ON SOME SUBLATTICES OF REGULAR OPERATORS ON BANACH LATTICES

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ABSTRACT. We give some sufficient conditions under which the linear span of positive compact (resp. Dunford-Pettis, weakly compact, AM-compact) operators cannot be a vector lattice without being a sublattice of the order complete vector lattice of all regular operators. Also, some interesting consequences are obtained.

1. INTRODUCTION AND NOTATION

The paper concerns the following natural and interesting problem: find conditions implying that a subspace $F$ of a vector lattice $E$ is a vector lattice, with respect to the order induced by $E$, but $F$ is not a sublattice of $E$. It is easy to see that such subspaces $F$ exist (consider the subspace $F \subset C([0,1]) = E$ consisting of linear functions where $C([0,1])$ is the vector lattice of continuous functions on $[0,1]$).

Recall that in [1], Abramovich and Wickstead constructed two compact operators $S$ and $T$ from a Banach lattice $E$ into an order complete Banach lattice $F$ such that $\pm S < T$ but the modulus $|S|$ is not compact. This means that the linear span of positive compact operators $K^r(E,F)$ is not a sublattice of the order complete vector lattice of all regular operators $L^r(E,F)$ i.e. the space of operators $T : E \rightarrow F$ such that $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive operators from $E$ into $F$. Also, in the same paper, Abramovich and Wickstead proved that $K^r(E,F)$ is not a vector lattice and they asked in ([1], p. 325) the following question:

Is it possible to construct Banach lattices $E$ and $F$ such that $F$ is order complete and $K^r(E,F)$ is a vector lattice without being a sublattice of $L^r(E,F)$?

First, we observe that the above question has a negative answer whenever $E$ and $F$ satisfy the necessary and sufficient conditions of Theorem 1 of Wickstead [11] i.e. if the topological dual $E'$ is discrete and its norm is order continuous, or the Banach lattice $F$ is discrete and its norm is order continuous, or the norms of $E$ and of its topological dual $E'$ are order continuous.

Second, we remark that the question of Abramovich and Wickstead [1] can be asked also for the class of Dunford-Pettis (resp. weakly compact, AM-compact) operators between Banach lattices.

Our objective in this paper is to formulate several sufficient conditions under which classes of regular and compact (resp. weakly compact, Dunford-Pettis, AM-compact) operators cannot be a vector lattice without being a sublattice in the space of all regular operators acting between suitable Banach lattices. More precisely, we will prove that if $E$ and $F$ are Banach lattices, then the linear span of positive compact (resp. Dunford-Pettis, weakly compact, AM-compact) operators between $E$ and $F$ cannot be a vector lattice without being a sublattice of $L^r(E,F)$ if the Banach lattice $E$ is discrete and its norm is order continuous, or the vector lattice $F$ is discrete, or the topological dual $E'$ is discrete.

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and $F$ is reflexive. As consequences, we show that the spaces $\mathcal{K}^\prime(l^1([0,1]), l^2([0,1]))$ and $\mathcal{W}^\prime(l^1([0,1]), l^\infty([0,1]))$ are not vector lattices. Also, we will establish that the spaces $\mathcal{D}^\prime(L^2([0,1]), l^\infty(L^2([0,1])))$ and $\mathcal{AM}^\prime(L^2([0,1]), l^\infty(L^2([0,1])))$ are not vector lattices.

To state our results we need to fix some notations and recall some definitions. Let $E$ be a vector lattice, then for any two elements $x, y \in E$ with $x \leq y$, the set $[x, y] = \{ z \in E : x \leq z \leq y \}$ is an order interval. A subset of $E$ is said to be order bounded if it is included in some order interval. An order ideal $B$ is a solid subspace of a vector lattice $E$ i.e. if $x \in B$ and $y \in E$ such that $|y| \leq |x|$, then $y \in B$. A principal ideal is any order ideal generated by a subset containing only one element $x$, this ideal will be denoted by $I_x$. A generalized sequence $(x_\alpha)$ is order convergent to $x \in E$ if there exists a generalized sequence $(y_\alpha)$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each $\alpha$, where the notation $y_\alpha \downarrow 0$ means that the sequence $(y_\alpha)$ is decreasing, its infimum exists and $\inf(y_\alpha) = 0$. A band is an order ideal which is order closed. The band generated by an element $x$ is called a principal band that we design by $B_x$.

