

## THE DIRECT AND INVERSE SPECTRAL PROBLEMS FOR $(2N + 1)$ -DIAGONAL COMPLEX TRANSPOSITION-ANTISYMMETRIC MATRICES

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*To Professor M. L. Gorbachuk on the occasion of his 70th birthday.*

ABSTRACT. We consider a difference equation associated with a semi-infinite complex  $(2N + 1)$ -diagonal transposition-antisymmetric matrix  $J = (g_{k,l})_{k,l=0}^{\infty}$  with  $g_{k,k+N} \neq 0$ ,  $k = 0, 1, 2, \dots$ ,  $(g_{k,l} = -g_{l,k})$ :  $\sum_{j=-N}^N g_{k,k+j} y_{k+j} = \lambda^N y_k$ ,  $k = 0, 1, 2, \dots$ , where  $y = (y_0, y_1, y_2, \dots)$  is an unknown vector,  $\lambda$  is a complex parameter,  $g_{k,l}$  and  $y_l$  with negative indices are equal to zero,  $N \in \mathbb{N}$ . We introduce a notion of the spectral function for this difference equation. We state and solve the direct and inverse problems for this equation.

### 1. INTRODUCTION

The aim of our present investigation is to give precise definitions and solve the direct and inverse spectral problems for  $(2N + 1)$ -diagonal semi-infinite complex antisymmetric matrices,  $N \in \mathbb{N}$ . The object of our study is a semi-infinite matrix  $J = (g_{k,l})_{k,l \in \mathbb{Z}_+}$ ,  $g_{k,l} \in \mathbb{C}$ :

$$(1) \quad g_{k,l} = 0, \quad k, l \in \mathbb{Z}_+ : |k - l| > N,$$

$$(2) \quad g_{k,l} = -g_{l,k}, \quad k, l \in \mathbb{Z}_+,$$

and

$$(3) \quad g_{k,k+N} \neq 0, \quad k \in \mathbb{Z}_+.$$

In particular, from relation (2) it follows that  $g_{k,k} = 0$ ,  $k \in \mathbb{Z}_+$ .

We recall that the classical Jacobi matrix is a semi-infinite matrix  $J = (g_{k,l})_{k,l \in \mathbb{Z}_+}$ ,  $g_{k,l} \in \mathbb{R}$ , such that (1) holds with  $N = 1$ ,

$$(4) \quad g_{k,l} = g_{l,k}, \quad k, l \in \mathbb{Z}_+,$$

and

$$(5) \quad g_{k,k+N} > 0, \quad k \in \mathbb{Z}_+.$$

This matrix defines a symmetric operator  $J$  in the space  $l^2 = \{(x_k)_{k \in \mathbb{Z}_+}, x_k \in \mathbb{C} : \sum_{k=0}^{\infty} |x_k|^2 < \infty\}$ . The operator  $J$  is a difference operator. Moreover, it has a cyclic vector  $e_0 = (1, 0, 0, \dots)$ . The direct spectral problem consists in constructing its spectral function  $\sigma(\lambda) = (E(\lambda)e_0, e_0)$ ,  $\{E_\lambda\}$  is a resolution of unity for a self-adjoint extension of  $J$ . The inverse problem is to get  $J$  from its spectral function. A detailed exposition of this classical subject can be found in books [1], [2], [3].

The direct and inverse problems for Jacobi matrices with matrix elements are described in [2], [3]. For Jacobi fields these problems were studied in [4], [5].

To the best of our knowledge, Guseynov was the first who stated and solved the direct and inverse problems for a non-symmetric semi-infinite matrix [6]. More precisely, he

considered a semi-infinite matrix  $J = (g_{k,l})_{k,l \in \mathbb{Z}_+}$ ,  $g_{k,l} \in \mathbb{C}$ , such that (1) holds with  $N = 1$  and (4) is true (with complex elements  $g_{k,l}$ ). For arbitrary  $N \in \mathbb{N}$  analogous results were obtained in [7]. However, there was used a different and more simple method for the reconstruction of  $J$  from its spectral function (we can call it a "complex" orthogonalization).

