ON SOME APPROXIMATIONS AND MAIN TOPOLOGICAL DESCRIPTIONS FOR SPECIAL CLASSES OF BANACH SPACES WITH INTEGRABLE DERIVATIVES

PAVLO KASYANOV, VALERIY MEL'NIK, AND ANNA MARIA PICCIRILLO

ABSTRACT. We consider some classes of Banach spaces with integrable derivatives. An important compactness lemma for nonreflexive spaces is obtained. However some main topological properties for the given spaces are obtained.

Method of monotony and method of compactness represent fundamental approaches to study nonlinear differential-operator equations, evolutionary inclusions and variational inequalities in Banach spaces. The general idea is the following: using the corresponding approximation scheme the approximate solutions of a problem are constructed, for them some approaching a priori estimations are established, at last they prove the existence of a sequence of approximate solutions, that converges to an exact solution of problem. In many cases the aim is obtained by using both a method of compactness and a method of monotonicity.

In the present paper we obtain a new of compact embedding theorems for Banach spaces, suggested by researches about differential-operational inclusions in function spaces. Moreover, we introduce some constructions to prove the convergence of Faedo–Galerkin method for evolution variation inequalities with w_λ–pseudomonotone maps [7], [8], [11], [14], [5].

In the following referring to Banach spaces X, Y, when we write

$$X \subset Y,$$

we mean the embedding in the set-theory sense and in the topological sense.

For $n \geq 2$ let $\{X_i\}_{i=1}^n$ be some family of Banach spaces.

Definition 1. *An interpolation family* refers to a family of Banach spaces $\{X_i\}_{i=1}^n$ such that for some locally convex linear topological space (LCLTS) Y we have

$$X_i \subset Y \quad \text{for all} \quad i = \overline{1, n}.$$

If $n = 2$, then the interpolation family is called *an interpolation pair*.

Further let $\{X_i\}_{i=1}^n$ be some interpolation family. By analogy with ([3], p. 23), in the linear variety $X = \cap_{i=1}^n X_i$ we consider the norm

$$(1) \qquad \|x\|_X := \sum_{i=1}^{n} \|x\|_{X_i} \quad \forall x \in X,$$

where $\|\cdot\|_{X_i}$ is the norm in X_i.

Proposition 1. *Let $\{X, Y, Z\}$ be an interpolation family. Then*

$$X \cap (Y \cap Z) = (X \cap Y) \cap Z = X \cap Y \cap Z, \quad X \cap Y = Y \cap X$$

Received by the editors 27/02/2007; Revised 18/12/2007.

2000 *Mathematics Subject Classification.* Primary 41A65; Secondary 46E40.

Key words and phrases. Some approximations for Banach spaces, embedding theorems, Faedo-Galerkin method.

both in the sense of equality of sets and in the sense of equality of norms.

We also consider the linear space

$$Z := \sum_{i=1}^{n} X_i = \left\{ \sum_{i=1}^{n} x_i \;\middle|\; x_i \in X_i,\ i = \overline{1,n} \right\}$$

with the norm

$$\|z\|_Z := \inf \left\{ \max_{i=\overline{1,n}} \|x_i\|_{X_i} \;\middle|\; x_i \in X_i,\ \sum_{i=1}^{n} x_i = z \right\} \quad \forall z \in Z. \tag{2}$$

Proposition 2. *Let $\{X_i\}_{i=1}^n$ be an interpolation family. Then $X = \cap_{i=1}^n X_i$ and $Z = \sum_{i=1}^n X_i$ are Banach spaces and it results in*

$$X \subset X_i \subset Z \quad \text{for all} \quad i = \overline{1,n}. \tag{3}$$

The proof is similar to the proof of the corresponding statement from ([3], chapter I).

Remark 1. ([3], p. 24). Let Banach spaces X and Y satisfy the following conditions:

$$X \subset Y, \quad X \text{ is dense in } Y,$$

$$\|x\|_Y \le \gamma \|x\|_X \quad \forall x \in X, \quad \gamma = \text{const}.$$

Then

$$Y^* \subset X^*, \quad \|f\|_{X^*} \le \gamma \|f\|_{Y^*} \quad \forall f \in Y^*.$$

Moreover, if X is reflexive, then Y^* is dense in X^*.

Let $\{X_i\}_{i=1}^n$ be an interpolation family such that the space $X := \cap_{i=1}^n X_i$ with the norm (1) is dense in X_i for all $i = \overline{1,n}$. Due to Remark 1, the space X_i^* can be considered as a subspace of X^*. Thus we can construct $\sum_{i=1}^n X_i^*$ and

$$\sum_{i=1}^{n} X_i^* \subset \left(\bigcap_{i=1}^{n} X_i \right)^*. \tag{4}$$

Under the given assumptions X is dense in $Z := \sum\limits_{i=1}^{n} X_i$ for every $i = \overline{1,n}$. So X_i is dense in Z, too. Thanks to Remark 1 we can consider the space Z^* as a subspace of X_i^* for all $i = \overline{1,n}$, and also as a subspace of $\cap_{i=1}^n X_i^*$, i.e.,

$$\left(\sum_{i=1}^{n} X_i \right)^* \subset \bigcap_{i=1}^{n} X_i^*. \tag{5}$$

Theorem 1. *Let $\{X_i\}_{i=1}^n$ be an interpolation family such that the space $X := \cap_{i=1}^n X_i$ with the norm (1) is dense in X_i for all $i = \overline{1,n}$. Then*

$$\sum_{i=1}^{n} X_i^* = \left(\bigcap_{i=1}^{n} X_i \right)^*$$

and

$$\left(\sum_{i=1}^{n} X_i \right)^* = \bigcap_{i=1}^{n} X_i^*$$

both in the sense of equality of sets and in the sense of equality of the norms.

The proof is similar to the proof of the corresponding Theorem I.5.13 from [3] and based on the next lemma.

Lemma 1. *Let* $f \in \cap_{i=1}^{n} X_i^*$. *Then for every* $k = \overline{2,n}$ *and* $x_i, y_i \in X_i$ $(i = \overline{1,k})$ *such that* $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i =: x$, *we have*

$$\sum_{i=1}^{k} f(x_i) = \sum_{i=1}^{k} f(y_i) =: f(x). \tag{6}$$

Now let Y be some Banach space, Y^* its topological conjugate space, S be some compact time interval. We consider the classes of functions defined on S with values in Y (or in Y^*).

The set $L_p(S;Y)$ of all Bochner measurable functions (see [3], [4]) for $1 \le p \le +\infty$ with natural linear operations and the norm

$$\|y\|_{L_p(S;Y)} = \left(\int_S \|y(t)\|_Y^p dt \right)^{1/p}$$

is a Banach space. For $p = +\infty$, $L_\infty(S;Y)$ with the norm

$$\|y\|_{L_\infty(S;Y)} = \operatorname*{vrai\,max}_{t \in S} \|y(t)\|_Y$$

is a Banach space.

The next theorem shows that under the assumption of reflexivity or separability of Y, the space $(L_p(S;Y))^*$ conjugate to $L_p(S;Y)$, $1 \le p < +\infty$, can be identified with $L_q(S;Y^*)$, where q is such that $p^{-1} + q^{-1} = 1$.

Theorem 2. ([4], Theorems 8.18.3, 8.20.5). *If the space* Y *is reflexive and* $1 \le p < +\infty$, *then each element* $f \in (L_p(S;Y))^*$ *has the unique representation*

$$f(y) = \int_S \langle \xi(t), y(t) \rangle_Y dt \quad \text{for every} \quad y \in L_p(S;Y)$$

with a function $\xi \in L_q(S;Y^*)$, $p^{-1} + q^{-1} = 1$. *The correspondence* $f \to \xi$, *with* $f \in (L_p(S;Y))^*$ *is linear and*

$$\|f\|_{(L_p(S;Y))^*} = \|\xi\|_{L_q(S;Y^*)}.$$

We remark that in the latter theorem, $(L_p(S;Y))^*$ is considered to be the Banach space of all linear continuous functionals that map $L_p(S;Y)$ into $\mathbb{R}$.

