Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 3, pp. 287–296

ON STABILITY, SUPERSTABILITY AND STRONG SUPERSTABILITY OF CLASSICAL SYSTEMS OF STATISTICAL MECHANICS

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Abstract. A detailed analysis of conditions on 2-body interaction potential, which ensure stability, superstability or strong superstability of statistical systems is given. We give a connection between conditions of superstability (strong superstability) and the problem of minimization of Riesz energy in bounded volumes.

1. Introduction

Stability **(S)** of the interaction is a necessary condition for a correct thermodynamic description of infinite statistical systems. This condition can be formulated in terms of an infinite system of inequalities on the interaction energy of an arbitrary finite subsystem, consisting of N particles, which are situated in the points $x_1, \ldots, x_N$ of the space $\mathbb{R}^d$.

(S) *Stability. There exists* $B \geq 0$ *such that*

$$U(x_1, \ldots, x_N) \geq -B\,N \tag{1.1}$$

for any $N \geq 2$ *and* $\{x_1, \ldots, x_N\}$.

In the present paper we consider an infinite system that consists of identical point particles interacting the via 2-body potential

$$V_2(x, y) = \Phi(|x - y|), \tag{1.2}$$

where $|x - y|$ means the Euclidean distance between points $x, y \in \mathbb{R}^d$. In this case,

$$U(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} \Phi(|x - y|). \tag{1.3}$$

One of the most important conditions is the *condition of integrability at infinity*. This means that, for any $R > 0$,

$$\int_{|x| \geq R} \Phi(|x|)\, dx < +\infty. \tag{1.4}$$

Conditions (1.1) and (1.4) are sufficient for constructing a Gibbs measure of an infinite system of particles in the area of small values of the parameters $\beta = \frac{1}{k_B T}$ and z, where T is temperature of a system and z is chemical activity, which is directly connected with the density of the system of particles (see for example [25], Ch. 4). In order to solve the problem of constructing a Gibbs state (Gibbs measure) of an infinite system for all positive values of the parameters β and z, it is necessary to impose more restrictive conditions on the interaction. Such a condition is the condition of superstability **(SS)** (see [8], [26]). At first we give several necessary definitions.

2000 *Mathematics Subject Classification.* 82B05, 82B21.
Key words and phrases. Continuous classical system, superstable interaction, minimal Riesz energy.

For each $\lambda \in \mathbb{R}_+$ one can define the partition $\overline{\Delta_\lambda}$ of the space $\mathbb{R}^d$ into cubes Δ with edge λ and center in $r \in \mathbb{Z}^d$,

$$\Delta = \Delta_\lambda(r) := \left\{ x \in \mathbb{R}^d \;\mid\; \lambda\left(r^i - 1/2\right) \le x^i < \lambda\left(r^i + 1/2\right) \right\}. \tag{1.5}$$

Let Γ be the phase space of an infinite statistical system of identical point particles. In the case of an equilibrium system, Γ coincides with the space of *configurations* (in our situation, coordinates of the particles) γ which are locally finite subsets of $\mathbb{R}^d$. In other words,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \,|\, |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \right\}, \tag{1.6}$$

where $\mathcal{B}_c(\mathbb{R}^d)$ is a set of all bounded Borel subsets of $\mathbb{R}^d$, and $|X|$ is cardinality of the set $X \Subset \mathbb{R}^d$. Let us also define the subset Γ_0 of all finite configurations,

$$\Gamma_0 = \coprod_{n \in \mathbb{N}_0} \Gamma^{(n)}, \quad \Gamma^{(n)} := \{ \gamma \in \Gamma \mid |\gamma| = n \}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.7}$$

Besides, let

$$\gamma_\Lambda := \gamma \cap \Lambda, \quad \gamma \in \Gamma, \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d). \tag{1.8}$$

