# ON THE APPROXIMATION TO SOLUTIONS OF OPERATOR EQUATIONS BY THE LEAST SQUARES METHOD 

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Dedicated to the centenary of S. I. Zuchovitsky.


#### Abstract

We consider the equation $A u=f$, where $A$ is a linear operator with compact inverse in a Hilbert space. For the approximate solution $u_{n}$ of this equation by the least squares method in a coordinate system that is an orthonormal basis of eigenvectors of a self-adjoint operator $B$ similar to $A(\mathcal{D}(A)=\mathcal{D}(B))$, we give a priori estimates for the asymptotic behavior of the expression $R_{n}=\left\|A u_{n}-f\right\|$ as $n \rightarrow \infty$. A relationship between the order of smallness of this expression and the degree of smoothness of the solution $u$ with respect to the operator $B$ (direct and converse theorems) is established.


0. Let $\mathfrak{H}$ be a complex separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, and let $A$ be a linear operator in $\mathfrak{H}, \overline{\mathcal{D}(A)}=\mathfrak{H}, 0 \in \rho(A)$, where $\mathcal{D}(\cdot)$ and $\rho(\cdot)$ are the domain and the resolvent set of an operator, respectively.

We consider the equation

$$
\begin{equation*}
A u=f, \quad f \in \mathfrak{H}, \tag{1}
\end{equation*}
$$

and seek an approximate solution $u_{n}$ of this equation by the least squares method [1], that is, in the form $u_{n}=\sum_{k=1}^{n} \alpha_{k} e_{k}$, where $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ is a given linearly independent system of vectors in $\mathcal{D}(A)$ (a so-called coordinate system), and $\alpha_{k} \in \mathbb{C}$ are such that

$$
R_{n}^{2}=\left\|A u_{n}-f\right\|^{2}
$$

assumes the least value. The numbers $\alpha_{k}$ are uniquely determined by the system of algebraic equations

$$
\sum_{k=1}^{n} \alpha_{k}\left(A e_{k}, A e_{i}\right)=\left(f, A e_{i}\right), \quad i=1, \ldots, n
$$

If the system $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ is complete in the Hilbert space

$$
\mathfrak{H}^{1}(A)=\mathcal{D}(A), \quad(f, g)_{\mathfrak{H}^{1}(A)}=(A f, A g), \quad f, g \in \mathcal{D}(A)
$$

then (e.g., see [1]),

$$
r_{n}=\left\|u-u_{n}\right\| \rightarrow 0, \quad R_{n}=\left\|A u_{n}-f\right\| \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty .
$$

As it will be shown below, the value $R_{n}$ can tend to zero at infinity in an arbitrary way. So, it is important to have a priori estimates for this value. As a rule, such estimates are of an asymptotic character and indicate an order of smallness of $R_{n}$. For the Ritz

[^0]method, such kind estimates were established under a certain choice of a coordinate system and some conditions of smoothness for the solution $u$ of equation (1) (see [1-3]).

The purpose of this paper is to obtain similar estimates for the least squares method. It should be noted that in a number of specific cases, such questions were being considered by many mathematicians (we refer to [4] for details). But their considerations concerned only direct theorems and only power decreasing of $R_{n}$. To do this in the general situation and to obtain not only direct but also converse theorems for an arbitrary order of convergence of $R_{n}$ to zero, we use the operator approach proposed in $[2,5,6]$, with some generalizations and refinements of the results contained there, which we shall present some later. The results of this paper is partially announced in [7].

We would like also to remark that the significant role in application of the operator theory to the approximation problems belongs to S. I. Zuchovitsky (in this connection see, for example, $[8,9])$.

1. Let $B$ be a closed densely defined operator in $\mathfrak{H}$. Denote by $C^{\infty}(B)$ the space of infinitely differentiable vectors of the operator $B$ :

$$
C^{\infty}(B)=\bigcap_{n \in \mathbb{N}_{0}=\mathbb{N} \cup\{0\}} \mathfrak{H}^{n}(B),
$$

where $\mathfrak{H}^{n}(B)=\mathcal{D}\left(B^{n}\right)$ is a Hilbert space with respect to the inner product

$$
(f, g)_{\mathfrak{H}^{n}(B)}=(f, g)+\left(B^{n} f, B^{n} g\right)
$$

Let $\left\{m_{n}\right\}_{n \in \mathbb{N}_{0}}$ be a nondecreasing sequence of positive numbers (there is no loss of generality in assuming that $m_{0}=1$ ). We put

$$
C_{\left\{m_{n}\right\}}(B)=\bigcup_{\alpha>0} C_{m_{n}}^{\alpha}(B), \quad C_{\left(m_{n}\right)}(B)=\bigcap_{\alpha>0} C_{m_{n}}^{\alpha}(B)
$$

where

$$
C_{m_{n}}^{\alpha}(B)=\left\{f \in C^{\infty}(B) \mid \exists c=c(f)>0, \forall k \in \mathbb{N}_{0}:\left\|B^{k} f\right\| \leq c \alpha^{k} m_{k}\right\}
$$

is a Banach space with the norm

$$
\|f\|_{C_{m_{n}}^{\alpha}(B)}=\sup _{k \in \mathbb{N}_{\mathrm{O}}} \frac{\left\|B^{k} f\right\|}{\alpha^{k} m_{k}} .
$$

The number

$$
\sigma\left(f, m_{n}, B\right)=\inf \left\{\alpha: f \in C_{m_{n}}^{\alpha}(B)\right\}=\varlimsup_{n \rightarrow \infty}\left(\frac{\left\|B^{n} f\right\|}{m_{n}}\right)^{1 / n}
$$

is called the type of the vector $f$ for the operator $B$ and the sequence $\left\{m_{n}\right\}_{n \in \mathbb{N}_{0}}$. It is evident that $\sigma\left(f, m_{n}, B\right)=0$ if $f \in C_{\left(m_{n}\right)}(B)$.

