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FINDING GENERALIZED WALRAS-WALD EQUILIBRIUM

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In memory of Professor Semion Israelevich Zuchovitsky.

Abstract. The Generalized Walras-Wald Equilibrium (GE) was introduced by S. I. Zuchovitsky et al. in 1973 (see [17]) as an alternative to Linear Programming (LP) approach for optimal resources allocation. There are two fundamental differences between the GE and LP approach for the best resources allocation. First, the prices for goods (products) are not fixed as it is in LP; they are functions of the production output. Second, the factors (resources) used in the production process are not fixed either; they are functions of the prices for resources. In this paper we show that under natural economic assumptions on both price and factor functions the GE exists and is unique. Finding GE is equivalent to solving a variational inequality with a strongly monotone operator. For solving the variational inequality we introduce projected pseudo-gradient method. We prove that under the same assumptions on price and factor functions the projected pseudo-gradient method converges globally with Q-linear rate. It allows estimating its computational complexity and finding parameters critical for the complexity bound. The method can be viewed as a natural pricing mechanism for establishing economics equilibrium.

1. Introduction

At the early 60th the emerging field of Modern Optimization became the main focus of S. I. research.

My late friend Matvey Primak and myself were very lucky: it was the starting point of our long lasting cooperation with S. I., who opened for us the beautiful World of Mathematics and helped us entering the World. For us it was a unique opportunity which would be impossible without S. I. In fact S. I. saved us from spiritual death and changed our lives forever.

In the late 30th L. V. Kantorovich in St. Petersburg (then Leningrad) discovered that a number of real life problems arising in technological, engineering and economic applications lead to finding a maximum (minimum) of a linear function under linear inequality constraints

$$(c, x^*) = \max \{(c, x) | Ax \le b, x \ge 0\} \tag{1}$$

where $A : \mathbf{R}^n \to \mathbf{R}^m; c, x \in \mathbf{R}^n, b \in \mathbf{R}^m$. This was the beginning of the Linear Programming (LP) era.

During the Second World War and immediately after it became clear that LP has important military applications. In 1947 George Dantzig developed Simplex method for solving LP, which became one of the ten best algorithms in the 20 century. Almost at the same time John von Neumann in a conversation with G. Dantzig introduced the LP Duality, which became the cornerstone of the LP theory.

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With each LP problem (1) is associated the so-called dual problem:

$$(b, \lambda^*) = \min \left\{ (b, \lambda) | A^T \lambda \geq c, \lambda \geq 0 \right\}. \tag{2}$$

The following Duality results are fundamental for LP

$$(c, x^*) = (b, \lambda^*) \tag{3}$$

and

$$(A^T \lambda^* - c, x^*) = 0, \quad (b - Ax^*, \lambda^*) = 0, \tag{4}$$

Since then the interest to LP grew dramatically and over the last 60 years LP became one of the most advanced fields of Modern Optimization.

In 1949 S. I. Zuchovitsky in Kiev introduced a method for finding Tchebyshev approximation for a system of linear equations $Ax = b$, i.e. finding

$$F(x^*) = \min\{F(x) | x \in \mathbf{R}^n\}, \tag{5}$$

where $F(x) = \max\{|(Ax - b)_i| \, |i = 1, \dots, m\}$ is a convex and piece-wise linear function. The method was a realization of Fourier's idea: in order to find $F(x^*)$ one has to start at any point on the descent edge of the piece-wise linear surface and following along the descent edges move from one vertex to another until the bottom of the surface will be reached.

S. I. was not aware of G. Dantzig's results, the "Iron Curtain" made it impossible. It will take another ten years before S. I. learned from E. Stiefel's paper [14] that his algorithm for solving the problem (5) is in fact the Simplex method for solving the following equivalent to (5) LP problem

$$y^* = \min\{y| \; |(Ax - b)_i| \leq y, i = 1, \dots, m\}. \tag{6}$$

From this point on Optimization became the main focus of S. I. research.

The efficiency of the original Simplex method has been drastically improved over the last sixty years and new very powerful Interior Point Methods for solving LP problems has emerged in the last 20 years.

In 1975 L. V. Kantorovich and T. C. Koopmans shared the Nobel Price in Economics "for their contributions to the theory of optimum allocation of limited resources".

The question however is: how adequate LP reflects the economic reality when it comes to the best allocation of limited resources.

Two fundamental LP assumptions are:

a) The price vector $c = (c_1, \dots, c_n)$ for produced goods is fixed and independent on the production output vector $x = (x_1, \dots, x_n)$.
b) The resource vector $b = (b_1, \dots, b_m)$ is fixed and independent on the prices $\lambda = (\lambda_1, \dots, \lambda_n)$ for the resources.

There are few essential difficulties associated with these assumptions.

1. Unfortunately such assumptions do not reflect the basic market law of supply and demand. Therefore the LP models might lead to solutions, which are not always practical.
2. Due to these assumptions a very small change of at least one component at the price vector c might lead to a drastic change of the primal solution.
3. The small variations of the resource vector b can lead to a dramatic change of the dual solution.

The purpose of the paper is developing an alternative to LP approach for resources allocation which is based on GE [17].

The fixed vector c is replaced by a vector-function $c(x)$: the prices for goods depends on the output $x = (x_1, \dots, x_n)$. The fixed factor vector b is replaced by a vector-function $b(\lambda)$: the factors availability depends on the prices $\lambda = (\lambda_1, \dots, \lambda_m)$ for the factors.

We introduce the notion of well defined vector-functions $c(x)$ and $b(\lambda)$ and show that for such function the GE equilibrium exists and unique.

Then we show that finding the GE is equivalent to solving a variation inequality with a strongly monotone operator on the Cartesian product of non-negative octants of the primal and dual spaces.

For solving the variation inequality we used the projected pseudo-gradient method and show that for well defined vector-functions $c(x)$ and $b(\lambda)$ the method globally converges with Q-linear rate. It allows estimating the computational complexity of the method.

The paper is organized as follows. In the following section we state the problem and describe the basic assumptions on the input data.

