

ON SOME SUBLATTICES OF REGULAR OPERATORS ON BANACH LATTICES

BELMESNAOUI AQZZOUZ AND REDOUANE NOUIRA

ABSTRACT. We give some sufficient conditions under which the linear span of positive compact (resp. Dunford-Pettis, weakly compact, AM-compact) operators cannot be a vector lattice without being a sublattice of the order complete vector lattice of all regular operators. Also, some interesting consequences are obtained.

1. INTRODUCTION AND NOTATION

The paper concerns the following natural and interesting problem: find conditions implying that a subspace F of a vector lattice E is a vector lattice, with respect to the order induced by E , but F is not a sublattice of E . It is easy to see that such subspaces F exist (consider the subspace $F \subset C([0, 1]) = E$ consisting of linear functions where $C([0, 1])$ is the vector lattice of continuous functions on $[0, 1]$).

Recall that in [1], Abramovich and Wickstead constructed two compact operators S and T from a Banach lattice E into an order complete Banach lattice F such that $\pm S < T$ but the modulus $|S|$ is not compact. This means that the linear span of positive compact operators $\mathcal{K}^r(E, F)$ is not a sublattice of the order complete vector lattice of all regular operators $\mathcal{L}^r(E, F)$ i.e. the space of operators $T : E \rightarrow F$ such that $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F . Also, in the same paper, Abramovich and Wickstead proved that $\mathcal{K}^r(E, F)$ is not a vector lattice and they asked in ([1], p. 325) the following question:

Is it possible to construct Banach lattices E and F such that F is order complete and $\mathcal{K}^r(E, F)$ is a vector lattice without being a sublattice of $\mathcal{L}^r(E, F)$?

First, we observe that the above question has a negative answer whenever E and F satisfy the necessary and sufficient conditions of Theorem 1 of Wickstead [11] i.e. if the topological dual E' is discrete and its norm is order continuous, or the Banach lattice F is discrete and its norm is order continuous, or the norms of E and of its topological dual E' are order continuous.

Second, we remark that the question of Abramovich and Wickstead [1] can be asked also for the class of Dunford-Pettis (resp. weakly compact, AM-compact) operators between Banach lattices.

Our objective in this paper is to formulate several sufficient conditions under which classes of regular and compact (resp. weakly compact, Dunford-Pettis, AM-compact) operators cannot be a vector lattice without being a sublattice in the space of all regular operators acting between suitable Banach lattices. More precisely, we will prove that if E and F are Banach lattices, then the linear span of positive compact (resp. Dunford-Pettis, weakly compact, AM-compact) operators between E and F cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E, F)$ if the Banach lattice E is discrete and its norm is order continuous, or the vector lattice F is discrete, or the topological dual E' is discrete

2000 *Mathematics Subject Classification.* 46A40, 46B40, 46B42.

Key words and phrases. Regular operator, order continuous norm, discrete Banach lattice.

and F is reflexive. As consequences, we show that the spaces $\mathcal{K}^r(l^1(l_2^{2^n}), L^2([0, 1]))$ and $\mathcal{W}^r(l^1(l_2^{2^n}), l^\infty(L^2([0, 1])))$ are not vector lattices. Also, we will establish that the spaces $\mathcal{D}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$ and $\mathcal{AM}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$ are not vector lattices.

To state our results we need to fix some notations and recall some definitions. Let E be a vector lattice, then for any two elements $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is an order interval. A subset of E is said to be order bounded if it is included in some order interval. An order ideal B is a solid subspace of a vector lattice E i.e. if $x \in B$ and $y \in E$ such that $|y| \leq |x|$, then $y \in B$. A principal ideal is any order ideal generated by a subset containing only one element x , this ideal will be denoted by I_x . A generalized sequence (x_α) is order convergent to $x \in E$ if there exists a generalized sequence (y_α) such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each α , where the notation $y_\alpha \downarrow 0$ means that the sequence (y_α) is decreasing, its infimum exists and $\inf(y_\alpha) = 0$. A band is an order ideal which is order closed. The band generated by an element x is called a principal band that we design by B_x .

A Banach lattice $(E, \|\cdot\|)$ is a Banach space such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. Finally, we note that the topological dual E' of a Banach lattice E , endowing with the dual norm, is a Banach lattice. For terminology which is not explained, we refer the reader to the book of Zaanen [12].

2. MAIN RESULTS

Let us recall that an operator $T : E \rightarrow F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E . An operator $T : E \rightarrow F$ is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F . It is well known that each positive linear mapping on a Banach lattice is continuous.

A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E , the sequence (x_α) converges to 0 for the norm $\|\cdot\|$. For example, the norm of the Banach lattice l^1 is order continuous but the norm of the Banach lattice l^∞ is not.

A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the sublattice generated by x . The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. For example, the Banach lattice l^1 is discrete but $C([0, 1])$ is not.

Recall that Krengel [9] is the first who constructed an operator $T : l^2 \rightarrow l^2$ (resp. $S : l^2 \rightarrow l^2$) such that T is compact but $T \notin \mathcal{L}^r(l^2, l^2)$ (resp. S is regular and compact but $|S|$ is not compact).

A regular operator $T : E \rightarrow F$ between two Banach lattices is said to be AM-compact if it carries order bounded subsets of E onto relatively compact subsets of F . Also, an operator T from a Banach space E into another F is said to be Dunford-Pettis if it carries weakly compact subsets of E onto compact subsets of F .