Recently, direct and inverse problems for the block Jacobi type unitary matrices and for the block Jacobi type bounded normal matrices were solved in [8], [9].

Spectral problems for generalized Jacobi matrices connected with the indefinite product inner spaces were studied in [10].

**Notations.** As usual, we denote by  $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$  the sets of real, complex, positive integer, integer, non-negative integer numbers, respectively. By  $\mathbb{P}$  we denote the set of all polynomials with complex coefficients.

2. THE DIRECT PROBLEM

Let us fix an arbitrary  $N \in \mathbb{N}$ . Consider a semi-infinite matrix  $J = (g_{k,l})_{k,l \in \mathbb{Z}_+}$ ,  $g_{k,l} \in \mathbb{C}$ , which satisfies relations (1)-(3). We can associate with  $J$  a difference equation:

$$(6) \quad \sum_{j=-N}^N g_{k,k+j} y_{k+j} = \lambda^N y_k, \quad k \in \mathbb{Z}_+,$$

where  $y = (y_0, y_1, y_2, \dots)$  is an unknown vector,  $\lambda$  is a complex parameter. Here we set  $g_{k,l}$  and  $y_l$  with negative indices equal to zero.

We set

$$(7) \quad p_k(\lambda) := \lambda^k, \quad k = 0, 1, \dots, N - 1.$$

From (6) it follows that

$$(8) \quad y_{k+N} = \frac{1}{g_{k,k+N}} \left( \lambda^N y_k - \sum_{j=-N}^{N-1} g_{k,k+j} y_{k+j} \right), \quad k \in \mathbb{Z}_+.$$

If we take  $p_k(\lambda)$ ,  $k = 0, 1, \dots, N - 1$ , as the initial values for difference equation (6), by virtue of (8) we can successively find a sequence  $\{p_k\}_{k \in \mathbb{Z}_+}$  which is a solution of (6).

Define a functional  $\sigma(u, v)$  on polynomials  $\{p_k\}_{k \in \mathbb{Z}_+}$ ,

$$(9) \quad \sigma(p_k(\lambda), p_l(\lambda)) = \delta_{k,l}, \quad k, l \in \mathbb{Z}_+.$$

Let  $u(\lambda), v(\lambda)$  be arbitrary complex polynomials. They can be written in the following form:

$$(10) \quad u(\lambda) = \sum_{k=0}^{\infty} \alpha_k p_k(\lambda), \quad v(\lambda) = \sum_{k=0}^{\infty} \beta_k p_k(\lambda), \quad \alpha_k, \beta_k \in \mathbb{C}.$$

Moreover, expansion (10) is unique,  $\alpha_k = 0$ ,  $k > \deg u$ , and  $\beta_k = 0$ ,  $k > \deg v$  (see [11, Lemma 1.1]).

We set

$$(11) \quad \sigma(u(\lambda), v(\lambda)) = \sum_{k=0}^{\infty} \alpha_k \beta_k.$$

Note that the sum in (11) is finite.

It is not hard to see that the functional  $\sigma$  is linear with respect to the both arguments and possesses the following property:

$$(12) \quad \sigma(u(\lambda), v(\lambda)) = \sigma(v(\lambda), u(\lambda)), \quad u, v \in \mathbb{P}.$$

Choose  $n, m \in \mathbb{Z}_+$ . By virtue of (6) and (9) we can write

$$(13) \quad \sigma(\lambda^N p_n(\lambda), p_m(\lambda)) = \left( \sum_{j=-N}^N g_{n,n+j} p_{n+j}, p_m(\lambda) \right) = \sum_{j=-N}^N g_{n,n+j} \delta_{n+j,m};$$

$$(14) \quad \sigma(p_n(\lambda), \lambda^N p_m(\lambda)) = \left( p_n(\lambda), \sum_{l=-N}^N g_{m,m+l} p_{m+l} \right) = \sum_{l=-N}^N g_{m,m+l} \delta_{n,m+l}.$$

From (2) it follows that

$$(15) \quad g_{k,l} = -g_{l,k}, \quad k, l \in \mathbb{Z},$$

since for negative indices the coefficients  $g_{i,j}$  are equal to zero.

