

G-FRAMES AND STABILITY OF G-FRAMES IN HILBERT SPACES

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ABSTRACT. Wenchang Sun in his paper [Wenchang Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), 437–452] has introduced *g*-frames which are generalized frames and include ordinary frames and many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. In this paper we develop the *g*-frame theory for separable Hilbert spaces and give characterizations of *g*-frames and we show that *g*-frames share many useful properties with frames. We present a version of the Paley-Wiener Theorem for *g*-frames which is in spirit close to results for frames, due to Ole Christensen.

1. INTRODUCTION

There are some generalizations of frames, for example bounded quasi-projectors [8] and frames of subspaces [3]. The mean of *g*-frames has been presented by W. Sun in [15]. This is an extension of frames that include all of the previous extensions of frames.

Through this paper, \mathcal{H} and \mathcal{K} are Hilbert spaces and $\{\mathcal{H}_i : i \in I\}$ is a sequence of Hilbert spaces, where I is a subset of \mathbb{Z} . $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from \mathcal{H} to \mathcal{H}_i .

Note that for any sequence $\{\mathcal{H}_i : i \in I\}$, we can assume that there exists a Hilbert space \mathcal{K} such that for all $i \in I$, $\mathcal{H}_i \subseteq \mathcal{K}$ (for example $\mathcal{K} = \bigoplus_{i \in I} \mathcal{H}_i$).

Definition 1.1. We call a sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a generalized frame, or simply a *g*-frame, for \mathcal{H} with respect to $\{\mathcal{H}_i : i \in I\}$ if there exist two positive constants A and B such that

$$(1.1) \quad A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

We call A and B the lower and upper *g*-frame bounds, respectively.

We call $\{\Lambda_i : i \in I\}$ a tight *g*-frame if $A = B$ and a Parseval *g*-frame if $A = B = 1$.

We call $\{\Lambda_i : i \in I\}$ an exact *g*-frame if it ceases to be a *g*-frame whenever any of its elements is removed.

We say simply a *g*-frame for \mathcal{H} whenever the space sequence $\{\mathcal{H}_i : i \in I\}$ is clear.

We say also a *g*-frame for \mathcal{H} with respect to \mathcal{K} whenever $\mathcal{H}_i = \mathcal{K}$, for each $i \in I$.

We say $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a *g*-frame sequence, if it is a *g*-frame for $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$.

We say $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a *g*-Bessel sequence with bound B , if we only have the upper bound in (1.1).

Notation 1.2. For each sequence $\{\mathcal{H}_i\}_{i \in I}$, we define the space $\left(\sum_{i \in I} \bigoplus \mathcal{H}_i\right)_{\ell_2}$ by

$$(1.2) \quad \left(\sum_{i \in I} \bigoplus \mathcal{H}_i\right)_{\ell_2} = \left\{ \{f_i\}_{i \in I} : f_i \in \mathcal{H}_i, i \in I \text{ and } \sum_{i \in I} \|f_i\|^2 < +\infty \right\}$$

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with the inner product defined by

$$\langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle.$$

It is clear that $\left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2}$ is a Hilbert space with pointwise operations.

2. CHARACTERIZATION OF g -FRAMES

In this section, we will try to characterize g -frames from the point of view of operator theory. We are starting with the definition of a synthesis operator for a g -frame. For this mean, we must show that the series appearing in the definition of a synthesis operator converges unconditionally. So we need the next lemma.

Lemma 2.1. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -Bessel sequence for \mathcal{H} with bound B . Then for each sequence $\{f_i\}_{i \in I} \in \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2}$, the series $\sum_{i \in I} \Lambda_i^*(f_i)$ converges unconditionally.*

Proof. Let $J \subseteq I$ with $|J| < \infty$, then

$$\begin{aligned} \left\| \sum_{i \in J} \Lambda_i^*(f_i) \right\| &= \sup_{\|g\|=1} \left| \left\langle \sum_{i \in J} \Lambda_i^*(f_i), g \right\rangle \right| \\ &\leq \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}} \sup_{\|g\|=1} \left(\sum_{i \in J} \|\Lambda_i g\|^2 \right)^{\frac{1}{2}} \leq \sqrt{B} \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

It follows that $\sum_{i \in I} \Lambda_i^*(f_i)$ is weakly unconditionally Cauchy and hence unconditionally convergent in \mathcal{H} (see [6], page 44, Theorems 6 and 8). \square

Definition 2.2. Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} . Then the *synthesis operator* for $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is the operator

$$T : \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2} \longrightarrow \mathcal{H}$$

defined by

$$T(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i).$$

We call the adjoint T^* of the synthesis operator the *analysis operator*.

The following proposition will provide a concrete formula for the analysis operator.

Proposition 2.3. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} . Then the analysis operator $T^* : \mathcal{H} \longrightarrow \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2}$ is given by*

$$T^*(f) = \{\Lambda_i f\}_{i \in I}.$$

Proof. For all $f \in \mathcal{H}$ and $\{g_i\}_{i \in I} \in \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2}$ we have

$$\begin{aligned} \langle T^* f, \{g_i\}_{i \in I} \rangle &= \langle f, T\{g_i\}_{i \in I} \rangle = \left\langle f, \sum_{i \in I} \Lambda_i^*(g_i) \right\rangle \\ &= \sum_{i \in I} \langle \Lambda_i f, g_i \rangle = \left\langle \{\Lambda_i f\}_{i \in I}, \{g_i\}_{i \in I} \right\rangle. \end{aligned}$$

So that $T^* f = \{\Lambda_i f\}_{i \in I}$. \square

The following proposition characterizes g -Bessel sequences.

Proposition 2.4. $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence for \mathcal{H} with bound B , if and only if the operator

$$T : \left(\sum_{i \in I} \bigoplus \mathcal{H}_i \right)_{\ell_2} \longrightarrow \mathcal{H}$$

defined by

$$T(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i)$$

is a well-defined and bounded operator with $\|T\| \leq \sqrt{B}$.

Proof. Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -Bessel sequence for \mathcal{H} with bound B . Then by Lemma 2.1, T is well-defined and $\|T\| \leq \sqrt{B}$.