In the cases when $m_{n}=n$ ! or $m_{n} \equiv 1$, we arrive at the well-known spaces of analytic $\left(C_{\{n!\}}(B)\right)$, entire $\left(C_{(n!)}(B)\right)$, and entire of exponential type $\left(C_{\{1\}}(B)\right)$ vectors of the operator $B$ (see [10-12]). If $\mathfrak{H}=L_{2}(a, b),-\infty<a<b<\infty$, and $B=\frac{d}{d x}, \mathcal{D}(B)=$ $W_{2}^{1}(a, b)$ (Sobolev space), then $C^{\infty}(B)$ is the set of all infinitely differentiable functions on $[a, b], C_{\{n!\}}(B)\left(C_{(n!)}(B)\right)$ is the space of analytic on $[a, b]$ (entire) functions, the spaces $C_{\left\{n^{n \beta}\right\}}(B)$ and $C_{\left(n^{n \beta}\right)}(B)(\beta>1)$ coincide with the usual Gevrey classes. If $\beta<1, C_{\left\{n^{n \beta}\right\}}(B)$ is the space of entire functions of order $\frac{1}{1-\beta}$. In particular, if $\beta=0$, we have the space $C_{\{1\}}(B)$ of entire functions $f$ of exponential type, whose exponential type is equal to $\sigma(f, 1, B)$.

Consider the functions

$$
\begin{equation*}
\rho_{1}(\lambda)=\sup _{n \in \mathbb{N}_{0}} \frac{\lambda^{n}}{m_{n}}, \quad \rho_{2}(\lambda)=\left(\sum_{n=0}^{\infty} \frac{\lambda^{2 n}}{m_{n}^{2}}\right)^{1 / 2}, \quad \rho_{3}(\lambda)=\sum_{n=0}^{\infty} \frac{\lambda^{n}}{m_{n}}, \quad \lambda>0 \tag{2}
\end{equation*}
$$

Under the condition

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sqrt[n]{m}_{n}=\infty \tag{3}
\end{equation*}
$$

the series for $\rho_{3}(\lambda)$ converges in the whole complex plane and gives an entire function. All the functions in (2) are not less than 1 and tend monotonically to $\infty$ as $\lambda \rightarrow \infty$. It is not hard also to verify that for any $\beta_{1}>\beta>1$,

$$
\begin{equation*}
\rho_{1}(\lambda) \leq \rho_{2}(\lambda) \leq \rho_{3}(\lambda) \leq c_{\beta} \rho_{2}(\beta \lambda) \leq c_{\beta, \beta_{1}} \rho_{1}\left(\beta_{1} \lambda\right), \quad \lambda>0 \tag{4}
\end{equation*}
$$

where

$$
c_{\beta}=\frac{\beta}{\left(\beta^{2}-1\right)^{1 / 2}}, \quad c_{\beta, \beta_{1}}=\frac{c(\beta) \beta_{1}}{\left(\beta_{1}^{2}-\beta^{2}\right)^{1 / 2}}, \quad 0<c(\beta)=\text { const. }
$$

Throughout further, $c$ denotes various numerical constants.
If, in addition, for the sequence $\left\{m_{n}\right\}_{n \in \mathbb{N}_{0}}$ there exists a number $h>1$ such that

$$
\begin{equation*}
\forall n \in \mathbb{N}_{0}: m_{n+1} \leq c h^{n} m_{n} \tag{5}
\end{equation*}
$$

(for example, the sequences $m_{n}=n^{n \beta}$ and $m_{n}=(n!)^{\beta}$ with $\beta>0$ satisfy (3) and (5)), then, as it is easily seen,

$$
\begin{equation*}
\exists c>0: \rho(\lambda) \geq c \lambda \rho\left(\alpha_{0} \lambda\right) \tag{6}
\end{equation*}
$$

where $\alpha_{0}=h^{-1}<1$. By $\rho(\lambda)$ we mean any of the three functions $\rho_{i}(\lambda)(i=1,2,3)$.
Proposition 1. Let the sequence $\left\{m_{n}\right\}_{n \in \mathbb{N}_{0}}$ satisfy the condition (3) and for any (some) $h>1$ the inequality (5) is fulfilled. Then for any (some) $a>1$,

$$
\begin{equation*}
\int_{0}^{\infty} \frac{\rho(\lambda)}{\rho(a \lambda)} d \lambda<\infty \tag{7}
\end{equation*}
$$

Proof. As it follows from (6),

$$
\exists c>0: \rho_{3}(a \lambda) \geq c \lambda^{2} \rho_{3}(\lambda)
$$

where $a=h^{2}$. Then

$$
\int_{1}^{\infty} \frac{\rho_{3}(\lambda)}{\rho_{3}(a \lambda)} d \lambda \leq \frac{1}{c} \int_{1}^{\infty} \frac{1}{\lambda^{2}} d \lambda<\infty
$$

Since the function $\frac{\rho(\lambda)}{\rho(a \lambda)}$ is continuous on $[0,1]$, the finiteness of the integral (7) is ensured.

Suppose the operator $B$ to be nonnegative and self-adjoint. Using the function $\rho(\lambda)$, we construct the family of Hilbert spaces associated with this function as follows:

$$
\forall a>0: \mathfrak{H}_{\rho}^{a}(B)=\mathcal{D}(\rho(a B)), \quad(f, g)_{\mathfrak{H}_{\rho}^{a}(B)}=(\rho(a B) f, \rho(a B) g) .
$$

Here

$$
\rho(a B)=\int_{0}^{\infty} \rho(a \lambda) d E_{\lambda}
$$

$E_{\lambda}=E([0, \lambda])$ is the spectral measure of $B$. Because of $\rho(\lambda) \geq 1$ and $\rho(\lambda) \rightarrow \infty$ monotonically as $\lambda \rightarrow \infty$, we have

$$
\forall f \in \mathfrak{H}_{\rho}^{b}(B):\|f\|_{\mathfrak{H}_{\rho}^{a}(B)} \leq\|f\|_{\mathfrak{H}_{\rho}^{b}(B)} \quad \text { as } \quad a<b,
$$

and these norms are compatible. So, the embedding

$$
\mathfrak{H}_{\rho}^{b}(B) \subseteq \mathfrak{H}_{\rho}^{a}(B)
$$

is dense and continuous. We put

$$
\mathfrak{H}_{\{\rho\}}(B)=\bigcup_{a>0} \mathfrak{H}_{\rho}^{a}(B) \quad \text { and } \quad \mathfrak{H}_{(\rho)}(B)=\bigcap_{a>0} \mathfrak{H}_{\rho}^{a}(B) .
$$

In view of (4),

$$
\mathfrak{H}_{\left\{\rho_{1}\right\}}(B)=\mathfrak{H}_{\left\{\rho_{2}\right\}}(B)=\mathfrak{H}_{\left\{\rho_{3}\right\}}(B), \quad \mathfrak{H}_{\left(\rho_{1}\right)}(B)=\mathfrak{H}_{\left(\rho_{2}\right)}(B)=\mathfrak{H}_{\left(\rho_{3}\right)}(B) .
$$