In the third section we provide some background related to John Nash equilibrium of n-person concave games and classical Walras-Wald equilibrium.

In section four we consider the existence and uniqueness of the GE and show that finding equilibrium is equivalent to solving a variation inequality for a strongly monotone operator.

In section five we consider the projected pseudo-gradient method, establish its global convergence with Q-linear rate and estimate its computational complexity.

We conclude the paper with some remarks related to the future research.

2. Statement of the problem and the basic assumptions

We consider an economy which produces n goods (products) by consuming m factors (resources) in the production process. There are three sets of data required for problem formulation.

1) The technological matrix $A : \mathbf{R}^n \to \mathbf{R}^m$ which "transforms" production factors into goods, i.e. $a_{ij} \geq 0$ defines the amount of factor $1 \leq i \leq m$ required to produce one item of good $1 \leq j \leq n$.
2) The prices vector-function $c(x) = (c_1(x), \dots, c_n(x))$, where $c_j(x)$ is the price for one item of good j under the production output $x = (x_1, \dots, x_j \dots, x_n)$.
3) The factors vector-function $b(\lambda) = (b_1(\lambda), \dots, b_i(\lambda) \dots, b_m(\lambda))$, where $b_i(\lambda)$ is the availability of the factor i under the prices vector $\lambda = (\lambda_1, \dots, \lambda_i, \dots, \lambda_n)$.

We will use the following assumptions.

A. The matrix A does not have zero rows or columns, which means that each factor is used for production of at least one of the goods and each good requires at least one of the production factors.

B. The price vector-function $c : \mathbf{R}^n_+ \to \mathbf{R}^n_+$ for goods are continuous and strongly monotone decreasing, i.e. there is such $\alpha > 0$ that

$$(c(x^2) - c(x^1), x^2 - x^1) \leq -\alpha \|x^1 - x^2\|^2, \quad \forall x^1, x^2 \in \mathbf{R}^n_+. \tag{7}$$

C. The factor vector-function $b : \mathbf{R}^m_+ \to \mathbf{R}^m_+$ is continuous and strongly monotone increasing, i.e. there is such $\beta > 0$ that

$$(b(\lambda^2) - b(\lambda^1), \lambda^2 - \lambda^1) \geq \beta \|\lambda^1 - \lambda^2\|^2, \quad \forall \lambda^1, \lambda^2 \in \mathbf{R}^m_+ \tag{8}$$

where $\|.\|$ is the Euclidean norm.

The assumption B means that an increase in the production x_j of any good $1 \leq j \leq n$ when the rest is fixed leads to the decrease of the price $c_j(x)$ for an item of good j, moreover the margin of the price decrease has a negative upper bound.

The assumption C means that the increase of the price λ_i of any factor $1 \leq i \leq m$ when the rest is fixed leads to the increase of the availability of the resource $b_i(\lambda)$ and the margin for the resource increase has a positive lower bound.

We will say that the price $c(x)$ and factor $b(x)$ vector-functions are well defined if (7) and (8) holds.

It is worth mentioning that (7) is true if $c(x)$ is a gradient of a strongly concave function defined on $\mathbf{R}^n_+$, whereas (8) is true if $b(\lambda)$ is a gradient of a strongly convex function on $\mathbf{R}^m_+$.

The GE problem consists of finding such $x^* \in \mathbf{R}^n_+$ and $\lambda^* \in \mathbf{R}^m_+$:

$$(c(x^*), x^*) = \max\{(c(x^*), X) | AX \leq b(\lambda^*), X \in \mathbf{R}^n_+\}), \tag{9}$$

$$(b(\lambda^*), \lambda^*) = \max\{(b(\lambda^*), \Lambda) | A^T\Lambda \geq c(x^*), \Lambda \in \mathbf{R}^m_+\}). \tag{10}$$

In the following section we provide some background, which helps understanding the relations between GE and both LP duality and the Classical Walras-Wald equilibrium, which is equivalent to J. Nash equilibrium in n-person concave game.

3. J. Nash equilibrium in n-person concave game and Walras-Wald equilibrium

The equilibrium in n-person concave game was introduced by J. Nash in 1950 (see [9]). In 1994 J. Nash received the Nobel Prize in Economics for his discovery.

For many years it was not clear at all whether J. Nash equilibrium has anything to do with economics equilibrium introduced as early as in 1874 by Leon Walras in his most admired work "Elements of Pure Economics". Moreover, it was even not clear whether Walras equations has a solution.

The first substantial contribution was due to Abraham Wald, who in the mid 30th proved the existence of Walras equilibrium under some special assumptions on the price vector-function $c(x)$. These assumptions unfortunately were hard to justify from the economics standpoint (see [5]).

In the mid 50th Harold Kuhn modified Walras-Wald model and proved under minimum assumptions on the input data the existence of the equilibrium. He used two basic tools: Kakutani's Theorem (1941) and LP Duality (3)–(4) (see [5] and references therein).

Our interest in J. Nash equilibrium was inspired by Ben Rosen's paper [13] in mid 60th, where he discovered some relations between J. Nash equilibrium and convex optimization. Soon after few numerical methods for finding J. Nash equilibrium developed (see S. I. et al. [15]–[18]). Moreover, we proved that H. Kuhn's version of Walras-Wald equilibrium is equivalent to J. Nash equilibrium in n-person concave game [16].

In this section we remind some basic facts and notations related to J. Nash equilibrium, which will be used later.

Let $\Omega_j \subset \mathbf{R}^m_j$ be a convex compact set, which defines the strategies vector $x_j = (x_{j1}, \ldots, x_{jm_j})$ for the players $1 \leq j \leq n$. Then the Cartesian product $\Omega = \Omega_1 \otimes \ldots \otimes \Omega_j \otimes \ldots \otimes \Omega_n$ define the feasible strategies set for the n–person concave game. The set $\Omega \in \mathbf{R}^m, m = \Sigma^n_{j=1} m_j$ is convex and bounded.

