ON THE APPROXIMATION TO SOLUTIONS OF OPERATOR EQUATIONS BY THE LEAST SQUARES METHOD

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ABSTRACT. We consider the equation Au = f, where A is a linear operator with compact inverse in a Hilbert space. For the approximate solution u_n of this equation by the least squares method in a coordinate system that is an orthonormal basis of eigenvectors of a self-adjoint operator B similar to $A (\mathcal{D}(A) = \mathcal{D}(B))$, we give a priori estimates for the asymptotic behavior of the expression $R_n = ||Au_n - f||$ as $n \to \infty$. A relationship between the order of smallness of this expression and the degree of smoothness of the solution u with respect to the operator B (direct and converse theorems) is established.

0. Let \mathfrak{H} be a complex separable Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$, and let A be a linear operator in \mathfrak{H} , $\overline{\mathcal{D}(A)} = \mathfrak{H}$, $0 \in \rho(A)$, where $\mathcal{D}(\cdot)$ and $\rho(\cdot)$ are the domain and the resolvent set of an operator, respectively.

We consider the equation

(1)

$$Au = f, \quad f \in \mathfrak{H},$$

and seek an approximate solution u_n of this equation by the least squares method [1], that is, in the form $u_n = \sum_{k=1}^n \alpha_k e_k$, where $\{e_k\}_{k \in \mathbb{N}}$ is a given linearly independent system of vectors in $\mathcal{D}(A)$ (a so-called coordinate system), and $\alpha_k \in \mathbb{C}$ are such that

$$R_n^2 = \|Au_n - f\|^2$$

assumes the least value. The numbers α_k are uniquely determined by the system of algebraic equations

$$\sum_{k=1}^{n} \alpha_k(Ae_k, Ae_i) = (f, Ae_i), \quad i = 1, \dots, n.$$

If the system $\{e_k\}_{k\in\mathbb{N}}$ is complete in the Hilbert space

$$(A) = \mathcal{D}(A), \quad (f,g)_{\mathfrak{H}^1(A)} = (Af, Ag), \quad f,g \in \mathcal{D}(A),$$

then (e.g., see [1]),

 \mathfrak{H}^1

$$r_n = ||u - u_n|| \to 0, \quad R_n = ||Au_n - f|| \to 0 \text{ as } n \to \infty.$$

As it will be shown below, the value R_n can tend to zero at infinity in an arbitrary way. So, it is important to have a priori estimates for this value. As a rule, such estimates are of an asymptotic character and indicate an order of smallness of R_n . For the Ritz

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method, such kind estimates were established under a certain choice of a coordinate system and some conditions of smoothness for the solution u of equation (1) (see [1–3]).

The purpose of this paper is to obtain similar estimates for the least squares method. It should be noted that in a number of specific cases, such questions were being considered by many mathematicians (we refer to [4] for details). But their considerations concerned only direct theorems and only power decreasing of R_n . To do this in the general situation and to obtain not only direct but also converse theorems for an arbitrary order of convergence of R_n to zero, we use the operator approach proposed in [2,5,6], with some generalizations and refinements of the results contained there, which we shall present some later. The results of this paper is partially announced in [7].

We would like also to remark that the significant role in application of the operator theory to the approximation problems belongs to S. I. Zuchovitsky (in this connection see, for example, [8,9]).

1. Let *B* be a closed densely defined operator in \mathfrak{H} . Denote by $C^{\infty}(B)$ the space of infinitely differentiable vectors of the operator *B*:

$$C^{\infty}(B) = \bigcap_{n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}} \mathfrak{H}^n(B),$$

where $\mathfrak{H}^n(B) = \mathcal{D}(B^n)$ is a Hilbert space with respect to the inner product

$$(f,g)_{\mathfrak{H}^n(B)} = (f,g) + (B^n f, B^n g).$$

Let $\{m_n\}_{n\in\mathbb{N}_0}$ be a nondecreasing sequence of positive numbers (there is no loss of generality in assuming that $m_0 = 1$). We put

$$C_{\{m_n\}}(B) = \bigcup_{\alpha>0} C^{\alpha}_{m_n}(B), \quad C_{(m_n)}(B) = \bigcap_{\alpha>0} C^{\alpha}_{m_n}(B),$$

where

 $C^{\alpha}_{m_n}(B) = \left\{ f \in C^{\infty}(B) \middle| \exists c = c(f) > 0, \ \forall k \in \mathbb{N}_0 : \|B^k f\| \le c\alpha^k m_k \right\}$

is a Banach space with the norm

$$\|f\|_{C^{\alpha}_{m_n}(B)} = \sup_{k \in \mathbb{N}_0} \frac{\|B^k f\|}{\alpha^k m_k}$$

The number

$$\sigma(f, m_n, B) = \inf\{\alpha : f \in C^{\alpha}_{m_n}(B)\} = \overline{\lim_{n \to \infty}} \left(\frac{\|B^n f\|}{m_n}\right)^{1/n}$$

is called the type of the vector f for the operator B and the sequence $\{m_n\}_{n\in\mathbb{N}_0}$. It is evident that $\sigma(f, m_n, B) = 0$ if $f \in C_{(m_n)}(B)$.

In the cases when $m_n = n!$ or $m_n \equiv 1$, we arrive at the well-known spaces of analytic $(C_{\{n!\}}(B))$, entire $(C_{(n!)}(B))$, and entire of exponential type $(C_{\{1\}}(B))$ vectors of the operator B (see [10–12]). If $\mathfrak{H} = L_2(a, b)$, $-\infty < a < b < \infty$, and $B = \frac{d}{dx}$, $\mathcal{D}(B) = W_2^1(a, b)$ (Sobolev space), then $C^{\infty}(B)$ is the set of all infinitely differentiable functions on [a, b], $C_{\{n!\}}(B)$ ($C_{(n!)}(B)$) is the space of analytic on [a, b] (entire) functions, the spaces $C_{\{n^{n\beta}\}}(B)$ and $C_{(n^{n\beta})}(B)$ ($\beta > 1$) coincide with the usual Gevrey classes. If $\beta < 1$, $C_{\{n^{n\beta}\}}(B)$ is the space of entire functions of order $\frac{1}{1-\beta}$. In particular, if $\beta = 0$, we have the space $C_{\{1\}}(B)$ of entire functions f of exponential type, whose exponential type is equal to $\sigma(f, 1, B)$.