In the case  $|n - m| > N$  we get zeros in the right-hand sides of (13) and (14).

In the case  $|r| \leq N$ ,  $r := n - m$ , we get  $g_{n,n-r}$  on the right in (13) and  $g_{m,m+r} = g_{n-r,n}$  on the right in (14). Using (15) we see that these values differ by the sign.

Consequently, we get

$$(16) \quad \sigma(\lambda^N p_n(\lambda), p_m(\lambda)) = -\sigma(p_n(\lambda), \lambda^N p_m(\lambda)), \quad n, m \in \mathbb{Z}_+.$$

Choose arbitrary complex polynomials  $u(\lambda), v(\lambda)$ . They possess representation (10). Using this representation and relation (16) we write

$$\begin{aligned} \sigma(\lambda^N u(\lambda), v(\lambda)) &= \sum_{k,l} \alpha_k \beta_l \sigma(\lambda^N p_k(\lambda), p_l(\lambda)) \\ &= - \sum_{k,l} \alpha_k \beta_l \sigma(p_k(\lambda), \lambda^N p_l(\lambda)) = -\sigma(u(\lambda), \lambda^N v(\lambda)). \end{aligned}$$

Hence, we get

$$(17) \quad \sigma(\lambda^N u(\lambda), v(\lambda)) = -\sigma(u(\lambda), \lambda^N v(\lambda)), \quad u, v \in \mathbb{P}.$$

**Definition 1.** A functional  $\sigma(u, v)$ ,  $u, v \in \mathbb{P}$ , which is linear with respect to both arguments and satisfies (9), will be called *the spectral function* of difference equation (6).

*The direct spectral problem* for difference equation (6) consists in finding answers to the following questions:

- (1) Does the spectral function exist?
- (2) If the spectral function exists, is it unique?
- (3) If the spectral function exists, how to find it (or them)?

The answer to the first question is surely affirmative as it follows from the preceding considerations.

The answer to the second question is given in the following theorem.

**Theorem 1.** *Let a difference equation of type (6) which corresponds to a semi-infinite complex antisymmetric matrix  $J$  be given. Then the spectral function of difference equation (6) exists and it is unique. Moreover, it satisfies relation (17).*

*Proof.* Existence of the spectral function  $\sigma$  and relation (17) were obtained above. Suppose that there exists another spectral function  $\sigma_1$ . For arbitrary complex polynomials  $u(\lambda), v(\lambda)$  we can write (10). By virtue of (9) we get

$$\sigma_1(u, v) = \sum_{k,l} \alpha_k \beta_l \sigma_1(p_k, p_l) = \sum_{k,l} \alpha_k \beta_l \delta_{k,l} = \sum_{k,l} \alpha_k \beta_l \sigma(p_k, p_l) = \sigma(u, v).$$

Consequently, the functionals  $\sigma$  and  $\sigma_1$  coincide.  $\square$

As for the third question in the direct spectral problem, the construction of the spectral function  $\sigma$  was given above.

### 3. THE INVERSE PROBLEM

*The inverse spectral problem* for difference equation (6) consists in answering the following questions.

- (1) Is it possible to reconstruct difference equation (6) using its spectral function? If it is, what is a procedure for the reconstruction?

- (2) What are necessary and sufficient conditions for a functional  $\sigma(u, v)$ ,  $u, v \in \mathbb{P}$ , linear in both arguments, to be the spectral function of a difference equation of type (6)?

An answer on the second question is given in the following theorem.