Conversely, let T be a well-defined and bounded operator with $\|T\| \leq \sqrt{B}$. Let $J \subseteq I$ with $|J| < +\infty$, then for each $f \in \mathcal{H}$,

$$\sum_{i \in J} \|\Lambda_i f\|^2 = \sum_{i \in J} \langle \Lambda_i^* \Lambda_i f, f \rangle = \left\langle T \{g_i\}_{i \in J}, f \right\rangle \leq \|T\| \|\{g_i\}_{i \in J}\| \|f\|$$

where

$$g_i := \begin{cases} 0 & \text{if } i \in I \setminus J \\ \Lambda_i f & \text{if } i \in J \end{cases}.$$

Hence

$$\sum_{i \in J} \|\Lambda_i f\|^2 \leq \|T\| \left(\sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}} \|f\|$$

and so

$$\sum_{i \in J} \|\Lambda_i f\|^2 \leq \|T\|^2 \|f\|^2.$$

Therefore

$$\sum_{i \in I} \|\Lambda_i f\|^2 \leq \|T\|^2 \|f\|^2.$$

It follows that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence for \mathcal{H} with bound B . \square

Definition 2.5. Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} , the operator

$$S : \mathcal{H} \longrightarrow \mathcal{H}, \quad S = TT^*$$

is called the g -frame operator of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$.

For any $f \in \mathcal{H}$ we have

$$Sf = \sum_{i \in I} \Lambda_i^* \Lambda_i f$$

and

$$\langle Sf, f \rangle = \left\langle \sum_{i \in I} \Lambda_i^* \Lambda_i f, f \right\rangle = \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle = \sum_{i \in I} \|\Lambda_i f\|^2.$$

Therefore

$$A \langle f, f \rangle \leq \langle Sf, f \rangle \leq B \langle f, f \rangle,$$

i.e.,

$$AI \leq S \leq BI.$$

Therefore S is a bounded, positive and invertible operator. In this case we have the reconstruction formula

$$f = \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} f, \quad f \in \mathcal{H}.$$

The well-known relation between a frame and the associated synthesis operator also hold in g -frames.

Proposition 2.6. *A sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} if and only if*

$$T : \{f_i\}_{i \in I} \rightarrow \sum_{i \in I} \Lambda_i^*(f_i)$$

is a well-defined and bounded mapping from $\left(\sum_{i \in I} \oplus \mathcal{H}_i\right)_{\ell_2}$ onto \mathcal{H} .

Proof. If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} , then $S = TT^*$ is invertible. So T is onto. Conversely, let T be well-defined, bounded and onto. Then by Proposition 2.4, $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence for \mathcal{H} . Therefore $T^*f = \{\Lambda_i f\}_{i \in I}$ for all $f \in \mathcal{H}$. Since T is onto, there exists an operator $T^\dagger : \mathcal{H} \rightarrow \left(\sum_{i \in I} \oplus \mathcal{H}_i\right)_{\ell_2}$ such that $TT^\dagger = I_{\mathcal{H}}$. Hence $(T^\dagger)^*T^* = I_{\mathcal{H}}$. Then for all $f \in \mathcal{H}$,

$$\|f\|^2 \leq \|(T^\dagger)^*\|^2 \|T^*f\|^2 = \|T^\dagger\|^2 \|T^*f\|^2 = \|T^\dagger\|^2 \sum_{i \in I} \|\Lambda_i f\|^2.$$

It follows that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} with lower g -frame bound $\|T^\dagger\|^{-2}$ and upper g -frame bound $\|T\|^2$. \square

Proposition 2.7. *A sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} if and only if*

$$S : f \rightarrow \sum_{i \in I} \Lambda_i^* \Lambda_i f$$

is a well-defined and bounded mapping from \mathcal{H} onto \mathcal{H} .

Proof. If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} , then S is the g -frame operator. Therefore S is well-defined, bounded and onto. Conversely, let S be a well-defined, bounded and onto operator. Since S is positive, (i.e., $\langle Sf, f \rangle \geq 0$ for all $f \in \mathcal{H}$) $\mathcal{R}_S^\perp = N(S)$. Then S is injective and therefore it is invertible. So $0 \notin \sigma(S)$. Let $C := \inf_{\|f\|=1} \langle Sf, f \rangle$. By Proposition 70.8 in [11], $C \in \sigma(S)$. Then $C > 0$. Hence for all $f \in \mathcal{H}$ we have

$$\sum_{i \in I} \|\Lambda_i f\|^2 = \langle Sf, f \rangle \geq C \|f\|^2$$

and

$$\sum_{i \in I} \|\Lambda_i f\|^2 = \langle Sf, f \rangle \leq \|S\| \|f\|^2.$$

Then $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} . \square

Corollary 2.8. *If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of operators, then T is well-defined, bounded and onto if and only if S is well-defined, bounded and onto.*

Similarly to g -frames, we can define g -complete, g -Riesz bases and g -orthonormal bases.

Definition 2.9. (i) We say that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g -complete, if $\{f : \Lambda_i f = 0, i \in I\} = \{0\}$.

(ii) We say that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthonormal basis for \mathcal{H} , if it satisfies the following

$$(2.1) \quad \langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in I, \quad g_i \in \mathcal{H}_i, \quad g_j \in \mathcal{H}_j$$

$$(2.2) \quad \sum_{i \in I} \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$

(iii) We say that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Riesz basis for \mathcal{H} , if it is g -complete and there exist constants $0 < A \leq B < \infty$, such that for any finite subset $J \subseteq I$ and $g_i \in \mathcal{H}_i, i \in J$,

$$(2.3) \quad A \sum_{i \in J} \|g_i\|^2 \leq \left\| \sum_{i \in J} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i \in J} \|g_i\|^2.$$

Proposition 2.10. *If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} , then $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$.*

Proof. Let $f \in \mathcal{H}$ and $f \perp \text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$. Then for all $i \in I$ and $g \in \mathcal{H}_i$,

$$\langle \Lambda_i f, g \rangle = \langle f, \Lambda_i^* g \rangle = 0.$$

Therefore $\Lambda_i f = 0$, for all $i \in I$. Since $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} , then $f = 0$. \square

The following proposition gives an equivalent condition for g -completeness of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$.

Proposition 2.11. *$\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g -complete if and only if*

$$\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}.$$

Proof. Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be g -complete. Since $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} \subseteq \mathcal{H}$, it is enough to prove that if $f \in \mathcal{H}$ and $f \perp \text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$, then $f = 0$. Let $f \in \mathcal{H}$ and $f \perp \text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$. Since for any $i \in I$, $f \perp \Lambda_i^* \Lambda_i(f)$, then for all $i \in I$,

$$\|\Lambda_i f\|^2 = \langle f, \Lambda_i^* \Lambda_i(f) \rangle = 0.$$

Therefore $f = 0$, by g -completeness of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$. Conversely, let $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$. Let $f \in \mathcal{H}$ and suppose that $\Lambda_i f = 0$ for all $i \in I$. Then for each $g \in \mathcal{H}_i$

$$\langle \Lambda_i f, g \rangle = \langle f, \Lambda_i^* g \rangle = 0,$$

hence $f \perp \text{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$. Therefore $f \perp \overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} = \mathcal{H}$. It shows that $f = 0$. Hence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is g -complete. \square

The following proposition shows that we can remove the second equality in 2.9.