Consider the functions

(2)
$$\rho_1(\lambda) = \sup_{n \in \mathbb{N}_0} \frac{\lambda^n}{m_n}, \quad \rho_2(\lambda) = \left(\sum_{n=0}^\infty \frac{\lambda^{2n}}{m_n^2}\right)^{1/2}, \quad \rho_3(\lambda) = \sum_{n=0}^\infty \frac{\lambda^n}{m_n}, \quad \lambda > 0.$$

231

Under the condition

(3)
$$\lim_{n \to \infty} \sqrt[n]{m_n} = \infty,$$

the series for $\rho_3(\lambda)$ converges in the whole complex plane and gives an entire function. All the functions in (2) are not less than 1 and tend monotonically to ∞ as $\lambda \to \infty$. It is not hard also to verify that for any $\beta_1 > \beta > 1$,

(4)
$$\rho_1(\lambda) \le \rho_2(\lambda) \le \rho_3(\lambda) \le c_\beta \rho_2(\beta\lambda) \le c_{\beta,\beta_1} \rho_1(\beta_1\lambda), \quad \lambda > 0,$$

where

$$c_{\beta} = \frac{\beta}{(\beta^2 - 1)^{1/2}}, \quad c_{\beta,\beta_1} = \frac{c(\beta)\beta_1}{(\beta_1^2 - \beta^2)^{1/2}}, \quad 0 < c(\beta) = \text{const.}$$

Throughout further, c denotes various numerical constants.

If, in addition, for the sequence
$$\{m_n\}_{n\in\mathbb{N}_0}$$
 there exists a number $h>1$ such that

(5)
$$\forall n \in \mathbb{N}_0 : m_{n+1} \le ch^n m_n$$

(for example, the sequences $m_n = n^{n\beta}$ and $m_n = (n!)^{\beta}$ with $\beta > 0$ satisfy (3) and (5)), then, as it is easily seen,

(6)
$$\exists c > 0 : \rho(\lambda) \ge c\lambda\rho(\alpha_0\lambda)$$

where $\alpha_0 = h^{-1} < 1$. By $\rho(\lambda)$ we mean any of the three functions $\rho_i(\lambda)$ (i = 1, 2, 3).

Proposition 1. Let the sequence $\{m_n\}_{n \in \mathbb{N}_0}$ satisfy the condition (3) and for any (some) h > 1 the inequality (5) is fulfilled. Then for any (some) a > 1,

(7)
$$\int_0^\infty \frac{\rho(\lambda)}{\rho(a\lambda)} d\lambda < \infty.$$

Proof. As it follows from (6),

$$\exists c > 0 : \rho_3(a\lambda) \ge c\lambda^2 \rho_3(\lambda),$$

where $a = h^2$. Then

$$\int_{1}^{\infty} \frac{\rho_{3}(\lambda)}{\rho_{3}(a\lambda)} d\lambda \leq \frac{1}{c} \int_{1}^{\infty} \frac{1}{\lambda^{2}} d\lambda < \infty$$

Since the function $\frac{\rho(\lambda)}{\rho(a\lambda)}$ is continuous on [0, 1], the finiteness of the integral (7) is ensured.

Suppose the operator B to be nonnegative and self-adjoint. Using the function $\rho(\lambda)$, we construct the family of Hilbert spaces associated with this function as follows:

$$\forall a > 0 : \mathfrak{H}^a_{\rho}(B) = \mathcal{D}(\rho(aB)), \quad (f,g)_{\mathfrak{H}^a_{\rho}(B)} = (\rho(aB)f, \rho(aB)g).$$

Here

$$\rho(aB) = \int_0^\infty \rho(a\lambda) \, dE_\lambda,$$

 $E_{\lambda} = E([0, \lambda])$ is the spectral measure of B. Because of $\rho(\lambda) \ge 1$ and $\rho(\lambda) \to \infty$ monotonically as $\lambda \to \infty$, we have

$$\forall f \in \mathfrak{H}^b_\rho(B) : \|f\|_{\mathfrak{H}^a_\rho(B)} \le \|f\|_{\mathfrak{H}^b_\rho(B)} \quad \text{as} \quad a < b,$$

and these norms are compatible. So, the embedding

$$\mathfrak{H}^b_\rho(B) \subseteq \mathfrak{H}^a_\rho(B)$$

is dense and continuous. We put

$$\mathfrak{H}_{\{\rho\}}(B) = \bigcup_{a>0} \mathfrak{H}_{\rho}^{a}(B) \text{ and } \mathfrak{H}_{(\rho)}(B) = \bigcap_{a>0} \mathfrak{H}_{\rho}^{a}(B).$$

In view of (4),

$$\mathfrak{H}_{\{\rho_1\}}(B) = \mathfrak{H}_{\{\rho_2\}}(B) = \mathfrak{H}_{\{\rho_3\}}(B), \quad \mathfrak{H}_{(\rho_1)}(B) = \mathfrak{H}_{(\rho_2)}(B) = \mathfrak{H}_{(\rho_3)}(B).$$

As it was shown in [13],

(8)
$$C_{\{m_n\}}(B) = \mathfrak{H}_{\{\rho\}}(B), \quad C_{(m_n)}(B) = \mathfrak{H}_{(\rho)}(B).$$