(SS) *Superstability. There exist $A > 0$, $B \ge 0$ and a partition $\overline{\Delta_\lambda}$ such that, for any $\gamma = \{x_1, \dots, x_N\} \in \Gamma_0$, the following holds:*

$$U(\gamma) \ge \sum_{\Delta \in \overline{\Delta_\lambda}} \left[A|\gamma_\Delta|^2 - B|\gamma_\Delta| \right]. \tag{1.9}$$

Remark 1.1. A slightly different definition was introduced by Ginibre (see [8]); an interaction is superstable if there exist two real constants $B \ge 0$ and $A_1 \ge 0$ such that for any $\gamma \in \Gamma_0$ the following is true:

$$U(\gamma) \ge A_1 \frac{|\gamma|^2}{\xi^d} - B|\gamma|, \tag{1.10}$$

where $\xi = \max_{\{x,y\} \subset \gamma} |x - y|$. Let us consider a box Λ with a volume $V = \mathrm{vol}(\Lambda)$ such that $\gamma \subset \Lambda$. Then condition (1.10) can be rewritten in the following form:

$$U(\gamma) \ge A_\Lambda \frac{|\gamma|^2}{V} - B|\gamma|, \tag{1.11}$$

where the constant A_Λ does not depend on the volume V for a given shape, but it may be shape dependent. It is easy to notice that if we consider the box Λ as a union of the cubes Δ defined by (1.5) and containing at least one point of the configuration γ, then using the Cauchy-Schwarz inequality one can write the following inequality:

$$|\gamma|^2 = \left(\sum_{\Delta \in \overline{\Delta_\lambda}} |\gamma_\Delta| \right)^2 \le \sum_{\Delta \in \overline{\Delta_\lambda} \cap \gamma} 1 \cdot \sum_{\Delta \in \overline{\Delta_\lambda}} |\gamma_\Delta|^2 = \frac{V}{\lambda^d} \sum_{\Delta \in \overline{\Delta_\lambda}} |\gamma_\Delta|^2.$$

So, condition (1.11) follows directly from (1.9) with $A_\Lambda = A\lambda^d$.

There is a stronger condition on the interaction than (1.9).

(SSS) *Strong superstability. There exist $A > 0$, $B \ge 0$, $p \ge 2$, and the partition $\overline{\Delta_{\lambda_0}}$ such that for any $\gamma = \{x_1, \dots, x_N\} \in \Gamma_0$ the following holds:*

$$U(\gamma) \ge \sum_{\Delta \in \overline{\Delta_\lambda}} \left[A|\gamma_\Delta|^p - B|\gamma_\Delta| \right] \tag{1.12}$$

for any $\lambda \le \lambda_0$.

V. M. Park (see [19]) was the first who used condition (1.12) with $p > 2$ for the proof of bounds on exponentials of local number operators of quantum systems of interacting Bose gas.

In connection with conditions (1.1), (1.9), (1.12) there is a problem of describing the behavior of interaction potentials, which would ensure stability, superstability, or strong superstability of the statistical systems. Putting $N = 2$ in (1.1)–(1.3) we deduce that the function Φ must be bounded from below,

$$\Phi(|x|) \geq -2B. \tag{1.13}$$

In addition to this, R. L. Dobrushin (see [6]) proposed a *necessary* condition of stability of interaction in the form

$$\int_{\mathbb{R}^d} \Phi(|x|)\, dx \geq 0. \tag{1.14}$$

Consequently, a positive part of the interaction must be large enough. As a rule, for neutral physical systems, the potential with the behavior as on the Figure 1 is considered.