Now let us consider a reflexive separable Banach space V with a norm $\|\cdot\|_V$ and a Hilbert space $(H, (\cdot,\cdot)_H)$ with a norm $\|\cdot\|_H$ and, for the given spaces, let the next conditions be satisfied:

$$\begin{gathered} V \subset H, \quad V \text{ is dense in } H, \\ \exists \gamma > 0 \ : \ \|v\|_H \le \gamma \|v\|_V \quad \forall v \in V. \end{gathered} \tag{7}$$

Due to Remark 1, under the given assumptions we can consider the space H^* conjugate to H as a subspace of V^* that is conjugate to V. Since V is reflexive, H^* is dense in V^* and

$$\|f\|_{V^*} \le \gamma \|f\|_{H^*} \quad \forall f \in H^*,$$

where $\|\cdot\|_{V^*}$ and $\|\cdot\|_{H^*}$ are the norms in V^* and in H^*, respectively.

Further, we identify the spaces H and H^*. Then we obtain

$$V \subset H \subset V^*$$

with a continuous and dense embedding.

Definition 2. The triple of spaces $(V; H; V^*)$, which satisfy the latter conditions, will be called *an evolution triple*.

Let us point out that identifying H with H^* and H^* with some subspace of V^*, an element $y \in H$ is identified with some $f_y \in V^*$ and

$$(y, x) = \langle f_y, x\rangle_V \quad \forall x \in V,$$

where $\langle\cdot,\cdot\rangle_V$ is the canonical pairing between V^* and V. Since an element y and f_y are identified, under condition (7), the pairing $\langle\cdot,\cdot\rangle_V$ will denote the inner product on H $(\cdot,\cdot)$.

We consider p_i, r_i, $i = 1,2$, such that $1 < p_i \le r_i \le +\infty$, $p_i < +\infty$. Let $q_i \ge r_i' \ge 1$ be defined by

$$p_i^{-1} + q_i^{-1} = r_i^{-1} + r_i'^{-1} = 1 \quad \forall i = 1,2.$$

Remark that $1/\infty = 0$.

Now we consider some Banach spaces that play an important role in the investigation of differential-operator equations and evolution variational inequalities in non-reflexive Banach spaces.

Referring to evolution triples $(V_i; H; V_i^*)$ $(i = 1,2)$ such that

(8) $\qquad$ the set $V_1 \cap V_2$ is dense in the spaces V_1, V_2 and H

we consider the functional Banach spaces (proposition 2)

$$X_i = X_i(S) = L_{q_i}(S; V_i^*) + L_{r_i'}(S; H), \quad i = 1,2$$

with the norms

$$\|y\|_{X_i} = \inf\Big\{ \max\{ \|y_1\|_{L_{q_i}(S;V_i^*)}; \|y_2\|_{L_{r_i'}(S;H)} \} \;\Big|\; y_1 \in L_{q_i}(S; V_i^*),\ y_2 \in L_{r_i'}(S;H),\ y = y_1 + y_2\Big\},$$

for all $y \in X_i$, and

$$X = X(S) = L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*) + L_{r_2'}(S; H) + L_{r_1'}(S; H)$$

with

$$\|y\|_X = \inf\Big\{ \max_{i=1,2}\{ \|y_{1i}\|_{L_{q_i}(S;V_i^*)}; \|y_{2i}\|_{L_{r_i'}(S;H)} \} \;\Big|\; y_{1i} \in L_{q_i}(S; V_i^*),\ y_{2i} \in L_{r_i'}(S; H),\ i = 1,2;\ y = y_{11} + y_{12} + y_{21} + y_{22}\Big\},$$

for each $y \in X$. Since $r_i < +\infty$, due to Theorem 1 and Theorem 2, the space X_i is reflexive. Analogously, if $\max\{r_1, r_2\} < +\infty$, the space X is reflexive.

Under the latter theorems we identify the space conjugate to $X_i(S)$, $X_i^* = X_i^*(S)$, with $L_{r_i}(S; H) \cap L_{p_i}(S; V_i)$, where

$$\|y\|_{X_i^*} = \|y\|_{L_{r_i}(S;H)} + \|y\|_{L_{p_i}(S;V_i)} \quad \forall y \in X_i^*,$$

and, respectively, the space conjugate to $X(S)$, $X^* = X^*(S)$, we identify with

$$L_{r_1}(S; H) \cap L_{r_2}(S; H) \cap L_{p_1}(S; V_1) \cap L_{p_2}(S; V_2),$$

where

$$\|y\|_{X^*(S)} = \|y\|_{L_{r_1}(S;H)} + \|y\|_{L_{r_2}(S;H)} + \|y\|_{L_{p_1}(S;V_1)} + \|y\|_{L_{p_2}(S;V_2)} \quad \forall y \in X^*.$$

On $X(S) \times X^*(S)$ we denote

$$\begin{aligned}\langle f, y\rangle = \langle f, y\rangle_S &= \int_S (f_{11}(\tau), y(\tau))_H\, d\tau + \int_S (f_{12}(\tau), y(\tau))_H\, d\tau \\ &+ \int_S \langle f_{21}(\tau), y(\tau)\rangle_{V_1} d\tau + \int_S \langle f_{22}(\tau), y(\tau)\rangle_{V_2} d\tau \\ &= \int_S (f(\tau), y(\tau))\, d\tau \quad \forall f \in X,\ y \in X^*,\end{aligned}$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S; H)$, $f_{2i} \in L_{q_i}(S; V_i^*)$, $i = 1,2$.

Let $V = V_1 \cap V_2$, $\mathcal{F}(V)$ be a filter of all finite-dimensional subspaces of V. Since V is separable, there exists a countable monotone increasing system of subspaces $\{H_i\}_{i\geq 1} \subset \mathcal{F}(V)$ complete in V, and consequently in H. On H_n we consider the inner product induced from H, which we denote again by $(\cdot,\cdot)$. Moreover let $P_n : H \to H_n \subset H$ be an orthogonal projection from H onto H_n defined by

$$\text{for every } h \in H, \quad P_n h = \arg\min_{h_n \in H_n} \|h - h_n\|_H.$$

Definition 3. We say that the triple $(\{H_i\}_{i\geq 1}; V; H)$ satisfies *condition (γ)*, if

$$\sup_{n\geq 1} \|P_n\|_{\mathcal{L}(V,V)} < +\infty,$$

i.e., there exists $C \geq 1$ such that, for every $v \in V$ and $n \geq 1$,

$$\|P_n v\|_V \leq C \cdot \|v\|_V. \tag{9}$$

Some constructions that satisfy condition (γ) were presented in [9].

Remark 2. It is easy to notice that there exists a complete orthonormal in H vector system $\{h_i\}_{i\geq 1} \subset V$ such that, for any $n \geq 1$, H_n is a linear capsule stretched on $\{h_i\}_{i=1}^n$. Then condition (γ) means that the given system is a Schauder basis in the space V (see [6], p. 403).

Remark 3. Due to the identification of H^* and H, it follows that H_n^* and H_n are identified too.

Remark 4. Since $P_n \in \mathcal{L}(V,V)$ for every $n \geq 1$, we get $P_n^* \in \mathcal{L}(V^*,V^*)$ and $\|P_n\|_{\mathcal{L}(V,V)} = \|P_n^*\|_{\mathcal{L}(V^*,V^*)}$. It is clear that, for every $h \in H$, $P_n h = P_n^* h$. Hence, we identify P_n with its conjugate P_n^* for every $n \geq 1$. Then, condition (γ) means that for every $v \in V$ and $n \geq 1$, we have

$$\|P_n v\|_V \leq C \cdot \|v\|_V \quad \text{and} \quad \|P_n v\|_{V^*} \leq C \cdot \|v\|_{V^*}. \tag{10}$$

For each $n \geq 1$, we consider the Banach spaces

$$X_n = X_n(S) = L_{q_0}(S; H_n) \subset X, \quad X_n^* = X_n^*(S) = L_{p_0}(S; H_n) \subset X^*,$$

where $p_0 := \max\{r_1, r_2\}$, $q_0^{-1} + p_0^{-1} = 1$ with the natural norms. The space $L_{q_0}(S; H_n)$ is isometrically isomorphic to the space X_n^* conjugate of X_n and, moreover, the map

$$X_n \times X_n^* \ni f, x \to \int_S (f(\tau), x(\tau))_{H_n} d\tau = \int_S (f(\tau), x(\tau))\, d\tau = \langle f, x \rangle_{X_n}$$

is the duality form on $X_n \times X_n^*$. This statement is correct due to

$$L_{q_0}(S; H_n) \subset L_{q_0}(S; H) \subset L_{r_1'}(S; H) + L_{r_2'}(S; H) + L_{q_1}(S; V_1^*) + L_{q_2}(S; V_2^*).$$

Let us remark that $\langle \cdot,\cdot \rangle_S \big|_{X_n(S)\times X_n^*(S)} = \langle \cdot,\cdot \rangle_{X_n(S)}$.