The payoff functions $\varphi_j : \Omega \to \mathbf{R}$ of the players $1 \leq j \leq n$, are continuous and $\varphi_j(x_1, \ldots, x_j, \ldots x_n)$ is concave in $x_j \in \Omega_j$.

The vector $x^* = (x^*_1, \ldots x^*_j, \ldots x^*_n) \in \Omega$ defines J. Nash equilibrium if for every $1 \leq j \leq n$ the following inequality holds:

$$\varphi_j(x^*_1, \ldots, x_j, \ldots, x^*_n) \leq \varphi_i(x^*_1, \ldots, x^*_j, \ldots, x^*_n), \quad \forall x_j \in \Omega_j \tag{11}$$

which means that for every player $1 \leq j \leq n$ violation of the equilibrium strategy can only reduce the correspondent payoff value if the rest accepted their equilibrium strategies. Along with J. Nash equilibrium, let's consider the normalized equilibrium for a concave n-person game. Let $x = (x_1, \ldots, x_j, \ldots x_n) \in \Omega$. On $\Omega \otimes \Omega$ we consider the following normalized payoff function:

$$\Phi(x, X) = \sum_{j=1}^{n} \varphi_j(x_1, \ldots, X_j, \ldots x_n)). \tag{12}$$

Vector $x^* \in \Omega$ is a normalized equilibrium in n-person concave game if

$$\Phi(x^*, x^*) = \max\{\Phi(x^*, X) | X \in \Omega\}. \tag{13}$$

It is clear that each normalized equilibrium $x^* = (x_1^*, \ldots, x_n^*)$ is J. Nash equilibrium, but the opposite generally speaking is not true.

By finding $x^* \in \Omega$ from (13) we find the J. Nash equilibrium in a concave n-person game, because it follows from (13) that if any players violates his equilibrium strategy x_j^* and the rest accept their equilibrium strategies, then the player $1 \leq j \leq n$ can only reduce its payoff, i.e. (11) holds.

Let's consider the vector-function

$$\begin{aligned} g(x) &= \nabla_x \Phi(x, X)|_{X=x} \\ &= (\nabla_{x_1} \varphi(X_1, x_2, \ldots, x_n), \ldots, \nabla_{x_n} \varphi(x_1, \ldots, x_{n-1}, X_n))|_{X=x} \\ &= (\nabla_{x_1} \varphi_1(x), \ldots, \nabla_{x_n} \varphi_n(x)), \end{aligned} \tag{14}$$

where

$$\nabla_{x_j} \varphi_j(x_1, \ldots X_j, \ldots x_n)|_{X_j = x_j} = \left(\frac{\partial \varphi_j(x)}{\partial x_{j1}}, \ldots, \frac{\partial \varphi_j(x)}{\partial x_{jm_j}} \right), \quad 1 \leq j \leq n. \tag{15}$$

The $g(x)$ is called pseudo-gradient of $\Phi(x, X)$ at $X = x$.

For any given $x \in \Omega$ the normalized payoff function $\Phi(x, X)$ is concave in $X \in \Omega$. Hence finding

$$\omega(x) = \operatorname{Arg\,max}\{\Phi(x, X) | X \in \Omega\} \tag{16}$$

is a convex optimization problem therefore the set $\omega(x) \subset \Omega$ is a convex compact set. Moreover, it follows from the continuity of $\Phi(x, X)$ that $\omega(x)$ is upper semi-continuous set value function on Ω (see [6]). Therefore the existence of a fixed point $x^* \in \Omega$ of the map $x \to \omega(x)$ i.e. $x^* \in \omega(x^*)$ is a direct consequence of the Shizuo Kakutani theorem (see [6]).

If $x^* \in \Omega$ is the normalized equilibrium, then finding

$$\Phi(x^*, x^*) = \max\{\Phi(x^*, X) | X \in \Omega\} \tag{17}$$

is a convex optimization problem.

The fact that $x^* \in \Omega$ is the solution of convex optimization problem (17) means that for the pseudo-gradient $g(x^*) = \nabla_x \Phi(x^*, X) | X = x^*$ the following inequality

$$(g(x^*), X - x^*) \leq 0, \quad \forall X \in \Omega \tag{18}$$

holds.

On the other hand if $x^* \in \Omega$ is a solution of the variational inequality (18) then x^* is J. Nash equilibrium in n-person concave game. Therefore finding J. Nash equilibrium is equivalent to solving the variational inequality (18).

At the same time (18) one can view as an optimality criteria for the normalized equilibrium.

The monotonicity of the nonlinear operator $g : \Omega \to \mathbf{R}^m$, i.e.

$$(g(x^1) - g(x^2), x^1 - x^2) \leq 0, \quad \forall (x^1, x^2) \in \Omega \times \Omega \tag{19}$$

is critical for solving the variational inequality (18).

The operator g is strongly monotone on Ω if there is $\gamma > 0$ such that

$$(g(x^1) - g(x^2), x^1 - x^2) \leq -\gamma \|x^1 - x^2\|^2, \quad \forall (x^1, x^2) \in \Omega \times \Omega. \tag{20}$$

We conclude the section by reminding that H. Kuhn's version of the classical Warlas-Wald equilibrium is equivalent to J. Nash equilibrium in n-person concave game (see [16]).

One obtains H. Kuhn's version from (9)–(10) by assuming that the factor vector-function $b(\lambda)$ is fixed, i.e. $b(\lambda) \equiv b$.

In other words Walras-Wald equilibrium is a pair $(x^*, \lambda^*) \in \mathbf{R}^n_+ \otimes \mathbf{R}^m_+$ such that

$$(c(x^*), x^*) = \max\{(c(x^*), X)|AX \le b, X \in \mathbf{R}^n_+\}, \tag{21}$$

$$(b, \lambda^*) = \min\{(b, \Lambda)|A^T\Lambda \ge c(x^*)|\Lambda \in \mathbf{R}^m_+\}. \tag{22}$$

Let's assume that the price for any good is not increasing function of the correspondent output when the production of other goods are fixed, i.e.

$$t_2 > t_1 > 0 \Rightarrow c_j(x_1, \dots, t_2, \dots, x_n) \le c_j(x_1, \dots, t_1, \dots, x_n), \quad 1 \le j \le n.$$

Then the function

$$\varphi_j(x_1, \dots, x_j, \dots, x_n) = \int_0^{x_j} c_j(x_1, \dots, t, \dots, x_n)\, dt \tag{23}$$

is concave in x_j.