2. As above, in this section the operator B is assumed to be nonnegative and selfadjoint.

For an arbitrary $f \in \mathfrak{H}$ we put

$$\mathcal{E}_r(f) = \mathcal{E}_r(f, B) = \inf_{g \in C_{\{1\}}(B): \sigma(g) \le r} \|f - g\|,$$

where $\sigma(g) = \sigma(g, 1, B)$ is the type of the vector g for the operator B and the sequence $m_n \equiv 1$. Since $C_1^r = E_r \mathfrak{H}$, we have

(9)
$$\mathcal{E}_{r}^{2}(f) = \|f - E_{r}f\|^{2} = \int_{r}^{\infty} d(E_{\lambda}f, f).$$

Theorem 1. Let $G(\lambda)$ be a continuous on $[0,\infty)$ and continuously differentiable on $(0,\infty)$ function such that $G(\lambda) \ge 0$ and $G'(\lambda) \ge 0$ as $\lambda > 0$. Then the equivalence

(10)
$$f \in \mathcal{D}(G(B)) \Longleftrightarrow \int_0^\infty G(\lambda) G'(\lambda) \mathcal{E}_\lambda^2(f) \, d\lambda < \infty$$

holds, and

(11)
$$\|G(B)f\|^2 = 2 \int_0^\infty G(\lambda)G'(\lambda)\mathcal{E}_{\lambda}^2(f) \, d\lambda + G^2(0)\mathcal{E}_0^2(f).$$

Proof. Let $f \in \mathcal{D}(G(B))$. As $G(\lambda)$ is nondecreasing on $(0, \infty)$, we have

$$\forall r > 0 : \mathcal{E}_r^2(f) = \int_r^\infty d(E_\lambda f, f) \le \frac{1}{G^2(r)} \int_r^\infty G^2(\lambda) \, d(E_\lambda f, f),$$

whence

(12)
$$G^2(r)\mathcal{E}_r^2(f) \to 0 \text{ as } r \to \infty.$$

Taking into account that

(13)
$$\int_{0}^{r} G^{2}(\lambda) d(E_{\lambda}f, f) = -\int_{0}^{r} G^{2}(\lambda) d\mathcal{E}_{\lambda}^{2}(f) \\ = -G^{2}(r)\mathcal{E}_{r}^{2}(f) + G^{2}(0)\mathcal{E}_{0}^{2}(f) + 2\int_{0}^{r} G(\lambda)G'(\lambda)\mathcal{E}_{\lambda}^{2}(f) d\lambda,$$

and there exists the limit of the integral in the left hand side of (13) as $r \to \infty$, we conclude, by virtue of (12), that the integral

$$\int_0^\infty G(\lambda) G'(\lambda) \mathcal{E}_\lambda^2(f) \, d\lambda$$

is finite, and the equality (11) is true. Conversely, let $\int_0^\infty G(\lambda)G'(\lambda)\mathcal{E}_{\lambda}^2(f) d\lambda < \infty$. Then the equality (13) implies

$$\int_0^r G^2(\lambda) \, d(E_\lambda f, f) \le G^2(0) \mathcal{E}_0^2(f) + 2 \int_0^r G(\lambda) G'(\lambda) \mathcal{E}_\lambda^2(f) \, d\lambda.$$

It follows from here that $\int_0^\infty G^2(\lambda) d(E_\lambda f, f) < \infty$, that is, $f \in \mathcal{D}(G(B))$. Then the equality (11) is a direct consequence of (13) and (12).

Corollary 1. Let $G(\lambda)$ satisfy the conditions of Theorem 1. Then

(14)
$$f \in \mathcal{D}(G(B)) \Longrightarrow \mathcal{E}_r(f) = o(G^{-1}(r)), \quad r \to \infty$$

Setting in Theorem 1 $G(\lambda) = \lambda^{\alpha}$ ($\alpha > 0$), we obtain the next assertion.

Corollary 2. For an arbitrary $\alpha > 0$,

$$\begin{array}{ll} f \in \mathcal{D}(B^{\alpha}) & \Longrightarrow & \mathcal{E}_{r}(f) = o(r^{-\alpha}), \\ f \in \mathcal{D}(B^{\alpha}) & \Longleftrightarrow & \int_{0}^{\infty} \lambda^{2\alpha - 1} \mathcal{E}_{\lambda}^{2}(f) \, d\lambda < \infty \end{array}$$

Moreover,

$$\|B^{\alpha}f\|^{2} = 2\alpha \int_{0}^{\infty} \lambda^{2\alpha-1} \mathcal{E}_{\lambda}^{2}(f) \, d\lambda$$

Note that the relation $\mathcal{E}_r(f) = o(r^{-\alpha})$ does not yet imply the inclusion $f \in \mathcal{D}(B^{\alpha})$. But the following statement is valid.

Corollary 3. If for some $\varepsilon > 0$,

$$r^{\alpha+\varepsilon}\mathcal{E}_r(f) = O(1) \quad (r \to \infty),$$

then $f \in \mathcal{D}(B^{\alpha})$.

Proof. Taking in (13) $G(\lambda) = \lambda^{\alpha}$, we obtain for any r > 0

$$\int_{0}^{r} \lambda^{2\alpha} d(E_{\lambda}f, f) = -r^{2\alpha} \mathcal{E}_{r}^{2}(f) + 2\alpha \int_{0}^{r} \lambda^{2\alpha-1} \mathcal{E}_{\lambda}^{2}(f) d\lambda$$
$$\leq 2\alpha \int_{0}^{r} \lambda^{2(\alpha+\varepsilon)} \mathcal{E}_{\lambda}^{2}(f) \frac{d\lambda}{\lambda^{1+2\varepsilon}}$$
$$\leq 2\alpha \sup_{\lambda \in [0,\infty)} \left(\lambda^{2(\alpha+\varepsilon)} \mathcal{E}_{\lambda}^{2}(f)\right) \int_{0}^{r} \frac{d\lambda}{\lambda^{1+2\varepsilon}},$$

whence $f \in \mathcal{D}(B^{\alpha})$.