Proposition 3. *For every $n \geq 1$ $X_n = P_n X$, i.e.,*

$$X_n = \{P_n f(\cdot) \mid f(\cdot) \in X\}.$$

Moreover, if the triple $(\{H_j\}_{j\geq 1}; V_i; H)$, $i = 1, 2$, satisfies the condition (γ) with $C = C_i$, then

$$\textit{for every } f \in X \textit{ and } n \geq 1 \quad \|P_n f\|_X \leq \max\{C_1, C_2\} \cdot \|f\|_X.$$

Proof. Let us fix an arbitrary number $n \geq 1$. For every $y \in X$ let $y_n(\cdot) := P_n y(\cdot)$, i.e., $y_n(t) = P_n y(t)$ for almost all $t \in S$. Since P_n is linear and continuous on V_1^*, on V_2^* and on H we have that $y_n \in X_n \subset X$. It is immediate that the inverse inclusion is valid.

Now let us prove the second part of this statement. We suppose that condition (γ) holds, let us fix $f \in X$ and $n \geq 1$. Then from condition (γ) it follows that for every $f_{1i} \in L_{r_i'}(S;H)$ and $f_{2i} \in L_{q_i}(S;V_i^*)$ such that $f = f_{11} + f_{12} + f_{21} + f_{22}$ we have

$$\|P_n f_{11}\|_{L_{r_1'}(S;H)} + \|P_n f_{12}\|_{L_{r_2'}(S;H)} + \|P_n f_{21}\|_{L_{q_1}(S;V_1^*)} + \|P_n f_{22}\|_{L_{q_2}(S;V_2^*)}$$
$$\leq \max\{C_1, C_2\} \cdot \left(\|f_{11}\|_{L_{r_1'}(S;H)} + \|f_{12}\|_{L_{r_2'}(S;H)} + \|f_{21}\|_{L_{q_1}(S;V_1^*)} + \|f_{22}\|_{L_{q_2}(S;V_2^*)} \right),$$

because $C_1, C_2 \geq 1$. Therefore, due to the definition of the norm in X we complete the proof. □

Proposition 4. *For every $n \geq 1$ it results in $X_n^* = P_n X^*$, i.e.,*

$$X_n^* = \{P_n y(\cdot) \mid y(\cdot) \in X^*\},$$

and

$$\langle f, P_n y\rangle = \langle f, y\rangle \quad \forall y \in X^* \text{ and } f \in X_n.$$

Furthermore, if the triple $(\{H_j\}_{j\geq 1}; V_i; H)$, $i = 1, 2$ satisfies condition (γ) with $C = C_i$, then we have

$$\|P_n y\|_{X^*} \leq \max\{C_1, C_2\} \cdot \|y\|_{X^*} \quad \forall y \in X^* \text{ and } n \geq 1.$$

Proof. For every $y \in X^*$, we set $y_n(\cdot) := P_n y(\cdot)$, i.e., $y_n(t) = P_n y(t)$ for a.e. $t \in S$. Since the operator P_n is linear and continuous on V_1, on V_2 and on H we have that $y_n \in {X^*}_n \subset X^*$. The inverse inclusion is obvious.

Due to condition (γ) and the definition of $\|\cdot\|_{L_{p_i}(S;V_i)}$ and $\|\cdot\|_{L_{r_i}(S;H)}$, it follows that

$$\|y_n\|_{L_{p_i}(S;V_i)} \leq C_i \cdot \|y\|_{L_{p_i}(S;V_i)} \quad \text{and} \quad \|y_n\|_{L_{r_i}(S;H)} \leq \|y\|_{L_{r_i}(S;H)}.$$

Thus $\|y_n\|_{X^*} \leq \max\{C_1, C_2\} \cdot \|y\|_{X^*}$.

Now let us show that, for every $f \in X_n$,

$$\langle f, y_n\rangle = \langle f, y\rangle.$$

Since $f \in L_{p_0}(S; H_n)$, we have

$$\langle f, y\rangle = \int_S (f(\tau), y(\tau))\, d\tau = \int_S (f(\tau), P_n y(\tau))\, d\tau$$
$$= \int_S (f(\tau), y_n(\tau))\, d\tau = \langle f, y_n\rangle,$$

because for every $n \geq 1$, $h \in H$ and $v \in H_n$, it follows that

$$(h - P_n h, v) = (h - P_n h, v)_H = 0.$$

The proposition is proved. □

Proposition 5. *Under the condition $\max\{r_1, r_2\} < +\infty$, the set $\bigcup_{n\geq 1} X_n^*$ is dense in $(X^*, \|\cdot\|_{X^*})$.*

Proof. a) At first we prove that the set $L_\infty(S;V)$ is dense in the space

$$(X^*, \|\cdot\|_{X^*}).$$

Let us fix $x \in X^*$. Then for every $n \geq 1$ we consider

$$x_n(t) := \begin{cases} x(t), & \|x(t)\|_V \leq n, \\ 0, & \text{elsewhere}. \end{cases} \tag{11}$$

Obviously, for all $n \geq 1$, $x_n \in L_\infty(S;V)$. The continuous embedding of V into H assures existence of some positive γ such that for $i = 1, 2$ and a.e. $t \in S$ we have

$$\begin{cases} \|x_n(t) - x(t)\|_H \leq \gamma \|x_n(t) - x(t)\|_V \to 0, \\ \|x_n(t) - x(t)\|_{V_i} \leq \|x_n(t) - x(t)\|_V \to 0 \quad \text{as} \quad n \to \infty, \end{cases} \tag{12}$$

(13) $$\|x_n(t)\|_H \le \|x(t)\|_H, \quad \|x_n(t)\|_{V_i} \le \|x(t)\|_{V_i}.$$

Further, let us set

$$\phi_H^n(t) = \|x_n(t) - x(t)\|_H^{p_0}, \quad \phi_{V_i}^n(t) = \|x_n(t) - x(t)\|_{V_i}^{p_i}.$$

So, from (12) and (13) we obtain

(14) $$\phi_H^n(t) \to 0, \quad \phi_{V_i}^n(t) \to 0 \ \text{ as } n \to \infty \quad \text{for a.e. } t \in S$$

and for almost every $t \in S$

(15) $$|\phi_H^n(t)| \le 2^{p_0}\|x(t)\|_H^{p_0} =: \phi_H(t), \quad |\phi_{V_i}^n(t)| \le 2^{p_i}\|x(t)\|_{V_i}^{p_i} =: \phi_{V_i}(t).$$

Since $x \in X^*$, we have ϕ_H, ϕ_{V_1}, $\phi_{V_2} \in L_1(S)$. Thus, due to (14) and (15), we can apply the Lebesgue theorem with integrable majorants ϕ_H, ϕ_{V_1} and ϕ_{V_2} respectively. Hence it follows that $\phi_H^n \to \bar{0}$ and $\phi_{V_i}^n \to \bar{0}$ in $L_1(S)$ for $i = 1, 2$. Consequently, $\|x_n - x\|_{X^*} \to 0$ as $n \to \infty$.