As it was shown in [13],

$$
\begin{equation*}
C_{\left\{m_{n}\right\}}(B)=\mathfrak{H}_{\{\rho\}}(B), \quad C_{\left(m_{n}\right)}(B)=\mathfrak{H}_{(\rho)}(B) \tag{8}
\end{equation*}
$$

2. As above, in this section the operator $B$ is assumed to be nonnegative and selfadjoint.

For an arbitrary $f \in \mathfrak{H}$ we put

$$
\mathcal{E}_{r}(f)=\mathcal{E}_{r}(f, B)=\inf _{g \in C_{\{1\}}(B): \sigma(g) \leq r}\|f-g\|,
$$

where $\sigma(g)=\sigma(g, 1, B)$ is the type of the vector $g$ for the operator $B$ and the sequence $m_{n} \equiv 1$. Since $C_{1}^{r}=E_{r} \mathfrak{H}$, we have

$$
\begin{equation*}
\mathcal{E}_{r}^{2}(f)=\left\|f-E_{r} f\right\|^{2}=\int_{r}^{\infty} d\left(E_{\lambda} f, f\right) \tag{9}
\end{equation*}
$$

Theorem 1. Let $G(\lambda)$ be a continuous on $[0, \infty)$ and continuously differentiable on $(0, \infty)$ function such that $G(\lambda) \geq 0$ and $G^{\prime}(\lambda) \geq 0$ as $\lambda>0$. Then the equivalence

$$
\begin{equation*}
f \in \mathcal{D}(G(B)) \Longleftrightarrow \int_{0}^{\infty} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda<\infty \tag{10}
\end{equation*}
$$

holds, and

$$
\begin{equation*}
\|G(B) f\|^{2}=2 \int_{0}^{\infty} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda+G^{2}(0) \mathcal{E}_{0}^{2}(f) \tag{11}
\end{equation*}
$$

Proof. Let $f \in \mathcal{D}(G(B))$. As $G(\lambda)$ is nondecreasing on $(0, \infty)$, we have

$$
\forall r>0: \mathcal{E}_{r}^{2}(f)=\int_{r}^{\infty} d\left(E_{\lambda} f, f\right) \leq \frac{1}{G^{2}(r)} \int_{r}^{\infty} G^{2}(\lambda) d\left(E_{\lambda} f, f\right)
$$

whence

$$
\begin{equation*}
G^{2}(r) \mathcal{E}_{r}^{2}(f) \rightarrow 0 \quad \text { as } \quad r \rightarrow \infty \tag{12}
\end{equation*}
$$

Taking into account that

$$
\begin{align*}
\int_{0}^{r} G^{2}(\lambda) d\left(E_{\lambda} f, f\right) & =-\int_{0}^{r} G^{2}(\lambda) d \mathcal{E}_{\lambda}^{2}(f)  \tag{13}\\
& =-G^{2}(r) \mathcal{E}_{r}^{2}(f)+G^{2}(0) \mathcal{E}_{0}^{2}(f)+2 \int_{0}^{r} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda
\end{align*}
$$

and there exists the limit of the integral in the left hand side of (13) as $r \rightarrow \infty$, we conclude, by virtue of (12), that the integral

$$
\int_{0}^{\infty} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda
$$

is finite, and the equality (11) is true.
Conversely, let $\int_{0}^{\infty} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda<\infty$. Then the equality (13) implies

$$
\int_{0}^{r} G^{2}(\lambda) d\left(E_{\lambda} f, f\right) \leq G^{2}(0) \mathcal{E}_{0}^{2}(f)+2 \int_{0}^{r} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda
$$

It follows from here that $\int_{0}^{\infty} G^{2}(\lambda) d\left(E_{\lambda} f, f\right)<\infty$, that is, $f \in \mathcal{D}(G(B))$. Then the equality (11) is a direct consequence of (13) and (12).

Corollary 1. Let $G(\lambda)$ satisfy the conditions of Theorem 1. Then

$$
\begin{equation*}
f \in \mathcal{D}(G(B)) \Longrightarrow \mathcal{E}_{r}(f)=o\left(G^{-1}(r)\right), \quad r \rightarrow \infty \tag{14}
\end{equation*}
$$

Setting in Theorem $1 G(\lambda)=\lambda^{\alpha}(\alpha>0)$, we obtain the next assertion.
Corollary 2. For an arbitrary $\alpha>0$,

$$
\begin{aligned}
& f \in \mathcal{D}\left(B^{\alpha}\right) \Longleftrightarrow \mathcal{E}_{r}(f)=o\left(r^{-\alpha}\right) \\
& f \in \mathcal{D}\left(B^{\alpha}\right) \Longleftrightarrow \\
& \int_{0}^{\infty} \lambda^{2 \alpha-1} \mathcal{E}_{\lambda}^{2}(f) d \lambda<\infty
\end{aligned}
$$

Moreover,

$$
\left\|B^{\alpha} f\right\|^{2}=2 \alpha \int_{0}^{\infty} \lambda^{2 \alpha-1} \mathcal{E}_{\lambda}^{2}(f) d \lambda
$$

Note that the relation $\mathcal{E}_{r}(f)=o\left(r^{-\alpha}\right)$ does not yet imply the inclusion $f \in \mathcal{D}\left(B^{\alpha}\right)$. But the following statement is valid.

Corollary 3. If for some $\varepsilon>0$,

$$
r^{\alpha+\varepsilon} \mathcal{E}_{r}(f)=O(1) \quad(r \rightarrow \infty)
$$

then $f \in \mathcal{D}\left(B^{\alpha}\right)$.
Proof. Taking in (13) $G(\lambda)=\lambda^{\alpha}$, we obtain for any $r>0$

$$
\begin{aligned}
\int_{0}^{r} \lambda^{2 \alpha} d\left(E_{\lambda} f, f\right) & =-r^{2 \alpha} \mathcal{E}_{r}^{2}(f)+2 \alpha \int_{0}^{r} \lambda^{2 \alpha-1} \mathcal{E}_{\lambda}^{2}(f) d \lambda \\
& \leq 2 \alpha \int_{0}^{r} \lambda^{2(\alpha+\varepsilon)} \mathcal{E}_{\lambda}^{2}(f) \frac{d \lambda}{\lambda^{1+2 \varepsilon}} \\
& \leq 2 \alpha \sup _{\lambda \in[0, \infty)}\left(\lambda^{2(\alpha+\varepsilon)} \mathcal{E}_{\lambda}^{2}(f)\right) \int_{0}^{r} \frac{d \lambda}{\lambda^{1+2 \varepsilon}}
\end{aligned}
$$

whence $f \in \mathcal{D}\left(B^{\alpha}\right)$.
Corollary 4. Let $G(\lambda)$ satisfy the conditions of Theorem 1, and

$$
\begin{equation*}
\int_{r_{0}}^{\infty} \frac{G^{\prime}(\lambda)}{G(a \lambda)} d \lambda<\infty \tag{15}
\end{equation*}
$$

with some $r_{0}>0, a>1$. Then

$$
\mathcal{E}_{\lambda}(f)=O\left(G^{-1}(a \lambda)\right)(\lambda \rightarrow \infty) \Longrightarrow f \in \mathcal{D}(G(B))
$$