Let $X = (x_1, \dots, x_j \dots, x_n) \in \Omega = \{x : AX \le b, X \in \mathbf{R}^n_+\}$, then we obtain Walras-Wald equilibrium by finding the normalized J. Nash equilibrium when the payoff functions of the players are defined by (23) i.e.

$$\Phi(x^*, x^*) = \max\left\{\sum_{j=1}^n \varphi_j(x_1^*, \dots, X_j, \dots, x_n^*)|X \in \Omega\right\}. \tag{24}$$

Therefore existence of the Warlas-Wald equilibrium is a direct consequence of existence of the normalized J. Nash equilibrium of the concave n-person game with payoff functions (23) and feasible strategies set Ω.

In fact, using the optimality criteria (18) and keeping in mind that $g(x^*) = c(x^*)$ we can rewrite the problem (24) as following LP problem

$$\max\{(c(x^*), X)|AX \le b, X \in \mathbf{R}^m_+\} = (c(x^*), x^*).$$

Therefore using LP Duality one obtains $\lambda^* \in \mathbf{R}^m_+$

$$(b, \lambda^*) = \min\{(b, \lambda)|A^T\lambda \ge c(x^*), \lambda \in \mathbf{R}^m_+\}$$

and

$$(c(x^*, x^*) = b(\lambda^*)),$$

i.e. by finding J. Nash equilibrium from (24) one finds the Walras-Wald equilibrium (21)–(22).

Hence finding Walras-Wald equilibrium leads to solving a variational inequality (18). Solving (18) is generally speaking more difficult then solving the Primal LP (1).

In the following section we consider the GE (9)–(10), which looks more difficult then (21)–(22) however for well defined $c(x)$ and $b(\lambda)$ finding GE turned out to be much easier then solving LP.

4. Existence and uniqueness of the GE

The GE problem was revisited in 1977 by M. Primak and the author in [10]. It was shown that finding GE is equivalent to solving a particular variational inequality. We will discuss it briefly in this section and use the equivalence later for establishing the existence and uniqueness GE as well as for developing the projection pseudo-gradient method for finding GE.

Let's fix $x \in \mathbf{R}^n_+$ and $\lambda \in \mathbf{R}^n_+$ and consider the following dual pair of LP

$$\max\{(c(x), X)|AX \le b(\lambda), X \in \mathbf{R}^n_+\}, \tag{25}$$

$$\min\{(b(\lambda), \Lambda)|A^T\Lambda \ge c(x), \Lambda \in \mathbf{R}^m_+\}. \tag{26}$$

Solving the dual LP pair (25)–(26) is equivalent to finding on $\mathbf{R}^n_+ \otimes \mathbf{R}^m_+$ a saddle point for the correspondent Lagrangean

$$\mathbf{L}(x,\lambda;X,\Lambda) = (c(x),X) - (\Lambda, AX - b(\lambda)), \tag{27}$$

i.e. finding

$$\max_{X\in\mathbf{R}^m_+} \min_{\Lambda\in\mathbf{R}^m_+} \mathbf{L}(x,\lambda;X,\Lambda) = \min_{\Lambda\in\mathbf{R}^n_+} \max_{X\in\mathbf{R}^n_+} \mathbf{L}(x,\lambda;X,\Lambda), \tag{28}$$

under fixed $x\in\mathbf{R}^n_+, \lambda\in\mathbf{R}^m_+$. The problem (28) is equivalent in turn to finding J. Nash equilibrium of two person concave game with the following payoff functions

$$\varphi_1(x,\lambda;X,\lambda) = (c(x),X) - (\lambda, AX - b(\lambda)) = (c(x) - A^T\lambda, X) + (\lambda, b(\lambda)) \tag{29}$$

and

$$\varphi_2(x,\lambda;x,\Lambda) = (Ax - b(\lambda), \Lambda), \tag{30}$$

where $X\in\mathbf{R}^n_+$ is the strategy of the first player and $\Lambda\in\mathbf{R}^m_+$ is the strategy of the second player.

Let $y=(x,\lambda), Y=(X,\Lambda)\in\mathbf{R}^n_+\otimes\mathbf{R}^n_+ = \Omega$. The correspondent normalized payoff function is defined as follows

$$\Phi(y,Y) = (c(x) - A^T\lambda, X) + (Ax - b(\lambda), \Lambda) + (\lambda, b(\lambda)). \tag{31}$$

Therefore finding the saddle point is equivalent to finding the J. Nash normalized equilibrium of two person concave game, i.e. such $\bar{y}\in\Omega$ that

$$\Phi(\bar{y},\bar{y}) = \max\{\Phi(\bar{y},Y) | Y\in\Omega\}. \tag{32}$$

Let us consider the corresponded pseudo-gradient

$$\nabla_Y\Phi(y,Y)|_{Y=y} = g(y) \equiv g(x,\lambda) = (c(x) - A^T\lambda, Ax - b(\lambda)). \tag{33}$$

So finding $\bar{y}\in\Omega$ from (32) is equivalent to solving the following variational inequality

$$(g(\bar{y}), Y - \bar{y}) \le 0, \quad \forall Y\in\Omega. \tag{34}$$

Let us fix $\bar{y}\in\Omega$. It follows from (34) that

$$\max\{(g(\bar{y}), Y - \bar{y}) | Y\in\Omega\} = (g(\bar{y}, \bar{y} - \bar{y})) = 0. \tag{35}$$

From (35) we obtain that $g(\bar{y})\le 0$, because by assuming that at least one component of the vector $g(\bar{y})$ is positive, we obtain

$$\max\{(g(\bar{y}), Y - \bar{y}) | Y\in\Omega\} = \infty. \tag{36}$$

Therefore if $\bar{y} = (\bar{x},\bar{\lambda})\ge 0$ solves the variational inequality (34), then

$$c(\bar{x}) - A^T\bar{\lambda} \le 0 \quad \text{and} \quad A\bar{x} - b(\bar{\lambda}) \le 0. \tag{37}$$