A Banach lattice $(E, \| \cdot \|)$ is a Banach space such that $E$ is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. Finally, we note that the topological dual $E'$ of a Banach lattice $E$, endowing with the dual norm, is a Banach lattice. For terminology which is not explained, we refer the reader to the book of Zaanen [12].

2. Main results

Let us recall that an operator $T : E \rightarrow F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. An operator $T : E \rightarrow F$ is regular if $T = T_1 - T_2$ where $T_1$ and $T_2$ are positive operators from $E$ into $F$. It is well known that each positive linear mapping on a Banach lattice is continuous.

A norm $\| \cdot \|$ of a Banach lattice $E$ is order continuous if for each generalized sequence $(x_\alpha)$ such that $x_\alpha \downarrow 0$ in $E$, the sequence $(x_\alpha)$ converges to $0$ for the norm $\| \cdot \|$. For example, the norm of the Banach lattice $l^1$ is order continuous but the norm of the Banach lattice $l^\infty$ is not.

A nonzero element $x$ of a vector lattice $E$ is discrete if the order ideal generated by $x$ equals the sublattice generated by $x$. The vector lattice $E$ is discrete, if it admits a complete disjoint system of discrete elements. For example, the Banach lattice $l^1$ is discrete but $c_0([0,1])$ is not.

Recall that Krengel [9] is the first who constructed an operator $T : l^2 \rightarrow l^2$ (resp. $S : l^2 \rightarrow l^2$) such that $T$ is compact but $T \notin \mathcal{L}^\prime(l^2, l^2)$ (resp. $S$ is regular and compact but $|S|$ is not compact).

A regular operator $T : E \rightarrow F$ between two Banach lattices is said to be AM-compact if it carries order bounded subsets of $E$ onto relatively compact subsets of $F$. Also, an operator $T$ from a Banach space $E$ into another $F$ is said to be Dunford-Pettis if it carries weakly compact subsets of $E$ onto compact subsets of $F$.

If $\mathcal{K}(E, F)$ (resp. $\mathcal{W}(E, F)$, $\mathcal{D}(E, F)$, $\mathcal{AM}(E, F)$) designs the subspace of all compact (resp. weakly compact, Dunford-Pettis, AM-compact) operators from $E$ into $F$, we denote by $\mathcal{K}^\prime(E, F)$ (resp. $\mathcal{W}^\prime(E, F)$, $\mathcal{D}^\prime(E, F)$, $\mathcal{AM}^\prime(E, F)$) the linear span of positive elements of $\mathcal{K}(E, F)$ (resp. $\mathcal{W}(E, F)$, $\mathcal{D}(E, F)$, $\mathcal{AM}(E, F)$).

If $\mathcal{B}(E, F)$ is any one of the subspaces $\mathcal{K}^\prime(E, F)$, $\mathcal{W}^\prime(E, F)$, $\mathcal{D}^\prime(E, F)$ or $\mathcal{AM}^\prime(E, F)$, our principal result is the following:

Theorem 2.1. Let $E$ and $F$ be two Banach lattices. Then $\mathcal{B}(E, F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^\prime(E, F)$ if one of the following conditions holds:

i) The Banach lattice $E$ is discrete and its norm is order continuous.
ii) The vector lattice $F$ is discrete.

iii) The topological dual $E'$ is discrete and $F$ is reflexive.

Proof. Let $S \in \mathcal{B}(E,F)$ and denote by $|S|$ and $T$ its modulus in $\mathcal{L}'(E,F)$ and $\mathcal{B}(E,F)$ respectively. It is clear that $\pm S \leq |S| \leq T$. Assume that $|S| \neq T$.

i. There exists some discrete element $x_0 \in E^+ = \{x \in E : 0 \leq x\}$ such that $|S|(x_0) < T(x_0)$. Let $R$ be the operator defined from $E$ into $F$ by the following formula:

\[ R(x) = T(x) - (T - |S|) \circ P_{x_0}(x), \]

where $P_{x_0}$ is the principal projection on the band generated by $x_0$. Then $|S| \leq R < T$ and $R \in \mathcal{B}(E,F)$. This gives a contradiction.

ii. There exists a discrete element $y_0$ of $F$ and there exists an element $x_0 \in E^+$ such that

\[ Q_{y_0}(|S|(x_0)) < Q_{y_0}(T(x_0)), \]

where $Q_{y_0}$ is the principal projection on the band generated by $y_0$. We consider the operator $R$ defined from $E$ into $F$ by

\[ R(x) = T(x) - Q_{y_0} \circ (T - |S|)(x). \]