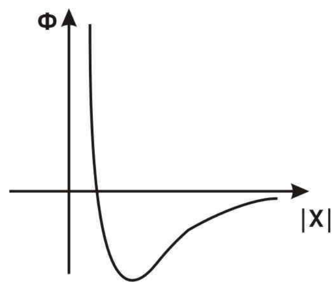

Fig. 1

The behavior of the potentials at infinity ($|x| \to \infty$) is determined by condition (1.4), but the behavior near the initial point depends on the chosen model and, as we will see later, it actually defines the interaction type **(S), (SS), (SSS)**. D. Ruelle was the first who introduced the conditions that ensure estimate (1.11) for systems of particles that are located in the cube Λ with volume V (see [24]). He proposed the potential Φ in the following form:

$$\Phi(|x|) = \Phi_1(|x|) + \Phi_2(|x|), \tag{1.15}$$

where Φ_1 is a Lebesgue measurable function with values in the closed interval $[0;\infty]$ and satisfies the condition (1.4); Φ_2 is a continuous function of positive type and

$$\widetilde{\Phi}_2(0) = \int_{\mathbb{R}^d} \Phi_2(x)\, dx > 0. \tag{1.16}$$

The above mentioned conditions and their direct consequence, the inequality (1.11), were used in works [24] for the proof of existence of a thermodynamic limit ($\Lambda \nearrow \mathbb{R}^d$) for free energy (canonical ensemble) and pressure (grand canonical ensemble). Later M. Fisher noticed (see remarks in [24]) that these results can be proved using less restrictive assumptions on the potential Φ,

$$\Phi(|x|) \geq \frac{c}{|x|^{d+\varepsilon}} \quad \text{for} \quad |x| < a_1, \tag{1.17}$$

$$\Phi(|x|) \geq -w \quad \text{for} \quad a_1 \leq |x| \leq a_2, \tag{1.18}$$

$$\Phi(|x|) \geq -\frac{c'}{|x|^{d+\varepsilon'}} \quad \text{for} \quad |x| > a_2, \tag{1.19}$$

where a_1, a_2, c, c', w, ε, ε' are some positive constants. See also the article [7] for systems of particles with different species and "charged" systems. As the authors have pointed out, conditions (1.17) – (1.19) ensure **(S)** stability of the system, in other words, condition

(1.1) is satisfied. In fact, these conditions guarantee also superstability of the interaction. But at that time such a notion was not yet introduced.

Independently and simultaneously, A. Ya. Povzner (communication at the Moscow State University seminar on Statistical Mechanics (1963)) found conditions on the potential, which would ensure existence of estimate (1.1) (and even (1.9)). One can find his arguments in [28] where they have been refined for the analysis of stability of classical statistical systems with highly singular potentials. Later R. L. Dobrushin has proposed a more general condition on the potential Φ, which also included, in contrast to (1.17), integrable at the origin potentials (see [6], formula (1.17)). Having modified Povzner conditions he proved that stability and existence of limit values of thermodynamic potentials follow from these conditions. In order to complete this short survey we have to mention a criterion of stability, which was proposed by A. G. Basuev [2]. Note that it is rather close to the Povzner's conditions (see also [20]).

In terms of usage of conditions (1.9), (1.11), it is important to obtain the optimal values of the constants A, B. In this area we have to mention the article [17] in which, for continuous $L^1(\mathbb{R}^d)$ potentials of positive type satisfying condition (1.16), the inequality (1.11) was proved with

$$A = \frac{1}{2}\Big(\widetilde{\Phi}(0) - \varepsilon\Big), \quad B = \frac{1}{2}\Phi(0), \quad \text{and} \quad V = V(\varepsilon)$$

for any small $\varepsilon > 0$. The constants A, B are best possible.

The purpose of the present article is not only to make an overview of the previous results, but to obtain some new sufficient conditions on the 2-body interaction potential, which make a system stable, superstable or strong superstable. It is important to notice that the remark about the possible behavior of singular potentials, which ensures the condition (1.12) for $p > 2$, was firstly proposed by D. Ruelle (see [25], Ch. 3, formula (2.28)). It seems to be just an intuitive assumption, which one can guess on the physical level of rigor, if we accept the following hypothesis: the configuration that minimizes energy of N particles that are located in a cube of volume V is uniformly distributed. This means that all particles are located in the sites of a lattice at the distances $\sim \left(\frac{V}{N}\right)^{\frac{1}{d}}$. Implicitly such estimate of the energy was calculated also by Dobrushin (see [5], formulas (4.1), (3.2)). Therefore, the present work can be considered as a new proof of Ruelle's conjecture [25]. We used rigorous results that have been obtained during last several years (see [10], [9], [11], [3]) and some facts of the classical potential theory (see, for example, [13]). Besides, exact values of the constants A and B in the Eqs. (1.9), (1.12) are established.

