathcal{A}_t], \quad t \in [0, T],$$

where  $\mathbb{E}[\cdot | \mathcal{A}_t]$  is a conditional expectation with respect to the  $\sigma$ -algebra  $\mathcal{A}_t$ . It is well known that  $\mathbb{E}[\cdot | \mathcal{A}_t]$  is the orthogonal projector in the space  $L^2(\Omega, \mathcal{A}, P)$  onto its subspace  $L^2(\Omega, \mathcal{A}_t, P)$  and, moreover, the corresponding projector-valued function  $\mathbb{R}_+ \ni t \mapsto E_t := \mathbb{E}[\cdot | \mathcal{A}_t]$  is a resolution of identity in  $L^2(\Omega, \mathcal{A}, P)$ , see e.g. [13, 3, 4, 12, 7]. In this way the normal martingale  $\{N_t\}_{t \in [0, T]}$  can be interpreted as an abstract martingale, i.e.,

$$[0, T] \ni t \mapsto N_t = \mathbb{E}[N_T | \mathcal{A}_t] = E_t N_T \in \mathcal{H}.$$

Hence, in the space  $L^2(\Omega, \mathcal{A}, P)$  we can construct the *H-stochastic integral* with respect to the normal martingale  $N_t$ . Let  $F \in L^2([0, T] \times \Omega, dt \times P)$  be a square integrable

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stochastic process adapted to the filtration  $\{\mathcal{A}_t\}_{t \in [0, T]}$ . We consider the operator-valued function  $[0, T] \ni t \mapsto A_F(t)$  whose values are operators  $A_F(t)$  of multiplication by the function  $F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P)$  in the space  $L^2(\Omega, \mathcal{A}, P)$ ,

$$L^2(\Omega, \mathcal{A}, P) \supset \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P).$$

In this paper we prove that the  $H$ -stochastic integral of  $[0, T] \ni t \mapsto A_F(t)$  coincides with the classical Itô stochastic integral  $\int_{[0, T]} F(t) dN_t$  of  $F$ . That is,

$$\int_{[0, T]} A_F(t) dN_t = \int_{[0, T]} F(t) dN_t.$$

In the last part of this note we show that the Itô integral in a Fock space is the  $H$ -stochastic integral and establish a connection of such an integral with the classical Itô stochastic integral. The corresponding results are given without proofs (the proofs will be given in a forthcoming publication). Note that the Itô integral in a Fock space is a useful tool in the quantum stochastic calculus, see e.g. [2] for more details.

We remark that in [3, 4] the authors gave a definition of the operator-valued stochastic integral

$$B := \int_{[0, T]} A(t) dE_t$$

for a family  $\{A(t)\}_{t \in [0, T]}$  of commuting normal operators in  $\mathcal{H}$ . Such an integral was defined using a spectral theory of commuting normal operators. It is clear that for a fixed vector  $M \in \text{Dom}(B) \subset \mathcal{H}$  the formula

$$\int_{[0, T]} A(t) dM_t := \left( \int_{[0, T]} A(t) dE_t \right) M$$

can be regarded as a definition of integral (1). In this way we obtain another definition of integral (1) different from the one we have proposed in this paper.

## 2. THE CONSTRUCTION OF THE $H$ -STOCHASTIC INTEGRAL

Let  $\mathcal{H}$  be a complex Hilbert space,  $\mathcal{L}(\mathcal{H})$  be a space of all bounded linear operators in  $\mathcal{H}$ ,  $M \neq 0$  be a fixed vector from  $\mathcal{H}$  and

$$[0, T] \ni t \mapsto E_t \in \mathcal{L}(\mathcal{H})$$

be a resolution of identity in  $\mathcal{H}$ , that is a right-continuous increasing family of orthogonal projections in  $\mathcal{H}$  such that  $E_T = 1$ . Note that the resolution of identity  $E$  can be regarded as a projector-valued measure  $\mathcal{B}([0, T]) \ni \alpha \mapsto E(\alpha) \in \mathcal{L}(\mathcal{H})$  on the Borel  $\sigma$ -algebra  $\mathcal{B}([0, T])$ . Namely, for any interval  $(s, t] \subset [0, T]$  we set

$$E((s, t]) := E_t - E_s, \quad E(\{0\}) := E_0, \quad E(\emptyset) := 0,$$

and extend this definition to all Borel subsets of  $[0, T]$ , see e.g. [6] for more details.

By definition, the  $\mathcal{H}$ -valued function

$$[0, T] \ni t \mapsto M_t := E_t M \in \mathcal{H}$$

is an *abstract martingale* in the Hilbert space  $\mathcal{H}$ .

In this section we give a definition of integral (1) for a certain class of operator-valued functions with respect to the abstract martingale  $M_t$ . A construction of such an integral is given step-by-step, beginning with the simplest class of operator-valued functions. Let us introduce the required class of simple functions.