Proof. Assume that the condition (15) is fulfilled, and $\mathcal{E}_{\lambda}(f)=O\left(G^{-1}(a \lambda)\right)$. Then for sufficiently large $\lambda$ (we may consider $\lambda>r_{0}$ )

$$
\exists c>0: \mathcal{E}_{\lambda}(f)<c G^{-1}(a \lambda)
$$

and, as it follows from (13), for $r>r_{0}$ we have

$$
\begin{aligned}
\int_{0}^{r} G^{2}(\lambda) d\left(E_{\lambda} f, f\right) & \leq G^{2}(0) \mathcal{E}_{0}^{2}(f)+2 \int_{0}^{r} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda \\
& \leq G^{2}(0) \mathcal{E}_{0}^{2}(f)+2 \int_{0}^{r_{0}} G(\lambda) G^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda+2 c \int_{r_{0}}^{r} \frac{G^{\prime}(\lambda)}{G(a \lambda)} d \lambda
\end{aligned}
$$

whence $\int_{0}^{\infty} G^{2}(\lambda) d\left(E_{\lambda} f, f\right)<\infty$, that is, $f \in \mathcal{D}(G(B))$.
Corollary 5. If the sequence $\left\{m_{n}\right\}_{n \in \mathbb{N}_{0}}$ satisfies the conditions (3) and (5) with $h>1$, then

$$
\rho\left(h^{2} \lambda\right) \mathcal{E}_{\lambda}(f)=O(1)(\lambda \rightarrow \infty) \Longrightarrow f \in \mathcal{D}(\rho(B))
$$

where, as before, $\rho(\lambda)$ stands for any function from (2).

Proof. In view of (4), it is sufficient to consider only the case $\rho(\lambda)=\rho_{3}(\lambda)$. Since $h>1$, we have $n \leq c h^{2 n}, n \in \mathbb{N}_{0}$. Hence, for $\lambda>1$,

$$
\rho_{3}^{\prime}(\lambda)=\sum_{n=1}^{\infty} \frac{n \lambda^{n-1}}{m_{n}} \leq c \sum_{n=1}^{\infty} \frac{\left(h^{2} \lambda\right)^{n}}{m_{n}}=c \rho_{3}\left(h^{2} \lambda\right)
$$

and, by Proposition 1,

$$
\int_{1}^{\infty} \rho_{3}(\lambda) \rho_{3}^{\prime}(\lambda) \mathcal{E}_{\lambda}^{2}(f) d \lambda \leq c \int_{1}^{\infty} \rho_{3}(\lambda) \rho_{3}\left(h^{2} \lambda\right) \frac{d \lambda}{\rho_{3}^{2}\left(h^{2} \lambda\right)}=c \int_{1}^{\infty} \frac{\rho_{3}(\lambda)}{\rho_{3}\left(h^{2} \lambda\right)} d \lambda<\infty
$$

In view of Theorem $1, f \in \mathcal{D}\left(\rho_{3}(B)\right)$.
Remark 1. It is not hard to see that the proof of Corollary 5 is suitable for the case where the function $\rho_{\mu}(\lambda)=\rho(\mu \lambda)(\mu>0)$ is taken instead of $\rho(\lambda)$, that is,

$$
\rho\left(h^{2} \mu \lambda\right) \mathcal{E}_{\lambda}(f)=O(1)(\lambda \rightarrow \infty) \Longrightarrow f \in \mathcal{D}(\rho(\mu B))
$$

For a nondecreasing on $[0, \infty)$ function $F(\lambda)$ such that $\lim _{\lambda \rightarrow \infty} F(\lambda)=\infty$, we put

$$
s(f, F, B)=\sup \left\{a \in(0, \infty): \int_{0}^{\infty} F^{2}(a \lambda) d\left(E_{\lambda} f, f\right)<\infty\right\}
$$

Theorem 2. Let $\rho(\lambda)$ be one of the functions $\rho_{i}(\lambda), i=1,2,3$. If $f \in C_{\left\{m_{n}\right\}}(B)$, then

$$
\begin{equation*}
\sigma\left(f, m_{n}, B\right)=\frac{1}{s(f, \rho, B)} \tag{16}
\end{equation*}
$$

Proof. We shall prove the theorem only for $\rho(\lambda)=\rho_{2}(\lambda)$. The rest follows from the inequalities (4).

Assume that $f \in C_{\left\{m_{n}\right\}}(B)$. Due to (8), $f \in \mathfrak{H}_{\left\{\rho_{2}\right\}}(B)$. So, $f \in \mathcal{D}\left(\rho_{2}(a B)\right)$ with some $a>0$, and

$$
\begin{align*}
\left\|\rho_{2}(a B) f\right\|^{2} & =\int_{0}^{\infty} \sum_{n=0}^{\infty} \frac{(a \lambda)^{2 n}}{m_{n}^{2}} d\left(E_{\lambda} f, f\right)=\sum_{n=0}^{\infty} \frac{a^{2 n}}{m_{n}^{2}} \int_{0}^{\infty} \lambda^{2 n} d\left(E_{\lambda} f, f\right) \\
& =\sum_{n=0}^{\infty} \frac{a^{2 n}}{m_{n}^{2}}\left\|B^{n} f\right\|^{2}<\infty \tag{17}
\end{align*}
$$