Proposition 2.12. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} and suppose that (2.1) holds. Then $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthogonal basis for \mathcal{H} .*

Proof. Let S be the g -frame operator of $\{\Lambda_i\}$ and let $\mathcal{K} = \{f \in \mathcal{H} : Sf = f\}$. It is clear that \mathcal{K} is a nonempty closed subspace of \mathcal{H} . We show that $\mathcal{K}^\perp = \{0\}$. Let $f \in \mathcal{H}$ and $j \in I$, then $\Lambda_j^* \Lambda_j f \in \mathcal{K}$ since for $i \neq j$ we have

$$\begin{aligned} \Lambda_i^* \Lambda_i \Lambda_j^* \Lambda_j f &= 0, \\ \Lambda_j^* \Lambda_j \Lambda_j^* \Lambda_j f &= \Lambda_j^* \Lambda_j f \end{aligned}$$

so

$$\sum_{i \in I} \Lambda_i^* \Lambda_i \Lambda_j^* \Lambda_j f = \Lambda_j^* \Lambda_j f.$$

Hence for $f \in \mathcal{K}^\perp$ and $g \in \mathcal{K}$ we have

$$\langle \Lambda_j^* \Lambda_j f, g \rangle = \langle f, \Lambda_j^* \Lambda_j g \rangle = 0, \quad j \in I.$$

Therefore

$$\Lambda_i^* \Lambda_i f = 0, \quad i \in I$$

and so $\|\Lambda_i f\| = 0$, for each $i \in I$. By definition of g -frame we conclude $f = 0$ and therefore $\mathcal{H} = \mathcal{K}$. So $S = I$ and hence for every $f \in \mathcal{H}$

$$\sum_{i \in I} \|\Lambda_i f\|^2 = \langle Sf, f \rangle = \langle f, f \rangle = \|f\|^2.$$

□

Corollary 2.13. $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthonormal basis for \mathcal{H} if and only if $\{\Lambda_i\}$ is a g -frame and for all $i, j \in I$ and $g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j$

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle.$$

Proposition 2.14. $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthonormal basis for \mathcal{H} if and only if

- (1) Λ_i^* is isometric for all $i \in I$;
- (2) $\bigoplus_{i \in I} \Lambda_i^*(\mathcal{H}_i) = \mathcal{H}$.

Proof. First assume that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthonormal basis for \mathcal{H} . Then for each $i \in I$ and $g \in \mathcal{H}_i$, we obtain from (2.1)

$$\langle \Lambda_i^* g, \Lambda_i^* g \rangle = \delta_{i,i} \langle g, g \rangle = \langle g, g \rangle.$$

Then Λ_i^* is isometric for all $i \in I$. Therefore $\Lambda_i^*(\mathcal{H}_i)$ is a closed subspace of \mathcal{H} , for all $i \in I$. From (2.2), we obtain

$$\langle f, f \rangle = \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle = \sum_{i \in I} \langle \Lambda_i^* \Lambda_i f, f \rangle = \left\langle \sum_{i \in I} \Lambda_i^* \Lambda_i f, f \right\rangle, \quad f \in \mathcal{H}.$$

Hence

$$f = \sum_{i \in I} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H}.$$

By letting $g_i = \Lambda_i f$, we have $f = \sum_{i \in I} \Lambda_i^* g_i$ and

$$\sum_{i \in I} \|\Lambda_i^* g_i\|^2 = \sum_{i \in I} \|g_i\|^2 = \sum_{i \in I} \|\Lambda_i f\|^2 = \|f\|^2.$$

Then $\mathcal{H} = \bigoplus_{i \in I} \Lambda_i^*(\mathcal{H}_i)$. Conversely, if (1) and (2) are satisfied, then it is clear that (2.1) holds, since for all $i \neq j$, $\Lambda_i^* \mathcal{H}_i \perp \Lambda_j^* \mathcal{H}_j$ and Λ_i^* is isometric for all $i \in I$. Let $f \in \mathcal{H}$, then we obtain from (2),

$$f = \sum_{i \in I} \Lambda_i^* g_i, \quad g_i \in \mathcal{H}_i, \quad i \in I,$$

and

$$\|f\|^2 = \sum_{i \in I} \|\Lambda_i^* g_i\|^2 = \sum_{i \in I} \|g_i\|^2.$$

Let $m \in I$, then for each $h \in \mathcal{H}$, we have

$$\langle \Lambda_m f, h \rangle = \left\langle \sum_{i \in I} \Lambda_m \Lambda_i^* g_i, h \right\rangle = \sum_{i \in I} \langle \Lambda_i^* g_i, \Lambda_m^* h \rangle = \langle \Lambda_m^* g_m, \Lambda_m^* h \rangle = \langle g_m, h \rangle.$$

Hence $g_m = \Lambda_m f$, for all $m \in I$. Therefore $f = \sum_{i \in I} \Lambda_i^* \Lambda_i f$, for all $f \in \mathcal{H}$. It follows that

$$\|f\|^2 = \sum_{i \in I} \|\Lambda_i f\|^2, \quad f \in \mathcal{H}.$$

□

Corollary 2.15. Every g -orthonormal basis for \mathcal{H} is a g -Riesz basis for \mathcal{H} with bounds $A = B = 1$.

In the following proposition we show that g -frames for \mathcal{H} with respect to $\{\mathcal{H}_i : i \in I\}$ are characterized as the families $\{\Theta_i V^*\}_{i \in I}$, where $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthonormal basis for \mathcal{H} with respect to $\{\mathcal{H}_i : i \in I\}$ and $V : \mathcal{H} \rightarrow \mathcal{H}$ is bounded and onto.

Proposition 2.16. *Let $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -orthonormal basis for \mathcal{H} with respect to $\{\mathcal{H}_i : i \in I\}$ and $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with respect to $\{\mathcal{H}_i : i \in I\}$. Then there is a bounded and onto operator $V : \mathcal{H} \rightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i V^*$, for all $i \in I$.*