For each point  $t \in [0, T]$ , we denote by

$$\mathcal{H}_M(t) := \text{span}\{M_{s_2} - M_{s_1} \mid (s_1, s_2] \subset (t, T]\} \subset \mathcal{H}$$

the linear span of the set  $\{M_{s_2} - M_{s_1} \mid (s_1, s_2] \subset (t, T]\}$  in  $\mathcal{H}$  and by

$$\mathcal{L}_M(t) = \mathcal{L}(\mathcal{H}_M(t) \rightarrow \mathcal{H})$$

the set of all linear operators in  $\mathcal{H}$  that continuously act from  $\mathcal{H}_M(t)$  to  $\mathcal{H}$ . The increasing family  $\mathcal{L}_M = \{\mathcal{L}_M(t)\}_{t \in [0, T]}$  will play here a role of the filtration  $\{\mathcal{A}_t\}_{t \in [0, T]}$  in the classical martingale theory.

For a fixed  $t \in [0, T)$ , a linear operator  $A$  in  $\mathcal{H}$  will be called  $\mathcal{L}_M(t)$ -measurable if

- (i)  $A \in \mathcal{L}_M(t)$  and, for all  $s \in [t, T)$ ,

$$\|A\|_{\mathcal{L}_M(t)} = \|A\|_{\mathcal{L}_M(s)} := \sup \left\{ \frac{\|Ag\|_{\mathcal{H}}}{\|g\|_{\mathcal{H}}} \mid g \in \mathcal{H}_M(s), g \neq 0 \right\}.$$

- (ii)  $A$  is partially commuting with the resolution of identity  $E$ . More precisely,

$$AE_s g = E_s A g, \quad g \in \mathcal{H}_M(t), \quad s \in [t, T].$$

Such a definition of  $\mathcal{L}_M(t)$ -measurability is motivated by a number of reasons:

- $\mathcal{L}_M(t)$ -measurability is a natural generalization of the usual  $\mathcal{A}_t$ -measurability in classical stochastic calculus, see Lemma 1 (Section 3) for more details;
- in some sense,  $\mathcal{L}_M(t)$ -measurability (for each  $t$ ) is the minimal restriction on the behavior of a simple operator-valued function  $[0, T] \ni t \mapsto A(t)$  that will allow us to obtain an analogue of the Itô isometry property (see inequality (4) below) and to extend the  $H$ -stochastic integral from a simple class of functions to a wider one.

In what follows, it is convenient for us to call  $\mathcal{L}_M(T)$ -measurable all linear operators in  $\mathcal{H}$ . Evidently, if a linear operator  $A$  in  $\mathcal{H}$  is  $\mathcal{L}_M(t)$ -measurable for some  $t \in [0, T]$  then  $A$  is  $\mathcal{L}_M(s)$ -measurable for all  $s \in [t, T]$ .

A family  $\{A(t)\}_{t \in [0, T]}$  of linear operators in  $\mathcal{H}$  will be called a *simple  $\mathcal{L}_M$ -adapted operator-valued function on  $[0, T]$*  if, for each  $t \in [0, T]$ , the operator  $A(t)$  is  $\mathcal{L}_M(t)$ -measurable and there exists a partition  $0 = t_0 < t_1 < \dots < t_n = T$  of  $[0, T]$  such that

$$(2) \quad A(t) = \sum_{k=0}^{n-1} A_k \varkappa_{(t_k, t_{k+1}]}(t), \quad t \in [0, T],$$

where  $\varkappa_\alpha(\cdot)$  is the characteristic function of the Borel set  $\alpha \in \mathcal{B}([0, T])$ .

Let  $S = S(M)$  denote the space of all simple  $\mathcal{L}_M$ -adapted operator-valued functions on  $[0, T]$ . For a function  $A \in S$  with representation (2) we define an  *$H$ -stochastic integral* of  $A$  with respect to the abstract martingale  $M_t$  through the formula

$$(3) \quad \int_{[0, T]} A(t) dM_t := \sum_{k=0}^{n-1} A_k (M_{t_{k+1}} - M_{t_k}) \in \mathcal{H}.$$

We can show that this definition does not depend on the choice of representation of the simple function  $A$  in the space  $S$ .

In the space  $S$  we introduce a quasinorm by setting

$$\|A\|_{S_2} := \left( \int_{[0, T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t) \right)^{\frac{1}{2}} := \left( \sum_{k=0}^{n-1} \|A_k\|_{\mathcal{L}_M(t_k)}^2 \mu((t_k, t_{k+1}]) \right)^{\frac{1}{2}}$$

for each  $A \in S$  with representation (2). Here the measure  $\mu$  is defined by the formula

$$\mathcal{B}([0, T]) \ni \alpha \mapsto \mu(\alpha) := \|M(\alpha)\|_{\mathcal{H}}^2 = (E(\alpha)M, M)_{\mathcal{H}} \in \mathbb{R}_+,$$

where  $M(\alpha) := E(\alpha)M$  for all  $\alpha \in \mathcal{B}([0, T])$ , in particular,

$$M((t_k, t_{k+1}]) := E((t_k, t_{k+1}])M = M_{t_{k+1}} - M_{t_k}, \quad (t_k, t_{k+1}] \subset [0, T].$$

The following statement is fundamental.