**Theorem 2.** *A functional  $\sigma(u, v)$ ,  $u, v \in \mathbb{P}$ , linear in both arguments and satisfying (12)<sup>1</sup> is a spectral function of a difference equation of type (6) corresponding to a semi-infinite complex antisymmetric matrix  $J$  iff the following is satisfied:*

- 1)  $\sigma(\lambda^N u(\lambda), v(\lambda)) = -\sigma(u(\lambda), \lambda^N v(\lambda))$ ,  $u, v \in \mathbb{P}$ ;
- 2)  $\sigma(\lambda^k, \lambda^l) = \delta_{k,l}$ ,  $k, l = 0, 1, \dots, N - 1$ ;
- 3) for an arbitrary polynomial  $u_k(\lambda)$  of degree  $k$ ,  $k \in \mathbb{Z}_+$ , there exists a polynomial  $\hat{u}_k(\lambda)$  of the same degree  $k$  such that

$$\sigma(u_k(\lambda), \hat{u}_k(\lambda)) \neq 0.$$

*Proof. Necessity.* Let  $\sigma$  be a spectral function of a difference equation of type (6) corresponding to a semi-infinite complex antisymmetric matrix  $J$ . Relation 1) follows from Theorem 1. Relation 2) follows from (9) since  $p_k(\lambda) = \lambda^k$ ,  $k = 0, 1, \dots, N - 1$ . Choose an arbitrary complex polynomial  $u(\lambda)$  of degree  $l$ ,  $l \in \mathbb{Z}_+$ . For  $u(\lambda)$  we can write decomposition (10) where  $\alpha_l \neq 0$ . Set  $\hat{u}(\lambda) = \sum_{k=0}^l \overline{\alpha_k} p_k(\lambda)$ . By virtue of (9) we get

$$\sigma(u(\lambda), \hat{u}(\lambda)) = \sum_{k=0}^l |\alpha_k|^2 > 0.$$

*Sufficiency.* Let  $\sigma(u, v)$ ,  $u, v \in \mathbb{P}$ , be a functional linear with respect to both arguments and satisfying relations 1), 2), 3). Set  $p_k(\lambda) = \lambda^k$ ,  $k = 0, 1, \dots, N - 1$ . For these polynomials relation (9) is true.

Suppose we have constructed polynomials  $p_0(\lambda), p_1(\lambda), \dots, p_{n-1}(\lambda)$ , where  $n \geq N$ , and for these polynomials relation (9) holds.

Consider an arbitrary complex monic polynomial of degree  $n$ ,

$$R_n(\lambda) = \lambda^n + u_{n-1}(\lambda),$$

where  $u_{n-1}$  is a complex polynomial of degree  $\leq n - 1$ . The polynomial  $u_{n-1}$  can be decomposed like in (10) and we get

$$R_n(\lambda) = \lambda^n + \sum_{j=0}^{n-1} \beta_{n,j} p_j(\lambda), \quad \beta_{n,j} \in \mathbb{C}.$$

Using orthogonality of polynomials  $p_j$  we can write

$$\sigma(R_n(\lambda), p_l(\lambda)) = \sigma(\lambda^n, p_l(\lambda)) + \beta_{n,l}, \quad 0 \leq l \leq n - 1.$$

Set

$$\beta_{n,l} = -\sigma(\lambda^n, p_l(\lambda)), \quad 0 \leq l \leq n - 1.$$

This yields

$$(18) \quad \sigma(R_n(\lambda), p_l(\lambda)) = 0, \quad 0 \leq l \leq n - 1.$$

In accordance with relation 3) there exists a polynomial  $\hat{R}_n(\lambda)$  of degree  $n$  such that  $\sigma(R_n(\lambda), \hat{R}_n(\lambda)) \neq 0$ . Set  $M_n := \sigma(R_n(\lambda), \hat{R}_n(\lambda))$ . For the polynomial  $\hat{R}_n$ , we can write

$$\hat{R}_n(\lambda) = \hat{\mu}_n R_n(\lambda) + \hat{u}_{n-1}(\lambda), \quad \hat{\mu}_n \in \mathbb{C}, \hat{\mu}_n \neq 0,$$

where  $\hat{u}_{n-1} \in \mathbb{P}$ ,  $\deg \hat{u}_{n-1} \leq n - 1$ .