For the same precedent reason, we obtain a contradiction.

iii. Since $|S| \leq T$, it follows that $|S'| \leq T'$ where $|S'|$ and $T'$ are the adjoint operator of $|S|$ and $T$ respectively. This implies the existence of some $f_0$ in $F$ such that

\[ |S'|(g_0) < T'(g_0). \]

In the same way as ii, there exists a discrete element $g_0$ in $E'$ such that

\[ P_{g_0} \circ |S'|(g_0) < P_{g_0} \circ T'(g_0), \]

where $P_{g_0}$ is the principal projection on the band generated by $g_0$. Now, we consider the operator $R$ defined from $F'$ into $E'$ by

\[ R = T' - (P_{g_0} \circ T' - P_{g_0} \circ |S'|). \]

We have $|S'| \leq R < T'$. In fact, for the first inequality, it is sufficient to composite with the projections on bands generated by discrete elements of $E$.

In other hand, the operator $(P_{g_0} \circ T' - P_{g_0} \circ |S'|)$ is of rank one, and hence there exists some $z \in F'' = F$ such that

\[ (P_{g_0} \circ T' - P_{g_0} \circ |S'|)(f) = z(f)g_0 = f(z)g_0 \]

for each $f \in F'$, where $F''$ is the topological bidual of $F$.

It is easy to prove that $(P_{g_0} \circ T' - P_{g_0} \circ |S'|)$ is the operator dual of the operator $K : E \rightarrow F$ defined by $K(x) = g_0(x)z$. Finally, $|S'| \leq (T - K)' < T'$ or again $|S| \leq (T - K) < T$. This is in contradiction with the fact that $T - K \in \mathcal{B}(E,F)$. □

An immediate consequence of Theorem 2.1 (i) or (iii), we obtain the following result of Abramovich and Wickstead ([1], Corollary 3):

Corollary 2.2. The space $\mathcal{K}'(l^1(l_2^n), L^2([0,1]))$ is not a vector lattice.

Our second consequence, follows from a combination of Theorem 2.1 (i) and Theorem 2.7 of [7].

Corollary 2.3. The space $\mathcal{W}'(l^1(l_2^n), l^\infty(L^2([0,1])))$ is not a vector lattice.

For Dunford-Pettis and AM-compact operators, we obtain the following results:

Theorem 2.4. The spaces $\mathcal{D}'(L^2([0,1]), l^\infty(L^2([0,1])))$ and $\mathcal{AM}'(L^2([0,1]), l^\infty(L^2([0,1])))$ are not vector lattices.

Proof. The proof follows along the lines of the proof of Theorem 2.1 (ii). In fact, for each $n \in \mathbb{N}^*$, let $Q_n$ be the projection operator from $l^\infty(L^2([0,1]))$ onto $L^2([0,1])$ defined by the following formula:
\[ Q_n((f_k)_{k \in \mathbb{N}^*}) = f_n \quad \text{for each} \quad (f_k)_{k \in \mathbb{N}^*} \in l^\infty(L^2([0,1])) \]
and let \( i_n \) be the operator defined from \( L^2([0,1]) \) into \( l^\infty(L^2([0,1])) \) by
\[ i_n(f) = (0,0,0,\ldots,0,f,0,0,\ldots). \]

Now, assume that there exists an element \( S \in \mathcal{F}(L^2([0,1]), l^\infty(L^2([0,1]))) \) such that its modulus \( T \) in \( \mathcal{F}(L^2([0,1]), l^\infty(L^2([0,1]))) \) exists and is different of its modulus \( |S| \) in \( L'(L^2([0,1]), l^\infty(L^2([0,1]))) \) where
\[ \mathcal{F}(L^2([0,1]), l^\infty(L^2([0,1]))) = D'(L^2([0,1]), l^\infty(L^2([0,1]))) \]
(resp. \( \mathcal{AM}'(L^2([0,1]), l^\infty(L^2([0,1]))) \)).