**Theorem 1.** *Let  $A, B \in S$  and  $a, b \in \mathbb{C}$ . Then*

$$\int_{[0, T]} (aA(t) + bB(t)) dM_t = a \int_{[0, T]} A(t) dM_t + b \int_{[0, T]} B(t) dM_t$$

and

$$(4) \quad \left\| \int_{[0,T]} A(t) dM_t \right\|_{\mathcal{H}}^2 \leq \int_{[0,T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t).$$

*Proof.* The first assertion is trivial.

Let us check inequality (4). Using (i), (ii) and properties of the resolution of identity  $E$ , for  $A \in S$  with representation (2), we obtain

$$\begin{aligned} \left\| \int_{[0,T]} A(t) dM_t \right\|_{\mathcal{H}}^2 &= \left( \int_{[0,T]} A(t) dM_t, \int_{[0,T]} A(t) dM_t \right)_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (A_k M(\Delta_k), A_m M(\Delta_m))_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (A_k E(\Delta_k) M, A_m E(\Delta_m) M)_{\mathcal{H}} \\ &= \sum_{k,m=0}^{n-1} (E(\Delta_k) A_k E(\Delta_k) M, E(\Delta_m) A_m E(\Delta_m) M)_{\mathcal{H}} \\ &= \sum_{k=0}^{n-1} (A_k E(\Delta_k) M, A_k E(\Delta_k) M)_{\mathcal{H}} = \sum_{k=0}^{n-1} \|A_k M(\Delta_k)\|_{\mathcal{H}}^2 \\ &\leq \sum_{k=0}^{n-1} \|A_k\|_{\mathcal{L}_M(t_k)}^2 \|M(\Delta_k)\|_{\mathcal{H}}^2 = \sum_{k=0}^{n-1} \|A_k\|_{\mathcal{L}_M(t_k)}^2 \mu(\Delta_k) \\ &= \int_{[0,T]} \|A(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t), \end{aligned}$$

where  $\Delta_k := (t_k, t_{k+1}]$  for all  $k \in \{0, \dots, n-1\}$ . □

Inequality (4) enables us to extend the  $H$ -stochastic integral to operator-valued functions  $[0, T] \ni t \mapsto A(t)$  which are not necessarily simple. Namely, denote by  $S_2 = S_2(M)$  a Banach space associated with the quasinorm  $\|\cdot\|_{S_2}$ . For its construction, it is first necessary to pass from  $S$  to the factor space

$$\dot{S} := S / \{A \in S \mid \|A\|_{S_2} = 0\}$$

and then to take the completion of  $\dot{S}$ . It is not difficult to see that elements of the space  $S_2$  are equivalence classes of operator-valued functions on  $[0, T]$  whose values are linear operators in the space  $\mathcal{H}$ .

An operator-valued function  $[0, T] \ni t \mapsto A(t)$  will be called *H-stochastic integrable* with respect to  $M_t$  if  $A$  belongs to the space  $S_2$ . It follows from the definition of the space  $S_2$  that for each  $A \in S_2$  there exists a sequence  $(A_n)_{n=0}^{\infty}$  of simple operator-valued functions  $A_n \in S$  such that

$$(5) \quad \int_{[0,T]} \|A(t) - A_n(t)\|_{\mathcal{L}_M(t)}^2 d\mu(t) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Due to (4), for such a sequence  $(A_n)_{n=0}^{\infty}$ , the limit

$$\lim_{n \rightarrow \infty} \int_{[0,T]} A_n(t) dM_t$$

exists in  $\mathcal{H}$  and does not depend on the choice of the sequence  $(A_n)_{n=0}^{\infty} \subset S$  satisfying (5). We denote this limit by

$$\int_{[0,T]} A(t) dM_t := \lim_{n \rightarrow \infty} \int_{[0,T]} A_n(t) dM_t$$

and call it the *H-stochastic integral of  $A \in S_2$*  with respect to the abstract martingale  $M_t$ . It is clear that for all  $A \in S_2$  the assertions of Theorem 1 still hold.

Note one simple property of the integral introduced above. Let  $U$  be some unitary operator acting from  $\mathcal{H}$  onto another complex Hilbert space  $\mathcal{K}$ . Then

$$[0, T] \ni t \mapsto G_t := UM_t \in \mathcal{K}$$

is an abstract martingale in the space  $\mathcal{K}$  because, for any  $t \in [0, T]$ ,

$$G_t = UM_t = X_t G, \quad X_t := UE_t U^{-1}, \quad G := UM \in \mathcal{K},$$

and  $X_t$  is a resolution of identity in the space  $\mathcal{K}$ .