b) Further, for some linear variety L from V we set

$$\Upsilon(L) := \{y \in (S \to L) \mid y \text{ is a simple function}\}$$

(see [3], p. 152). Let us show that the set $\Upsilon(V)$ is dense in the normalized space $(L_\infty(S,V), \|\cdot\|_{X^*})$. Let x be fixed in $L_\infty(S,V)$; so, there exists a sequence $\{x_n\}_{n\ge1} \subset \Upsilon(V)$ such that

(16) $$x_n(t) \to x(t) \text{ in } V \quad \text{as } n \to \infty \quad \text{for a.e. } t \in S.$$

Since $x \in L_\infty(S,V)$, we have $\operatorname{ess\,sup}_{t\in S} \|x(t)\|_V =: c < +\infty$. For every $n \ge 1$, let us introduce

(17) $$y_n(t) := \begin{cases} x_n(t), & \|x_n(t)\|_V \le 2c, \\ \bar{0}, & \text{else.} \end{cases}$$

From (16) and (17) it follows that $y_n \in \Upsilon(V)$ as $n \ge 1$ and, moreover,

$$y_n(t) \to x(t) \text{ in } V \quad \text{as } n \to \infty \quad \text{for a.e. } t \in S.$$

Hence, taking into account that $V \subset H$ as $i = 1, 2$, for a.e. $t \in S$ we obtain the following convergences

$$y_n(t) \to x(t) \text{ in } H, \quad y_n(t) \to x(t) \text{ in } V_1, \quad y_n(t) \to x(t) \text{ in } V_2 \quad \text{as } n \to \infty.$$

As in a), assuming that

$$\phi_H \equiv \phi_{V_1} \equiv \phi_{V_2} \equiv \max\{(3c)^{p_1}, (3c)^{p_2}, (3c\gamma)^{p_0}\} \in L_1(S))$$

we obtain that $y_n \to x$ in X^* as $n \to \infty$. So, $\Upsilon(V)$ is dense in

$$(L_\infty(S,V), \|\cdot\|_{X^*}).$$

c) Since the set $\operatorname{span}\{h_n\}_{n\ge1} = \bigcup_{n\ge1} H_n$ is dense in $(V, \|\cdot\|_V)$ and $V \subset H$ with continuous embedding it follows that the set

$$\Upsilon\Big(\bigcup_{n\ge1} H_n\Big) = \bigcup_{n\ge1} \Upsilon(H_n)$$

is dense in $(\Upsilon(V), \|\cdot\|_{X^*})$.

In order to complete the proof we point out that, for every $n \ge 1$, $\Upsilon(H_n) \subset {X^*}_n$. The proposition is proved. □

Now we consider the Banach space $W^* = \{y \in X^* | \, y' \in X\}$ with the norm

$$\|y\|_{W^*} = \|y\|_{X^*} + \|y'\|_X,$$

where the derivative y' of an element $y \in X^*$ is taken in the sense of the scalar distribution space $\mathcal{D}^*(S; V^*) = \mathcal{L}(\mathcal{D}(S); V_w^*)$, where V_w^* is equal to V^* with the topology $\sigma(V^*; V)$ [12].

Together with $W^* = W^*(S)$, we consider the Banach space

$$W_i^* = W_i^*(S) = \{y \in L_{p_i}(S; V_i) | \, y' \in X(S)\}, \quad i = 1, 2,$$

with the norm

$$\|y\|_{W_i^*} = \|y\|_{L_{p_i}(S;V_i)} + \|y'\|_X \quad \forall y \in W_i^*.$$

We also consider the space $W_0^* = W_0^*(S) = W_1^*(S) \cap W_2^*(S)$ with the norm

$$\|y\|_{W_0^*} = \|y\|_{L_{p_1}(S;V_1)} + \|y\|_{L_{p_2}(S;V_2)} + \|y'\|_X \quad \forall y \in W_0^*.$$

The space W^* is continuously embedded in W_i^* for $i = \overline{0, 2}$.

Theorem 3. *It follows that $W_i^* \subset C(S; V^*)$, where the embedding is continuous for $i = \overline{0, 2}$.*

The proof is similar to the corresponding proof for lemma IV.1.11 from [3].

Remark 5. From the definition of the norms in the spaces W^* and W_0^*, we obtain that $W^* \subset C(S; V^*)$ with the continuous embedding for the compact S in the natural topology of the space W^*.

Having in mind applications to evolution equations and inclusions we need to give some generalization and some improvement of the results in [3], [13], [10].

Theorem 4. *The set $C^1(S; V) \cap W_0^*$ is dense in W_0^*.*

The proof is similar to the proof for the Lemma IV.1.12 from [3].

Theorem 5. *$W_0^* \subset C(S; H)$ with continuous embedding. Moreover, for every $y, \xi \in W_0^*$ and $s, t \in S$, the next integration by parts formula takes place:*

$$\big(y(t), \xi(t)\big) - \big(y(s), \xi(s)\big) = \int_s^t \Big\{ (y'(\tau), \xi(\tau)) + (y(\tau), \xi'(\tau)) \Big\} d\tau. \tag{18}$$

In particular, when $y = \xi$, we have

$$\frac{1}{2}\Big(\|y(t)\|_H^2 - \|y(s)\|_H^2 \Big) = \int_s^t (y'(\tau), y(\tau)) \, d\tau.$$

Proof. Similarly to the proof for the Theorem IV.1.17 from [3] we consider $S = [a, b]$ for some

$$-\infty < a < b < +\infty.$$

The validity of formula (18) for $y, \xi \in C^1(S; V)$ is checked by a direct calculation. Now let $\varphi \in C^1(S)$ be fixed such that $\varphi(a) = 0$ and $\varphi(b) = 1$. Moreover, for $y \in C^1(S; V)$ let $\xi = \varphi y$ and $\eta = y - \varphi y$. Then, due to (18),

$$\big(\xi(t), y(t)\big) = \int_a^t \Big\{ \varphi'(s)\big(y(s), y(s)\big) + 2\varphi(s)(y'(s), y(s)) \Big\} ds,$$

$$-\big(\eta(t), y(t)\big) = \int_t^b \Big\{ -\varphi'(s)\big(y(s), y(s)\big) + 2(1 - \varphi(s))(y'(s), y(s)) \Big\} ds.$$

This shows that for $\xi_i \in L_{q_i}(S;V_i^*)$ and $\eta_i \in L_{r_i'}(S;H)$ $(i = 1,2)$ such that $y' = \xi_1 + \xi_2 + \eta_1 + \eta_2$,

$$\begin{aligned}
\|y(t)\|_H^2 &= \int_t^b \Big\{\varphi'(s)\big(y(s),y(s)\big) + 2\varphi(s)(y'(s),y(s))\Big\} ds - 2\int_t^b (y'(s),y(s))\,ds \\
&\le \max_{s\in S}|\varphi'(s)|\cdot\|y\|_{C(S;V^*)}\cdot\|y\|_{L_1(S;V)} + 2\int_S (\varphi(s)-1)(y'(s),y(s))\,ds \\
&\le \max_{s\in S}|\varphi'(s)|\cdot\|y\|_{C(S;V^*)}\cdot\|y\|_{L_1(S;V)} \\
&+ 2\max_{s\in S}|\varphi(s)-1|\cdot\Big(\|\xi_1\|_{L_{q_1}(S;V_1^*)}\|y\|_{L_{p_1}(S;V_1)} + \|\xi_2\|_{L_{q_2}(S;V_2^*)}\|y\|_{L_{p_2}(S;V_2)} \\
&+ \|\eta_1\|_{L_{r_1'}(S;H)}\|y\|_{L_{r_1}(S;H)} + \|\eta_2\|_{L_{r_2'}(S;H)}\|y\|_{L_{r_2}(S;H)}\Big) \\
&\le \max_{s\in S}|\varphi'(s)|\cdot\|y\|_{C(S;V^*)}\cdot\Big(\|y\|_{L_{p_1}(S;V_1)}\mathrm{mes}(S)^{1/q_1} + \|y\|_{L_{p_2}(S;V_2)}\mathrm{mes}(S)^{1/q_2}\Big) \\
&+ 2\max_{s\in S}|\varphi(s)-1|\cdot\Big(\|\xi_1\|_{L_{q_1}(S;V_1^*)} + \|\xi_2\|_{L_{q_2}(S;V_2^*)} \\
&+ \|\eta_1\|_{L_{r_1'}(S;H)} + \|\eta_2\|_{L_{r_2'}(S;H)}\Big)\cdot\Big(\|y\|_{L_{p_1}(S;V_1)} + \|y\|_{L_{p_2}(S;V_2)} \\
&+ \|y\|_{C(S;H)}\mathrm{mes}(S)^{1/r_1} + \|y\|_{C(S;H)}\mathrm{mes}(S)^{1/r_2}\Big).
\end{aligned}$$