2. Notations and main results

Following [13] let us propose several new notations, some of them will be introduced in accordance with Section 1 of the present article. Let K be a compact set in $\mathbb{R}^d$. For any configuration $\gamma_K(|\gamma_K| = N)$ in K we define the Riesz s-energy

$$E_s^{(N)}(\gamma_K) := \sum_{\{x,y\}\subset\gamma_K} \frac{1}{|x-y|^s}, \quad s > 0. \tag{2.1}$$

In the case $s < d$ consider *the energy integral*

$$I_s(\mu; K) := \frac{1}{2}\int\int_{K\times K} \frac{1}{|x-y|^s}\,\mu(dx)\mu(dy) \tag{2.2}$$

w. r. t. some probability measure μ, support of which is K ($\mu(K) = 1$).

One of the most important problems in modern potential theory is to find a measure μ^* that minimizes the integral (2.2).

There is a fact (see [13], Ch. 2) that if the configuration $\gamma_K^{min} = \{\xi_1, \dots, \xi_N\}$ minimizes the energy (2.2), then the sequence of measures

$$\mu_N(\cdot) := \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_i}(\cdot), \tag{2.3}$$

where δ_{ξ_i} is a point Dirac measure, converges in the weak-star topology to the measure μ^* (minimizing measure of the integral (2.2)).

The sequence $e_{s,K}^{(N)} = \frac{E_s^{(N)}(\gamma_K^{min})}{N^2}$ is monotonically increasing and

$$\lim_{N\to\infty} e_{s,K}^{(N)} = \lim_{N\to\infty} \frac{E_s^{(N)}(\gamma_K^{min})}{N^2} = I_s(\mu^*) < \infty. \tag{2.4}$$

There are two different behaviors of the minimizing configurations in the limit $N \to \infty$: 1) if $s \le d-2$, then $\operatorname{supp}\mu_N \subset \partial K$, $\operatorname{supp}\mu^* \subset \partial K$, where ∂K is the border of the compact set K; 2) for $d-2 < s < d$ $\operatorname{supp}\mu^* \subset K$.

Let $K = \mathcal{B}^d(0;r)$ be a d-dimensional ball with radius r and $\partial K = S^d(0;r)$ be a surface of the corresponding sphere. Consider the following cases:

1) for $s \le d-2$, the minimizing measure is distributed uniformly on the surface of the ball $\mathcal{B}^d(0;r)$ and

$$\mu^*(dx; \mathcal{B}^d(0;r)) = \frac{m(dx)|_{S^d(0;r)}}{m(S^d(0;r))}, \quad m(S^d(0;r)) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} r^{d-1}; \tag{2.5}$$

2) for $d-2 < s < d$,

$$\mu^*(dx; \mathcal{B}^d(0;r) = \frac{A(d;s)}{(r^2-x^2)^{\frac{d-s}{2}}} m(dx), \quad A(d;s) = \frac{\Gamma\left(1+\frac{s}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(1-\frac{d-s}{2}\right)}, \tag{2.6}$$

where $m(\cdot)$ is the Lebesgue measure on $\mathbb{R}^d$.

