Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 4, pp. 297–301

ON SOME SUBLATTICES OF REGULAR OPERATORS ON BANACH LATTICES

BELMESNAOUI AQZZOUZ AND REDOUANE NOUIRA

Abstract. We give some sufficient conditions under which the linear span of positive compact (resp. Dunford-Pettis, weakly compact, AM-compact) operators cannot be a vector lattice without being a sublattice of the order complete vector lattice of all regular operators. Also, some interesting consequences are obtained.

1. Introduction and notation

The paper concerns the following natural and interesting problem: find conditions implying that a subspace F of a vector lattice E is a vector lattice, with respect to the order induced by E, but F is not a sublattice of E. It is easy to see that such subspaces F exist (consider the subspace $F \subset C([0,1]) = E$ consisting of linear functions where $C([0,1])$ is the vector lattice of continuous functions on $[0,1]$).

Recall that in [1], Abramovich and Wickstead constructed two compact operators S and T from a Banach lattice E into an order complete Banach lattice F such that $\pm\ S < T$ but the modulus $|S|$ is not compact. This means that the linear span of positive compact operators $\mathcal{K}^r(E,F)$ is not a sublattice of the order complete vector lattice of all regular operators $\mathcal{L}^r(E,F)$ i.e. the space of operators $T : E \longrightarrow F$ such that $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Also, in the same paper, Abramovich and Wickstead proved that $\mathcal{K}^r(E,F)$ is not a vector lattice and they asked in ([1], p. 325) the following question:

Is it possible to construct Banach lattices E and F such that F is order complete and $\mathcal{K}^r(E,F)$ is a vector lattice without being a sublattice of $\mathcal{L}^r(E,F)$?

First, we observe that the above question has a negative answer whenever E and F satisfy the necessary and sufficient conditions of Theorem 1 of Wickstead [11] i.e. if the topological dual E' is discrete and its norm is order continuous, or the Banach lattice F is discrete and its norm is order continuous, or the norms of E and of its topological dual E' are order continuous.

Second, we remark that the question of Abramovich and Wickstead [1] can be asked also for the class of Dunford-Pettis (resp. weakly compact, AM-compact) operators between Banach lattices.

Our objective in this paper is to formulate several sufficient conditions under which classes of regular and compact (resp. weakly compact, Dunford-Pettis, AM-compact) operators cannot be a vector lattice without being a sublattice in the space of all regular operators acting between suitable Banach lattices. More precisely, we will prove that if E and F are Banach lattices, then the linear span of positive compact (resp. Dunford-Pettis, weakly compact, AM-compact) operators between E and F cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E,F)$ if the Banach lattice E is discrete and its norm is order continuous, or the vector lattice F is discrete, or the topological dual E' is discrete

2000 *Mathematics Subject Classification.* 46A40, 46B40, 46B42.
Key words and phrases. Regular operator, order continuous norm, discrete Banach lattice.

and F is reflexive. As consequences, we show that the spaces $\mathcal{K}^r(l^1(l_2^{2^n}), L^2([0,1]))$ and $\mathcal{W}^r(l^1(l_2^{2^n}), l^\infty(L^2([0,1])))$ are not vector lattices. Also, we will establish that the spaces $\mathcal{D}^r(L^2([0,1]), l^\infty(L^2([0,1])))$ and $\mathcal{AM}^r(L^2([0,1]), l^\infty(L^2([0,1])))$ are not vector lattices.

To state our results we need to fix some notations and recall some definitions. Let E be a vector lattice, then for any two elements $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is an order interval. A subset of E is said to be order bounded if it is included in some order interval. An order ideal B is a solid subspace of a vector lattice E i.e. if $x \in B$ and $y \in E$ such that $|y| \leq |x|$, then $y \in B$. A principal ideal is any order ideal generated by a subset containing only one element x, this ideal will be denoted by I_x. A generalized sequence (x_α) is order convergent to $x \in E$ if there exists a generalized sequence (y_α) such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each α, where the notation $y_\alpha \downarrow 0$ means that the sequence (y_α) is decreasing, its infimum exists and $\inf(y_\alpha) = 0$. A band is an order ideal which is order closed. The band generated by an element x is called a principal band that we design by B_x.

A Banach lattice $(E, \|.\|)$ is a Banach space such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. Finally, we note that the topological dual E' of a Banach lattice E, endowing with the dual norm, is a Banach lattice. For terminology which is not explained, we refer the reader to the book of Zaanen [12].