Let an operator-valued function  $[0, T] \ni t \mapsto A(t)$  be  $H$ -stochastic integrable with respect to  $M_t$ . One can show that the operator-valued function  $[0, T] \ni t \mapsto UA(t)U^{-1}$  is  $H$ -stochastic integrable with respect to  $G_t$  and

$$U \left( \int_{[0, T]} A(t) dM_t \right) = \int_{[0, T]} UA(t)U^{-1} dG_t \in \mathcal{K}.$$

### 3. THE ITÔ STOCHASTIC INTEGRAL AS AN $H$ -STOCHASTIC INTEGRAL

Let  $(\Omega, \mathcal{A}, P)$  be a complete probability space and  $\{\mathcal{A}_t\}_{t \in [0, T]}$  be a right continuous filtration. Suppose that the  $\sigma$ -algebra  $\mathcal{A}_0$  contains all  $P$ -null sets of  $\mathcal{A}$  and  $\mathcal{A} = \mathcal{A}_T$ . Moreover, we assume that  $\mathcal{A}_0$  is trivial, i.e., every set  $\alpha \in \mathcal{A}_0$  has probability 0 or 1.

Let  $N = \{N_t\}_{t \in [0, T]}$  be a *normal martingale* on  $(\Omega, \mathcal{A}, P)$  with respect to  $\{\mathcal{A}_t\}_{t \in [0, T]}$ . That is,  $N_t \in L^2(\Omega, \mathcal{A}_t, P)$  for all  $t \in [0, T]$  and

$$\mathbb{E}[N_t - N_s | \mathcal{A}_s] = 0, \quad \mathbb{E}[(N_t - N_s)^2 | \mathcal{A}_s] = t - s$$

for all  $s, t \in [0, T]$  such that  $s < t$ . Without loss of generality one can assume that  $N_0 = 0$ . Note that there are many examples of normal martingales, — the Brownian motion, the compensated Poisson process, the Azéma martingales and others, see for instance [10, 8, 12].

We will denote by  $L_a^2([0, T] \times \Omega)$  the set of all functions (equivalence classes), adapted to the filtration  $\{\mathcal{A}_t\}_{t \in [0, T]}$ , from the space

$$L^2([0, T] \times \Omega) := L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{A}, dt \times P)$$

where  $dt$  is the Lebesgue measure on  $\mathcal{B}([0, T])$ .

Let us show that the *Itô stochastic integral*  $\int_{[0, T]} F(t) dN_t$  of  $F \in L_a^2([0, T] \times \Omega)$  with respect to the normal martingale  $N$  can be considered as an  $H$ -stochastic integral (see e.g. [15, 16] for the definition and properties of the classical Itô integral). To this end, we set  $\mathcal{H} := L^2(\Omega, \mathcal{A}, P)$  and consider, in this space, the resolution of identity

$$[0, T] \ni t \mapsto E_t := \mathbb{E}[\cdot | \mathcal{A}_t] \in \mathcal{L}(\mathcal{H})$$

generated by the filtration  $\{\mathcal{A}_t\}_{t \in [0, T]}$ . Let  $M := N_T \in L^2(\Omega, \mathcal{A}, P)$ , then the corresponding abstract martingale

$$[0, T] \ni t \mapsto N_t := E_t N_T = \mathbb{E}[N_T | \mathcal{A}_t] \in \mathcal{H}$$

is our normal martingale. Note also that

$$\mu([0, t]) = \|N([0, t])\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \|N_t\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \mathbb{E}[N_t^2] = \mathbb{E}[N_t^2 | \mathcal{A}_0] = t,$$

i.e.,  $\mu$  is the Lebesgue measure on  $\mathcal{B}([0, T])$ .

In the context of this section,  $\mathcal{L}_M(t)$ -measurability is equivalent to the usual  $\mathcal{A}_t$ -measurability. More precisely, the following result holds.

**Lemma 1.** *Let  $t \in [0, T]$ . For given  $F \in L^2(\Omega, \mathcal{A}, P)$  the operator  $A_F$  of multiplication by the function  $F$  in the space  $L^2(\Omega, \mathcal{A}, P)$  is  $\mathcal{L}_N(t)$ -measurable if and only if the function  $F$  is  $\mathcal{A}_t$ -measurable, i.e.,  $F = \mathbb{E}[F | \mathcal{A}_t]$ . Moreover, if  $F \in L^2(\Omega, \mathcal{A}, P)$  is an  $\mathcal{A}_t$ -measurable function then*

$$(6) \quad \|A_F\|_{\mathcal{L}_N(t)} = \|A_F\|_{\mathcal{L}_N(s)} = \|F\|_{L^2(\Omega, \mathcal{A}, P)}, \quad s \in [t, T].$$

*Proof.* Suppose  $F \in L^2(\Omega, \mathcal{A}, P)$  is an  $\mathcal{A}_t$ -measurable function. Let us show that the operator  $A_F$  is  $\mathcal{L}_N(t)$ -measurable.