Using (18) we get

$$(19) \quad M_n = \sigma(R_n(\lambda), \hat{R}_n(\lambda)) = \sigma(R_n(\lambda), \hat{\mu}_n R_n(\lambda) + \hat{u}_{n-1}(\lambda)) = \hat{\mu}_n \sigma(R_n(\lambda), R_n(\lambda)).$$

<sup>1</sup>this was missed in [7, Theorem 2].

Therefore,

$$(20) \quad \sigma(R_n(\lambda), R_n(\lambda)) = \frac{M_n}{\widehat{\mu}_n}.$$

Set

$$(21) \quad \mu_n = \sqrt{\frac{\widehat{\mu}_n}{M_n}},$$

where we take an arbitrary branch of the square root.

We set

$$(22) \quad p_n(\lambda) = \mu_n R_n(\lambda).$$

By virtue of (20) and (18) we get that relation (9) holds for polynomials  $p_0, p_1, \dots, p_n$ .

Applying this procedure successively for  $n = N, N+1, N+2, \dots$ , we obtain a sequence of polynomials  $\{p_k(\lambda)\}_{k \in \mathbb{Z}_+}$ ,  $\deg p_k = k$ , such that relation (9) holds.

Fix an arbitrary  $k \in \mathbb{Z}_+$ . For the polynomial  $\lambda^N p_k(\lambda)$  we can write the following decomposition:

$$(23) \quad \lambda^N p_k(\lambda) = \sum_{l=0}^{k+N} \xi_{k,l} p_l(\lambda), \quad \xi_{k,l} \in \mathbb{C}, \quad \xi_{k,k+N} \neq 0.$$

By virtue of (9) we get

$$(24) \quad \xi_{k,l} = \sigma(\lambda^N p_k(\lambda), p_l(\lambda)), \quad 0 \leq l \leq k+N.$$

On the other hand, using relation 1) from the statement of the theorem and relation (12) we can write

$$(25) \quad \begin{aligned} \sigma(\lambda^N p_k(\lambda), p_l(\lambda)) &= -\sigma(p_k(\lambda), \lambda^N p_l(\lambda)) = -\sigma(\lambda^N p_l(\lambda), p_k(\lambda)) \\ &= \begin{cases} -\xi_{l,k}, & l+N \geq k \\ 0, & l+N < k \end{cases}, \quad 0 \leq l \leq k+N. \end{aligned}$$

From (24), (25) it follows that

$$(26) \quad \xi_{k,l} = 0 \quad \text{if } l < k-N;$$

$$(27) \quad \xi_{k,l} = -\xi_{l,k} \quad \text{if } k-N \leq l \quad (0 \leq l \leq k+N).$$

Consequently, relation (23) can be written in the following form:

$$(28) \quad \lambda^N p_k(\lambda) = \sum_{l=k-N}^{k+N} \xi_{k,l} p_l(\lambda),$$

where  $\xi_{k,l}$  and  $p_l$  with negative indices  $l$  are set to be equal to zeros.

Changing the index in the sum in (28)  $l = k+j$ ;  $j = l-k$ ; we can write

$$(29) \quad \lambda^N p_k(\lambda) = \sum_{j=-N}^N \xi_{k,k+j} p_{k+j}(\lambda).$$

Set  $J := (g_{k,l})_{k,l \in \mathbb{Z}_+}$ , where

$$g_{k,l} = \begin{cases} \xi_{k,l}, & l \leq k+N, \\ 0, & \text{otherwise.} \end{cases}$$

Relation (26) yields that the matrix  $J$  is  $(2N+1)$ -diagonal. From (27) we get that  $J$  is antisymmetric.