Note that solving (35) is equivalent to finding

$$\max\left\{\sum_{j=1}^n (c(\bar{x}) - A^T\bar{\lambda}))_j X_j + \sum_{i=1}^m (A(\bar{x}) - b(\bar{\lambda}))_i \Lambda_i | X_j \ge 0,\right.$$
$$\left. j=1,\dots,n,\ \Lambda_i\ge 0,\ i=1,\dots,m\right\}.$$

Therefore for $1\le j\le n$ we have

$$(c(\bar{x}) - A^T\bar{\lambda})_j < 0 \Rightarrow \bar{x}_j = 0, \quad \bar{x}_j > 0 \Rightarrow (c(\bar{x}) - A^T\bar{\lambda}) = 0, \tag{38}$$

and for $1\le i\le m$ and we have

$$(A\bar{x} - b(\lambda))_i < 0 \Rightarrow \bar{\lambda} = 0, \quad \bar{\lambda}_i > 0 \Rightarrow (A\bar{x} - b(\bar{\lambda}))_i = 0. \tag{39}$$

Hence $\bar{y} = (\bar{x},\bar{\lambda})\in\Omega$ is primal-dual feasible solution which, satisfied the complementarily condition (38)–(39).

Therefore vector $Y = \bar{y}$ is the solution for of the following primal-dual LP

$$\max\{(c(\bar{x}), X) | AX \leq b(\bar{\lambda}), X \in \mathbf{R}^n_+\} = (c(\bar{x}), \bar{x}), \tag{40}$$

$$\min\{(b(\bar{\lambda}), \Lambda) | A^T \Lambda \geq c(\bar{x}), \Lambda \in \mathbf{R}^m_+\} = (b(\bar{\lambda}), \bar{\lambda}), \tag{41}$$

i.e. $\bar{y} = y^*$. On the other hand it is easy to see that GE y^* which is defined by (9)–(10), solves the variational inequality (34). Therefore finding GE y^* is equivalent to solving variational inequality (34).

Now let us to show that $y^* \in \Omega$ exists.

The arguments used for proving the existence of J. Nash normalized equilibrium in section 3 can't be used in case of (32), because the feasible set Ω is unbounded, and Kakutani Theorem can't be applied.

It turns out that if $c(x)$ and $b(\lambda)$ are well defined, then in spite of unboundedness Ω the GE exists. We start with the following technical Lemma.

Lemma 1. *If vector-functions $c(x)$ and $b(\lambda)$ are well defined, then the pseudo-gradient $g : \Omega \to \mathbf{R}^{m+n}$ is a strongly monotone operator, i.e. for $\gamma = \min\{\alpha, \beta\} > 0$ the following inequality holds:*

$$\big(g(y^1) - g(y^2), y^1 - y^2\big) \leq -\gamma \|y^1 - y^2\|^2 \tag{42}$$

for any pair $(y^1; y^2) \in \Omega \otimes \Omega$.

Proof. Let $y^1 = (x^1, \lambda^1), y^2 = (x^2, \lambda^2) \in \Omega$ then

$$\begin{aligned}
&(g(y^1) - g(y^2), y^1 - y^2) \\
&\quad = (c(x^1) - A^T\lambda^1 - c(x^2) + A^T\lambda^2, x^1 - x^2) \\
&\quad + (A(x^1) - b(\lambda^1) - Ax^2 + b(\lambda^2), \lambda^1 - \lambda^2) \\
&\quad = (c(x^1) - c(x^2), x^1 - x^2) - (A^T(\lambda^1 - \lambda^2), x^1 - x^2) \\
&\quad + (A(x^1 - x^2), \lambda^1 - \lambda^2) - (b(\lambda^1) - b(\lambda^2), \lambda^1 - \lambda^2) \\
&\quad = (c(x^1) - c(x^2), x^1 - x^2) - (b(\lambda^1) - b(\lambda^2), \lambda^1 - \lambda^2).
\end{aligned}$$

Invoking (7) and (8) we obtain (42). □

We are ready to prove existence and uniqueness of GE.

Theorem 1. *If $c(x)$ and $b(\lambda)$ are well defined, then the GE $y^* = (x^*, \lambda^*)$ exists and unique.*

Proof. Let us consider $y_0 \in \Omega : \|y_0\| \leq 1$ and a large enough number $M > 0$. Instead of (32) we consider the following equilibrium problem:

$$\Phi(y^*_M, y^*_M) = \max\{\Phi(y^*_M, Y) | Y \in \Omega_M\} \tag{43}$$

where $\Omega_M = \{Y \in \Omega : \|Y\| \leq M\}$. The normalized function $\Phi(y, Y)$ defined by (31) is linear in $Y \in \Omega$ and Ω_M is a convex compact set. Therefore for a given $y \in \Omega_M$ the set

$$\omega(y) = \text{Argmax}\{\Phi(y, Y) | Y \in \Omega_M\} \tag{44}$$

is a solution set of convex optimization problem. Therefore for any given $y \in \Omega_M$ the set $\omega(y) \subset \Omega_M$ is a convex compact. Moreover the map $y \to \omega(y)$ is upper semi-continuous. In fact, let us consider a sequence $\{y^s\} \subset \Omega_M : y^s \to \bar{y}$ and any sequence of images $\{z^s \in \omega(y^s)\}$ converging to $\bar{z}$.

Due to the continuity of $\Phi(y, Y)$ in y and Y we have $\bar{z} \in \omega(\bar{y})$. Therefore $y \to \omega(y)$ maps a convex compact Ω_M into itself and the map is upper semi–continuous, therefore Kakutani's Theorem can be applied.

Therefore there exists $y^*_M \in \Omega_M : y^*_M \in \omega(y^*_M)$.