Corollary 4. Let $G(\lambda)$ satisfy the conditions of Theorem 1, and

(15)
$$\int_{r_0}^{\infty} \frac{G'(\lambda)}{G(a\lambda)} d\lambda < \infty$$

with some $r_0 > 0, a > 1$. Then

$$\mathcal{E}_{\lambda}(f) = O(G^{-1}(a\lambda)) \ (\lambda \to \infty) \Longrightarrow f \in \mathcal{D}(G(B)).$$

Proof. Assume that the condition (15) is fulfilled, and $\mathcal{E}_{\lambda}(f) = O(G^{-1}(a\lambda))$. Then for sufficiently large λ (we may consider $\lambda > r_0$)

$$\exists c > 0 : \mathcal{E}_{\lambda}(f) < cG^{-1}(a\lambda),$$

and, as it follows from (13), for $r > r_0$ we have

$$\begin{split} \int_0^r G^2(\lambda) \, d(E_\lambda f, f) &\leq G^2(0) \mathcal{E}_0^2(f) + 2 \int_0^r G(\lambda) G'(\lambda) \mathcal{E}_\lambda^2(f) \, d\lambda \\ &\leq G^2(0) \mathcal{E}_0^2(f) + 2 \int_0^{r_0} G(\lambda) G'(\lambda) \mathcal{E}_\lambda^2(f) \, d\lambda + 2c \int_{r_0}^r \frac{G'(\lambda)}{G(a\lambda)} \, d\lambda, \\ &\text{nce } \int_0^\infty G^2(\lambda) \, d(E_\lambda f, f) < \infty, \text{ that is, } f \in \mathcal{D}(G(B)). \end{split}$$

whence $\int_0^\infty G^2(\lambda) d(E_\lambda f, f) < \infty$, that is, $f \in \mathcal{D}(G(B))$.

Corollary 5. If the sequence $\{m_n\}_{n \in \mathbb{N}_0}$ satisfies the conditions (3) and (5) with h > 1, then

$$\rho(h^2\lambda)\mathcal{E}_{\lambda}(f) = O(1) \ (\lambda \to \infty) \Longrightarrow f \in \mathcal{D}(\rho(B)),$$

where, as before, $\rho(\lambda)$ stands for any function from (2).

Proof. In view of (4), it is sufficient to consider only the case $\rho(\lambda) = \rho_3(\lambda)$. Since h > 1, we have $n \le ch^{2n}$, $n \in \mathbb{N}_0$. Hence, for $\lambda > 1$,

$$\rho_3'(\lambda) = \sum_{n=1}^{\infty} \frac{n\lambda^{n-1}}{m_n} \le c \sum_{n=1}^{\infty} \frac{(h^2\lambda)^n}{m_n} = c\rho_3(h^2\lambda),$$

and, by Proposition 1,

$$\int_{1}^{\infty} \rho_{3}(\lambda) \rho_{3}'(\lambda) \mathcal{E}_{\lambda}^{2}(f) \, d\lambda \leq c \int_{1}^{\infty} \rho_{3}(\lambda) \rho_{3}(h^{2}\lambda) \frac{d\lambda}{\rho_{3}^{2}(h^{2}\lambda)} = c \int_{1}^{\infty} \frac{\rho_{3}(\lambda)}{\rho_{3}(h^{2}\lambda)} \, d\lambda < \infty.$$
view of Theorem 1, $f \in \mathcal{D}(\rho_{3}(B))$.

Remark 1. It is not hard to see that the proof of Corollary 5 is suitable for the case where the function $\rho_{\mu}(\lambda) = \rho(\mu\lambda)$ ($\mu > 0$) is taken instead of $\rho(\lambda)$, that is,

$$\rho(h^2\mu\lambda)\mathcal{E}_{\lambda}(f) = O(1) \ (\lambda \to \infty) \Longrightarrow f \in \mathcal{D}(\rho(\mu B)).$$

For a nondecreasing on $[0,\infty)$ function $F(\lambda)$ such that $\lim_{\lambda\to\infty} F(\lambda) = \infty$, we put

$$s(f, F, B) = \sup\{a \in (0, \infty) : \int_0^\infty F^2(a\lambda) \, d(E_\lambda f, f) < \infty\}.$$

Theorem 2. Let $\rho(\lambda)$ be one of the functions $\rho_i(\lambda)$, i = 1, 2, 3. If $f \in C_{\{m_n\}}(B)$, then

(16)
$$\sigma(f, m_n, B) = \frac{1}{s(f, \rho, B)}.$$

Proof. We shall prove the theorem only for $\rho(\lambda) = \rho_2(\lambda)$. The rest follows from the inequalities (4).

Assume that $f \in C_{\{m_n\}}(B)$. Due to (8), $f \in \mathfrak{H}_{\{\rho_2\}}(B)$. So, $f \in \mathcal{D}(\rho_2(aB))$ with some a > 0, and

(17)
$$\|\rho_{2}(aB)f\|^{2} = \int_{0}^{\infty} \sum_{n=0}^{\infty} \frac{(a\lambda)^{2n}}{m_{n}^{2}} d(E_{\lambda}f, f) = \sum_{n=0}^{\infty} \frac{a^{2n}}{m_{n}^{2}} \int_{0}^{\infty} \lambda^{2n} d(E_{\lambda}f, f)$$
$$= \sum_{n=0}^{\infty} \frac{a^{2n}}{m_{n}^{2}} \|B^{n}f\|^{2} < \infty.$$