First, we prove that  $A_F \in \mathcal{L}_N(t)$ . Taking into account that  $F$  is an  $\mathcal{A}_t$ -measurable function,  $\{N_t\}_{t \in [0, T]}$  is the normal martingale and the  $\sigma$ -algebra  $\mathcal{A}_0$  is trivial, for any interval  $(s_1, s_2] \subset (t, T]$ , we obtain

$$\begin{aligned} \|A_F(N_{s_2} - N_{s_1})\|_{L^2(\Omega, \mathcal{A}, P)}^2 &= \|F(N_{s_2} - N_{s_1})\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2] \\ &= \mathbb{E}[F^2(N_{s_2} - N_{s_1})^2 | \mathcal{A}_0] = \mathbb{E}[F^2 \mathbb{E}[(N_{s_2} - N_{s_1})^2 | \mathcal{A}_{s_1}] | \mathcal{A}_0] \\ &= \mathbb{E}[F^2 \mathbb{E}[(N_{s_2} - N_{s_1})^2 | \mathcal{A}_{s_1}]] = \mathbb{E}[F^2](s_2 - s_1) \\ &= \mathbb{E}[F^2] \mathbb{E}[(N_{s_2} - N_{s_1})^2] \\ &= \|F\|_{L^2(\Omega, \mathcal{A}, P)}^2 \|N_{s_2} - N_{s_1}\|_{L^2(\Omega, \mathcal{A}, P)}^2. \end{aligned}$$

We can similarly show that

$$\|A_F G\|_{L^2(\Omega, \mathcal{A}, P)}^2 = \|F\|_{L^2(\Omega, \mathcal{A}, P)}^2 \|G\|_{L^2(\Omega, \mathcal{A}, P)}^2$$

for all  $G \in \mathcal{H}_N(t) = \text{span}\{N_{s_2} - N_{s_1} \mid (s_1, s_2] \subset (t, T]\}$ . Hence  $A_F \in \mathcal{L}_N(t)$  and, moreover, equality (6) takes place.

Let us check that  $A_F$  is partially commuting with  $E$ , i.e.,

$$A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t), \quad s \in [t, T].$$

Since  $F \in L^2(\Omega, \mathcal{A}, P)$  is an  $\mathcal{A}_t$ -measurable function and  $FG \in L^2(\Omega, \mathcal{A}, P)$ , for any  $s \in [t, T]$  and any function  $G \in \mathcal{H}_N(t)$ , we have

$$A_F E_s G = F E_s G = F \mathbb{E}[G | \mathcal{A}_s] = \mathbb{E}[FG | \mathcal{A}_s] = E_s A_F G.$$

Thus, the first part of the lemma is proved.

Let us prove the converse statement of the lemma: if for a given  $F \in L^2(\Omega, \mathcal{A}, P)$  the operator  $A_F$  is  $\mathcal{L}_N(t)$ -measurable then  $F$  is an  $\mathcal{A}_t$ -measurable function.

Since  $A_F$  is an  $\mathcal{L}_N(t)$ -measurable operator, we see that for any  $s \in [t, T]$

$$A_F E_s G = E_s A_F G, \quad G \in \mathcal{H}_N(t),$$

or, equivalently,

$$(7) \quad A_F \mathbb{E}[G | \mathcal{A}_s] = \mathbb{E}[A_F G | \mathcal{A}_s], \quad G \in \mathcal{H}_N(t).$$

Let  $s \in (t, T)$  and  $(s_1, s_2] \subset (t, s]$ . We take

$$G := N_{s_2} - N_{s_1} \in \mathcal{H}_N(t).$$

Evidently,  $G$  is an  $\mathcal{A}_s$ -measurable function and

$$A_F \mathbb{E}[G | \mathcal{A}_s] = A_F G = FG, \quad \mathbb{E}[A_F G | \mathcal{A}_s] = \mathbb{E}[FG | \mathcal{A}_s] = G \mathbb{E}[F | \mathcal{A}_s].$$

Hence, using (7), we obtain

$$FG = G \mathbb{E}[F | \mathcal{A}_s].$$

As a result,

$$F = \mathbb{E}[F | \mathcal{A}_s], \quad s \in (t, T].$$

Since the resolution of identity  $[0, T] \ni s \mapsto E_s = \mathbb{E}[\cdot | \mathcal{A}_s] \in \mathcal{L}(\mathcal{H})$  is a right-continuous function, the latter equality still holds for  $s = t$ , and therefore  $F$  is an  $\mathcal{A}_t$ -measurable function.  $\square$

As a simple consequence of Lemma 1 we obtain the following result.