Let's show that the constraint $\|Y\| \leq M$ is irrelevant in problem (43). Using the bound (42) for $y^1 = y_0$ and $y^2 = y_M^*$ one obtains

$$(45)\quad \gamma\|y_0 - y_M^*\|^2 \leq (g(y_M^* - g(y_0)), y_0 - y_M^*) = (g(y_M^*), y_0 - y_M^*) + (g(y_0), y_M^* - y_0)\,.$$

Vector y_M^* is the solution of variational inequality (34) when Ω is replaced by Ω_M, hence

$$(46)\qquad (y(y_M^*), y_0 - y_M^*) \leq 0.$$

It follows from (45)–(46) and Cauchy-Schwarz inequality that

$$(47)\qquad \gamma\|y_0 - y_M^*\|^2 \leq |(g(y_0), y_M^* - y_0)| \leq \|g(y_0)\|\|y_M^* - y_0\|$$

or

$$(48)\qquad \|y_0 - y_M^*\| \leq \gamma^{-1}\|g(y_0)\|.$$

Therefore,

$$\|y_M^*\| \leq \|y_0\| + \|y_M^* - y_0\| \leq 1 + \gamma^{-1}\|g(y_0)\|.$$

Hence for $M > 0$ large enough, the inequality $\|Y\| \leq M$ is irrelevant and can be removed from constraints set in Ω (43).

In other words $y_M^* = y^* = (x^*, \lambda^*)$ is the GE. It turns out that if the vector-function $c(x)$ and $b(\lambda)$ are well defined, then GE not only exists, but it is unique as well.

In fact, assuming that $\bar{y} \in \Omega, y^* \in \Omega$ are two different solutions of the variational inequality (34) then one obtains $(g(\bar{y}), y^* - \bar{y}) \leq 0$ and $(g(y^*), \bar{y} - y^*) \leq 0$, therefore

$$(g(\bar{y}) - g(y^*), \bar{y} - y^*) \geq 0.$$

On the other hand it follows from (42) for $y^1 = \bar{y}$ and $y^2 = y^*$ that

$$(49)\qquad (g(\bar{y}) - g(y^*), \bar{y} - y^*) \leq -\gamma\|\bar{y} - y^*\|^2.$$

The contradiction proves uniqueness of the GE y^*. □

5. Projected pseudo-gradient method for finding GE

In this section we introduce the projected pseudo-gradient method for finding GE and show its global convergence with Q-linear rate. We estimate the ratio through the basic parameters of the input data. The global convergence with Q-linear rate allows estimate the computational complexity of the method.

For well defined $c(x)$ and $b(\lambda)$ the pseudo-gradient

$$(50)\qquad g(y) = (c(x) - A^T\lambda, Ax - b(\lambda))$$

is strongly monotone, i.e. (42) holds. Therefore finding GE can be reduced to solving the variational inequality (34) for a strongly monotone operator $g : \Omega \to \mathbf{R}^{n+m}$.

We start by reminding basic facts related to projection operation (see [4], [8]).

Let Q be a closed convex set in $\mathbf{R}^n$. Then for each $x \in \mathbf{R}^n$ there is a unique point $y \in Q$ such, that

$$(51)\qquad \|x - y\| \leq \|x - z\|, \quad \forall z \in Q.$$

The vector y is called the projection of x on Q, i.e.

$$(52)\qquad y = P_Q x$$

and $P_Q : \mathbf{R}^n \to Q$ defined by (52) is called the projection operator. Let $x_0 \in Q$, then finding projection $y = P_Q x$ is equivalent to solving the following convex optimization problem

$$(53)\qquad d(y) = \min\{d(z) = \|z - x\|^2 | z \in Q\},$$

where $Q = \{z : \|z - x\| \leq \|x_0 - x\|\}$.

The problem (53) has a compact convex feasible set Q and strongly convex and continuous objective function $d(z)$. Therefore the projection $y = P_Q x$ exist and unique for any given $x \in \mathbf{R}^n$.

Let's consider the optimality criteria for the projection y in (53). Keeping in mind that $\nabla d(z) = 2(z - x)$ and the fact that y is the closest to the vector x point in Q we obtain

$$(y - x, z - y) \geq 0 \quad \text{or} \quad (y, z - y) \geq (x, z - y), \quad \forall z \in Q. \tag{54}$$

Now we would like to remind few properties of the projection operator P_Q which will be used later.

Lemma 2. *Let Q be a closed convex set. Then the projection operator P_Q is non-expansive, that is*

$$\|P_Q x_1 - P_Q x_2\| \leq \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbf{R}^n. \tag{55}$$

Proof. Let $y_1 = P_Q x_1$ and $y_2 = P_Q x_2$, then using the optimality criteria (54) for y_1 and y_2 we obtain

$$(y_1, z - y_1) \geq (x_1, z - y_1), \quad \forall z \in Q$$

and

$$(y_2, z - y_2) \geq (x_2, z - y_2), \quad \forall z \in Q.$$

From the first inequality one obtains

$$(y_1, y_2 - y_1) \geq (x_1, y_2 - y_1). \tag{56}$$

From the second we have

$$(y_2, y_1 - y_2) \geq (x_2, y_1 - y_2). \tag{57}$$

Combining (56) and (57) and using Caushy-Schwarz inequality we obtain

$$\|y_1 - y_2\|^2 \leq (x_1 - x_2, y_1 - y_2) \leq \|x_1 - x_2\| \|y_1 - y_2\|.$$

Hence

$$\|y_1 - y_2\| \leq \|x_1 - x_2\|.$$

□

Lemma 3. *Let Q be a closed convex set in $\mathbf{R}^n$. Then x^* is a solution of the variational inequality*

$$(g(x^*), x - x^*) \leq 0, \quad \forall x \in Q \tag{58}$$

if and only if for any $t > 0$ x^* is a fixed point of the map $P_Q(I + tg) : Q \to Q$, i.e.

$$x^* = P_Q(x^* + tg(x^*)).$$

Proof. If x^* is the solution of the variational inequality then (58) holds. Multiplying the inequality by $t > 0$ and adding to both sides $(x^*, x - x^*)$ one obtains

$$(x^* + tg(x^*), x - x^*) \leq (x^*, x - x^*), \quad \forall x \in Q.$$

It follows from (54) that $x^* = P_\Omega(x^* + tg(x^*))$. On the other hand if $x^* = P_Q(x^* + tg(x^*))$ then from (54) taking $y = x^*$ and $x = x^* + tg(x^*)$ we obtain $(x^*, x - x^*) \geq (x^* + tg(x^*), x - x^*)$, i.e. $(g(x^*), x - x^*) \leq 0, \forall x \in Q$. □

We are ready to describe the projected pseudo-gradient method.