Therefore

$$\forall n \in \mathbb{N}_0 : \frac{a^n \|B^n f\|^2}{m_n} < c$$

that is, $f \in C_{m_n}^{1/a}$. The latter means that $\sigma(f, m_n, B) \leq \frac{1}{a}$. As a is an arbitrary positive number for which $f \in \mathcal{D}(\rho_2(aB))$, we can conclude that

$$\sigma(f, m_n, B) \le \frac{1}{s(f, \rho_2, B)}.$$

Conversely, let $a \geq \sigma(f, m_n, B)$. Then

$$\forall \varepsilon > 0: \sum_{n=0}^{\infty} \frac{\|B^n f\|^2}{(a+\varepsilon)^{2n} m_n^2} < \infty$$

and the equalities (17) show that $\frac{1}{a+\varepsilon} \leq s(f, \rho_2, B)$, whence

$$\frac{1}{\sigma(f, m_n, B)} \le s(f, \rho_2, B).$$

It follows from here that the formula (16) is true.

In

3. Now we turn to equation (1), where A is a linear operator in \mathfrak{H} , $\overline{\mathcal{D}(A)} = \mathfrak{H}$, and $0 \in \rho(A)$. We say that a closed linear operator B in \mathfrak{H} is similar to A if $\mathcal{D}(B) = \mathcal{D}(A)$. If $0 \in \rho(B)$, then by closed graph theorem, the operators BA^{-1} and AB^{-1} are defined and continuous on the whole space \mathfrak{H} . In what follows, we assume everywhere that:

 1^{\diamond} . B is a positive definite self-adjoint operator similar to A;

 2^{\diamond} . the operator B^{-1} is compact, and the spectrum $\lambda_k = \lambda_k(B), k \in \mathbb{N}$, of the operator B is simple (the eigenvalues λ_k are arranged in ascending order);

 3^{\diamond} . the orthonormal basis of eigenvectors of the operator B is taken as the coordinate system $\{e_k\}_{k\in\mathbb{N}}$ in the least squares method in approximate solving equation (1).

The relation

$$\left\| A\left(f - \sum_{k=1}^{n} a_k e_k\right) \right\| = \left\| AB^{-1}B\left(f - \sum_{k=1}^{n} a_k e_k\right) \right\|$$
$$\leq \|AB^{-1}\| \left\| B\left(f - \sum_{k=1}^{n} a_k e_k\right) \right\|$$

shows that the system $\{e_k\}_{k\in\mathbb{N}}$ is complete in $\mathfrak{H}^1(A)$, and therefore,

$$||Au_n - f|| \to 0, \ n \to \infty,$$

where $u_n = \sum_{k=1}^n \alpha_k e_k$ is the approximate solution of (1) by means of the least square method with the coordinate system $\{e_k\}_{k \in \mathbb{N}}$. The next theorem shows that this convergence may be as slow as desired.

Theorem 3. Let A be a densely defined linear operator in \mathfrak{H} such that $0 \in \rho(A)$. Suppose also that the conditions $1^{\diamond}-3^{\diamond}$ hold. Then whatever decreasing number sequence $\{\gamma_n\}_{n\in\mathbb{N}}$, $\gamma_n > 0, \gamma_n \to 0 \ (n \to \infty)$ may be, there exists a vector $f \in \mathfrak{H}$ such that

$$\forall n \in \mathbb{N} : R_n = \|Au_n - f\| = \gamma_n.$$

Proof. First we shall prove that for an arbitrary Hilbert space \mathfrak{H} and any orthonormal basis $\{g_k\}_{k\in\mathbb{N}}$ in \mathfrak{H} , there exists a vector $v \in \mathfrak{H}$ such that

$$\forall n \in \mathbb{N} : \left\| v - \sum_{k=1}^{n} (v, g_k) g_k \right\| = \gamma_n.$$

Really, put $\beta_1 = 0$, $\beta_i^2 = \gamma_{i-1}^2 - \gamma_i^2$ (i = 2, 3, ...). Then the series $\sum_{k=1}^n \beta_i g_i$ converges in \mathfrak{H} to a certain element $v \in \mathfrak{H}$, and it is the Fourier series of this element $(\beta_i = (v, g_i))$. So,

$$\left\| v - \sum_{k=1}^{n} (v, g_k) g_k \right\| = \left(\sum_{k=1}^{n} \beta_i^2 \right)^{1/2} = \gamma_n.$$

Taking the space $\mathfrak{H}^1(A)$ as \mathfrak{H} and the sequence $\{\tilde{e}_k\}_{k\in\mathbb{N}}$ obtained in the process of orthogonalization in $\mathfrak{H}^1(A)$ of the coordinate system $\{e_k\}_{k\in\mathbb{N}}$ as $\{g_k\}_{k\in\mathbb{N}}$, we conclude that there exists a vector $u \in \mathfrak{H}^1(A)$ such that

$$\forall n \in \mathbb{N} : \left\| u - \sum_{k=1}^{n} (u, \tilde{e}_k)_{\mathfrak{H}^1(A)} \tilde{e}_k \right\|_{\mathfrak{H}^1(A)} = \left\| A \left(u - \sum_{k=1}^{n} (u, \tilde{e}_k)_{\mathfrak{H}^1(A)} \tilde{e}_k \right) \right\| = \gamma_n.$$

Setting f = Au, we get

$$R_{n} = \|Au_{n} - f\| = \min_{\{a_{k}\}_{k \in \mathbb{N}}} \left\| f - \sum_{k=1}^{n} a_{k} Ae_{k} \right\| = \min_{\{a_{k}\}_{k \in \mathbb{N}}} \left\| A\left(u - \sum_{k=1}^{n} a_{k} e_{k} \right) \right\|$$
$$= \min_{\{a_{k}\}_{k \in \mathbb{N}}} \left\| u - \sum_{k=1}^{n} a_{k} e_{k} \right\|_{\mathfrak{H}^{1}(A)} = \min_{\{a_{k}\}_{k \in \mathbb{N}}} \left\| u - \sum_{k=1}^{n} a_{k} \tilde{e}_{k} \right\|_{\mathfrak{H}^{1}(A)}$$
$$= \left\| u - \sum_{k=1}^{n} (u, \tilde{e}_{k})_{\mathfrak{H}^{1}(A)} \tilde{e}_{k} \right\|_{\mathfrak{H}^{1}(A)} = \gamma_{n}.$$

In connection with this, the question arises, what properties the solution $u = A^{-1}f$ has to possess in order that the value $R_n = ||Au_n - f||$ be of a certain order of decreasing to zero as $n \to \infty$. The answer is given by the theorem below.