2. Main results

Let us recall that an operator $T : E \longrightarrow F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. An operator $T : E \longrightarrow F$ is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous.

A norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$. For example, the norm of the Banach lattice l^1 is order continuous but the norm of the Banach lattice l^∞ is not.

A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the sublattice generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. For example, the Banach lattice l^1 is discrete but $C([0,1])$ is not.

Recall that Krengel [9] is the first who constructed an operator $T : l^2 \longrightarrow l^2$ (resp. $S : l^2 \longrightarrow l^2$) such that T is compact but $T \notin \mathcal{L}^r(l^2, l^2)$ (resp. S is regular and compact but $|S|$ is not compact).

A regular operator $T : E \longrightarrow F$ between two Banach lattices is said to be AM-compact if it carries order bounded subsets of E onto relatively compact subsets of F. Also, an operator T from a Banach space E into another F is said to be Dunford-Pettis if it carries weakly compact subsets of E onto compact subsets of F.

If $\mathcal{K}(E,F)$ (resp. $\mathcal{W}(E,F)$, $\mathcal{D}(E,F)$, $\mathcal{AM}(E,F)$) designs the subspace of all compact (resp. weakly compact, Dunford-Pettis, AM-compact) operators from E into F, we denote by $\mathcal{K}^r(E,F)$ (resp. $\mathcal{W}^r(E,F)$, $\mathcal{D}^r(E,F)$, $\mathcal{AM}^r(E,F)$) the linear span of positive elements of $\mathcal{K}(E,F)$ (resp. $\mathcal{W}(E,F)$, $\mathcal{D}(E,F)$, $\mathcal{AM}(E,F)$).

If $\mathcal{B}(E,F)$ is any one of the subspaces $\mathcal{K}^r(E,F)$, $\mathcal{W}^r(E,F)$, $\mathcal{D}^r(E,F)$ or $\mathcal{AM}^r(E,F)$), our principal result is the following:

Theorem 2.1. *Let E and F be two Banach lattices. Then $\mathcal{B}(E,F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E,F)$ if one of the following conditions holds:*

i) *The Banach lattice E is discrete and its norm is order continuous.*

ii) *The vector lattice F is discrete.*
iii) *The topological dual E' is discrete and F is reflexive.*

Proof. Let $S \in \mathcal{B}(E,F)$ and denote by $|S|$ and T its modulus in $\mathcal{L}^r(E,F)$ and $\mathcal{B}(E,F)$ respectively. It is clear that $\pm\, S \leq |S| \leq T$. Assume that $|S| \neq T$.

i. There exists some discrete element $x_0 \in E^+ = \{x \in E : 0 \leq x\}$ such that $|S|(x_0) < T(x_0)$. Let R be the operator defined from E into F by the following formula:

$$R(x) = T(x) - (T - |S|) \circ P_{x_0}(x),$$

where P_{x_0} is the principal projection on the band generated by x_0. Then $|S| \leq R < T$ and $R \in \mathcal{B}(E,F)$. This gives a contradiction.

ii. There exists a discrete element y_0 of F and there exists an element $x_0 \in E^+$ such that

$$Q_{y_0}\left(|S|(x_0)\right) < Q_{y_0}\left(T(x_0)\right),$$

where Q_{y_0} is the principal projection on the band generated by y_0. We consider the operator R defined from E into F by

$$R(x) = T(x) - Q_{y_0} \circ (T - |S|)(x).$$

For the same precedent reason, we obtain a contradiction.

iii. Since $|S| \leq T$, it follows that $|S|' \leq T'$ where $|S|'$ and T' are the adjoint operator of $|S|$ and T respectively. This implies the existence of some f_0 in F such that

$$|S|'(g_0) < T'(g_0).$$

In the same way as **ii**, there exists a discrete element g_0 in E' such that

$$P_{g_0} \circ |S|'(g_0) < P_{g_0} \circ T'(g_0),$$

where P_{g_0} is the principal projection on the band generated by g_0. Now, we consider the operator R defined from F' into E' by

$$R = T' - (P_{g_0} \circ T' - P_{g_0} \circ |S|').$$

We have $|S|' \leq R < T'$. In fact, for the first inequality, it is sufficient to composite with the projections on bands generated by discrete elements of E.

In other hand, the operator $(P_{g_0} \circ T' - P_{g_0} \circ |S|')$ is of rank one, and hence there exists some $z \in F'' = F$ such that

$$(P_{g_0} \circ T' - P_{g_0} \circ |S|')(f) = z(f)g_0 = f(z)g_0$$

for each $f \in F'$, where F'' is the topological bidual of F.