Therefore

$$
\forall n \in \mathbb{N}_{0}: \frac{a^{n}\left\|B^{n} f\right\|^{2}}{m_{n}}<c
$$

that is, $f \in C_{m_{n}}^{1 / a}$. The latter means that $\sigma\left(f, m_{n}, B\right) \leq \frac{1}{a}$. As $a$ is an arbitrary positive number for which $f \in \mathcal{D}\left(\rho_{2}(a B)\right)$, we can conclude that

$$
\sigma\left(f, m_{n}, B\right) \leq \frac{1}{s\left(f, \rho_{2}, B\right)}
$$

Conversely, let $a \geq \sigma\left(f, m_{n}, B\right)$. Then

$$
\forall \varepsilon>0: \sum_{n=0}^{\infty} \frac{\left\|B^{n} f\right\|^{2}}{(a+\varepsilon)^{2 n} m_{n}^{2}}<\infty
$$

and the equalities (17) show that $\frac{1}{a+\varepsilon} \leq s\left(f, \rho_{2}, B\right)$, whence

$$
\frac{1}{\sigma\left(f, m_{n}, B\right)} \leq s\left(f, \rho_{2}, B\right)
$$

It follows from here that the formula (16) is true.
3. Now we turn to equation (1), where $A$ is a linear operator in $\mathfrak{H}, \overline{\mathcal{D}(A)}=\mathfrak{H}$, and $0 \in \rho(A)$. We say that a closed linear operator $B$ in $\mathfrak{H}$ is similar to $A$ if $\mathcal{D}(B)=\mathcal{D}(A)$. If $0 \in \rho(B)$, then by closed graph theorem, the operators $B A^{-1}$ and $A B^{-1}$ are defined and continuous on the whole space $\mathfrak{H}$. In what follows, we assume everywhere that:
$1^{\diamond} . B$ is a positive definite self-adjoint operator similar to $A$;
$2^{\diamond}$. the operator $B^{-1}$ is compact, and the spectrum $\lambda_{k}=\lambda_{k}(B), k \in \mathbb{N}$, of the operator $B$ is simple (the eigenvalues $\lambda_{k}$ are arranged in ascending order);
$3^{\diamond}$. the orthonormal basis of eigenvectors of the operator $B$ is taken as the coordinate system $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ in the least squares method in approximate solving equation (1).

The relation

$$
\begin{aligned}
\left\|A\left(f-\sum_{k=1}^{n} a_{k} e_{k}\right)\right\| & =\left\|A B^{-1} B\left(f-\sum_{k=1}^{n} a_{k} e_{k}\right)\right\| \\
& \leq\left\|A B^{-1}\right\|\left\|B\left(f-\sum_{k=1}^{n} a_{k} e_{k}\right)\right\|
\end{aligned}
$$

shows that the system $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ is complete in $\mathfrak{H}^{1}(A)$, and therefore,

$$
\left\|A u_{n}-f\right\| \rightarrow 0, n \rightarrow \infty
$$

where $u_{n}=\sum_{k=1}^{n} \alpha_{k} e_{k}$ is the approximate solution of (1) by means of the least square method with the coordinate system $\left\{e_{k}\right\}_{k \in \mathbb{N}}$. The next theorem shows that this convergence may be as slow as desired.

Theorem 3. Let $A$ be a densely defined linear operator in $\mathfrak{H}$ such that $0 \in \rho(A)$. Suppose also that the conditions $1^{\diamond-3}$ hold. Then whatever decreasing number sequence $\left\{\gamma_{n}\right\}_{n \in \mathbb{N}}$, $\gamma_{n}>0, \gamma_{n} \rightarrow 0(n \rightarrow \infty)$ may be, there exists a vector $f \in \mathfrak{H}$ such that

$$
\forall n \in \mathbb{N}: R_{n}=\left\|A u_{n}-f\right\|=\gamma_{n}
$$

Proof. First we shall prove that for an arbitrary Hilbert space $\mathfrak{H}$ and any orthonormal basis $\left\{g_{k}\right\}_{k \in \mathbb{N}}$ in $\mathfrak{H}$, there exists a vector $v \in \mathfrak{H}$ such that

$$
\forall n \in \mathbb{N}:\left\|v-\sum_{k=1}^{n}\left(v, g_{k}\right) g_{k}\right\|=\gamma_{n}
$$

Really, put $\beta_{1}=0, \beta_{i}^{2}=\gamma_{i-1}^{2}-\gamma_{i}^{2}(i=2,3, \ldots)$. Then the series $\sum_{k=1}^{n} \beta_{i} g_{i}$ converges in $\mathfrak{H}$ to a certain element $v \in \mathfrak{H}$, and it is the Fourier series of this element $\left(\beta_{i}=\left(v, g_{i}\right)\right)$. So,

$$
\left\|v-\sum_{k=1}^{n}\left(v, g_{k}\right) g_{k}\right\|=\left(\sum_{k=1}^{n} \beta_{i}^{2}\right)^{1 / 2}=\gamma_{n} .
$$

Taking the space $\mathfrak{H}^{1}(A)$ as $\mathfrak{H}$ and the sequence $\left\{\tilde{e}_{k}\right\}_{k \in \mathbb{N}}$ obtained in the process of orthogonalization in $\mathfrak{H}^{1}(A)$ of the coordinate system $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ as $\left\{g_{k}\right\}_{k \in \mathbb{N}}$, we conclude that there exists a vector $u \in \mathfrak{H}^{1}(A)$ such that

$$
\forall n \in \mathbb{N}:\left\|u-\sum_{k=1}^{n}\left(u, \tilde{e}_{k}\right)_{\mathfrak{H}^{1}(A)} \tilde{e}_{k}\right\|_{\mathfrak{H}^{1}(A)}=\left\|A\left(u-\sum_{k=1}^{n}\left(u, \tilde{e}_{k}\right)_{\mathfrak{H}^{1}(A)} \tilde{e}_{k}\right)\right\|=\gamma_{n}
$$

Setting $f=A u$, we get

$$
\begin{aligned}
R_{n}=\left\|A u_{n}-f\right\| & =\min _{\left\{a_{k}\right\}_{k \in \mathbb{N}}}\left\|f-\sum_{k=1}^{n} a_{k} A e_{k}\right\|=\min _{\left\{a_{k}\right\}_{k \in \mathbb{N}}}\left\|A\left(u-\sum_{k=1}^{n} a_{k} e_{k}\right)\right\| \\
& =\min _{\left\{a_{k}\right\}_{k \in \mathbb{N}}}\left\|u-\sum_{k=1}^{n} a_{k} e_{k}\right\|_{\mathfrak{H}^{1}(A)}=\min _{\left\{a_{k}\right\}_{k \in \mathbb{N}}}\left\|u-\sum_{k=1}^{n} a_{k} \tilde{e}_{k}\right\|_{\mathfrak{H}^{1}(A)} \\
& =\left\|u-\sum_{k=1}^{n}\left(u, \tilde{e}_{k}\right)_{\mathfrak{H}^{1}(A)} \tilde{e}_{k}\right\|_{\mathfrak{H}^{1}(A)}=\gamma_{n} .
\end{aligned}
$$

In connection with this, the question arises, what properties the solution $u=A^{-1} f$ has to possess in order that the value $R_{n}=\left\|A u_{n}-f\right\|$ be of a certain order of decreasing to zero as $n \rightarrow \infty$. The answer is given by the theorem below.