Let $y^0 = (x^0, \lambda^0) \in \mathbf{R}^n_{++} \otimes \mathbf{R}^m_{++}$ be a starting point. Assuming that approximation $y^s = (x^s, \lambda^s)$ has been found already. The next approximation one finds by formula

$$y^{s+1} = P_\Omega(y^s + tg(y^s)). \tag{59}$$

Keeping in mind that $\Omega = \mathbf{R}^n_+ \otimes \mathbf{R}^m_+$ the method (59) translates into the following formulas for goods and prices update:

$$x_j^{s+1} = \left[x_j^s + t(c(x^s) - A^T\lambda^s)_j\right]_+, \quad j = 1,\dots,n, \tag{60}$$

$$\lambda_i^{s+1} = [x_i^s + t(Ax^s - b(\lambda^s))_i]_+, \quad i = 1,\dots,m, \tag{61}$$

where $[z]_+ = \begin{cases} z & z \geq 0, \\ 0 & z < 0. \end{cases}$

Before we specify the step length $t > 0$ let consider the update formulas (60) and (61).

It follows from (60) that if the current price $c_j(x^s)$ exceeds the current expenses $(A^T\lambda^s)_j$ then the production of the good $1 \leq j \leq n$ has to be increased. If the current price $c_j(x^s)$ is less then current expenses $(A^T\lambda^s)_j$ then the production of the good has to be reduced.

It follows from (61) that if the current consumption $(Ax^s)_i$ of factor $1 \leq i \leq m$ exceeds availability $b_i(\lambda^s)$ of this factor, then one has to increase the price for the factor.

If availability $b_i(\lambda^s)$ of the factor $1 \leq i \leq m$ exceeds consumption $(Ax^s)_i$ of the factor, then the price should be reduced.

The projection operation keeps the new primal-dual approximation (x^{s+1}, λ^{s+1}) in Ω.

We remind that for well defined $c(x)$ and $b(\lambda)$ the pseudo-gradient $g(y)$ is a strongly monotone on $\Omega = \mathbf{R}^n_+ \times \mathbf{R}^m_+$ i.e. (42) holds.

Let us also assume that pseudo-gradient $g(y)$ satisfies the Lipschitz condition as well, i.e. there is $L > 0$ such that

$$\|g(y_1) - g(y_2)\| \leq L\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \Omega. \tag{62}$$

Theorem 2. *If $c(x)$ and $b(\lambda)$ are well defined and (62) holds then for any given $0 < t < 2\gamma L^{-2}$ the projected pseudo-gradient method (60) globally converges to GE with Q-linear rate and the ratio $0 < q(t) = 1 - 2t\gamma + t^2L^2 < 1$. Also the following bound*

$$\|y^{s+1} - y^*\| \leq (1 - (\gamma L^{-1})^2)^{1/2}\|y^s - y^*\| \tag{63}$$

holds for $t = \gamma L^{-2} = \min_{t>0} q(t)$.

Proof. Let

$$\|y^{s+1} - y^*\|^2 = \|P_\Omega(y^s + tg(y^s)) - P_\Omega(y^* + tg(y^*))\|^2.$$

Using Lemma 2 one obtains

$$\|P_\Omega(y^s + tg(y^s)) - P_\Omega(y^* + tg(y^*))\|^2 \leq \|y^s + tg(y^s) - (y^* + tg(y^*))\|.$$

Then

$$\begin{aligned} \|y^{s+1} - y^*\|^2 &\leq \|y^s + tg(y^s) - (y^* + tg(y^*))\|^2 \\ &= (y^s - y^* + t(g(y^s) - g(y^*)), y^s - y^* + t(g(y^s) - g(y^*)))) \\ &= \|y^s - y^*\|^2 - 2t(y^s - y^*, g(y^s) - g(y^*)) + t^2\|g(y^s) - g(y^*)\|^2. \end{aligned}$$

Using the strong monotonicity (42) and Lipschitz condition (63) one obtains

$$\begin{aligned} \|y^{s+1} - y^*\|^2 &\leq \|y^s - y^*\|^2 - 2t\gamma\|y^s - y^*\|^2 + t^2L^2\|y^s - y^*\|^2 \\ &= (1 - 2t\gamma + t^2L^2)\|y^s - y^*\|^2 = q(t)\|y^s - y^*\|^2. \end{aligned}$$

Therefore for any $0 < t < 2\gamma L^{-2}$ one obtains $0 < q(t) < 1$.

Hence for any $0 < t < 2\gamma L^{-2}$ the projected pseudo-gradient method globally converges with Q-linear rate and the ratio $0 < q(t) < 1$. Also $q(\gamma L^{-2}) = \min_{t>0} q(t) = 1 - (\gamma L^{-1})^2$, i.e. the bound (63) holds. $\square$

The projected pseudo-gradient method (60) converges to GE (x^*, λ^*) :

$$x_j^* > 0 \Rightarrow c_j(x^*) = (A^T\lambda^*),$$
$$x_j^* = 0 \Leftarrow (A^T\lambda^*)_j > c_j(x^*),$$
$$\lambda_i^* > 0 \Rightarrow (Ax^*)_i = b_i(\lambda^*),$$
$$\lambda_i^* = 0 \Leftarrow (Ax^*)_i < b_i(\lambda^*).$$

Therefore if the market "is cleared" from goods for which the production cost exceeds the price and from factors, which are exceeds the needs for them, then for the remaining goods and factors we obtain the following equilibrium equations

$$(A^T\lambda)_j = c_j(x^*), \quad (Ax^*)_i = b_i(\lambda^*)$$

and

$$(c(x^*), x^*) = (b(\lambda^*), \lambda^*),$$

which is in fact the Generalized Walras law.