Proof. Let

$$V : \mathcal{H} \rightarrow \mathcal{H}, \quad Vf = \sum_{i \in I} \Lambda_i^* \Theta_i f,$$

then V is well defined, since for any finite $J \subseteq I$ and $f \in \mathcal{H}$,

$$\left\| \sum_{i \in J} \Lambda_i^* \Theta_i f \right\| = \sup_{\|h\|=1} \left| \left\langle \sum_{i \in J} \Lambda_i^* \Theta_i f, h \right\rangle \right| \leq \sqrt{B} \left(\sum_{i \in J} \|\Theta_i f\|^2 \right)^{\frac{1}{2}},$$

where B is an upper g -frame bound for $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$. Since for all $f \in \mathcal{H}$, $\sum_{i \in I} \|\Theta_i f\|^2 = \|f\|^2$, it follows that $\sum_{i \in I} \Lambda_i^* \Theta_i f$ is weakly unconditionally Cauchy and hence unconditionally convergent in \mathcal{H} (see[6], Page 44, Theorems 6 and 8). Also it is clear that $\|V\| \leq \sqrt{B}$. Since $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthonormal basis, then $\Theta_i \Theta_j^* g = \delta_{ij} g$ and

$$V \Theta_j^* g = \sum_{i \in I} \Lambda_i^* \Theta_i \Theta_j^* g = \Lambda_j^* \Theta_j \Theta_j^* g = \Lambda_j^* g,$$

for all $g \in \mathcal{H}_j, j \in I$. Therefore

$$\Theta_j V^* = \Lambda_j, \quad j \in I.$$

Now we show that V is onto. Let T_Λ and T_Θ be the synthesis operators for $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ respectively. Let $f \in \mathcal{H}$. Then there exists $\{g_i\}_{i \in I} \in \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2}$ such that $\sum_{i \in I} \Lambda_i^* g_i = f$, since T_Λ is onto. Let $g = T_\Theta(\{g_i\}_{i \in I})$. Then

$$Vg = \sum_{i \in I} V \Theta_i^* g_i = \sum_{i \in I} \Lambda_i^* g_i = f.$$

Therefore V is onto. □

Corollary 2.17. *If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a Parseval g -frame for \mathcal{H} , then V is co-isometric.*

Corollary 2.18. *If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Riesz basis for \mathcal{H} , then V is invertible.*

Proof. Let $f \in \ker V$. Then $T_\Theta^* f = 0$, since $V = T_\Lambda T_\Theta^*$ and T_Λ is one to one. Therefore $\|f\|^2 = \|T_\Theta^* f\|^2 = 0$, so $f = 0$. □

Corollary 2.19. *If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -orthonormal basis for \mathcal{H} , then V is unitary.*

Proof. By Corollaries 2.15 and 2.18, V is invertible. For each $f \in \mathcal{H}$ we have

$$\|f\|^2 = \sum_{i \in I} \|\Lambda_i f\|^2 = \sum_{i \in I} \|\Theta_i V^* f\|^2 = \|V^* f\|^2.$$

Therefore $VV^* = I_{\mathcal{H}}$. It follows that V is unitary. □

Han and Larson in [10] have proved that for any Parseval frame $\{f_i\}_{i \in I}$ for a Hilbert space \mathcal{H} there is a Hilbert space $\mathcal{H} \subseteq \mathcal{K}$ and an orthonormal basis $\{e_i\}_{i \in I}$ for \mathcal{K} such that $Pe_i = f_i$, for all $i \in I$, where P is the orthogonal projection of \mathcal{K} onto \mathcal{H} . Asgari and Khosravi in [1] have extended this result for frame of subspaces and the following theorem is its extension to g -frames.

Theorem 2.20. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a Parseval g -frame for \mathcal{H} . Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis of subspaces $\{\mathcal{K}_i\}_{i \in I}$ for \mathcal{K} such that for each $i \in I$, $P(\mathcal{K}_i) = \Lambda_i^*(\mathcal{H}_i)$, where P is the orthogonal projection from \mathcal{K} onto \mathcal{H} .*

Proof. For each $i \in I$, let $\{e_{ij}\}_{j \in J_i}$ be an orthonormal basis for \mathcal{H}_i and for each $i \in I, j \in J_i$, let E_{ij} be an element of $\left(\sum_{i \in I} \oplus \mathcal{H}_i\right)_{\ell_2}$ defined by

$$(E_{ij})_k = \begin{cases} e_{ij} & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}.$$

It is clear that $\{E_{ij}\}_{i \in I, j \in J_i}$ is an orthonormal basis for $\left(\sum_{i \in I} \oplus \mathcal{H}_i\right)_{\ell_2}$. For each $i \in I$, let $\mathcal{K}_i = \overline{\text{span}}\{E_{ij}\}_{j \in J_i}$, then

$$\bigoplus_{i \in I} \mathcal{K}_i = \left(\sum_{i \in I} \oplus \mathcal{H}_i\right)_{\ell_2}.$$

Let $\mathcal{K} = \bigoplus_{i \in I} \mathcal{K}_i$ and let Θ be the analysis operator of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$. Since $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a Parseval g -frame, then Θ is an isometry. So we can embed \mathcal{H} into \mathcal{K} by identifying \mathcal{H} with $\Theta(\mathcal{H})$. Let $P : \mathcal{K} \rightarrow \Theta(\mathcal{H})$ be the orthogonal projection. Then for each $i \in I, j \in J_i$ and $f \in \mathcal{H}$ we have

$$\langle \Theta f, P(E_{ij}) \rangle = \langle \Theta f, E_{ij} \rangle = \left\langle \{\Lambda_i f\}_{i \in I}, E_{ij} \right\rangle = \langle \Lambda_i f, e_{ij} \rangle = \langle f, \Lambda_i^* e_{ij} \rangle = \langle \Theta f, \Theta \Lambda_i^* e_{ij} \rangle.$$

Thus $P(E_{ij}) - \Theta \Lambda_i^* e_{ij} \perp \Theta(\mathcal{H})$. Since $P(E_{ij}) - \Theta \Lambda_i^* e_{ij} \in \Theta(\mathcal{H})$, then $P(E_{ij}) = \Theta \Lambda_i^* e_{ij}$. Thus $P(\mathcal{K}_i) = \Theta \Lambda_i^*(\mathcal{H}_i)$. \square

The next result will turn out to be useful for finding a lower bound for $\sum_{i \in I} \|\Lambda_i f\|^2$.

Proposition 2.21. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two g -Bessel sequences for \mathcal{H} with bounds A and B respectively and let T_Λ and T_Θ be their analysis operators such that $T_\Theta T_\Lambda^* = I_{\mathcal{H}}$. Then both $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ are g -frames.*

Proof. For any $f \in \mathcal{H}$ we have

$$\begin{aligned} \|f\|^4 &= \langle f, f \rangle^2 = \langle T_\Lambda^* f, T_\Theta^* f \rangle^2 \leq \|T_\Lambda^* f\|^2 \|T_\Theta^* f\|^2 = \|\{\Lambda_i f\}_{i \in I}\|^2 \|\{\Theta_i f\}_{i \in I}\|^2 \\ &= \left(\sum_{i \in I} \|\Lambda_i f\|^2\right) \left(\sum_{i \in I} \|\Theta_i f\|^2\right) \leq \left(\sum_{i \in I} \|\Lambda_i f\|^2\right) B \|f\|^2. \end{aligned}$$

So $\frac{1}{B} \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2$. Therefore $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame. Similarly $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame. \square

The next proposition shows that with a given g -frame, we can produce another g -frame by using its g -frame operator.