Theorem 4. Let $\left\{e_{k}\right\}_{k \in \mathbb{N}}$ be the orthonormal basis of eigenvectors of the operator $B$, and let $u_{n}$ be the approximate solution of equation (1) in the least squares method with the coordinate system $\left\{e_{k}\right\}_{k \in \mathbb{N}}$. Then for any $\alpha \geq 0$

$$
\begin{equation*}
u=A^{-1} f \in \mathcal{D}\left(B^{\alpha+1}\right) \Longrightarrow R_{n}=o\left(\lambda_{n+1}^{-\alpha}\right) \tag{18}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\exists \varepsilon>0: R_{n}=o\left(\lambda_{n+1}^{-(\alpha+\varepsilon)}\right) \Longrightarrow u \in \mathcal{D}\left(B^{\alpha+1}\right) \tag{19}
\end{equation*}
$$

Proof. We put $v_{n}=\sum_{k=1}^{n}\left(u, e_{k}\right) e_{k}$. Then

$$
\begin{aligned}
\left\|A u_{n}-f\right\| & \leq\left\|A v_{n}-f\right\|=\left\|A\left(v_{n}-u\right)\right\|=\left\|A B^{-1} B\left(v_{n}-u\right)\right\| \\
& \leq\left\|A B^{-1}\right\|\left\|B v_{n}-B u\right\|=\left\|A B^{-1}\right\|\left\|\sum_{k=1}^{n}\left(B u, e_{k}\right) e_{k}-B u\right\|
\end{aligned}
$$

As the resolution of identity $E_{\lambda}$ of the operator $B$ has the form

$$
E_{\lambda} g=\sum_{k: \lambda_{k} \leq \lambda}\left(g, e_{k}\right) e_{k}
$$

and, because of (9),

$$
\mathcal{E}_{\lambda_{n+1}}(g)=\left\|g-E_{\lambda_{n+1}} g\right\|=\left\|g-\sum_{k=1}^{n}\left(g, e_{k}\right) e_{k}\right\|
$$

one can conclude that

$$
\begin{equation*}
\left\|A u_{n}-f\right\| \leq\left\|A B^{-1}\right\| \mathcal{E}_{\lambda_{n+1}}(B u) \tag{20}
\end{equation*}
$$

Then the relation (18) follows from Corollary 2.
Now assume that the equality in the left hand side of (19) is fulfilled. Then

$$
\begin{aligned}
\mathcal{E}_{\lambda_{n+1}}(B u) & =\left\|B u-\sum_{k=1}^{n}\left(B u, e_{k}\right) e_{k}\right\|=\left\|B u-\sum_{k=1}^{n}\left(u, e_{k}\right) B e_{k}\right\| \\
& \leq\left\|B\left(u-\sum_{k=1}^{n}\left(u, e_{k}\right) e_{k}\right)\right\| \leq\left\|B A^{-1}\right\|\left\|A\left(u_{n}-u\right)\right\|=o\left(\lambda_{n+1}^{-(\alpha+\varepsilon)}\right)
\end{aligned}
$$

By Corollary $3, B u \in \mathcal{D}\left(B^{\alpha}\right)$, whence $u \in \mathcal{D}\left(B^{\alpha+1}\right)$.

It should be noted that there exist examples verifying that the equality

$$
\left\|A u_{n}-f\right\|=o\left(\lambda_{n+1}^{-\alpha}\right)
$$

does not yet imply the inclusion $u \in \mathcal{D}\left(B^{\alpha+1}\right)$.
According to [14], for the operator $B$ and a vector $g \in \mathfrak{H}$, we put

$$
\omega_{k}(t, g, B)=\sup _{0 \leq \tau \leq t}\left\|\Delta_{\tau}^{k} g\right\|, \quad k \in \mathbb{N}
$$

where

$$
\Delta_{\tau}^{k}=(U(\tau)-I)^{k}=\sum_{j=0}^{k} C_{k}^{j} U(j \tau), \quad k \in \mathbb{N}_{0}, \quad \tau \in \mathbb{R}_{+}=[0, \infty)
$$

and $U(\tau)=e^{i \tau B}$ is a group of unitary operators in $\mathfrak{H}$ with the generating operator $i B$.
It follows from this definition that $\omega_{k}(t, g, B)$ possesses the following properties:

1) $\omega_{k}(0, g, B)=0$;
2) for a fixed $g$, the function $\omega_{k}(t, g, B)$ is nondecreasing on $\mathbb{R}_{+}$;
3) $\forall \alpha, t>0: \omega_{k}(\alpha t, g, B) \leq(1+\alpha)^{k} \omega_{k}(t, g, B)$;
4) for any fixed $t \in \mathbb{R}_{+}$, the function $\omega_{k}(t, g, B)$ is continuous with respect to $g$.

As was proved in [3], the following statement holds.
Proposition 2. Let $g \in \mathcal{D}\left(B^{\alpha}\right), \alpha>0$. Then

$$
\forall k \in \mathbb{N}: \mathcal{E}_{r}(g, B) \leq \frac{\sqrt{k+1}}{2^{k} r^{\alpha}} \omega_{k}\left(\frac{\pi}{r}, B^{\alpha} g, B\right)
$$

Conversely, if $\omega(t), t \in[0, \infty)$, is a function of continuity module type, that is:
(i) $\omega(t)$ is continuous and nondecreasing on $\mathbb{R}_{+}$,
(ii) $\omega(0)=0$,
(iii) $\exists c>0, \forall t>0: \omega(2 t) \leq c \omega(t)$,
and if, in addition, the condition

$$
\int_{0}^{1} \frac{\omega(t)}{t} d t<\infty
$$

is satisfied, then

$$
\exists c>0, \quad \forall r>0: \mathcal{E}_{r}(g, B) \leq \frac{c}{r^{\alpha}} \omega\left(\frac{1}{r}\right) \Longrightarrow g \in \mathcal{D}\left(B^{\alpha}\right)
$$

Using this proposition, on the basis of (20), we arrive at the following conclusion.
Theorem 5. If $u \in \mathcal{D}\left(B^{\alpha+1}\right), \alpha>0$, then

$$
\forall k \in \mathbb{N}:\left\|A u_{n}-f\right\| \leq \frac{\sqrt{k+1}}{2^{k} \lambda_{n+1}^{\alpha}} \omega_{k}\left(\frac{\pi}{\lambda_{n+1}}, B^{\alpha+1} g, B\right)
$$

Conversely, let for $\omega(t)$ the conditions of Proposition 2 be fulfilled. If $u \in \mathcal{D}(B)$ and

$$
\|A u-f\| \leq \lambda_{n+1}^{-\alpha} \omega\left(\frac{1}{\lambda_{n+1}}\right) \quad(\alpha>0)
$$

then $u \in \mathcal{D}\left(B^{\alpha+1}\right)$.
Now we pass to the approximation by means of the least squares method of the solution $u$ to equation (1) in assumption that $u$ belongs to some class of infinitely differentiable vectors for the operator $B$. First of all, note that Theorem 4 gives as a consequence such a result.