Then there exists an element \( x_0 \in (L^2([0,1]))^+ \) and there exists some \( n \in \mathbb{N}^* \) such that
\[ Q_n \circ |S|(x_0) < Q_n \circ T(x_0). \]
Consider the operator \( R \) defined from \( L^2([0,1]) \) into \( l^\infty(L^2([0,1])) \) by
\[ R = T - i_n \circ Q_n \circ (T - |S|). \]
We have
\[ 0 < Q_n \circ (T - |S|) < Q_n \circ T \]
as operators from \( L^2([0,1]) \) into \( L^2([0,1]) \). By applying Theorem 4.4 of Kalton-Saab [8] (resp. Theorem 2.1 of [2]) related to the domination problem for Dunford-Pettis (resp. AM-compact) operators, we conclude that \( Q_n \circ (T - |S|) \) is Dunford-Pettis (resp. AM-compact). Hence, the operator \( R \) is Dunford-Pettis (resp. AM-compact) too. But
\[ |S| \leq R < T, \]
this presents a contradiction.

On the other hand, it follows from ([5], Theorem 2.1) that the subspaces \( D'(L^2([0,1]), l^\infty(L^2([0,1]))) \) and \( \mathcal{AM}'(L^2([0,1]), l^\infty(L^2([0,1]))) \) are not sublattices of \( L'(L^2([0,1]), l^\infty(L^2([0,1]))) \). This completes the proof. \( \square \)

As consequence for the linear span of positive compact operators, it follows from Theorem 1 of [11] and Theorem 2.1:

**Corollary 2.5.** Let \( E \) and \( F \) be Banach lattices. Then \( \mathcal{K}'(E,F) \) cannot be a vector lattice without being a sublattice of \( L'(E,F) \), if one of the following conditions holds:

1) The Banach lattice \( E \) is discrete and its norm is order continuous.
2) The vector lattice \( F \) is discrete.
3) The topological dual \( E' \) is discrete and \( F \) is reflexive.
4) the Banach lattice \( E' \) is discrete and its norm is order continuous
5) the norms of \( E' \) and \( F \) are order continuous.

Recall that a Banach lattice \( E \) is reflexive, if and only if the norms of its topological dual \( E' \) and of its topological bidual \( E'' \) are order continuous ([10], Theorem 5.16).

The following result for the linear span of positive weakly compact operators is a consequence of Theorem 7 of [3], Theorem 5.16 of [10] and Theorem 2.1:

**Corollary 2.6.** Let \( E \) and \( F \) be Banach lattices. Then \( \mathcal{W}'(E,F) \) cannot be a vector lattice without being a sublattice of \( L'(E,F) \), if one of the following conditions holds:

1) The Banach lattice \( E \) is discrete and its norm is order continuous.
2) The vector lattice \( F \) is discrete.
3) the norm of \( E' \) is order continuous
4) the norm of \( F \) is order continuous.

To give the following consequence, recall that the lattice operations in a Banach lattice \( E \) are weakly sequentially continuous if the sequence \( (|x_n|) \) converges to 0 for the weak topology \( \sigma(E,E') \) whenever the sequence \( (x_n) \) converges to 0 for \( \sigma(E,E') \). For example,
the lattice operations of a AM-space are weakly sequentially continuous but the lattice operations of the Banach lattice $L^2$ are not.

Note that in ([6], Corollary 2.2), we have proved that if $E$ is a Banach lattice such that its topological dual $E'$ is discrete, then the lattice operations of $E$ are weakly sequentially continuous.

The following result for the linear span of positive Dunford-Pettis operators is a consequence of Theorem 2 of [11], Corollary 2.2 of [6] and Theorem 2.1:

**Corollary 2.7.** Let $E$ and $F$ be Banach lattices. Then $D'(E, F)$ cannot be a vector lattice without being a sublattice of $L'(E, F)$, if one of the following conditions holds:

1) The vector lattice $F$ is discrete.
2) the lattice operations in $E$ are weakly sequentially continuous.
3) the norm of $F$ is order continuous.

Finally, we have the following result for the linear span of positive AM-compact operators is a consequence of Corollary 2.14 and Theorem 2.15 of [4], Theorem 1.2 of [2] and Theorem 2.1:

**Corollary 2.8.** Let $E$ and $F$ be Banach lattices. Then $AM'(E, F)$ cannot be a vector lattice without being a sublattice of $L'(E, F)$, if one of the following conditions holds:

1) The Banach lattice $E$ is discrete and its norm is order continuous.
2) The vector lattice $F$ is discrete.
3) The topological dual $E'$ is discrete.
4) the norm of $F$ is order continuous.

**References**

5. B. Aqzzouz, R. Nouira, Order $σ$-completeness of the linear span of positive AM-compact operators (submitted).

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