**Theorem 2.** *Let  $F$  belong to  $L^2([0, T] \times \Omega)$ . The family  $\{A_F(t)\}_{t \in [0, T]}$  of the operators  $A_F(t)$  of multiplication by  $F(t) = F(t, \cdot) \in L^2(\Omega, \mathcal{A}, P)$  in the space  $L^2(\Omega, \mathcal{A}, P)$ ,*

$$L^2(\Omega, \mathcal{A}, P) \supset \text{Dom}(A_F(t)) \ni G \mapsto A_F(t)G := F(t)G \in L^2(\Omega, \mathcal{A}, P),$$

*is  $H$ -stochastic integrable with respect to the normal martingale  $N$  (i.e. belongs to  $S_2$ ) if and only if  $F$  belongs to the space  $L_a^2([0, T] \times \Omega)$ .*

The next theorem shows that the Itô stochastic integral with respect to the normal martingale  $N$  can be interpreted as an  $H$ -stochastic integral.

**Theorem 3.** *Let  $F \in L_a^2([0, T] \times \Omega)$  and  $\{A_F(t)\}_{t \in [0, T]}$  be the corresponding family of the operators  $A_F(t)$  of multiplication by  $F(t)$  in the space  $L^2(\Omega, \mathcal{A}, P)$ . Then*

$$\int_{[0, T]} A_F(t) dN_t = \int_{[0, T]} F(t) dN_t.$$

*Proof.* Taking into account Theorem 2, Lemma 1 and the definitions of the integrals

$$\int_{[0, T]} A_F(t) dN_t \quad \text{and} \quad \int_{[0, T]} F(t) dN_t,$$

it is sufficient to prove Theorem 3 for simple functions  $F \in L_a^2([0, T] \times \Omega)$ . But in this case Theorem 3 is obvious.  $\square$

#### 4. THE ITÔ INTEGRAL IN A FOCK SPACE AS AN $H$ -STOCHASTIC INTEGRAL

Let us recall the definition of the Itô integral in a Fock space, see e.g. [2] for more details. We denote by  $\mathcal{F}$  the symmetric Fock space over the real separable Hilbert space  $L^2([0, T]) := L^2([0, T], dt)$ . By definition (see e.g. [5]),

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{F}_n n!,$$

where  $\mathcal{F}_0 := \mathbb{C}$  and, for each  $n \in \mathbb{N}$ ,  $\mathcal{F}_n := (L_{\mathbb{C}}^2([0, T]))^{\widehat{\otimes} n}$  is an  $n$ -th symmetric tensor power  $\widehat{\otimes}$  of the complex Hilbert space  $L_{\mathbb{C}}^2([0, T])$ . Thus, the Fock space  $\mathcal{F}$  is the complex Hilbert space of sequences  $f = (f_n)_{n=0}^{\infty}$  such that  $f_n \in \mathcal{F}_n$  and

$$\|f\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{F}_n}^2 n! < \infty.$$

We denote by  $L^2([0, T]; \mathcal{F})$  the Hilbert space of all  $\mathcal{F}$ -valued functions

$$[0, T] \ni t \mapsto f(t) \in \mathcal{F}, \quad \|f\|_{L^2([0, T]; \mathcal{F})} := \left( \int_{[0, T]} \|f(t)\|_{\mathcal{F}}^2 dt \right)^{\frac{1}{2}} < \infty$$

with the corresponding scalar product. A function  $f(\cdot) = (f_n(\cdot))_{n=0}^{\infty} \in L^2([0, T]; \mathcal{F})$  is called *Itô integrable* if, for almost all  $t \in [0, T]$ ,

$$f(t) = (f_0(t), \varkappa_{[0, t]} f_1(t), \dots, \varkappa_{[0, t]^n} f_n(t), \dots).$$

We denote by  $L_a^2([0, T]; \mathcal{F})$  the set of all Itô integrable functions.

Let  $f$  belong to the space  $L_{a, s}^2([0, T]; \mathcal{F})$  of all *simple Itô integrable functions*. That is,  $f$  belongs to  $L_a^2([0, T]; \mathcal{F})$  and there exists a partition  $0 = t_0 < t_1 < \dots < t_n = T$  of  $[0, T]$  such that

$$f(t) = \sum_{k=0}^{n-1} f^{(k)} \varkappa_{(t_k, t_{k+1}]}(t) \in \mathcal{F}$$

for almost all  $t \in [0, T]$ . The *Itô integral*  $\mathbb{I}(f)$  of such a function  $f$  is defined by the formula

$$\mathbb{I}(f) := \sum_{k=0}^{n-1} f^{(k)} \diamond (0, \varkappa_{(t_k, t_{k+1}]}, 0, 0, \dots) \in \mathcal{F},$$

where the symbol  $\diamond$  denotes the Wick product in the Fock space  $\mathcal{F}$ . Let us recall that for given  $f = (f_n)_{n=0}^{\infty}$  and  $g = (g_n)_{n=0}^{\infty}$  from  $\mathcal{F}$  the Wick product  $f \diamond g$  is defined by

$$f \diamond g := \left( \sum_{m=0}^n f_m \widehat{\otimes} g_{n-m} \right)_{n=0}^{\infty},$$

provided the latter sequence belongs to the Fock space  $\mathcal{F}$ .