The corresponding values of the energy integral (2.2) are the following:

1) for $s \le d-2$,

$$I_s(\mu^*; \mathcal{B}^d(0;r)) = \frac{1}{r^s} \frac{2^{d-s-3}\,\Gamma\left(\frac{d-s-1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(d-1-\frac{s}{2}\right)}, \tag{2.7}$$

2) if $d-2 < s < d$,

$$I_s(\mu^*; \mathcal{B}^d(0;r)) = \frac{1}{r^s} \frac{\Gamma\left(1+\frac{s}{2}\right)\,\Gamma\left(\frac{d-s}{2}\right)}{2\,\Gamma\left(1+\frac{d}{2}\right)}. \tag{2.8}$$

For details, see [13].

The cases $s = d$ and $s > d$ are essentially different from the case $s < d$, which is considered in the classical potential theory. The construction of the minimizing measure and the estimates for the minimal energy of the configuration if $s \ge d$ are proposed in [9]–[11] (see, also, [3]). Let us formulate the most important points.

1) The energy integral $I_s(\mu) = +\infty$ for all probability measures on the compact set $K \subset \mathbb{R}^d$.

2) For any arbitrary compact set $K \subset \mathbb{R}^d$, the following is true:

$$\mu_N(\cdot) \to \frac{m(\cdot)|_K}{m(K)} \tag{2.9}$$

or, in other words, the point particles are asymptotically uniformly distributed.

3) If $s = d$ the following holds:

$$C_d = \lim_{N\to\infty} \frac{E_s^N\left(\gamma_K^{min}\right)}{N^2 \ln N} = \frac{\varphi_0}{\lambda^s} \frac{\pi^{\frac{d}{2}}}{d \cdot \Gamma\left(\frac{d}{2}\right)}. \tag{2.10}$$

4) If $s > d$, then

$$\lim_{N\to\infty} \frac{E_s^N\left(\gamma_K^{min}\right)}{N^{1+\frac{s}{d}}} = \frac{\varphi_0}{\lambda^s}\frac{C_{s,d}}{2}. \tag{2.11}$$

In the case $d = 1$ and $K = [0,1]$, we have $C_{s,1} = 2\xi(s)$, where $\xi(s)$ is the classical Riemann zeta-function.

5) Let K be a d-dimensional cube with edge λ. Then if $s > d$, the following holds:

$$E_s^N\left(\gamma_K\right) \geq \frac{\varphi_0}{\lambda^s}\frac{1}{2^{2s+1}}\left(\frac{2\pi^{\frac{d}{2}}}{d\cdot\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{s}{d}} N^{1+\frac{s}{d}}. \tag{2.12}$$

(A): Assumption on the interaction potential. *In this article we consider a general type of potentials Φ that are continuous on $\mathbb{R}_+ \setminus \{0\}$, and for which there exists $\lambda > 0$, $R > \lambda$, $\varphi_0 > 0$, $\varphi_1 > 0$, and $\epsilon > 0$ such that*

$$1)\ \Phi(|x|) \equiv \Phi^-(|x|) \geq -\frac{\varphi_1}{|x|^{d+\epsilon}} \text{ for } |x| \geq R,, \tag{2.13}$$

$$2)\ \Phi(|x|) \equiv \Phi^+(|x|) \geq \frac{\varphi_0}{|x|^s},\ s \geq 0 \text{ for } |x| \leq \lambda, \tag{2.14}$$

where

$$\Phi^+(|x|) := \max\{0, \Phi(|x|)\}, \quad \Phi^-(|x|) := \min\{0, \Phi(|x|)\}. \tag{2.15}$$

In contrast to [9], [13], we also consider the case $s = 0$, which looks probably trivial from the point of view of potential theory, but it will take place also in our description (see Remark 2.1 below).

Now we can formulate the following theorems.