Polynomials  $\{p_k\}_{k \in \mathbb{Z}_+}$  form a solution of difference equation (6) corresponding to  $J$ . The initial conditions are  $p_k(\lambda) = \lambda^k$ ,  $k = 0, 1, \dots, N-1$ . Since relation (9) is true, the functional  $\sigma$  is a spectral function of difference equation (6).  $\square$

Let difference equation (6) corresponding to a semi-infinite matrix  $J = (g_{k,l})_{k,l \in \mathbb{Z}_+}$ ,  $g_{k,l} \in \mathbb{C}$ , which satisfies relations (1)–(3), be given. Let  $\{p_k\}_{k \in \mathbb{Z}_+}$  be the corresponding polynomials and  $\sigma$  be a spectral function of (6). Denote the leading coefficient of  $p_k(\lambda)$  by  $\mu_k$ ,  $k \in \mathbb{Z}_+$ . Note that

$$(30) \quad \mu_j = 1, \quad j = 0, 1, \dots, N-1.$$

From relation (8) it follows that

$$(31) \quad \mu_j = \frac{1}{g_{j-N,j}} \mu_{j-N}, \quad j \geq N.$$

Let us prove that

$$(32) \quad \mu_j = \prod_{m=1}^{\lfloor \frac{j}{N} \rfloor} \frac{1}{g_{j-mN, j-(m-1)N}}, \quad j \geq N.$$

For an arbitrary  $j \in \mathbb{Z}_+$ ,  $j \geq N$  we can write  $j = lN + r$ ,  $l \in \mathbb{N}$ ,  $0 \leq r \leq N-1$ . Namely,  $l = \lfloor \frac{j}{N} \rfloor$ .

We shall prove relation (32) using the induction argument. For numbers  $j \geq N$  such that  $\lfloor \frac{j}{N} \rfloor = 1$  we have  $j = N + r$ ,  $0 \leq r \leq N-1$ . By virtue of relations (31) and (30) we conclude

$$(33) \quad \mu_j = \frac{1}{g_{r,j}} \mu_r = \frac{1}{g_{r,j}} = \prod_{m=1}^{\lfloor \frac{j}{N} \rfloor} \frac{1}{g_{j-mN, j-(m-1)N}},$$

and therefore (32) is true in this case.

Suppose that (32) is true for  $\lfloor \frac{j}{N} \rfloor = s$ ,  $s \geq 1$ . For numbers  $j \geq N$  such that  $\lfloor \frac{j}{N} \rfloor = s+1$  we have  $j = (s+1)N + r$ ,  $0 \leq r \leq N-1$ . By virtue of relations (31) and (32) we obtain

$$\begin{aligned} \mu_j &= \frac{1}{g_{j-N,j}} \mu_{j-N} = \frac{1}{g_{j-N,j}} \prod_{m=1}^{\lfloor \frac{j-N}{N} \rfloor} \frac{1}{g_{j-N-mN, j-N-(m-1)N}} \\ &= \frac{1}{g_{j-N,j}} \prod_{m=1}^{\lfloor \frac{j}{N} \rfloor - 1} \frac{1}{g_{j-(m+1)N, j-mN}} = \frac{1}{g_{j-N,j}} \prod_{t=2}^{\lfloor \frac{j}{N} \rfloor} \frac{1}{g_{j-tN, j-(t-1)N}} = \prod_{t=1}^{\lfloor \frac{j}{N} \rfloor} \frac{1}{g_{j-tN, j-(t-1)N}}, \end{aligned}$$

where  $t = m+1$ .

Hence, relation (32) is true for  $\lfloor \frac{j}{N} \rfloor = s+1$ . By induction we obtain that relation (32) holds.

Note that by virtue of (33) for  $j = N + r$ ,  $0 \leq r \leq N-1$ , we have

$$(34) \quad g_{r,j} = \frac{1}{\mu_j}.$$

If we shall construct polynomials using the spectral function  $\sigma$  like in the proof of Theorem 2, we can take different values of the leading coefficient in (21) which differ by the sign. Consequently, according to relation (34) we can obtain different difference equations of type (6) having the spectral function  $\sigma$ .