The Itô integral  $\mathbb{I}(f)$  of a simple function  $f \in L^2_{a,s}([0, T]; \mathcal{F})$  has the isometry property

$$\|\mathbb{I}(f)\|_{\mathcal{F}}^2 = \int_{[0, T]} \|f(t)\|_{\mathcal{F}}^2 dt,$$

see e.g. [2, 1]. Hence, extending the mapping

$$L^2_a([0, T]; \mathcal{F}) \supset L^2_{a,s}([0, T]; \mathcal{F}) \ni f \mapsto \mathbb{I}(f) \in \mathcal{F}$$

by continuity we obtain a definition of the Itô integral  $\mathbb{I}(f)$  for each  $f \in L^2_a([0, T]; \mathcal{F})$  (we keep the same notation  $\mathbb{I}$  for the extension).

Let us show that the Itô integral  $\mathbb{I}(f)$  of  $f \in L^2_a([0, T]; \mathcal{F})$  can be considered as an  $H$ -stochastic integral. To do this we set  $\mathcal{H} := \mathcal{F}$  and consider in this space the resolution of identity

$$[0, T] \ni t \mapsto \mathcal{X}_t f := (f_0, \varkappa_{[0, t]} f_1, \dots, \varkappa_{[0, t]^n} f_n, \dots) \in \mathcal{L}(\mathcal{F}), \quad f = (f_n)_{n=0}^\infty \in \mathcal{F}.$$

Let  $Z := (0, 1, 0, 0, \dots) \in \mathcal{F}$  and

$$[0, T] \ni t \mapsto Z_t := \mathcal{X}_t Z = (0, \varkappa_{[0, t]}, 0, 0, \dots) \in \mathcal{F}$$

be the corresponding abstract martingale in the Fock space  $\mathcal{F}$ . Note that now

$$\mu([0, t]) := \|Z_t\|_{\mathcal{F}}^2 = \|\varkappa_{[0, t]}\|_{L^2_c([0, T])}^2 = t, \quad t \in [0, T],$$

i.e.,  $\mu$  is the Lebesgue measure on  $\mathcal{B}([0, T])$ .

We have the following analogues of Theorems 2 and 3.

**Theorem 4.** *A function  $f \in L^2([0, T]; \mathcal{F})$  belongs to the space  $L^2_a([0, T]; \mathcal{F})$  if and only if the corresponding operator-valued function  $[0, T] \ni t \mapsto A_f(t)$  whose values are operators  $A_f(t)$  of Wick multiplication by  $f(t) \in \mathcal{F}$  in the Fock space  $\mathcal{F}$ ,*

$$\mathcal{F} \supset \text{Dom}(A_f(t)) \ni g \mapsto A_f(t)g := f(t) \diamond g \in \mathcal{F},$$

*belongs to the space  $S_2$ .*

**Theorem 5.** *Let  $f \in L^2_a([0, T]; \mathcal{F})$  and  $\{A_f(t)\}_{t \in [0, T]}$  be the corresponding family of the operators  $A_f(t)$  of Wick multiplication by  $f(t) \in \mathcal{F}$  in the Fock space  $\mathcal{F}$ . Then*

$$\mathbb{I}(f) = \int_{[0, T]} A_f(t) dZ_t.$$

Taking into account Theorem 5, in what follows we will denote the Itô integral  $\mathbb{I}(f)$  of  $f \in L^2_a([0, T]; \mathcal{F})$  by  $\int_{[0, T]} f(t) dZ_t$ . Note that this integral can be expressed in terms of the Fock space  $\mathcal{F}$ . Namely, for any  $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L^2_a([0, T]; \mathcal{F})$ , we have

$$(8) \quad \int_{[0, T]} f(t) dZ_t = (0, \hat{f}_1, \dots, \hat{f}_n, \dots) \in \mathcal{F},$$

where, for each  $n \in \mathbb{N}$  and almost all  $(t_1, \dots, t_n) \in [0, T]^n$ ,

$$\hat{f}_n(t_1, \dots, t_n) := \frac{1}{n} \sum_{k=1}^n f_{n-1}(t_k; t_1, \dots, \cancel{t_k}, \dots, t_n),$$

i.e.,  $\hat{f}_n$  is the symmetrization of  $f_{n-1}(t; t_1, \dots, t_{n-1})$  with respect to  $n$  variables.