Proposition 2.22. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with frame operator S and frame bounds A, B . Then for each $\alpha \in \mathbb{R}$, $\{\Lambda_i S^\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} and its frame operator is $S^{2\alpha+1}$.*

Proof. Since $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} , then

$$A\langle S^{2\alpha}f, f \rangle = A\|S^\alpha f\|^2 \leq \sum_i \|\Lambda_i S^\alpha f\|^2 \leq B\|S^\alpha f\|^2 = B\langle S^{2\alpha}f, f \rangle.$$

For $\alpha \geq 0$,

$$\begin{aligned} A\|S^\alpha f\|^2 &= A\langle S^{2\alpha}f, f \rangle \geq AA^{2\alpha}\langle f, f \rangle = A^{2\alpha+1}\|f\|^2, \\ B\|S^\alpha f\|^2 &= B\langle S^{2\alpha}f, f \rangle \leq BB^{2\alpha}\langle f, f \rangle = B^{2\alpha+1}\|f\|^2. \end{aligned}$$

Therefore

$$A^{2\alpha+1}\|f\|^2 \leq \sum_i \|\Lambda_i S^\alpha f\|^2 \leq B^{2\alpha+1}\|f\|^2.$$

Similarly for $\alpha < 0$,

$$B^{2\alpha+1}\|f\|^2 \leq \sum_i \|\Lambda_i S^\alpha f\|^2 \leq A^{2\alpha+1}\|f\|^2.$$

Now, let S^\diamond be the frame operator of $\{\Lambda_i S^\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, then for any $f \in \mathcal{H}$,

$$\begin{aligned} S^\diamond(f) &= \sum_i (\Lambda_i S^\alpha)^* (\Lambda_i S^\alpha) f = \sum_i S^\alpha \Lambda_i^* \Lambda_i S^\alpha f \\ &= S^\alpha \sum_i \Lambda_i^* \Lambda_i S^\alpha f = S^\alpha S S^\alpha f = S^{2\alpha+1} f. \end{aligned}$$

So $S^\diamond = S^{2\alpha+1}$. □

Corollary 2.23. *By assumptions of Proposition 2.22, $\{\Lambda_i S^{-\frac{1}{2}} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a Parseval g -frame.*

Proposition 2.24. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with frame operator S and $U : \mathcal{H} \rightarrow \mathcal{K}$ be a bounded and onto operator. Then $\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{K} and its frame operator is USU^* .*

Proof. Let A, B be the frame bounds for $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$. Since U is bounded and onto, there exists bounded operator $U^\dagger : \mathcal{K} \rightarrow \mathcal{H}$ such that $UU^\dagger = I_{\mathcal{K}}$. Thus $(U^\dagger)^* U^* = I_{\mathcal{K}}$. Hence for any $f \in \mathcal{K}$, we have

$$A\|(U^\dagger)^* U^* f\|^2 \leq A\|U^* f\|^2 \leq \sum_{i \in I} \|\Lambda_i U^* f\|^2 \leq B\|U^* f\|^2 \leq B\|U^*\|^2 \|f\|^2.$$

Therefore $\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{K} with frame bounds $A\|U^\dagger\|^{-2}$ and $B\|U\|^2$. Now, let S_U be the frame operator for $\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\}$, then for each $f \in \mathcal{K}$,

$$S_U f = \sum_{i \in I} (\Lambda_i U^*)^* (\Lambda_i U^*) f = U \sum_{i \in I} \Lambda_i^* \Lambda_i U^* f = USU^* f.$$

Hence $S_U = USU^*$. □

Corollary 2.25. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} and let $U : \mathcal{H} \rightarrow \mathcal{K}$ be a bounded operator with closed range. Then $\{\Lambda_i U^* \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{R}_U .*

Corollary 2.26. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} and let $U : \mathcal{H} \rightarrow \mathcal{K}$ be a bounded and invertible operator. Then $\{\Lambda_i U^{-1} \in \mathcal{L}(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{K} .*

Corollary 2.27. *Every g -Riesz basis for \mathcal{H} is a g -frame for \mathcal{H} .*

3. PERTURBATION OF g -FRAMES

Perturbation of frames has been discussed in [2]. However there are similar theorems about perturbation of frames for measurable spaces (see [9], [13]). Stability of g -frames and their duals has been investigated by W. Sun [16]. In this manuscript we give other perturbations of g -frames. At first we need the following lemma which is proved in [2].

Lemma 3.1. *Let U be a linear operator on a Banach space X and assume that there exist $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|x - Ux\| \leq \lambda_1 \|x\| + \lambda_2 \|Ux\|$$

for all $x \in X$. Then U is bounded and invertible. Moreover

$$\frac{1 - \lambda_1}{1 + \lambda_2} \|x\| \leq \|Ux\| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \|x\|$$

and

$$\frac{1 - \lambda_2}{1 + \lambda_1} \|x\| \leq \|U^{-1}x\| \leq \frac{1 + \lambda_2}{1 - \lambda_1} \|x\|$$

for all $x \in X$.

Theorem 3.2. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with bounds A, B and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a sequence of operators such that for any finite subset $J \subseteq I$ and for each $f \in \mathcal{H}$,*

$$(3.1) \quad \left\| \sum_{i \in J} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \right\| \leq \lambda \left\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \right\| + \mu \left\| \sum_{i \in J} \Theta_i^* \Theta_i f \right\| + \gamma \left(\sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}},$$

where $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$. Then $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} with frame bounds

$$(3.2) \quad A \frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu} \quad \text{and} \quad B \frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1 - \mu}.$$

Proof. Assume that $J \subseteq I$ and $|J| < +\infty$. For each $f \in \mathcal{H}$, we have

$$\begin{aligned} \left\| \sum_{i \in J} \Theta_i^* \Theta_i f \right\| &\leq \left\| \sum_{i \in J} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f) \right\| + \left\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \right\| \\ &\leq (1 + \lambda) \left\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \right\| + \mu \left\| \sum_{i \in J} \Theta_i^* \Theta_i f \right\| + \gamma \left(\sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\left\| \sum_{i \in J} \Theta_i^* \Theta_i f \right\| \leq \frac{1 + \lambda}{1 - \mu} \left\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \right\| + \frac{\gamma}{1 - \mu} \left(\sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}}.$$

Also

$$\begin{aligned} \left\| \sum_{i \in J} \Lambda_i^* \Lambda_i f \right\| &= \sup_{\|g\|=1} \left| \left\langle \sum_{i \in J} \Lambda_i^* \Lambda_i f, g \right\rangle \right| = \sup_{\|g\|=1} \left| \left\langle \sum_{i \in J} \Lambda_i f, \Lambda_i g \right\rangle \right| \\ &\leq \left(\sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}} \sup_{\|g\|=1} \left(\sum_{i \in J} \|\Lambda_i g\|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}} \sqrt{B} \leq B \|f\|. \end{aligned}$$