Theorem 4. Let $\{e_k\}_{k\in\mathbb{N}}$ be the orthonormal basis of eigenvectors of the operator B, and let u_n be the approximate solution of equation (1) in the least squares method with the coordinate system $\{e_k\}_{k\in\mathbb{N}}$. Then for any $\alpha \geq 0$

(18)
$$u = A^{-1}f \in \mathcal{D}(B^{\alpha+1}) \Longrightarrow R_n = o(\lambda_{n+1}^{-\alpha}).$$

Moreover,

(19)
$$\exists \varepsilon > 0 : R_n = o(\lambda_{n+1}^{-(\alpha+\varepsilon)}) \Longrightarrow u \in \mathcal{D}(B^{\alpha+1}).$$

Proof. We put $v_n = \sum_{k=1}^n (u, e_k) e_k$. Then

$$||Au_n - f|| \le ||Av_n - f|| = ||A(v_n - u)|| = ||AB^{-1}B(v_n - u)||$$

$$\le ||AB^{-1}|| ||Bv_n - Bu|| = ||AB^{-1}|| \left\| \sum_{k=1}^n (Bu, e_k)e_k - Bu \right\|.$$

As the resolution of identity E_{λ} of the operator B has the form

$$E_{\lambda}g = \sum_{k:\lambda_k \leq \lambda} (g, e_k)e_k,$$

and, because of (9),

$$\mathcal{E}_{\lambda_{n+1}}(g) = \|g - E_{\lambda_{n+1}}g\| = \left\|g - \sum_{k=1}^{n} (g, e_k)e_k\right\|,$$

one can conclude that

(20)
$$||Au_n - f|| \le ||AB^{-1}||\mathcal{E}_{\lambda_{n+1}}(Bu)|$$

Then the relation (18) follows from Corollary 2.

Now assume that the equality in the left hand side of (19) is fulfilled. Then

$$\mathcal{E}_{\lambda_{n+1}}(Bu) = \left\| Bu - \sum_{k=1}^{n} (Bu, e_k) e_k \right\| = \left\| Bu - \sum_{k=1}^{n} (u, e_k) Be_k \right\|$$
$$\leq \left\| B\left(u - \sum_{k=1}^{n} (u, e_k) e_k \right) \right\| \leq \|BA^{-1}\| \|A(u_n - u)\| = o(\lambda_{n+1}^{-(\alpha + \varepsilon)}).$$

By Corollary 3, $Bu \in \mathcal{D}(B^{\alpha})$, whence $u \in \mathcal{D}(B^{\alpha+1})$.

237

It should be noted that there exist examples verifying that the equality

$$||Au_n - f|| = o(\lambda_{n+1}^{-\alpha})$$

does not yet imply the inclusion $u \in \mathcal{D}(B^{\alpha+1})$.

According to [14], for the operator B and a vector $g \in \mathfrak{H}$, we put

$$\omega_k(t, g, B) = \sup_{0 \le \tau \le t} \|\Delta_{\tau}^k g\|, \quad k \in \mathbb{N},$$

where

$$\Delta_{\tau}^{k} = (U(\tau) - I)^{k} = \sum_{j=0}^{k} C_{k}^{j} U(j\tau), \quad k \in \mathbb{N}_{0}, \quad \tau \in \mathbb{R}_{+} = [0, \infty),$$

and $U(\tau) = e^{i\tau B}$ is a group of unitary operators in \mathfrak{H} with the generating operator iB.

It follows from this definition that $\omega_k(t, g, B)$ possesses the following properties:

- 1) $\omega_k(0,g,B) = 0;$
- 2) for a fixed g, the function $\omega_k(t, g, B)$ is nondecreasing on \mathbb{R}_+ ;
- 3) $\forall \alpha, t > 0 : \omega_k(\alpha t, g, B) \le (1 + \alpha)^k \omega_k(t, g, B);$
- 4) for any fixed $t \in \mathbb{R}_+$, the function $\omega_k(t, g, B)$ is continuous with respect to g.

As was proved in [3], the following statement holds.

Proposition 2. Let $g \in \mathcal{D}(B^{\alpha}), \ \alpha > 0$. Then

$$\forall k \in \mathbb{N} : \mathcal{E}_r(g, B) \le \frac{\sqrt{k+1}}{2^k r^{\alpha}} \omega_k(\frac{\pi}{r}, B^{\alpha}g, B).$$

Conversely, if $\omega(t)$, $t \in [0, \infty)$, is a function of continuity module type, that is:

- (i) $\omega(t)$ is continuous and nondecreasing on \mathbb{R}_+ ,
- (*ii*) $\omega(0) = 0$,
- (iii) $\exists c > 0, \forall t > 0 : \omega(2t) \le c\omega(t),$

and if, in addition, the condition

$$\int_0^1 \frac{\omega(t)}{t} \, dt < \infty,$$

is satisfied, then

$$\exists c > 0, \quad \forall r > 0 : \mathcal{E}_r(g, B) \le \frac{c}{r^{\alpha}} \omega\left(\frac{1}{r}\right) \Longrightarrow g \in \mathcal{D}(B^{\alpha}).$$

Using this proposition, on the basis of (20), we arrive at the following conclusion.