It is easy to prove that $(P_{g_0} \circ T' - P_{g_0} \circ |S|')$ is the operator dual of the operator $K : E \longrightarrow F$ defined by $K(x) = g_0(x)z$. Finally, $|S|' \leq (T-K)' < T'$ or again $|S| \leq (T-K) < T$. This is in contradiction with the fact that $T - K \in \mathcal{B}(E,F)$. □

An immediate consequence of Theorem 2.1 (i) or (iii), we obtain the following result of Abramovich and Wickstead ([1], Corollary 3):

Corollary 2.2. *The space $\mathcal{K}^r(l^1(l_2^{2^n}), L^2([0,1]))$ is not a vector lattice.*

Our second consequence, follows from a combination of Theorem 2.1 (i) and Theorem 2.7 of [7].

Corollary 2.3. *The space $\mathcal{W}^r(l^1(l_2^{2^n}), l^\infty(L^2([0,1])))$ is not a vector lattice.*

For Dunford-Pettis and AM-compact operators, we obtain the following results:

Theorem 2.4. *The spaces $\mathcal{D}^r(L^2([0,1]), l^\infty(L^2([0,1])))$ and $\mathcal{AM}^r(L^2([0,1]), l^\infty(L^2([0,1])))$ are not vector lattices.*

Proof. The proof follows along the lines of the proof of Theorem 2.1 (ii). In fact, for each $n \in \mathbb{N}^*$, let Q_n be the projection operator from $l^\infty(L^2([0,1]))$ onto $L^2([0,1])$ defined by the following formula:

$$Q_n((f_k)_{k\in\mathbb{N}^*}) = f_n \quad \text{for each} \quad (f_k)_{k\in\mathbb{N}^*} \in l^\infty(L^2([0,1]))$$

and let i_n be the operator defined from $L^2([0,1])$ into $l^\infty(L^2([0,1]))$ by

$$i_n(f) = (0,0,0,\ldots,0,f,0,0,\ldots).$$

Now, assume that there exists an element $S \in \mathcal{F}(L^2([0,1]), l^\infty(L^2([0,1])))$ such that its modulus T in $\mathcal{F}(L^2([0,1]), l^\infty(L^2([0,1])))$ exists and is different of its modulus $|S|$ in $\mathcal{L}^r(L^2([0,1]), l^\infty(L^2([0,1])))$ where

$$\mathcal{F}(L^2([0,1]), l^\infty(L^2([0,1]))) = \mathcal{D}^r(L^2([0,1]), l^\infty(L^2([0,1])))$$

(resp. $\mathcal{AM}^r(L^2([0,1]), l^\infty(L^2([0,1])))$).

Then there exists an element $x_0 \in \left(L^2([0,1])\right)^+$ and there exists some $n \in \mathbb{N}^*$ such that

$$Q_n \circ |S|(x_0) < Q_n \circ T(x_0).$$

Consider the operator R defined from $L^2([0,1])$ into $l^\infty(L^2([0,1]))$ by

$$R = T - i_n \circ Q_n \circ (T - |S|).$$

We have

$$0 < Q_n \circ (T - |S|) < Q_n \circ T$$

as operators from $L^2([0,1])$ into $L^2([0,1])$. By applying Theorem 4.4 of Kalton-Saab [8] (resp. Theorem 2.1 of [2]) related to the domination problem for Dunford-Pettis (resp. AM-compact) operators, we conclude that $Q_n \circ (T - |S|)$ is Dunford-Pettis (resp. AM-compact). Hence, the operator R is Dunford-Pettis (resp. AM-compact) too. But $|S| \le R < T$, this presents a contradiction.

On the other hand, it follows from ([5], Theorem 2.1) that the subspaces $\mathcal{D}^r(L^2([0,1]), l^\infty(L^2([0,1])))$ and $\mathcal{AM}^r(L^2([0,1]), l^\infty(L^2([0,1])))$ are not sublattices of $\mathcal{L}^r(L^2([0,1]), l^\infty(L^2([0,1])))$. This completes the proof. □

As consequence for the linear span of positive compact operators, it follows from Theorem 1 of [11] and Theorem 2.1:

Corollary 2.5. *Let E and F be Banach lattices. Then $\mathcal{K}^r(E,F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E,F)$, if one of the following conditions holds:*
1) The Banach lattice E is discrete and its norm is order continuous.
2) The vector lattice F is discrete.
3) The topological dual E' is discrete and F is reflexive.
4) the Banach lattice E' is discrete and its norm is order continuous
5) the norms of E' and F are order continuous.