Hence, due to Theorem 3, the definition of $\|\cdot\|_X$, taking $\varphi(t) = \frac{t-a}{b-a}$ for all $t \in S$ in last inequality we obtain

$$\|y\|_{C(S;H)}^2 \le C_2\cdot\|y\|_{W_0^*}^2 + C_3\cdot\|y\|_{W_0^*}\cdot\|y\|_{C(S;H)}, \tag{19}$$

where C_1 is the constant from the inequality $\|y\|_{C(S;V^*)} \le C_1\cdot\|y\|_{W_0^*}$ for every $y \in W_0^*$,

$$C_2 = 2 + \frac{C_1}{\min\{\mathrm{mes}(S)^{1/p_1}, \mathrm{mes}(S)^{1/p_2}\}}, \quad C_3 = 2\cdot\max\Big\{\mathrm{mes}(S)^{1/\min\{r_1,r_2\}}, 1\Big\}.$$

Remark that $\frac{1}{+\infty} = 0$, $C_2, C_3 > 0$. From (19) it obviously follows that

$$\|y\|_{C(S;H)} \le C_4\cdot\|y\|_{W_0^*} \quad \text{for all } y \in C^1(S;V), \tag{20}$$

where $C_4 = \frac{C_3+\sqrt{C_3^2+4C_2}}{2}$ does not depend on y.

Now let us apply Theorem 4. For arbitrary $y \in W_0^*$, let $\{y_n\}_{n\ge1}$ be a sequence of elements from $C^1(S;V)$ converging to y in W_0^*. Then, in virtue of relation (20), we have

$$\|y_n - y_k\|_{C(S;H)} \le C_4\|y_n - y_k\|_{W_0^*} \to 0$$

and, therefore, the sequence $\{y_n\}_{n\ge1}$ converges in $C(S;H)$ and its limit $\chi \in C(S;H)$ such that, for a.e. $t \in S$, $\chi(t) = y(t)$. So, we have $y \in C(S;H)$ and now the embedding $W_0^* \subset C(S;H)$ is proved. If we pass to the limit in (20) with $y = y_n$ as $n \to \infty$ we obtain validity of the given estimate for all $y \in W_0^*$. This proves continuity of the embedding of W^* into $C(S;H)$.

Now let us prove formula (18). For every $y, \xi \in W_0^*$ and for corresponding approximating sequences $\{y_n, \xi_n\}_{n\ge1} \subset C^1(S;V)$ we pass to the limit in (18) with $y = y_n$, $\xi = \xi_n$ as $n \to \infty$. In virtue of Lebesgue's theorem and $W_0^* \subset C(S;V^*)$ with continuous embedding, formula (18) is true for every $y \in W_0^*$.

The theorem is proved. □

Since $W^* \subset W_0^*$ with continuous embedding and due to the latter theorem the next statement is true.

Corollary 1. *$W^* \subset C(S;H)$ with continuous embedding. Moreover, for every $y, \xi \in W^*$ and $s, t \in S$ formula (18) takes place.*

For every $n \geq 1$ let us define the Banach space $W_n^* = \{y \in X_n^* | \ y' \in X_n\}$ with the norm

$$\|y\|_{W_n^*} = \|y\|_{X_n^*} + \|y'\|_{X_n},$$

where the derivative y' is considered in the sense of the scalar distributions space $\mathcal{D}^*(S; H_n)$. Since

$$\mathcal{D}^*(S; H_n) = \mathcal{L}(\mathcal{D}(S); H_n) \subset \mathcal{L}(\mathcal{D}(S); V_\omega^*) = \mathcal{D}^*(S; V^*),$$

it is possible to consider the derivative of an element $y \in X_n^*$ in the sense of $\mathcal{D}^*(S; V^*)$. Remark that, for every $n \geq 1$, $W_n^* \subset W_{n+1}^* \subset W^*$.

Proposition 6. *For every $y \in X^*$ and $n \geq 1$, we have $P_n y' = (P_n y)'$, where the derivative of an element $x \in X^*$ is understood in the sense of the scalar distributions space $\mathcal{D}^*(S; V^*)$.*

Remark 6. We remark that in virtue of the previous assumptions, the derivatives of an element $x \in X_n^*$ in taken the sense of $\mathcal{D}(S; V^*)$ and in the sense of $\mathcal{D}(S; H_n)$ coincide.

Proof. It is sufficient to show that, for every $\varphi \in D(S)$, $P_n y'(\varphi) = (P_n y)'(\varphi)$. In virtue of the definition of the derivative in the sense of $\mathcal{D}^*(S; V^*)$, we have

$$\forall \varphi \in D(S) \quad P_n y'(\varphi) = -P_n y(\varphi') = -P_n \int_S y(\tau)\varphi'(\tau) d\tau =$$

$$= -\int_S P_n y(\tau)\varphi'(\tau) d\tau = -(P_n y)(\varphi') = (P_n y)'(\varphi).$$

The proposition is proved. □

Due to propositions 4, 3, 6 we have the following.

Proposition 7. *For every $n \geq 1$ $W_n^* = P_n W^*$, i.e.,*

$$W_n^* = \{P_n y(\cdot) \mid y(\cdot) \in W^*\}.$$

Moreover, if the triple $(\{H_i\}_{i \geq 1}; V_j; H)$, $j = 1, 2$ satisfies condition (γ) with $C = C_j$. Then for every $y \in W^*$ and $n \geq 1$

$$\|P_n y(\cdot)\|_{W^*} \leq \max\{C_1, C_2\} \cdot \|y(\cdot)\|_{W^*}.$$

Theorem 6. *Let the triple $(\{H_i\}_{i \geq 1}; V_j; H)$, $j = 1, 2$ satisfy condition (γ) with $C = C_j$. We consider bounded in X^* set $D \subset X^*$ and $E \subset X$ that is bounded in X. For every $n \geq 1$, let us us consider*

$$D_n := \left\{ y_n \in X_n^* \,\middle|\, y_n \in D \text{ and } y_n' \in P_n E \right\} \subset W_n^*.$$

Then

$$\|y_n\|_{W^*} \leq \|D\|_+ + C \cdot \|E\|_+ \quad \text{for all } n \geq 1 \text{ and } y_n \in D_n, \tag{21}$$

where $C = \max\{C_1, C_2\}$, $\|D\|_+ = \sup_{y \in D} \|y\|_{X^*}$ and $\|E\|_+ = \sup_{f \in E} \|f\|_X$.

Remark 7. Due to Proposition 3, D_n is well-defined and $D_n \subset W_n^*$.

Remark 8. A priori estimates (like (21)) appear when studying solvability of differential–operator equations, inclusions and evolutional variational inequalities in Banach spaces with maps of w_λ-pseudomonotone type by using Faedo–Galerkin method (see [7], [8]) at boundary transition, when it is necessary obtain a priori estimates for approximate solutions y_n in X^* and its derivatives y_n' in X.

Proof. Due to proposition 3, for every $n \geq 1$ and $y_n \in D_n$,

$$\|y_n\|_{W^*} = \|y_n\|_{X^*} + \|y_n'\|_X \leq \|D\|_+ + \|P_n E\|_+ \leq \|D\|_+ + \max\{C_1, C_2\} \cdot \|E\|_+.$$

The theorem is proved. □

Further, let B_0, B_1, B_2 be some Banach spaces such that

$$B_0, B_2 \text{ are reflexive}, \quad B_0 \subset B_1 \text{ with compact embedding}, \tag{22}$$

$$B_0 \subset B_1 \subset B_2 \text{ with continuous embedding}. \tag{23}$$

The next result and its proof is some variations on Theorem 5 and Theorem 7 from [13].