So, the spectral function does not uniquely define difference equation (6).

Let us define the following sequence of signs:

$$S := \{\pm, \pm, \pm, \dots\},$$

where "+" is placed in the  $k$ -th place if

$$(35) \quad \text{Arg} \prod_{m=1}^{\lfloor \frac{k}{N} \rfloor + 1} \frac{1}{g_{k-(m-1)N, k-(m-2)N}} \in [0, \pi),$$

and "-" otherwise,  $k \in \mathbb{Z}_+$ .

In other words, relation (35) means that  $\text{Arg} \mu_{k+N} \in [0, \pi)$ .

The spectral function  $\sigma$  and the sequence of signs  $S$  uniquely define difference equation (6). Let us prove this. We can construct polynomials  $\{P_k\}_{k \in \mathbb{Z}_+}$  (we denote them by  $P_k$  instead of  $p_k$  to avoid the confusion) using the procedure in the proof of Theorem 1. In relation (21) there we take the value of the square root with the argument in  $[0, \pi)$  if "+" stands in  $S$  in the  $(n - N)$ -th place and with the argument in  $[\pi, 2\pi)$  otherwise. Let us show that  $P_k = p_k$ ,  $k \in \mathbb{Z}_+$ . From the initial conditions it follows that this is true for  $k = 0, 1, \dots, N - 1$ . Suppose it is true for  $k = 0, 1, \dots, n - 1$ ,  $n \geq N$ . To avoid the confusion we denote the leading coefficient of  $P_k$  by  $\tilde{\mu}_k$ ,  $k \in \mathbb{Z}_+$ . We can write the following decomposition for the polynomial  $p_n(\lambda)$ :

$$(36) \quad p_n(\lambda) = \mu_n \lambda^n + \sum_{k=0}^{n-1} \gamma_{n,k} p_k(\lambda) = \mu_n \lambda^n + \sum_{k=0}^{n-1} \gamma_{n,k} P_k(\lambda), \quad \gamma_{n,k} \in \mathbb{C}.$$

By virtue of orthogonality of polynomials we get

$$0 = \sigma(p_n(\lambda), p_k(\lambda)) = \mu_n \sigma(\lambda^n, p_k(\lambda)) + \gamma_{n,k}, \quad 0 \leq k \leq n - 1.$$

Thus, we have

$$\gamma_{n,k} = -\mu_n \sigma(\lambda^n, p_k(\lambda)) = -\mu_n \sigma(\lambda^n, P_k(\lambda)), \quad 0 \leq k \leq n - 1.$$

Therefore, we get

$$p_n(\lambda) = \mu_n \left( \lambda^n - \sum_{k=0}^{n-1} \sigma(u^n, P_k(u)) P_k(\lambda) \right) = \mu_n R_n(\lambda).$$

Consequently, we obtain

$$(37) \quad p_n(\lambda) = \frac{\mu_n}{\tilde{\mu}_n} P_n(\lambda).$$

Using orthonormality we get

$$1 = \sigma(p_n, p_n) = \sigma\left(\frac{\mu_n}{\tilde{\mu}_n} P_n, \frac{\mu_n}{\tilde{\mu}_n} P_n\right) = \left(\frac{\mu_n}{\tilde{\mu}_n}\right)^2 \sigma(P_n, P_n) = \left(\frac{\mu_n}{\tilde{\mu}_n}\right)^2.$$

Hence, we have  $(\mu_n)^2 = (\tilde{\mu}_n)^2$ . In accordance with our choice of the branch of the square root in the construction of  $P_n$  we get  $\mu_n = \tilde{\mu}_n$ . So, we get  $p_n = P_n$ . By induction we obtain  $P_k = p_k$ ,  $k \in \mathbb{Z}_+$ .