Theorem 2.1. *Let the interaction potential satisfy the conditions* **(A)**. *Then for $0 \leq s < d$ any $\gamma \in \Gamma_0$ and sufficiently small $\varepsilon > 0$ there exists a constant $B = B(\varepsilon)$ such that the following inequality holds:*

$$U(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 2}} \left(I_s(\mu^*;\Delta)\varphi_0 - \frac{v_0}{2} - \varepsilon\right)|\gamma_\Delta|^2 - B|\gamma|, \tag{2.16}$$

where

$$v_0 = v_0(\lambda) := \sup_{x\in\mathbb{R}^d} \sum_{\Delta\in\overline{\Delta_\lambda}} \sup_{y\in\Delta} \left|\Phi^-(|x-y|)\right|. \tag{2.17}$$

Corollary 2.1. *In the case $0 \leq s < d$, the potential Φ yields the condition* **(SS)** *(see* (1.9)*) if the following holds:*

$$I_s(\mu^*;\Delta)\varphi_0 > \frac{v_0}{2}. \tag{2.18}$$

Remark 2.1. The condition (2.18) can be rewritten in a simpler form if we consider that the minimal Riesz energy of the configuration with a fixed number of particles $|\gamma_\Delta|$ in the cube $\Delta \in \overline{\Delta_\lambda}$ is always bigger than the minimal Riesz energy of the configuration with the same number of particles in the described ball with radius $r = \frac{\sqrt{d}\lambda}{2}$. Consequently, one can substitute the formulas (2.7), (2.8) with $r = \frac{\sqrt{d}\lambda}{2}$ for $I_s(\mu^*;\Delta)$ in the l.h.s of (2.18) (for the cases $s \leq d-2, d-2 < s < d$, respectively). The r.h.s of (2.18) can be changed by $\frac{C}{\lambda^d}$, where the constant satisfies

$$C \approx \int_{\mathbb{R}^d} \left|\Phi^-(|x|)\right|\, dx$$

for sufficiently small λ. Then for the given configuration in the d-dimensional space and for the potential that satisfies condition (2.14), the system is superstable if there exists λ (in other words a partition $\overline{\Delta_\lambda}$ of the space $\mathbb{R}^d$) such that condition (2.18) is satisfied. The set of potentials that satisfy condition **(SS)** is not empty since one can choose φ_0 sufficiently large as to make (2.18) hold for any fixed $\lambda > 0$. For the case $s = 0$, $I_0(\mu^*; \Delta) = 1/2$.

Theorem 2.2. *Let the interaction potential satisfy conditions* **(A)***. Then, for $s = d$, any $\gamma \in \Gamma_0$ and sufficiently small $\varepsilon > 0$ there exists a constant $B = B(\varepsilon)$ such that the following inequality holds:*

$$U(\gamma) \geq \sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| \geq 2}} \Big(C_d \ln |\gamma_\Delta| - \frac{v_0}{2} - \varepsilon \ln |\gamma_\Delta| \Big) |\gamma_\Delta|^2 - B|\gamma|, \tag{2.19}$$

where (see [9]*)*

$$C_d = \frac{1}{\lambda^d} \frac{\pi^{\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)} \varphi_0. \tag{2.20}$$

Theorem 2.3. *Let the interaction potential satisfy conditions* **(A)***. Then, for $s > d$ and any $\gamma \in \Gamma_0$ there exists a constant $B = B(\varepsilon)$ such that the following inequality holds:*

$$U(\gamma) \geq \sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| \geq 2}} \Big(C_{s,d} \, |\gamma_\Delta|^{1+\frac{s}{d}} - \frac{v_0}{2} |\gamma_\Delta|^2 \Big) - B|\gamma|, \tag{2.21}$$

where (*see* [9])

$$C_{s,d} = \frac{1}{\lambda^s} \frac{1}{2^{2s+1}} \left(\frac{2\pi^{\frac{d}{2}}}{d\,\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{s}{d}} \varphi_0. \tag{2.22}$$