The pseudo-gradient method (60)–(61) one can view as a projected explicit Euler method for the following system of differential equations

$$\frac{dx}{dt} = c(x) - A^T\lambda,$$
$$\frac{d\lambda}{dt} = Ax - b(\lambda).$$

If vector functions $c(x)$ and $b(\lambda)$ are well defined and Lipschitz condition (62) holds then for any given $0 < t < 2\gamma L^{-2}$ correspondent trajectory $\{y^s = (x^s, \lambda^s)\}_{s=0}^{\infty}$ converges to GE (x^*, λ^*) and the following bound holds

$$\|y^{s+1} - y^*\| \le q\|y^s - y^*\|, \quad s \ge 1 \tag{64}$$

where $0 < q := (q(t))^{1/2} = (1 - 2t\gamma + t^2L^2)^{1/2} < 1$.

Let small enough $0 < \epsilon << 1$ be the required accuracy. Due to (64) it takes $O((\ln\epsilon)(\ln q)^{-1})$ projected pseudo-gradient steps to get an approximation for GE (x^*, λ^*) with accuracy $\epsilon > 0$. The number of operation per step is bounded by $O(n^2)$ (assuming that $n > m$), therefore the total number of operations for finding the GE with given accuracy $\epsilon > 0$ is

$$\mathbf{N} = O(n^2 \ln\epsilon(\ln q)^{-1}).$$

It means that for well defined $c(x)$ and $b(\lambda)$ finding GE might required much less computational effort then solving an LP.

6. Concluding remarks

Replacing the factor vector b by the vector-function $b(\lambda)$ leeds to "symmetrization" of the classical Walras-Wald equilibrium. The symmetrization is not only justifiable from the economic standpoint, it also eliminates both the combinatorial nature of LP as well as the basic difficulties associated with finding classical Walras-Wald equilibrium by solving variational inequality (21). In fact instead of solving variational inequality (21) on a polyhedron, we end up solving the variational inequality (34) on $\mathbf{R}_+^n \otimes \mathbf{R}_+^m$.

What is even more important, the symmetrization leads to projected pseudo-gradient method (59), which is not only attractive numerically, but also can be considered as a natural pricing mechanism for finding economic equilibrium.

All these nice featured of the projected pseudo-gradient method are due to our assumptions that vector-function $c(x)$ and $b(\lambda)$ are well defined and smooth enough.

If it is not the case then other methods for solving the variational inequality (34) can be used (see [7]).

Lately the interest to projection gradient type methods has been revitalized and number of important results were obtained [1]–[3]. Some of the methods developed in [1]–[3] can be used for solving variational inequality (34).

Our main goal however is keeping the algorithm simple and efficient with full understanding of the pricing mechanism they represent.

We hope that the dual Nonlinear rescaling methods in general (see [12]) and the dual MBF method (see [11]) in particular can be used as prototypes for such algorithms. Some new results in this direction will be covered in the upcoming paper.

References

1. A. Auslender, M. Teboulle, *Interior projection-like methods for monotone variational inequalities*, Math. Program., Ser. A **104** (2005), 39-68.
2. A. Auslender, M. Teboulle, *Projected subgradient methods with non-Euclidean distances for non-differentiable convex minimization and variational inequalities*, Math. Program., Ser. B, DOI 10.1007/s10107-007-0147-z, 2007.
3. A. Beck, M. Teboulle, *A linearly convegent dual based gradient projection algorithm for quadratically constrained convex optimization*, Mathematics of Operations Research **31** (2006), no. 2, 398–417.
4. A. A. Goldstein, *Convex programming in Hilbert space*, Bull. Amer. Math. Soc. **70** (1964), 709–710.
5. H. Kuhn, *On a theorem of Wald*, Annals of Math. Studies **38** (1956), 265–274.
6. K. Lancaster, *Mathematical Economics*, Dover Publication, New York, 1987.
7. T. Larsson, M. Patriksson, *A class of gap functions for variational inequalities*, Mathematical Programming **64** (1994). no. 1, 53–79.
8. E. S. Levitin, B. T. Polyak, *Constrained minimization methods*, Zh. Vychisl. Mat. i Mat. Fiz. **6** (1966), no. 5, 787–823. (Russian)
9. J. F. Nash, *Non-cooperative games*, Ann. Math. **54** (1951), 286–295.
10. R. Polyak and M. Primak, *Methods of controlling sequences for solving equilibrium problems*, Kibernetika (1977), no. 2, 92–102.
11. R. Polyak, *Modified barrier functions* (*theory and methods*), Mathematical Programming **54** (1992), no. 2, 177–222.
12. R. Polyak, *Nonlinear rescaling as interior quadratic prox method in convex optimization*, Computational Optimization and Applications **35** (2006), no. 3, 347–373.
13. J. B. Rosen, *Existence and uniqueness of equilibrium point for concave n-person games*, Econometrica **33** (1965), no. 3, 520–534.
14. E. Stiefel, *Note on Jourdan elimination, linear programming and Chebyshev approximation*, Numerische Mathematic **2** (1960).
15. S. I. Zuchovitsky, R. A. Polyak, M. E. Primak, *Two methods for finding equilibrium points in N-person concave games*, Soviet Math. Dokl. **10** (1969), no. 2.
16. S. I. Zuchovitsky, R. A. Polyak, M. E. Primak, *N-person concave game and one production model*, Soviet Math. Dokl. **11** (1970), no. 2.
17. S. I. Zuchovitsky, R. A. Polyak, M. E. Primak, *One general equilibrium problem*, Soviet Math. Dokl. **14** (1973), no. 2.
18. S. I. Zuchovitsky, R. A. Polyak, M. E. Primak, *Multiperson concave games: numerical methods*, Matekon **9** (1973).

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