Therefore for all $f \in \mathcal{H}$

$$\left\| \sum_{i \in J} \Theta_i^* \Theta_i f \right\| \leq \left(\frac{1 + \lambda}{1 - \mu} \sqrt{B} + \frac{\gamma}{1 - \mu} \right) \left(\sum_{i \in J} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}},$$

so $\sum_{i \in I} \Theta_i^* \Theta_i f$ is unconditionally convergent. Let

$$G : \mathcal{H} \longrightarrow \mathcal{H}, \quad G(f) = \sum_{i \in I} \Theta_i^* \Theta_i f, \quad f \in \mathcal{H}.$$

Then G is well-defined and bounded operator with

$$\|G\| \leq \frac{1+\lambda}{1-\mu} B + \frac{\gamma\sqrt{B}}{1-\mu}$$

and for each $f \in \mathcal{H}$, we have

$$\sum_{i \in I} \|\Theta_i f\|^2 = \langle Gf, f \rangle \leq \|G\| \|f\|^2.$$

It follows that $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence for \mathcal{H} .

Let S be the g -frame operator of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ then we obtain from (3.1)

$$\|Sf - Gf\| \leq \lambda \|Sf\| + \mu \|Gf\| + \gamma \left(\sum_{i \in I} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}}, \quad f \in \mathcal{H}.$$

Therefore

$$\begin{aligned} \|f - GS^{-1}f\| &\leq \lambda \|f\| + \mu \|GS^{-1}f\| + \gamma \left(\sum_{i \in I} \|\Lambda_i S^{-1}f\|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\lambda + \frac{\gamma}{\sqrt{A}} \right) \|f\| + \mu \|GS^{-1}f\|, \end{aligned}$$

since $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$, then by Lemma 3.1, GS^{-1} and consequently G is invertible and

$$\|G^{-1}\| \leq \|S^{-1}\| \|SG^{-1}\| \leq \frac{1+\mu}{A \left(1 - \left(\lambda + \frac{\gamma}{\sqrt{A}} \right) \right)}.$$

Hence by Proposition 2.7, $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} . It is clear that the optimal lower bound and optimal upper bound of $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ are $\|G^{-1}\|^{-1}$ and $\|G\|$, respectively. Then we can obtain the required frame bounds in (3.2). \square

Corollary 3.3. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with bounds A, B and let $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of operators. If there exists a constant $0 < R < A$ such that*

$$\sum_{i \in I} \|\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f\| \leq R \|f\|$$

for all $f \in \mathcal{H}$, then $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame with g -frame bounds $A - R$ and $\min\left\{1 + R\sqrt{\frac{B}{A}}, R + B\right\}$.

Proof. It is clear that $\sum_{i \in I} \Theta_i^* \Theta_i f$ is converges for each $f \in \mathcal{H}$. So we have

$$\left\| \sum_{i \in I} \Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f \right\| \leq R \|f\| \leq \frac{R}{\sqrt{A}} \left(\sum_{i \in I} \|\Lambda_i f\|^2 \right)^{\frac{1}{2}}, \quad f \in \mathcal{H}.$$

By letting, $\lambda = \mu = 0$ and $\gamma = R/\sqrt{A}$ in Theorem 3.2, $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ will be a g -frame for \mathcal{H} with lower bound $A - R$ and upper bound $1 + R\sqrt{\frac{B}{A}}$. Also we have

$$\left\| \sum_{i \in I} \Theta_i^* \Theta_i f \right\| \leq R \|f\| + \left\| \sum_{i \in I} \Lambda_i^* \Lambda_i f \right\| \leq (R + B) \|f\|$$

for all $f \in \mathcal{H}$. Hence the upper g -frame bound for $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is

$$\min\left\{1 + R\sqrt{\frac{B}{A}}, R + B\right\}. \quad \square$$

Corollary 3.4. Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} with frame bounds A, B and let $\{g_i\}_{i \in I}$ be a sequence of \mathcal{H} . Suppose that there exist constants λ, μ and γ with $0 \leq \max\{\mu, \lambda + \frac{\gamma}{\sqrt{A}}\} < 1$ such that for each $f \in \mathcal{H}$ and every finite subset $J \subseteq I$,

$$(3.3) \quad \left\| \sum_{i \in J} \langle f, f_i \rangle f_i - \langle f, g_i \rangle g_i \right\| \leq \lambda \left\| \sum_{i \in J} \langle f, f_i \rangle f_i \right\| + \mu \left\| \sum_{i \in J} \langle f, g_i \rangle g_i \right\| + \gamma \left(\sum_{i \in J} |\langle f, f_i \rangle| \right)^{\frac{1}{2}}.$$

Then $\{g_i\}_{i \in I}$ is a frame for \mathcal{H} with frame bounds

$$A \frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu} \quad \text{and} \quad B \frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1 - \mu}.$$

Proof. For each $i \in I$, let

$$\Lambda_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Lambda_i(f) = \langle f, f_i \rangle$$

and

$$\Theta_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Theta_i(f) = \langle f, g_i \rangle.$$

Then $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathbb{C}) : i \in I\}$ is a g -frame for \mathcal{H} and (3.3) means (3.1). Therefore the result follows from Theorem 3.2. \square

The following Theorem is another version of perturbation of g -frames.

Theorem 3.5. Let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with bounds A, B and let $\{\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of operators such that for any $J \subseteq I$ with $|J| < +\infty$,

$$(3.4) \quad \left\| \sum_{i \in J} (\Lambda_i^* f_i - \Theta_i^* f_i) \right\| \leq \lambda \left\| \sum_{i \in J} \Lambda_i^* f_i \right\| + \mu \left\| \sum_{i \in J} \Theta_i^* f_i \right\| + \gamma \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}}, \quad (f_i \in \mathcal{H}_i)$$

where $0 \leq \max\{\lambda + \frac{\gamma}{A}, \mu\} < 1$. Then $\{\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} with g -frame bounds

$$A \left(\frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu} \right)^2 \quad \text{and} \quad B \left(\frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1 - \mu} \right)^2.$$

Proof. It is clear that if $J \subseteq I$ with $|J| < +\infty$, then for all $i \in J$ and $f_i \in \mathcal{H}_i$, we have

$$\begin{aligned} \left\| \sum_{i \in J} \Theta_i^* f_i \right\| &\leq \left\| \sum_{i \in J} (\Lambda_i^* f_i - \Theta_i^* f_i) \right\| + \left\| \sum_{i \in J} \Lambda_i^* f_i \right\| \\ &\leq (\lambda + 1) \left\| \sum_{i \in J} \Lambda_i^* f_i \right\| + \mu \left\| \sum_{i \in J} \Theta_i^* f_i \right\| + \gamma \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