Corollary 6. The following equivalence holds:

$$
u \in C^{\infty}(B) \Longleftrightarrow \forall \alpha>0: \lim _{n \rightarrow \infty} \lambda_{n+1}^{\alpha} R_{n}=0
$$

For more smooth $u$, the assertion below is valid.
Theorem 6. Let the conditions $1^{\diamond}-3^{\diamond}$ hold, and let the sequence $\left\{m_{n}\right\}_{n \in \mathbb{N}_{0}}$ satisfy (3) and the condition (5) with any $h>1$. Then

$$
\begin{gather*}
u \in C_{\left\{m_{n}\right\}}(B), \quad \sigma\left(u, m_{n}, B\right)=\sigma \Longleftrightarrow \forall \varepsilon>0: \lim _{n \rightarrow \infty} \rho\left(\frac{\lambda_{n+1}}{\sigma+\varepsilon}\right) R_{n}=0,  \tag{21}\\
\varlimsup_{n \rightarrow \infty} \rho\left(\frac{\lambda_{n+1}}{\sigma-\varepsilon}\right) R_{n}=\infty ; \\
u \in C_{\left(m_{n}\right)}(B) \Longleftrightarrow \forall \varepsilon>0 \lim _{n \rightarrow \infty} \rho\left(\frac{\lambda_{n+1}}{\varepsilon}\right) R_{n}=0, \tag{22}
\end{gather*}
$$

where $\rho(\lambda)$ is any of the three functions $\rho_{i}(\lambda), i=1,2,3$ from (2).
Proof. Let $u \in C_{\left\{m_{n}\right\}}(B)$. Then

$$
\forall \alpha>\sigma, \quad \forall k \in \mathbb{N}_{0},
$$

$$
\exists c=c(\alpha)>0:\left\|B^{k} B u\right\|=\left\|B^{k+1} u\right\| \leq c \alpha^{k+1} m_{k+1} \leq c \alpha^{k+1} h^{k} m_{k}=c(\alpha h)^{k} m_{k}
$$

which implies that $\sigma\left(B u, m_{n}, B\right) \leq \alpha h$. As $h>1$ is arbitrary, we have $\sigma\left(B u, m_{n}, B\right) \leq \sigma$. But, by the definition of $\sigma$, for any $\varepsilon \in(0, \sigma)$, there exists a subsequence $k_{i} \in \mathbb{N}_{0}$ such that

$$
\left\|B^{k_{i}} u\right\| \geq c(\sigma-\varepsilon)^{k_{i}} m_{k_{i}}
$$

Therefore

$$
\left\|B^{k_{i}-1} B u\right\| \geq c(\sigma-\varepsilon)^{k_{i}-1} m_{k_{i}-1}
$$

Thus,

$$
\sigma\left(B u, m_{n}, B\right)=\sigma\left(u, m_{n}, B\right)=\sigma
$$

Since $B u \in C_{\left\{m_{n}\right\}}(B)=\mathfrak{H}_{\{\rho\}}(B)$ and because of Theorem 2, we have

$$
\sigma\left(B u, m_{n}, B\right)=s^{-1}(B u, \rho, B)
$$

It follows from the definition of $s(u, \rho, B)$ that

$$
\begin{equation*}
\forall \varepsilon \in(0, \sigma): B u \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma+\varepsilon}\right)\right), \quad B u \notin \mathcal{D}\left(\rho\left(\frac{B}{\sigma-\varepsilon}\right)\right) \tag{23}
\end{equation*}
$$

By Corollary 1,

$$
\begin{equation*}
\mathcal{E}_{r}(B u) \rho\left(\frac{r}{\sigma+\varepsilon}\right) \rightarrow 0 \quad \text { as } \quad r \rightarrow \infty \tag{24}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\varlimsup_{r \rightarrow \infty} \mathcal{E}_{r}(B u) \rho\left(\frac{r}{\sigma-\varepsilon}\right)=\infty \tag{25}
\end{equation*}
$$

Indeed, assume that for some $\varepsilon>0$ and $r \geq r_{0}$,

$$
\mathcal{E}_{r}(B u) \rho\left(\frac{r}{\sigma-\varepsilon}\right) \leq c<\infty
$$

As it follows from Remark 1, for any $\varepsilon^{\prime}<\varepsilon$ we have

$$
c \geq \mathcal{E}_{r}(B u) \rho\left(\frac{r}{\sigma-\varepsilon}\right)=\mathcal{E}_{r}(B u) \rho\left(\frac{\sigma-\varepsilon^{\prime}}{\sigma-\varepsilon} \cdot \frac{r}{\sigma-\varepsilon^{\prime}}\right) \Longrightarrow B u \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma-\varepsilon^{\prime}}\right)\right)
$$

contrary to (23).
According to (20), $\left\|A u_{n}-f\right\| \leq c \mathcal{E}_{\lambda_{n+1}}(B u)$. By (24),

$$
\left\|A u_{n}-f\right\| \rho\left(\frac{\lambda_{n+1}}{\sigma+\varepsilon}\right) \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty
$$

Since

$$
\begin{align*}
\mathcal{E}_{\lambda_{n+1}}(B u) & =\left\|\sum_{k=1}^{n}\left(B u, e_{k}\right) e_{k}-B u\right\|=\left\|B\left(u-u_{n}\right)\right\|  \tag{26}\\
& \leq\left\|B A^{-1}\right\|\left\|A u_{n}-f\right\|
\end{align*}
$$

the formula (25) implies the relation

$$
\varlimsup_{n \rightarrow \infty}\left\|A u_{n}-f\right\| \rho\left(\frac{\lambda_{n+1}}{\sigma-\varepsilon}\right)=\infty
$$