Recall that a Banach lattice E is reflexive, if and only if the norms of its topological dual E' and of its topological bidual E'' are order continuous ([10], Theorem 5.16).

The following result for the linear span of positive weakly compact operators is a consequence of Theorem 7 of [3], Theorem 5.16 of [10] and Theorem 2.1:

Corollary 2.6. *Let E and F be Banach lattices. Then $\mathcal{W}^r(E,F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E,F)$, if one of the following conditions holds:*
1) The Banach lattice E is discrete and its norm is order continuous.
2) The vector lattice F is discrete.
3) the norm of E' is order continuous
4) the norm of F is order continuous.

To give the following consequence, recall that the lattice operations in a Banach lattice E are weakly sequentially continuous if the sequence $(|x_n|)$ converges to 0 for the weak topology $\sigma(E,E')$ whenever the sequence (x_n) converges to 0 for $\sigma(E,E')$. For example, the lattice operations of a AM-space are weakly sequentially continuous but the lattice operations of the Banach lattice L^2 are not.

Note that in ([6], Corollary 2.2), we have proved that if E is a Banach lattice such that its topological dual E' is discrete, then the lattice operations of E are weakly sequentially continuous.

The following result for the linear span of positive Dunford-Pettis operators is a consequence of Theorem 2 of [11], Corollary 2.2 of [6] and Theorem 2.1:

Corollary 2.7. *Let E and F be Banach lattices. Then $\mathcal{D}^r(E,F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E,F)$, if one of the following conditions holds:*

1) The vector lattice F is discrete.

2) the lattice operations in E are weakly sequentially continuous.

3) the norm of F is order continuous.

Finally, we have the following result for the linear span of positive AM-compact operators is a consequence of Corollary 2.14 and Theorem 2.15 of [4], Theorem 1.2 of [2] and Theorem 2.1:

Corollary 2.8. *Let E and F be Banach lattices. Then $\mathcal{AM}^r(E,F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E,F)$, if one of the following conditions holds:*

1) The Banach lattice E is discrete and its norm is order continuous.

2) The vector lattice F is discrete.

3) The topological dual E' is discrete.

4) the norm of F is order continuous.

References

1. Y. A. Abramovich and A. W. Wickstead, *Solutions of several problems in the theory of compact positive operators*, Proc. Amer. Math. Soc. **123** (1995), no. 10, 3021–3026.
2. C. D. Aliprantis and O. Burkinshaw, *Positive compact operators on Banach lattices*, Math. Z. **174** (1980), 289–298.
3. C. D. Aliprantis and O. Burkinshaw, *On weakly compact operators on Banach lattices*, Proc. Amer. Math. Soc. **83** (1981), no. 3, 573–578.
4. B. Aqzzouz and R. Nouira, *Sur l'idéal d'ordre des opérateurs AM-compact*, Proc. Amer. Math. Soc. **134** (2006), 3515–3523.
5. B. Aqzzouz, R. Nouira, *Order σ-completeness of the linear span of positive AM-compact operators* (submitted).
6. B. Aqzzouz, R. Nouira, and L. Zraoula, *Compactness of AM-compact operators on Banach lattices* (to appear in Annales des Sciences Mathématiques du Québec).
7. Z. L. Chen and A. W. Wickstead, *Vector lattices of weakly compact operators on Banach lattices*, Trans. Amer. Math. Soc. **352** (1999), no. 1, 397–412.
8. N. J. Kalton and P. Saab, *Ideal properties of regular operators between Banach lattices*, Illinois Journal of Math. **29** (1985), no. 3, 382–400.
9. U. Krengel, *Remark on the modulus of compact operators*, Bull. Amer. Math. Soc. **72** (1966), 132–133.
10. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin—New York, 1974.
11. A. W. Wickstead, *Converses for the Dodds-Fremlin and Kalton-Saab theorems*, Math. Proc. Camb. Phil. Soc. **120** (1996), no. 1, 175–179.
12. A. C. Zaanen, *Riesz spaces* II, North Holland Publishing Company, 1983.

Université Mohammed V-Souissi, Faculté des Sciences Economiques, Juridiques et Sociales, Département d'Economie, B.P. 5295, Sala Eljadida, Morocco
E-mail address: baqzzouz@hotmail.com

Université Ibn Tofail, Faculté des Sciences, Département de Mathématiques, B.P. 133, Kénitra, Morocco

Received 08/01/2008