So we have

$$\left\| \sum_{i \in J} \Theta_i^* f_i \right\| \leq \frac{\lambda + 1}{1 - \mu} \left\| \sum_{i \in J} \Lambda_i^* f_i \right\| + \frac{\gamma}{1 - \mu} \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}}.$$

Since

$$\begin{aligned} \left\| \sum_{i \in J} \Lambda_i^* f_i \right\| &= \sup_{\|g\|=1} \left| \left\langle \sum_{i \in J} \Lambda_i^* f_i, g \right\rangle \right| \\ &\leq \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}} \sup_{\|g\|=1} \left(\sum_{i \in J} \|\Lambda_i g\|^2 \right)^{\frac{1}{2}} \leq \sqrt{B} \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

then

$$(3.5) \quad \left\| \sum_{i \in J} \Theta_i^* f_i \right\| \leq \left(\frac{\lambda+1}{1-\mu} \sqrt{B} + \frac{\gamma}{1-\mu} \right) \left(\sum_{i \in J} \|f_i\|^2 \right)^{\frac{1}{2}}.$$

Let

$$T : \left(\sum_{i \in I} \bigoplus \mathcal{H}_i \right)_{\ell_2} \longrightarrow \mathcal{H}, \quad T(\{f_i\}_{i \in I}) = \sum_{i \in I} \Theta_i^* f_i.$$

By (3.5), T is well-defined and bounded with

$$\|T\| \leq \frac{\lambda+1}{1-\mu} \sqrt{B} + \frac{\gamma}{1-\mu}.$$

Then by Proposition 2.4, $\{\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence for \mathcal{H} and

$$T^* : \mathcal{H} \longrightarrow \left(\sum_{i \in I} \bigoplus \mathcal{H}_i \right)_{\ell_2}, \quad T^*(f) = \{\Theta_i f\}_{i \in I}.$$

Let U and S be the synthesis operator and g -frame operator for $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, respectively. Let $G = TU^*S^{-1}$. Then we have from (3.4)

$$\begin{aligned} & \left\| \sum_{i \in I} (\Lambda_i^* \Lambda_i S^{-1} f - \Theta_i^* \Lambda_i S^{-1} f) \right\| \\ & \leq \lambda \left\| \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} f \right\| + \mu \left\| \sum_{i \in I} \Theta_i^* \Lambda_i S^{-1} f \right\| + \gamma \left(\sum_{i \in I} \|\Lambda_i S^{-1} f\|^2 \right)^{\frac{1}{2}}, \quad f \in \mathcal{H}. \end{aligned}$$

Since for all $f \in \mathcal{H}$, $f = \sum_{i \in I} \Lambda_i^* \Lambda_i S^{-1} f$, so we have

$$\begin{aligned} \|f - Gf\| & \leq \lambda \|f\| + \mu \|Gf\| + \gamma \left(\sum_{i \in I} \|\Lambda_i S^{-1} f\|^2 \right)^{\frac{1}{2}} \\ & \leq \lambda \|f\| + \mu \|Gf\| + \frac{\gamma}{\sqrt{A}} \|f\| \leq \left(\lambda + \frac{\gamma}{\sqrt{A}} \right) \|f\| + \mu \|Gf\|. \end{aligned}$$

Since $0 \leq \max\{\lambda + \frac{\gamma}{\sqrt{A}}, \mu\} < 1$, by Lemma 3.1, $G = TU^*S^{-1}$ and consequently TU^* is invertible and

$$\|G^{-1}\| \leq \frac{1+\mu}{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}.$$

Let $f \in \mathcal{H}$, then $f = GG^{-1}f = \sum_{i \in I} \Theta_i^* \Lambda_i S^{-1} G^{-1}f$. Hence for each $f \in \mathcal{H}$, we have

$$\begin{aligned} \|f\|^4 & = |\langle f, f \rangle|^2 = \left| \left\langle \sum_{i \in I} \Theta_i^* \Lambda_i S^{-1} G^{-1}f, f \right\rangle \right|^2 \leq \sum_{i \in I} \|\Lambda_i S^{-1} G^{-1}f\|^2 \sum_{i \in I} \|\Theta_i f\|^2 \\ & \leq \frac{1}{A} \|G^{-1}f\|^2 \sum_{i \in I} \|\Theta_i f\|^2 \leq \frac{1}{A} \left(\frac{1+\mu}{1 - (\lambda + \frac{\gamma}{\sqrt{A}})} \right)^2 \|f\|^2 \sum_{i \in I} \|\Theta_i f\|^2. \end{aligned}$$

Therefore

$$\sum_{i \in I} \|\Theta_i f\|^2 \geq A \left(\frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu} \right)^2 \|f\|^2, \quad f \in \mathcal{H}.$$

Then $\{\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} with the required g -frame bounds. \square

The following corollary has been proved in [2].

Corollary 3.6. *Let $\{f_i\}_{i \in I}$ be a frame for \mathcal{H} with bounds A, B . Let $\{g_i\}_{i \in I}$ be a sequence in \mathcal{H} and assume that there exist non-negative constants λ, μ and γ such that*

$$\max \left\{ \mu, \lambda + \frac{\gamma}{\sqrt{A}} \right\} < 1$$

and

$$(3.6) \quad \left\| \sum_i c_i f_i - c_i g_i \right\| \leq \lambda \left\| \sum_i c_i f_i \right\| + \mu \left\| \sum_i c_i g_i \right\| + \gamma \left(\sum_i |c_i|^2 \right)^{\frac{1}{2}}$$

for all finite scalar sequences $\{c_i\}_i$. Then $\{g_i\}_{i \in I}$ is a frame for \mathcal{H} with frame bounds

$$A \left(\frac{1 - (\lambda + \frac{\gamma}{\sqrt{A}})}{1 + \mu} \right)^2 \quad \text{and} \quad B \left(\frac{1 + \lambda + \frac{\gamma}{\sqrt{B}}}{1 - \mu} \right)^2.$$

Proof. For each $i \in I$, let

$$\Lambda_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Lambda_i(f) = \langle f, f_i \rangle$$

and

$$\Theta_i : \mathcal{H} \longrightarrow \mathbb{C}, \quad \Theta_i(f) = \langle f, g_i \rangle.$$

Then $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathbb{C}) : i \in I\}$ is a g -frame for \mathcal{H} and (3.6) means (3.4). Therefore the result follows from Theorem 3.5. \square

Proposition 3.7. *Let $\{\Lambda_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with bounds A, B and let $\{\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of operators. If there exists a $0 < R < A$ such that*

$$\sum_{i \in I} \|\Lambda_i f - \Theta_i f\|^2 \leq R \|f\|^2$$

for all $f \in \mathcal{H}$. Then $\{\Theta_i \in L(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} with bounds $(\sqrt{A} - \sqrt{R})^2$ and $(\sqrt{B} + \sqrt{R})^2$.