Theorem 7. *Let conditions (22)–(23) for B_0, B_1, B_2 be satisfied, $p_0, p_1 \in [1; +\infty)$, S be a finite time interval and $K \subset L_{p_1}(S; B_0)$ be such that*

a)

$$K \text{ is bounded in } L_{p_1}(S; B_0);$$

b) *for every $\varepsilon > 0$ there exists such $\delta > 0$ that from $0 < h < \delta$ it follows that*

$$\int_S \|u(\tau) - u(\tau+h)\|_{B_2}^{p_0} d\tau < \varepsilon \quad \forall u \in K. \tag{24}$$

Then K is precompact in $L_{\min\{p_0;p_1\}}(S; B_1)$.

Furthermore, if K is bounded in $L_q(S; B_1)$ for some $q > 1$, then K is precompact in $L_p(S; B_1)$ for every $p \in [1, q)$.

Remark 9. Further we consider that every element $x \in (S \to B_i)$ is equal to $\bar{0}$ outside the interval S.

Proof. At the beginning we consider the first case. For our goal it is enough to show, that it is possible to choose a Cauchy subsequence from every sequence $\{y_n\}_{n\geq 1} \subset K$ in $L_{\min\{p_0;p_1\}}(S; B_1)$. Due to Lemma 9 from [13] it is sufficient to prove this statement for $L_{\min\{p_0;p_1\}}(S; B_2)$.

For every $x \in K$ $\forall h > 0$ $\forall t \in S$ we put

$$x_h(t) := \frac{1}{h} \int_t^{t+h} x(\tau)\, d\tau,$$

where the integral is regarded in the Bochner sense. We point out that $\forall h > 0$ $x_h \in C(S; B_0) \subset C(S; B_2)$.

Fixing a positive number ε we construct, for the set

$$K \subset L_{p_0}(S; B_0) \subset L_{p_0}(S; B_2),$$

a final ε-net in $L_{p_0}(S; B_2)$. For $\varepsilon > 0$, we choose $\delta > 0$ from (24). Then, for every fixed h $(0 < h < \delta)$, we have

$$\|x_h(t+u) - x_h(t)\|_{B_2} = \frac{1}{h} \left\| \int_{t+u}^{t+u+h} x(\tau)\, d\tau - \int_t^{t+h} x(\tau)\, d\tau \right\|_{B_2}$$
$$= \frac{1}{h} \left\| \int_t^{t+h} x(\tau+u)\, d\tau - \int_t^{t+h} x(\tau)\, d\tau \right\|_{B_2} \leq \frac{1}{h} \int_t^{t+h} \|x(\tau+u) - x(\tau)\|_{B_2} d\tau.$$

Moreover, from the Hölder inequality we obtain

$$\frac{1}{h} \int_t^{t+h} \|x(\tau+u) - x(\tau)\|_{B_2} d\tau \leq \left(\frac{1}{h}\right)^{\frac{1}{p_0}} \left(\int_t^{t+h} \|x(\tau+u) - x(\tau)\|_{B_2}^{p_0} d\tau \right)^{\frac{1}{p_0}}$$
$$\leq \left(\frac{1}{h}\right)^{\frac{1}{p_0}} \left(\int_0^T \|x(\tau+u) - x(\tau)\|_{B_2}^{p_0} d\tau \right)^{\frac{1}{p_0}} < \left(\frac{\varepsilon}{h}\right)^{\frac{1}{p_0}}$$
$$\forall x \in K \quad \forall\, 0 < u < \delta \quad \forall t \in S.$$

Therefore the family of functions $\{x_h\}_{x\in K}$ is equicontinuous.

Since $\forall x \in K\ \forall t \in S$,

$$\|x_h(t)\|_{B_2} = \frac{1}{h}\left\| \int_t^{t+h} x(\tau)\,d\tau \right\|_{B_2} \le \frac{1}{h}\int_t^{t+h} \|x(\tau)\|_{B_2} d\tau$$
$$\le \left(\frac{1}{h}\right)^{\frac{1}{p_1}} \left(\int_t^{t+h} \|x(\tau)\|_{B_2}^{p_1} d\tau\right)^{\frac{1}{p_1}} \le \left(\frac{1}{h}\right)^{\frac{1}{p_1}} \left(\int_0^{T} \|x(\tau)\|_{B_2}^{p_1} d\tau\right)^{\frac{1}{p_1}} \le \left(\frac{C}{h}\right)^{\frac{1}{p_1}},$$

the family of functions $\{x_h\}_{x\in K}$ is uniformly bounded, because the constant $C \ge 0$ does not depend on $x \in K$. Hence, $\forall h : 0 < h < \delta$ the family of functions $\{x_h\}_{x\in K}$ is precompact in $C(S; B_2)$, so in $L_{\min\{p_0,p_1\}}(S; B_2)$ too.

On the other hand, $\forall\, 0 < h < \delta\ \forall x \in K\ \forall t \in S$,

$$\|x(t) - x_h(t)\|_{B_2} \le \frac{1}{h}\int_t^{t+h} \|x(t) - x(\tau)\|_{B_2} d\tau$$
$$\le \frac{1}{h}\int_0^{h} \|x(t) - x(t+\tau)\|_{B_2} d\tau \le \left(\frac{1}{h}\right)^{\frac{1}{p_0}} \left(\int_0^{h} \|x(t) - x(t+\tau)\|_{B_2}^{p_0} d\tau\right)^{\frac{1}{p_0}}.$$

From here, taking into account inequality (24) we obtain

$$\left(\int_0^T \|x(t) - x_h(t)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} \le \left(\int_0^T \frac{1}{h}\int_0^h \|x(t) - x(t+\tau)\|_{B_2}^{p_0} d\tau dt\right)^{\frac{1}{p_0}}$$
$$= \left(\frac{1}{h}\int_0^h \int_0^T \|x(t) - x(t+\tau)\|_{B_2}^{p_0} dt\, d\tau\right)^{\frac{1}{p_0}} < \left(\frac{1}{h}\int_0^h \varepsilon\, d\tau\right)^{\frac{1}{p_0}} = \varepsilon^{\frac{1}{p_0}}.$$

Hence, in virtue of precompactness of the system $\{x_h\}_{x\in K}$ in $L_{\min\{p_0,p_1\}}(S; B_2)$ $\forall\, 0 < h < \delta$ we have that K is a precompact set in $L_{\min\{p_0,p_1\}}(S; B_2)$.

Let us consider the second case. Assume that for some $q > 1$ the set K is bounded in $L_q(S; B_1)$. Similarly to the previous case, it is enough to show that for every $p \in [1; q)$ and $\{y_n\}_{n\ge1} \subset K$ there exists a subsequence $\{y_{n_k}\}_{k\ge1} \subset \{y_n\}_{n\ge1}$ and $y \in L_p(S; B_1)$ such that

$$y_{n_k} \to y \quad \text{in} \quad L_p(S; B_1) \quad \text{as} \quad k \to \infty.$$

Because $y_n \to y$ in $L_{\min\{p_0,p_1\}}(S; B_1)$, passing to a subsequence for $n \to \infty$, we have $\exists \{y_{n_k}\}_{k\ge1} \subset \{y_n\}_{n\ge1}$ such that $\lambda(B_{n_k}) \to 0$ as $k \to \infty$, where $B_n := \{t \in S \mid \|y_n(t) - y(t)\|_{B_1} \ge 1\}$ for every $n \ge 1$, λ is the Lebesgue measure on S. Then, for every $k \ge 1$,

$$\int_S \|y_{n_k}(s) - y(s)\|_{B_1}^p ds = \int_{A_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p ds + \int_{B_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p ds$$
$$\le \int_{A_{n_k}} \|y_{n_k}(s) - y(s)\|_{B_1}^p ds + \left(\int_S \|y_{n_k}(s) - y(s)\|_{B_1}^q ds\right)^{\frac{p}{q}} (\lambda(B_{n_k}))^{\frac{q-p}{q}} =: I_{n_k} + J_{n_k},$$

where $A_n = S \setminus B_n$ for every $n \ge 1$.