Conversely, let the relations in the right hand side of (21) be fulfilled with some $\sigma>0$. Because of (26),

$$
\rho\left(\frac{\lambda_{n+1}}{\sigma+\varepsilon}\right) \mathcal{E}_{\lambda_{n+1}}(B u) \leq \rho\left(\frac{\lambda_{n+1}}{\sigma+\varepsilon}\right)\left\|A u_{n}-f\right\|\left\|B A^{-1}\right\| \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty
$$

In the same way as above, one can show that

$$
B u \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma+\varepsilon^{\prime}}\right)\right)\left(\varepsilon^{\prime}>\varepsilon\right)
$$

and therefore,

$$
u \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma+\varepsilon^{\prime}}\right)\right)
$$

So,

$$
s(u, \rho, B) \geq \frac{1}{\sigma}
$$

Furthermore, due to (20),

$$
\forall \varepsilon>0: \varlimsup_{n \rightarrow \infty} \rho\left(\frac{\lambda_{n+1}}{\sigma-\varepsilon}\right) \mathcal{E}_{\lambda_{n+1}}(B u)=\infty
$$

which guarantees, by Corollary 1 , that

$$
B u \notin \mathcal{D}\left(\rho\left(\frac{B}{\sigma-\varepsilon}\right)\right)
$$

The latter means that

$$
s(B u, \rho, B)=s(u, \rho, B)=\sigma^{-1}
$$

that is equivalent to $\sigma\left(u, m_{n}, B\right)=\sigma$.
The relation (22) follows from the previous one, because in this case $\sigma\left(u, m_{n}, B\right)=0$ for any $u \in C_{\left(m_{n}\right)}(B)$.
4. As an example, we consider the case

$$
\begin{gathered}
\mathfrak{H}=L_{2}(0, \pi), \quad A=(-1)^{m} \frac{d^{2 m}}{d t^{2 m}}+\sum_{k=0}^{2 m-1} p_{k}(t) \frac{d^{k}}{d t^{k}} \\
\mathcal{D}(A)=\left\{v(\cdot) \in W_{2}^{2 m}[0, \pi] \mid v^{(2 k)}(0)=v^{(2 k)}(\pi)=0, k=0, \ldots, m-1\right\}
\end{gathered}
$$

where $p_{k}(\cdot) \in C[0, \pi]$. It is also assumed that the equation $A v=0$ has only the trivial solution.

We define the operator $B$ as

$$
B=(-1)^{m} \frac{d^{2 m}}{d t^{2 m}}, \quad \mathcal{D}(B)=\mathcal{D}(A)
$$

This operator is self-adjoint and positive definite, its spectrum $\left\{\lambda_{k}=k^{2 m}\right\}_{k \in \mathbb{N}}$ is discrete and simple, and the functions $\sqrt{\frac{2}{\pi}} \sin k t$ form an orthonormal basis of eigenvectors of $B$ in $L_{2}(0, \pi)$.

It can readily be shown that under the conditions

$$
\begin{equation*}
p_{k} \in C^{2 m j}[0, \pi] \tag{27}
\end{equation*}
$$

and

$$
\begin{align*}
& p_{k}^{(2 r-1)}(0)=p_{k}^{(2 r-1)}(\pi)=0 \quad \text { for even } k \quad(1 \leq r \leq m j)  \tag{28}\\
& p_{k}^{(2 r)}(0)=p_{k}^{(2 r)}(\pi)=0 \quad \text { for odd } k \quad(0 \leq r \leq(m-1) j) \tag{29}
\end{align*}
$$

the relation

$$
\mathcal{D}\left(A^{j+1}\right)=\mathcal{D}\left(B^{j+1}\right)
$$

is true.
On the basis of Corollary 2, we arrive at the following conclusion.
Proposition 3. If the coefficients $p_{k}(t)$ satisfy the conditions (27)-(29) and if

$$
\begin{equation*}
f(\cdot) \in C^{2 m j}[0, \pi] \quad \text { and } \quad f^{(2 k)}(0)=f^{(2 k)}(\pi)=0 \quad(k=0,1, \ldots, m j-1) \tag{30}
\end{equation*}
$$

then

$$
(n+1)^{2 m j}\left\|A u_{n}-f\right\|_{L_{2}(0, \pi)} \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty
$$

or, equivalently,

$$
\begin{equation*}
(n+1)^{2 m j}\left\|u_{n}-u\right\|_{W_{2}^{2 m}[0, \pi]} \rightarrow 0 \quad(n \rightarrow \infty) \tag{31}
\end{equation*}
$$

The validity of (31) for $m=1$ and $j=1$ was established in [15].
Corollary 6 and Theorem 6 for $m_{n}=n$ ! imply the following assertion.
Proposition 4. Suppose that all $p_{k}(\cdot)$ belong to $C^{\infty}[0, \pi]$ and satisfy the conditions (27)-(29) for each $r \in \mathbb{N}_{0}$. Then the inclusion $f(\cdot) \in C^{\infty}[0, \pi]$ and the condition (30) for $k \in \mathbb{N}_{0}$ are equivalent to the following statement:

$$
\forall \alpha>0 \quad n^{\alpha}\left\|A u_{n}-u\right\|_{L_{2}(0, \pi)} \rightarrow 0 \quad(n \rightarrow \infty)
$$

If, in addition, the functions $p_{k}(t)$ are analytic on $[0, \pi]$, then the assertion

$$
\exists \alpha>0 \quad e^{\alpha n}\left\|A u_{n}-u\right\|_{L_{2}(0, \pi)} \rightarrow 0 \quad(n \rightarrow \infty)
$$

is equivalent to the analyticity of the function $f(t)$ on $[0, \pi]$ and the validity of the condition (30) for all $k \in \mathbb{N}_{0}$. If the $p_{k}(t)$ are entire functions, then for the assertion

$$
\forall \alpha>0 \quad e^{\alpha n}\left\|A u_{n}-u\right\|_{L_{2}(0, \pi)} \rightarrow 0 \quad(n \rightarrow \infty)
$$

to be true, it is necessary and sufficient that the function $f(t)$ be entire and satisfy the condition (30) for all $k \in \mathbb{N}_{0}$.

It should be noted that the expression $\left\|A u_{n}-u\right\|_{L_{2}(0, \pi)}$ in the last Proposition can be replaced by $\left\|u_{n}-f\right\|_{W_{2}^{2 m}[0, \pi]}$.

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