Theorem 5. If $u \in \mathcal{D}(B^{\alpha+1})$, $\alpha > 0$, then

$$\forall k \in \mathbb{N} : \|Au_n - f\| \le \frac{\sqrt{k+1}}{2^k \lambda_{n+1}^{\alpha}} \omega_k \left(\frac{\pi}{\lambda_{n+1}}, B^{\alpha+1}g, B\right).$$

Conversely, let for $\omega(t)$ the conditions of Proposition 2 be fulfilled. If $u \in \mathcal{D}(B)$ and

$$\|Au - f\| \le \lambda_{n+1}^{-\alpha} \omega\left(\frac{1}{\lambda_{n+1}}\right) \ (\alpha > 0),$$

then $u \in \mathcal{D}(B^{\alpha+1})$.

Now we pass to the approximation by means of the least squares method of the solution u to equation (1) in assumption that u belongs to some class of infinitely differentiable vectors for the operator B. First of all, note that Theorem 4 gives as a consequence such a result.

Corollary 6. The following equivalence holds:

$$u \in C^{\infty}(B) \iff \forall \alpha > 0 : \lim_{n \to \infty} \lambda_{n+1}^{\alpha} R_n = 0.$$

For more smooth u, the assertion below is valid.

Theorem 6. Let the conditions $1^{\diamond}-3^{\diamond}$ hold, and let the sequence $\{m_n\}_{n\in\mathbb{N}_0}$ satisfy (3) and the condition (5) with any h > 1. Then

(21)
$$u \in C_{\{m_n\}}(B), \quad \sigma(u, m_n, B) = \sigma \iff \forall \varepsilon > 0 : \lim_{n \to \infty} \rho\left(\frac{\lambda_{n+1}}{\sigma + \varepsilon}\right) R_n = 0,$$
$$\lim_{n \to \infty} \rho\left(\frac{\lambda_{n+1}}{\sigma - \varepsilon}\right) R_n = \infty;$$

(22)
$$u \in C_{(m_n)}(B) \iff \forall \varepsilon > 0 \lim_{n \to \infty} \rho\left(\frac{\lambda_{n+1}}{\varepsilon}\right) R_n = 0,$$

where $\rho(\lambda)$ is any of the three functions $\rho_i(\lambda)$, i = 1, 2, 3 from (2).

Proof. Let $u \in C_{\{m_n\}}(B)$. Then

$$\forall \alpha > \sigma, \quad \forall k \in \mathbb{N}_0,$$
$$\exists c = c(\alpha) > 0 : \|B^k B u\| = \|B^{k+1} u\| \le c \alpha^{k+1} m_{k+1} \le c \alpha^{k+1} h^k m_k = c(\alpha h)^k m_k,$$

which implies that $\sigma(Bu, m_n, B) \leq \alpha h$. As h > 1 is arbitrary, we have $\sigma(Bu, m_n, B) \leq \sigma$. But, by the definition of σ , for any $\varepsilon \in (0, \sigma)$, there exists a subsequence $k_i \in \mathbb{N}_0$ such that

$$||B^{k_i}u|| \ge c(\sigma - \varepsilon)^{k_i}m_{k_i}.$$

Therefore

$$||B^{k_i-1}Bu|| \ge c(\sigma-\varepsilon)^{k_i-1}m_{k_i-1}$$

Thus,

$$\sigma(Bu, m_n, B) = \sigma(u, m_n, B) = \sigma_{-}$$

Since $Bu \in C_{\{m_n\}}(B) = \mathfrak{H}_{\{\rho\}}(B)$ and because of Theorem 2, we have

$$\sigma(Bu, m_n, B) = s^{-1}(Bu, \rho, B).$$

It follows from the definition of $s(u, \rho, B)$ that

(23)
$$\forall \varepsilon \in (0, \sigma) : Bu \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma + \varepsilon}\right)\right), \quad Bu \notin \mathcal{D}\left(\rho\left(\frac{B}{\sigma - \varepsilon}\right)\right).$$

By Corollary 1,

(24)
$$\mathcal{E}_r(Bu)\rho\left(\frac{r}{\sigma+\varepsilon}\right) \to 0 \quad \text{as} \quad r \to \infty.$$

Moreover,

(25)
$$\overline{\lim_{r \to \infty}} \mathcal{E}_r(Bu) \rho\left(\frac{r}{\sigma - \varepsilon}\right) = \infty.$$

Indeed, assume that for some $\varepsilon > 0$ and $r \ge r_0$,

$$\mathcal{E}_r(Bu)
ho\left(\frac{r}{\sigma-\varepsilon}\right) \le c < \infty.$$

As it follows from Remark 1, for any $\varepsilon' < \varepsilon$ we have

$$c \ge \mathcal{E}_r(Bu)\rho\left(\frac{r}{\sigma-\varepsilon}\right) = \mathcal{E}_r(Bu)\rho\left(\frac{\sigma-\varepsilon'}{\sigma-\varepsilon}\cdot\frac{r}{\sigma-\varepsilon'}\right) \Longrightarrow Bu \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma-\varepsilon'}\right)\right),$$

contrary to (23).