Proof. By the triangle inequality in $(\sum_{i \in I} \oplus \mathcal{H}_i)_{\ell_2}$, we have

$$\left\| \{\Theta_i f\}_{i \in I} \right\|_{\ell_2} \leq \left\| \{\Theta_i f - \Lambda_i f\}_{i \in I} \right\|_{\ell_2} + \left\| \{\Lambda_i f\}_{i \in I} \right\|_{\ell_2}, \quad f \in \mathcal{H}$$

therefore

$$\sum_{i \in I} \|\Theta_i f\|^2 \leq (\sqrt{B} + \sqrt{R})^2 \|f\|^2, \quad f \in \mathcal{H}.$$

Also for each $f \in \mathcal{H}$

$$\left\| \{\Theta_i f\}_{i \in I} \right\|_{\ell_2} \geq \left\| \{\Lambda_i f\}_{i \in I} \right\|_{\ell_2} - \left\| \{\Theta_i f - \Lambda_i f\}_{i \in I} \right\|_{\ell_2}$$

thus

$$\sum_{i \in I} \|\Theta_i f\|^2 \geq (\sqrt{A} - \sqrt{R})^2 \|f\|^2, \quad f \in \mathcal{H}.$$

\square

The following theorem is a generalization of a result in [5].

Theorem 3.8. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} with respect to $\{\mathcal{H}_i : i \in I\}$ and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of operators. If*

$$K : \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell_2} \longrightarrow \mathcal{H}, \quad K \left(\{f_i\}_{i \in I} \right) = \sum_{i \in I} (\Lambda_i^* - \Theta_i^*) f_i$$

is a well-defined and compact operator, then $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame sequence.

Proof. Since $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame for \mathcal{H} and the perturbation operator K is bounded, then the synthesis g -frame operator U for $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is well-defined and bounded. Therefore by Proposition 2.4, $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence for \mathcal{H} . Let T and S be synthesis operator and g -frame operator for $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, respectively. Then $U = T - K$ and

$$UU^* = (T - K)(T^* - K^*) = S\left(I + S^{-1}(-TK^* - KT^* + KK^*)\right).$$

Since K is compact operator, then $S^{-1}(-TK^* - KT^* + KK^*)$ is a compact operator and by Theorem 4.23 of [14] the operator $I + S^{-1}(-TK^* - KT^* + KK^*)$ has closed range. Composing this with S , we see that UU^* also has closed range. Hence \mathcal{R}_U is closed. So $\mathcal{R}_U = \overline{\text{span}}\{\Theta_i^*(\mathcal{H}_i)\}_{i \in I}$. Therefore the result follows from Proposition 2.6. \square

Corollary 3.9. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} . Let J be a finite subset of I such that for each $j \in J$, $\dim \mathcal{H}_j < \infty$. Then*

$$\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \setminus J\}$$

is a g -frame sequence.

Proof. Let $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a sequence of operators such that $\Theta_i = 0$, if $i \in J$ and $\Theta_i = \Lambda_i$, if $i \notin J$. Then the operator

$$K : \left(\sum_{i \in I} \bigoplus \mathcal{H}_i\right)_{\ell_2} \longrightarrow \mathcal{H}, \quad K(\{f_i\}_{i \in I}) = \sum_{i \in I} (\Lambda_i^* - \Theta_i^*)f_i = \sum_{i \in J} \Lambda_i^* f_i$$

has finite dimensional range and hence is compact. Then by Theorem 3.10, $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I \setminus J\}$ is a g -frame sequence. \square

Theorem 3.10. *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g -frame for \mathcal{H} and let $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a family of operators. If*

$$K : \mathcal{H} \longrightarrow \mathcal{H}, \quad Kf = \sum_{i \in I} (\Lambda_i^* \Lambda_i f - \Theta_i^* \Theta_i f)$$

is a well-defined and compact operator, then $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -frame sequence.

Proof. Let S be the g -frame operator of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$, then $S^\circ = S - K$ is a bounded operator. Therefore for each $f \in \mathcal{H}$

$$\sum_{i \in I} \|\Theta_i f\|^2 = \langle S^\circ f, f \rangle \leq \|S^\circ\| \|f\|^2.$$

Then $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g -Bessel sequence. Let

$$T : \left(\sum_{i \in I} \bigoplus \mathcal{H}_i\right)_{\ell_2} \longrightarrow \mathcal{H}, \quad T(\{f_i\}_{i \in I}) = \sum_{i \in I} \Theta_i^* f_i.$$

By Proposition 2.4, T is bounded and $S^\circ = TT^*$. Since $S^{-1}K$ is compact, so S° has closed range. Therefore $\mathcal{R}_{S^\circ} = \mathcal{R}_T = \overline{\text{span}}\{\Theta_i^*(\mathcal{H}_i)\}_{i \in I}$. Hence the result follows from Proposition 2.7. \square

REFERENCES

1. M. S. Asgari and A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl. **308** (2005), 541–553.
2. P. G. Casazza and O. Christensen, *Perturbation of operators and application to frame theory*, J. Fourier Anal. Appl. **3** (1997), 543–557.
3. P. G. Casazza and Gitta Kutyniok, *Frames of subspaces*, Contemp. Math. **345** (2004), 87–113.
4. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2002.
5. O. Christensen and C. Heil, *Perturbations of Banach frames and atomic decompositions*, Math. Nachr. **185** (1997), 33–47.

6. J. Diestel, *Sequences and Series in Banach Spaces*, Springer-Verlag, New York, 1984.
7. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.
8. M. Fornasier, *Decompositions of Hilbert space: local construction of global frames*, Proc. Int. Conf. Constrictive Function Theory, Varna 2002, B. Bojanov Ed., DARBA, Sofia, 2003, pp. 275–281.
9. Jean-Pierre Gabardo and D. Han, *Frames associated with measurable space*, Adv. Comput. Math. **18** (2003), 127–147.
10. D. Han and D. Larson, *Frames, bases and group representation*, Memoirs Amer. Math. Soc. **147** (2000).
11. H. Heuser, *Functional Analysis*, John Wiley, New York, 1982.
12. G. Kaiser, *A Friendly Guide to Wavelets*, Birkhäuser, Boston, 1995.
13. A. Rahimi, A. Najati and Y. N. Dehghan, *Continuous frames in Hilbert spaces*, Methods Funct. Anal. Topology **12** (2006), no. 2, 170–182.
14. W. Rudin, *Functional Analysis*, McGraw Hill, New York, 1991.
15. Wenchang Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), 437–452.
16. Wenchang Sun, *Stability of g-frames*, J. Math. Anal. Appl. **326** (2007), 858–868.

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