It is clear that $J_{n_k} \to 0$ as $k \to \infty$. Let us consider I_{n_k}. Since $\{y_{n_k}\}_{k\ge1}$ is precompact in $L_{\min\{p_0,p_1\}}(S; B_1)$, there exists $\{y_{m_k}\}_{k\ge1} \subset \{y_{n_k}\}_{k\ge1}$ such that $y_{m_k}(t) \to y(t)$ in B_1 as $k \to \infty$ almost everywhere in S. Setting

$$\forall k \ge 1 \quad \forall t \in S \qquad \varphi_{m_k}(t) := \begin{cases} \|y_{m_k}(t) - y(t)\|_{B_1}^p, & t \in A_n \\ 0, & \text{otherwise,} \end{cases}$$

using definition of A_{m_k}, we see that the sequence $\{\varphi_{m_k}\}_{k\ge1}$ satisfies the conditions of the Lebesgue theorem with the integrable majorant $\phi \equiv 1$. So $\varphi_{m_k} \to \overline{0}$ in $L_1(S)$ as $k \to \infty$. Thus, within to a subsequence, $y_n \to y$ in $L_q(S; B_1)$.

The theorem is proved. □

Let Banach spaces B_0, B_1, B_2 satisfy all assumptions (22)–(23), $p_0, p_1 \in [1; +\infty)$ be arbitrary numbers. We consider the set with the natural operations,

$$W = \{v \in L_{p_0}(S; B_0) \,|\, v' \in L_{p_1}(S; B_2)\},$$

where the derivative v' of an element $v \in L_{p_0}(S; B_0)$ is considered in the sense of the scalar distribution space $\mathcal{D}(S; B_2)$. It is clear that

$$W \subset L_{p_0}(S; B_0).$$

We remark that the set W with the natural operations and the graph norm

$$\|v\|_W = \|v\|_{L_{p_0}(S;B_0)} + \|v'\|_{L_{p_1}(S;B_2)}$$

is a Banach space. Under conditions (22)–(23), $W \subset C(S; B_2)$ with the continuous embedding. (The proof clearly follows from the proof for Theorem 1.11 from ([3], p. 173) using Theorem 2.2 from ([2], p. 19).)

The next result represents some generalization of the compactness lemma [10] (Theorem 1.5.1, p. 70) to the case $p_0, p_1 \in [1; +\infty)$. A Similar proposition formulated in [13] on p. 89 (see (10.6)) without a valid proof. The author remarks that the proof is based on Theorem 7 (an analogue of Theorem 7) and on Lemma 4 (an analogue of Lemma 2). The proof of the given lemma is based on the inequalities (1.3)–(1.5) that are given in author's paper without a substantiation (the reference to paper [13] is groundless). We remark that the proof of an analogue of (25) for the spaces $L_p(S; B_2)$ with $p > 1$ is easier and essentially differs from the proof for the case $p = 1$. We remark also that we do not assume the that (10.1) from ([13], p. 87) holds. So, we try to give a formal proof for an analogue of (10.6) given in paper [13].

Theorem 8. *Under conditions (22)–(23), for all $p_0, p_1 \in [1; +\infty)$, the Banach space W is compactly embedded in $L_{p_0}(S; B_1)$.*

Proof. At the beginning we prove that the embedding of W into $L_1(S; B_2)$ is compact.

For every $y \in W$ and $h \in \mathbb{R}$ let us take

$$y_h(t) = \begin{cases} y(t+h), & \text{if } t+h \in S, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

In virtue of the continuous embedding $W \subset C(S; B_2)$, the given definition is correct.

Lemma 2. *For every $y \in W$ and $h \in \mathbb{R}$,*

$$\|y - y_h\|_{L_1(S;B_2)} \leq h \cdot \|y'\|_{L_1(S;B_2)}. \tag{25}$$

Proof. Let $y \in W$ be fixed. Then

$$\|y - y_h\|_{L_1(S;B_2)} = \int_S \|y(t+h) - y(t)\|_{B_2} dt = \int_S \Big\| \int_t^{t+h} y'(\tau) d\tau \Big\|_{B_2} dt.$$

Let us put

$$g_y(t) = \int_t^{t+h} y'(\tau)\, d\tau = y(t+h) - y(t) \quad \forall t \in S, \quad i = 1, 2.$$

Since the embedding $W \subset C(S; B_2)$ is continuous, $g_y \in C(S; B_2)$. Hence, as S is a compact set, we have that $g_y \in L_1(S; B_2)$. Therefore, due to Proposition ([1], p. 191) with $X = L_1(S; B_2)$ and using Theorem 2 it follows that there exists $h_y \in L_\infty(S; B_2^*) \equiv X^*$ such that

$$\int_S \|g_y(t)\|_{B_2} dt = \int_S \langle h_y(t), g_y(t) \rangle_{B_2} dt \quad \text{and} \quad \|h_y\|_{L_\infty(S;B_2^*)} = 1.$$

Hence,

$$\begin{aligned}
&\int_S \Big\| \int_t^{t+h} y'(\tau)\, d\tau \Big\|_{B_2} dt = \int_S \|g_y(t)\|_{B_2}\, dt = \int_S \langle h_y(t), g_y(t)\rangle_{B_2}\, dt \\
&= \int_S \Big\langle h_y(t), \int_t^{t+h} y'(\tau) d\tau \Big\rangle_{B_2} dt = \int_S \int_t^{t+h} \langle h_y(t), y'(\tau)\rangle_{B_2}\, d\tau dt \\
&= \int_S \int_{\tau-h}^{\tau} \langle h_y(t), y'(\tau)\rangle_{B_2}\, dt\, d\tau = \int_S \Big\langle \int_{\tau-h}^{\tau} h_y(t) dt, y'(\tau) \Big\rangle_{B_2} d\tau \\
&\leq \operatorname*{ess\,sup}_{t\in S} \|h_y(t)\|_{B_2^*} \cdot h \cdot \int_S \|y'(\tau)\|_{B_2} d\tau \leq h \cdot \|y'\|_{L_1(S;B_2)}.
\end{aligned}$$

So, we have obtained the necessary estimate (25).

The lemma is proved. □

Let us continue the proof of the theorem. Let $K \subset W$ be an arbitrary bounded set. Then, for some $C > 0$,

$$\|y\|_{L_{p_0}(S;B_0)} \leq C, \quad \|y'\|_{L_{p_1}(S;B_2)} \leq C \quad \forall y \in K. \tag{26}$$

In order to prove precompactness of K in $L_1(S; B_1)$, let us apply Theorem 7 with $B_0 = B_0$, $B_1 = B_1$, $B_2 = B_2$, $p_0 = 1$, $p_1 = p_1$. Due to estimates (25) and (26), all conditions of the theorem are satisfied. So, the set K is precompact in $L_1(S; B_1)$ and, hence, in $L_1(S; B_2)$. In virtue of $W \subset C(S; B_2)$ with the continuous embedding and the Lebesgue theorem it follows that the set K is precompact in $L_{p_0}(S; B_0)$. Hence, due to Lemma 9 from [13] we obtain the necessary statement.

The theorem is proved. □

Proposition 8. *Let Banach spaces B_0, B_1, B_2 satisfy conditions (22)–(23), $p_0, p_1 \in [1; +\infty)$, $\{u_h\}_{h\in I} \subset L_{p_1}(S; B_0)$, where $I = (0, \delta) \subset \mathbb{R}_+$, $S = [a, b]$ such that*

- a) *$\{u_h\}_{h\in I}$ is bounded in $L_{p_1}(S; B_0)$;*
- b) *there exists $c : I \to \mathbb{R}_+$ such that $\underline{\lim}_{n\to\infty} c(\frac{b-a}{2^n}) = 0$ and*

$$\forall h \in I \quad \int_S \|u_h(t) - u_h(t+h)\|^{p_0}_{B_2} dt \leq c(h) h^{p_0}.$$

Then there exists $\{h_n\}_{n\geq 1} \subset I$ ($h_n \searrow 0+$ as $n \to \infty$) such that $\{u_{h_n}\}_{n\geq 1}$ converges in $L_{\min\{p_0,p_1\}}(S; B_1)$.