From relation (6) using orthonormality of polynomials we get

$$(38) \quad g_{k,k+j} = \sigma(\lambda^k p_k(\lambda), p_{k+j}(\lambda)), \quad k \in \mathbb{Z}_+, \quad -N \leq j \leq N : \quad k + j \geq 0.$$

Consequently, all coefficients  $g_{k,l}$  of difference equation (6) are uniquely determined by the polynomials  $\{p_k\}_{k \in \mathbb{Z}_+}$  and  $\sigma$ .

The given above procedure for the reconstruction of difference equation (6) from its spectral function and  $S$  gives an answer to the remaining questions concerning the inverse spectral problem.

*Example.* Consider a semi-infinite three-diagonal antisymmetric matrix  $J = (g_{k,l})_{k,l \in \mathbb{Z}_+}$ ,

$$(39) \quad g_{k,l} = \begin{cases} c, & l = k + 1, \\ -c, & k = l + 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $c$  is a complex parameter.

The corresponding polynomials satisfy the following difference equation:

$$(40) \quad -cp_{k-1}(\lambda) + cp_{k+1}(\lambda) = \lambda p_k(\lambda), \quad k \in \mathbb{Z}_+,$$

where  $p_{-1} = 0$ ,  $p_1 = 1$ .

Let  $U_k(x) = \frac{\sin((k+1)\arccos x)}{\sqrt{1-x^2}}$ ,  $k \in \mathbb{Z}_+$ ,  $x \in \mathbb{C}$ , be Chebyshev's polynomials of the second kind. They satisfy the following difference equation:

$$(41) \quad U_{k-1}(x) + U_{k+1}(x) = 2xU_k(x), \quad k \in \mathbb{Z}_+,$$

where  $U_{-1} = 0$ ,  $U_1 = 1$ .

For an arbitrary  $\lambda \in \mathbb{C}$  we can write

$$(42) \quad \begin{aligned} U_{k-1}\left(\frac{i}{2c}\lambda\right) + U_{k+1}\left(\frac{i}{2c}\lambda\right) &= \frac{i}{c}\lambda U_k\left(\frac{i}{2c}\lambda\right), \quad k \in \mathbb{Z}_+, \\ \frac{c}{i}U_{k-1}\left(\frac{i}{2c}\lambda\right) + \frac{c}{i}U_{k+1}\left(\frac{i}{2c}\lambda\right) &= \lambda U_k\left(\frac{i}{2c}\lambda\right), \quad k \in \mathbb{Z}_+. \end{aligned}$$

Let us multiply both sides of relation (42) by  $(-i)^k$ . We get

$$(43) \quad -c(-i)^{k-1}U_{k-1}\left(\frac{i}{2c}\lambda\right) - c(-i)^{k-1}U_{k+1}\left(\frac{i}{2c}\lambda\right) = \lambda(-i)^k U_k\left(\frac{i}{2c}\lambda\right), \quad k \in \mathbb{Z}_+.$$

Comparing relations (43) and (40) we see that

$$(44) \quad p_k(\lambda) = (-i)^k U_k\left(\frac{i}{2c}\lambda\right), \quad k \in \mathbb{Z}_+.$$

Note that polynomials  $U_k$  satisfy the orthonormality relations

$$(45) \quad \int_{-1}^1 U_k(x)U_l(x) \frac{dx}{\sqrt{1-x^2}} = \delta_{k,l}, \quad k, l \in \mathbb{Z}_+.$$

Using the change of the variable  $x = \frac{i}{2c}\lambda$  we obtain some orthonormality relations for the polynomials  $p_k$ ,

$$(46) \quad \frac{i}{2c} \int_{-\frac{2c}{i}}^{\frac{2c}{i}} p_k(\lambda)p_l(\lambda) \frac{d\lambda}{\sqrt{1 + \frac{\lambda^2}{4c^2}}} = (-i)^{k+l} \delta_{k,l}, \quad k, l \in \mathbb{Z}_+.$$

However, relation (46) does not give an integral representation for the spectral function  $\sigma$  corresponding to  $J$  and we should use the general definition of the spectral function.

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