Remark 2.2. In the case where $s = d$, the system of particles is superstable **(SS)** for all partitions $\overline{\Delta_\lambda}$, since for any $\varepsilon > 0$ and v_0 one can find $N_0 \geq 2$ and $B = B(N_0)$ such that for $N > N_0$

$$C_d \ln N > \frac{v_0}{2}. \tag{2.23}$$

In the case $s > d$, system of particles is strong superstable **(SSS)**, since one can always choose sufficiently small $\lambda > 0$ and some $A = A(\lambda)$ such that

$$C_{s,d} |\gamma_\Delta|^{1+\frac{s}{d}} - \frac{v_0}{2} |\gamma_\Delta|^2 \geq A |\gamma_\Delta|^{1+\frac{s}{d}} \tag{2.24}$$

for $|\gamma_\Delta| \geq 2$.

3. Proofs of the results

3.1. **Proof of Theorem 2.1.** We have for any $\gamma \in \Gamma_0$ and any partition $\overline{\Delta_\lambda}$ that

$$U(\gamma) = \sum_{\{x,y\} \subset \gamma} \Phi(|x-y|) = \sum_{\Delta \in \overline{\Delta_\lambda} : |\gamma_\Delta| \geq 2} U(\gamma_\Delta) + \sum_{\{\Delta, \Delta'\} \subset \overline{\Delta_\lambda}} \sum_{\substack{x \in \gamma_\Delta \\ y \in \gamma_{\Delta'}}} \Phi(|x-y|). \tag{3.1}$$

Taking into account the assumptions **(A)** on the interaction potential, definitions (2.1), (2.17), and the inequality $|\gamma_\Delta| \, |\gamma_{\Delta'}| \leq \frac{1}{2} \left(|\gamma_\Delta|^2 + |\gamma_{\Delta'}|^2 \right)$ we obtain from (3.1) that

$$U(\gamma) \geq \sum_{\Delta \in \overline{\Delta_\lambda} : |\gamma_\Delta| \geq 2} \left[E_s^{(N_\Delta(\gamma))} (\gamma_\Delta^{\min}) \varphi_0 - \frac{v_0}{2} |\gamma_\Delta|^2 \right] - \frac{v_0}{2} |\gamma|, \; N_\Delta(\gamma) = |\gamma_\Delta|. \tag{3.2}$$

For fixed $\varepsilon > 0$ let us define N_0 such that $I_s(\mu^*;\Delta) - e_{s,\Delta}^{(N)} > \varepsilon$ if $N < N_0$ and $I_s(\mu^*;\Delta) - e_{s,\Delta}^{(N)} < \varepsilon$ if $N \geq N_0$ (see (2.4)). Let's also define a sequence

$$B_N = \begin{cases} \left(e_{s,\Delta}^{(N_0)} - e_{s,\Delta}^{(N)}\right) \cdot N_0, & N \leq N_0, \\ 0, & N > N_0. \end{cases} \tag{3.3}$$

For $N \leq N_0$ we have $e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)} \leq 0$ and $N^2 \leq N\, N_0$. As a result we have the following:

1) if $N \leq N_0$, then

$$\left(e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)}\right) N^2 \geq \left(e_{s,\Delta}^{(N)} - e_{s,\Delta}^{(N_0)}\right) N_0\, N = -B_N\, N;$$

2) if $N > N_0$, then

$$e_{s,\Delta}^{(N)}\, N^2 \geq e_{s,\Delta}^{(N_0)}\, N^2.$$

Then, for any $N \geq 2$,

$$e_{s,\Delta}^{(N)} \cdot N^2 \geq e_{s,\Delta}^{(N_0)} \cdot N^2 - B_N \cdot N, \tag{3.4}$$

Because of $B_2 > B_N$, for any $N \geq 2$, we deduce from (3.4) that for all $N \geq 2$

$$\begin{aligned} e_{s,\Delta}^{(N)}\, N^2 &\geq e_{s,\Delta}^{(N_0)}\, N^2 - B_2 N \\ &= I_s(\mu^*,\Delta) N^2 + \left(e_{s,\Delta}^{(N_0)} - I_s(\mu^*,\Delta)\right) \cdot N^2 - B_2 N \\ &\geq (I_s(\mu^*,\Delta) - \varepsilon) \cdot N^2 - B_2 N. \end{aligned} \tag{3.5}$$