## 5. A CONNECTION BETWEEN THE CLASSICAL ITÔ INTEGRAL AND THE ITÔ INTEGRAL IN THE FOCK SPACE

As before, let  $(\Omega, \mathcal{A}, P)$  be a complete probability space with a right continuous filtration  $\{\mathcal{A}_t\}_{t \in [0, T]}$ ,  $\mathcal{A}_0$  be the trivial  $\sigma$ -algebra containing all  $P$ -null sets of  $\mathcal{A}$  and  $\mathcal{A} = \mathcal{A}_T$ .



Let  $N = \{N_t\}_{t \in [0, T]}$  be a normal martingale on  $(\Omega, \mathcal{A}, P)$  with respect to  $\{\mathcal{A}_t\}_{t \in [0, T]}$ ,  $N_0 = 0$ . It is known that the mapping

$$\mathcal{F} \ni f = (f_n)_{n=0}^\infty \mapsto If := \sum_{n=0}^\infty I_n(f_n) \in L^2(\Omega, \mathcal{A}, P)$$

is well-defined and isometric. Here  $I_0(f_0) := f_0$  and, for each  $n \in \mathbb{N}$ ,

$$I_n(f_n) := n! \int_0^T \int_0^{t_n} \cdots \left( \int_0^{t_2} f_n(t_1, \dots, t_n) dN_{t_1} \right) \cdots dN_{t_{n-1}} dN_{t_n}$$

is an  $n$ -iterated Itô integral with respect to  $N$ . We suppose that the normal martingale  $N$  has the *chaotic representation property* (CRP). In other words, we assume that the mapping  $I : \mathcal{F} \rightarrow L^2(\Omega, \mathcal{A}, P)$  is a unitary. Note that

$$N_t = IZ_t \in L^2(\Omega, \mathcal{A}, P), \quad t \in [0, T],$$

i.e.,  $N$  is the  $I$ -image of the abstract martingale  $[0, T] \ni t \mapsto Z_t = (0, \varkappa_{[0, t]}, 0, 0, \dots) \in \mathcal{F}$ .

The Brownian motion, the compensated Poisson process and some Azéma martingales are examples of normal martingales which possess the CRP, see e.g. [10, 11].

We note that the spaces  $L^2([0, T] \times \Omega)$  and  $L^2([0, T]; \mathcal{F})$  can be understood as the tensor products  $L^2([0, T]) \otimes L^2(\Omega, \mathcal{A}, P)$  and  $L^2([0, T]) \otimes \mathcal{F}$ , respectively. Therefore,

$$1 \otimes I : L^2([0, T]; \mathcal{F}) \rightarrow L^2([0, T] \times \Omega)$$

is a unitary operator.

The next result gives a relationship between the classical Itô integral with respect to the normal martingale with CRP and the Itô integral in the Fock space  $\mathcal{F}$ .

**Theorem 6.** *We have*

$$L_a^2([0, T] \times \Omega) = (1 \otimes I)L_a^2([0, T]; \mathcal{F})$$

and, for arbitrary  $f \in L_a^2([0, T]; \mathcal{F})$ ,

$$I\left(\int_{[0, T]} f(t) dZ_t\right) = \int_{[0, T]} If(t) dN_t.$$

Since  $N$  has CRP, for any  $F \in L_a^2([0, T] \times \Omega)$  there exists a uniquely defined vector  $f(\cdot) = (f_n(\cdot))_{n=0}^\infty \in L_a^2([0, T]; \mathcal{F})$  such that

$$F(t) = If(t) = \sum_{n=0}^\infty I_n(f_n(t))$$

for almost all  $t \in [0, T]$ . Hence, using Theorem 6 and equality (8) we obtain

$$\int_{[0, T]} F(t) dN_t = I\left(\int_{[0, T]} f(t) dZ_t\right) = \sum_{n=1}^\infty I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P).$$

It should be noticed that the right hand side of the latter equality was used by Hitsuda [9] and Skorohod [14] to define an extension of the Itô integral. Namely, a function

$$F(\cdot) = \sum_{n=0}^\infty I_n(f_n(\cdot)) \in L^2([0, T] \times \Omega)$$

is Hitsuda-Skorohod integrable if and only if

$$\sum_{n=1}^\infty I_n(\hat{f}_n) \in L^2(\Omega, \mathcal{A}, P) \quad \text{or, equivalently,} \quad \sum_{n=1}^\infty \|\hat{f}_n\|_{\mathcal{F}_n}^2 n! < \infty.$$

The corresponding *Hitsuda-Skorohod integral*  $\mathbb{I}_{\text{HS}}(F)$  of  $F$  is defined by the formula

$$\mathbb{I}_{\text{HS}}(F) := \sum_{n=1}^\infty I_n(\hat{f}_n).$$

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