According to (20), $||Au_n - f|| \leq c \mathcal{E}_{\lambda_{n+1}}(Bu)$. By (24),

$$||Au_n - f|| \rho\left(\frac{\lambda_{n+1}}{\sigma + \varepsilon}\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Since

(26)
$$\mathcal{E}_{\lambda_{n+1}}(Bu) = \left\| \sum_{k=1}^{n} (Bu, e_k) e_k - Bu \right\| = \|B(u - u_n)\| \\ \leq \|BA^{-1}\| \|Au_n - f\|,$$

the formula (25) implies the relation

$$\overline{\lim_{n \to \infty}} \|Au_n - f\|\rho\left(\frac{\lambda_{n+1}}{\sigma - \varepsilon}\right) = \infty.$$

Conversely, let the relations in the right hand side of (21) be fulfilled with some $\sigma > 0$. Because of (26),

$$\rho\left(\frac{\lambda_{n+1}}{\sigma+\varepsilon}\right)\mathcal{E}_{\lambda_{n+1}}(Bu) \le \rho\left(\frac{\lambda_{n+1}}{\sigma+\varepsilon}\right) \|Au_n - f\| \|BA^{-1}\| \to 0 \quad \text{as} \quad n \to \infty.$$

In the same way as above, one can show that

$$Bu \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma+\varepsilon'}\right)\right) \ (\varepsilon' > \varepsilon),$$

and therefore,

$$u \in \mathcal{D}\left(\rho\left(\frac{B}{\sigma + \varepsilon'}\right)\right)$$

So,

$$s(u,\rho,B) \ge \frac{1}{\sigma}$$

Furthermore, due to (20),

$$\forall \varepsilon > 0 : \overline{\lim_{n \to \infty}} \rho\left(\frac{\lambda_{n+1}}{\sigma - \varepsilon}\right) \mathcal{E}_{\lambda_{n+1}}(Bu) = \infty,$$

which guarantees, by Corollary 1, that

$$Bu \notin \mathcal{D}\left(\rho\left(\frac{B}{\sigma-\varepsilon}\right)\right).$$

The latter means that

$$s(Bu,\rho,B)=s(u,\rho,B)=\sigma^{-1}$$

that is equivalent to $\sigma(u, m_n, B) = \sigma$.

The relation (22) follows from the previous one, because in this case $\sigma(u, m_n, B) = 0$ for any $u \in C_{(m_n)}(B)$.

4. As an example, we consider the case

$$\mathfrak{H} = L_2(0,\pi), \quad A = (-1)^m \frac{d^{2m}}{dt^{2m}} + \sum_{k=0}^{2m-1} p_k(t) \frac{d^k}{dt^k},$$

$$\mathcal{D}(A) = \{ v(\cdot) \in W_2^{2m}[0,\pi] | v^{(2k)}(0) = v^{(2k)}(\pi) = 0, \ k = 0, \dots, m-1 \}$$

where $p_k(\cdot) \in C[0,\pi]$. It is also assumed that the equation Av = 0 has only the trivial solution.

We define the operator B as

$$B = (-1)^m \frac{d^{2m}}{dt^{2m}}, \quad \mathcal{D}(B) = \mathcal{D}(A).$$

This operator is self-adjoint and positive definite, its spectrum $\{\lambda_k = k^{2m}\}_{k \in \mathbb{N}}$ is discrete and simple, and the functions $\sqrt{\frac{2}{\pi}} \sin kt$ form an orthonormal basis of eigenvectors of Bin $L_2(0,\pi)$.

239

It can readily be shown that under the conditions

$$(27) p_k \in C^{2mj}[0,\pi]$$

and (28)

$$p_k^{(2r-1)}(0) = p_k^{(2r-1)}(\pi) = 0$$
 for even $k \quad (1 \le r \le mj),$

$$p_k^{(2r)}(0) = p_k^{(2r)}(\pi) = 0$$
 for odd k $(0 \le r \le (m-1)j),$

the relation

$$\mathcal{D}(A^{j+1}) = \mathcal{D}(B^{j+1})$$

is true.

On the basis of Corollary 2, we arrive at the following conclusion.

Proposition 3. If the coefficients $p_k(t)$ satisfy the conditions (27)–(29) and if

(30)
$$f(\cdot) \in C^{2mj}[0,\pi]$$
 and $f^{(2k)}(0) = f^{(2k)}(\pi) = 0$ $(k = 0, 1, \dots, mj - 1),$

then

$$(n+1)^{2mj} ||Au_n - f||_{L_2(0,\pi)} \to 0 \quad \text{as} \quad n \to \infty,$$

or, equivalently,

(31)
$$(n+1)^{2mj} \|u_n - u\|_{W_2^{2m}[0,\pi]} \to 0 \quad (n \to \infty).$$

The validity of (31) for m = 1 and j = 1 was established in [15].

Corollary 6 and Theorem 6 for $m_n = n!$ imply the following assertion.

Proposition 4. Suppose that all $p_k(\cdot)$ belong to $C^{\infty}[0,\pi]$ and satisfy the conditions (27)–(29) for each $r \in \mathbb{N}_0$. Then the inclusion $f(\cdot) \in C^{\infty}[0,\pi]$ and the condition (30) for $k \in \mathbb{N}_0$ are equivalent to the following statement:

$$\forall \alpha > 0 \quad n^{\alpha} \|Au_n - u\|_{L_2(0,\pi)} \to 0 \quad (n \to \infty)$$

If, in addition, the functions $p_k(t)$ are analytic on $[0, \pi]$, then the assertion

 $\exists \alpha > 0 \quad e^{\alpha n} \|Au_n - u\|_{L_2(0,\pi)} \to 0 \quad (n \to \infty)$

is equivalent to the analyticity of the function f(t) on $[0, \pi]$ and the validity of the condition (30) for all $k \in \mathbb{N}_0$. If the $p_k(t)$ are entire functions, then for the assertion

$$\forall \alpha > 0 \quad e^{\alpha n} \|Au_n - u\|_{L_2(0,\pi)} \to 0 \quad (n \to \infty)$$

to be true, it is necessary and sufficient that the function f(t) be entire and satisfy the condition (30) for all $k \in \mathbb{N}_0$.

It should be noted that the expression $||Au_n - u||_{L_2(0,\pi)}$ in the last Proposition can be replaced by $||u_n - f||_{W_2^{2m}[0,\pi]}$.

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