The inequality (3.5) proves Theorem (2.1) for the partition $\overline{\Delta_\lambda}$ such that for a given $\gamma \in \Gamma_0$ there exists at least one cube with $|\gamma_\Delta| \geq 2$. In this case,

$$B = B_2(\varepsilon) = \left(e_{s,\Delta}^{(N_0)} - e_{s,\Delta}^{(2)}\right) \cdot N_0, \quad N_0 = N_0(\varepsilon). \tag{3.6}$$

For $\gamma \in \Gamma_0$ with $|\gamma_\Delta| = 1$ or 0 it is clear that $B_2 = v_0/2$. So, one can choose

$$B = \max\left\{\left(e_{s,\Delta}^{(N_0)} - e_{s,\Delta}^{(2)}\right) \cdot N_0;\ \frac{v_0}{2}\right\}. \tag{3.7}$$

This ends the proof. □

3.2. **Proofs of Theorem 2.2 and Theorem 2.3.** In our case, K is a d-dimensional cube Δ with edge λ. As in the previous case we start from (3.1), (3.2). For fixed $\varepsilon > 0$ let us define N_0 such that

$$\left| C_d - \frac{E_s^N\left(\gamma_K^{min}\right)}{N^2 \ln N} \right| > \varepsilon$$

if $N < N_0$ and

$$\left| C_d - \frac{E_s^N\left(\gamma_K^{min}\right)}{N^2 \ln N} \right| < \varepsilon$$

if $N \geq N_0$ (the constant C_d is taken from (2.10)). Using (3.1), (3.2), (2.10) and neglecting in (3.1) part of the interaction energy

$$\sum_{\substack{\Delta \in \overline{\Delta_\lambda}, \\ |\gamma_\Delta| < N_0}} U(\gamma_\Delta)$$

one can write an estimate for the total energy of the system in the following form:

$$U(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 2}} \left[C_d \ln|\gamma_\Delta| - \frac{v_0}{2} - \varepsilon \ln|\gamma_\Delta|\right]|\gamma_\Delta|^2 - \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|=1}} \frac{v_0}{2}|\gamma_\Delta|^2 - \sum_{i=2}^{N_0-1}\sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|=i}} [C_d - \varepsilon]\,|\gamma_\Delta|^2 \ln|\gamma_\Delta|. \tag{3.8}$$

The number of cubes with $|\gamma_\Delta| = i$ is not more than $\frac{|\gamma|}{i}$. That is why

$$\sum_{i=2}^{N_0-1}\sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|=i}} [C_d - \varepsilon]\,|\gamma_\Delta|^2 \ln|\gamma_\Delta| \leq |\gamma| \sum_{i=2}^{N_0-1} \frac{[C_d - \varepsilon]\, i^2 \ln i}{i}. \tag{3.9}$$

As a result we can put

$$B = \frac{v_0}{2} + \sum_{i=2}^{N_0-1} [C_d - \varepsilon]\, i \ln i. \tag{3.10}$$

This ends the proof. □

Remark 3.1. The proof of the Theorem 2.3 is very similar to the previous proof of the Theorem 2.2. In this case, according to (2.14), the minimal energy of $N_\Delta(\gamma)$ particles which are situated in the d-dimensional cube $\Delta \in \overline{\Delta}_\lambda$ can be estimated from below by the inequality (2.12) (see [10] and [3]). Substituting this inequality into (3.2) we obtain Eq. (2.21) with $B = \frac{v_0}{2}$.

Acknowledgments. We are grateful to Prof. Dr. V. A. Zagrebnov and Dr. D. L. Finkelshtein for reading the manuscript and valuable remarks.

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Received 14/01/2008; Revised 05/06/2008