Remark 10. We assume that $u_h(t) = \overline{0}$ for $t > b$.

Remark 11. Without loss of generality let us consider $S = [0, 1]$.

Proof. At first we prove this statement for $L_{p_0}(S; B_2)$. In virtue of the Minkowski inequality, for every $h = \frac{1}{2^N} \in I$ and $k \geq 1$,

$$\begin{aligned}
&\Big(\int_0^1 \|u_h(t) - u_{\frac{h}{2^k}}(t)\|^{p_0}_{B_2} dt \Big)^{\frac{1}{p_0}} \leq \Big(\int_0^1 \|u_h(t) - u_h(t+h)\|^{p_0}_{B_2} dt \Big)^{\frac{1}{p_0}} \\
&+ \Big(\int_0^1 \|u_h(t+h) - u_{\frac{h}{2^k}}(t+h)\|^{p_0}_{B_2} dt \Big)^{\frac{1}{p_0}} + \Big(\int_0^1 \|u_{\frac{h}{2^k}}(t+h) - u_{\frac{h}{2^k}}(t)\|^{p_0}_{B_2} dt \Big)^{\frac{1}{p_0}}
\end{aligned}$$

$$\leq c^{\frac{1}{p_0}}(h)h + \left(\int_h^1 \|u_h(t) - u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} + \sum_{i=0}^{2^k-1} \left(\int_0^1 \left\|u_{\frac{h}{2^k}}\left(t + \frac{i+1}{2^k}h\right)\right.\right.$$
$$\left.\left.-u_{\frac{h}{2^k}}\left(t + \frac{i}{2^k}h\right)\right\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} \leq c^{\frac{1}{p_0}}(h)h + 2^k \frac{h}{2^k} c^{\frac{1}{p_0}}(h/2^k)$$
$$+ \left(\int_h^1 \|u_h(t) - u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} \leq h\left(c^{\frac{1}{p_0}}(h) + c^{\frac{1}{p_0}}(h/2^k)\right)$$
$$+ \left(\int_h^1 \|u_h(t) - u_h(t+h)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} + \left(\int_h^1 \|u_h(t+h) - u_{\frac{h}{2^k}}(t+h)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}}$$
$$+ \left(\int_h^1 \|u_{\frac{h}{2^k}}(t+h) - u_{\frac{h}{2^k}}(t)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} \leq \cdots \leq 2h\left(c^{\frac{1}{p_0}}(h) + c^{\frac{1}{p_0}}(h/2^k)\right)$$
$$+ \left(\int_{2h}^1 \|u_h(t) - u_h(t+h)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} \leq \cdots \leq 2^N h\left(c^{\frac{1}{p_0}}(h) + c^{\frac{1}{p_0}}(h/2^k)\right)$$
$$= c^{\frac{1}{p_0}}(h) + c^{\frac{1}{p_0}}(h/2^k).$$

So, for every $N \geq 1$ and $k \geq 1$, it results in

$$\left(\int_0^1 \|u_{1/2^N}(t) - u_{1/2^{N+k}}(t)\|_{B_2}^{p_0} dt\right)^{\frac{1}{p_0}} \leq c^{\frac{1}{p_0}}\left(\frac{1}{2^N}\right) + c^{\frac{1}{p_0}}\left(\frac{1}{2^{N+k}}\right).$$

In virtue of assumption b) we can choose $\{h_n\}_{n\geq 1} \subset \left\{\frac{1}{2^m}\right\}_{m\geq 1} \bigcap I$ such that $c(h_n) \to 0$ as $n \to \infty$. So, the sequence $\{u_{h_n}\}_{n\geq 1}$ is fundamental in $L_{p_0}(S; B_2)$. Because of $B_0 \subset B_1$ with compact embedding, the sequence $\{u_{h_n}\}_{n\geq 1}$ is bounded in $L_{\min\{p_0,p_1\}}(S; B_0)$; due to the lemma 9 from [13] it follows that $\{u_{h_n}\}_{n\geq 1}$ is fundamental in $L_{\min\{p_0,p_1\}}(S; B_1)$.

The proposition is proved. □

Now we combine all the results to obtain a necessary a priori estimate.

Theorem 9. *Let all conditions of Theorem 6 be satisfied and $V \subset H$ with compact embedding. Then (21) holds and the set*

$$\bigcup_{n\geq 1} D_n \text{ is bounded in } C(S; H) \text{ and precompact in } L_p(S; H)$$

for every $p \geq 1$.

Proof. Estimate (21) follows from Theorem 6. Now we use the compactness Theorem 8 with $B_0 = V$, $B_1 = H$, $B_2 = V^*$, $p_0 = 1$, $p_1 = 1$. Remark that $X^* \subset L_1(S; V)$ and $X \subset L_1(S; V^*)$ with continuous embedding. Hence, the set

$$\bigcup_{n\geq 1} D_n \text{ is precompact in } L_1(S; H).$$

In virtue of (21) and Theorem 5 on continuous embedding of W^* in $C(S; H)$, it follows that the set

$$\bigcup_{n\geq 1} D_n \text{ is bounded in } C(S; H).$$

Further, by using standard conclusions and the Lebesgue theorem we obtain the necessary statement.

The theorem is proved. □

Bibliography

1. J. P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, Mir, Moscow, 1988. (Russian)
2. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei R.S.R, Bucharest, 1976.
3. H. Gaevsky, K. Greger, K. Zaharias, *Nonlinear Operator Equations and Operator-Differential Equations*, Mir, Moscow, 1978. (Russian)
4. R. Edwards, *Functional Analysis*, Mir, Moscow, 1969. (Russian)
5. V. I. Ivanenko, V. S. Mel'nik, *Variational Methods in Optimal Control Problems for Systems with Distributed Parameters*, Kiev, Naukova Dumka, 1988. (Russian)
6. R. C. James, *Superreflexive spaces with basis*, Pacific J. Math. **41** (1972), no. 2, 409–419.
7. P. O. Kasyanov, *Galerkin's method for one class differential-operator inclusions*, Reports Nation. Acad. Sci. Ukraine (2005), no. 9, 20–24. (Ukrainian)
8. P. O. Kasyanov, V. S. Mel'nik, *Faedo-Galerkin method for differential-operator inclusions in Banach spaces with maps of w_{λ_0}-pseudomonotone type*, Zb. prac' Inst. mat. NAN Ukr. **2** (2005), no. 1, 103–126. (Ukrainian)
9. P. O. Kasyanov, V. S. Mel'nik, A. M. Piccirillo, *On the Schauder basis in some Banach spaces*, Reports Nation. Acad. Sci. Ukraine (2006), no. 4, 31–35.
10. J. L. Lions, *Quelques Méthodes de Résolution des Problemès aux Limites non Lineaires*, Paris, Dunod, Gauthier-Villars, 1969.
11. V. S. Mel'nik, L. Toscano, *About nonlinear differential-operator equations in Banach spaces with maps of pseudo-monotone type*, System Research & Information Technologies (2004), no. 3, 63–81.
12. M. Reed, B. Simon, *Methods of Modern Mathematical Physics,* Part I, Moscow, Mir, 1976. (Russian)
13. J. Simon, *Compact sets in the space $L^p(0,T;B)$*, J. Annali di Matematica Pura ed Applicata **146** (1986), no. 1, 65–96.
14. M. Z. Zgurovskiy, V. S. Mel'nik, A. N. Novikov, *Applied Methods of Analysis and Control of Nonlinear Processes and Fields*, Kyiv, Naukova Dumka, 2004. (Russian).

Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail address: kasyanov@univ.kiev.ua

Institute for Applied and System Analysis National Academy of Sciences of Ukraine, Kyiv, Ukraine
E-mail address: VSMelnik@ukr.net

Second University of Naples, Aversa, Italy
E-mail address: annamaria.piccirillo@unina2.it