If $\mathcal{K}(E, F)$ (resp. $\mathcal{W}(E, F)$, $\mathcal{D}(E, F)$, $\mathcal{AM}(E, F)$) designs the subspace of all compact (resp. weakly compact, Dunford-Pettis, AM-compact) operators from E into F , we denote by $\mathcal{K}^r(E, F)$ (resp. $\mathcal{W}^r(E, F)$, $\mathcal{D}^r(E, F)$, $\mathcal{AM}^r(E, F)$) the linear span of positive elements of $\mathcal{K}(E, F)$ (resp. $\mathcal{W}(E, F)$, $\mathcal{D}(E, F)$, $\mathcal{AM}(E, F)$).

If $\mathcal{B}(E, F)$ is any one of the subspaces $\mathcal{K}^r(E, F)$, $\mathcal{W}^r(E, F)$, $\mathcal{D}^r(E, F)$ or $\mathcal{AM}^r(E, F)$, our principal result is the following:

Theorem 2.1. *Let E and F be two Banach lattices. Then $\mathcal{B}(E, F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E, F)$ if one of the following conditions holds:*

- i) *The Banach lattice E is discrete and its norm is order continuous.*

- ii) *The vector lattice F is discrete.*
- iii) *The topological dual E' is discrete and F is reflexive.*

Proof. Let $S \in \mathcal{B}(E, F)$ and denote by $|S|$ and T its modulus in $\mathcal{L}^r(E, F)$ and $\mathcal{B}(E, F)$ respectively. It is clear that $\pm S \leq |S| \leq T$. Assume that $|S| \neq T$.

i. There exists some discrete element $x_0 \in E^+ = \{x \in E : 0 \leq x\}$ such that $|S|(x_0) < T(x_0)$. Let R be the operator defined from E into F by the following formula:

$$R(x) = T(x) - (T - |S|) \circ P_{x_0}(x),$$

where P_{x_0} is the principal projection on the band generated by x_0 . Then $|S| \leq R < T$ and $R \in \mathcal{B}(E, F)$. This gives a contradiction.

ii. There exists a discrete element y_0 of F and there exists an element $x_0 \in E^+$ such that

$$Q_{y_0}(|S|(x_0)) < Q_{y_0}(T(x_0)),$$

where Q_{y_0} is the principal projection on the band generated by y_0 . We consider the operator R defined from E into F by

$$R(x) = T(x) - Q_{y_0} \circ (T - |S|)(x).$$

For the same precedent reason, we obtain a contradiction.

iii. Since $|S| \leq T$, it follows that $|S|' \leq T'$ where $|S|'$ and T' are the adjoint operator of $|S|$ and T respectively. This implies the existence of some f_0 in F such that

$$|S|'(f_0) < T'(f_0).$$

In the same way as **ii**, there exists a discrete element g_0 in E' such that

$$P_{g_0} \circ |S|'(f_0) < P_{g_0} \circ T'(f_0),$$

where P_{g_0} is the principal projection on the band generated by g_0 . Now, we consider the operator R defined from F' into E' by

$$R = T' - (P_{g_0} \circ T' - P_{g_0} \circ |S|').$$

We have $|S|' \leq R < T'$. In fact, for the first inequality, it is sufficient to composite with the projections on bands generated by discrete elements of E .

In other hand, the operator $(P_{g_0} \circ T' - P_{g_0} \circ |S|')$ is of rank one, and hence there exists some $z \in F'' = F$ such that

$$(P_{g_0} \circ T' - P_{g_0} \circ |S|')(f) = z(f)g_0 = f(z)g_0$$

for each $f \in F'$, where F'' is the topological bidual of F .

It is easy to prove that $(P_{g_0} \circ T' - P_{g_0} \circ |S|')$ is the operator dual of the operator $K : E \rightarrow F$ defined by $K(x) = g_0(x)z$. Finally, $|S|' \leq (T - K)' < T'$ or again $|S| \leq (T - K) < T$. This is in contradiction with the fact that $T - K \in \mathcal{B}(E, F)$. \square

An immediate consequence of Theorem 2.1 (i) or (iii), we obtain the following result of Abramovich and Wickstead ([1], Corollary 3):

Corollary 2.2. *The space $\mathcal{K}^r(l^1(l_2^{2^n}), L^2([0, 1]))$ is not a vector lattice.*

Our second consequence, follows from a combination of Theorem 2.1 (i) and Theorem 2.7 of [7].

Corollary 2.3. *The space $\mathcal{W}^r(l^1(l_2^{2^n}), l^\infty(L^2([0, 1])))$ is not a vector lattice.*

For Dunford-Pettis and AM-compact operators, we obtain the following results:

Theorem 2.4. *The spaces $\mathcal{D}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$ and $\mathcal{AM}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$ are not vector lattices.*

Proof. The proof follows along the lines of the proof of Theorem 2.1 (ii). In fact, for each $n \in \mathbb{N}^*$, let Q_n be the projection operator from $l^\infty(L^2([0, 1]))$ onto $L^2([0, 1])$ defined by the following formula:

$$Q_n((f_k)_{k \in \mathbb{N}^*}) = f_n \quad \text{for each } (f_k)_{k \in \mathbb{N}^*} \in l^\infty(L^2([0, 1]))$$

and let i_n be the operator defined from $L^2([0, 1])$ into $l^\infty(L^2([0, 1]))$ by

$$i_n(f) = (0, 0, 0, \dots, 0, f, 0, 0, \dots).$$

Now, assume that there exists an element $S \in \mathcal{F}(L^2([0, 1]), l^\infty(L^2([0, 1])))$ such that its modulus T in $\mathcal{F}(L^2([0, 1]), l^\infty(L^2([0, 1])))$ exists and is different of its modulus $|S|$ in $\mathcal{L}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$ where

$$\mathcal{F}(L^2([0, 1]), l^\infty(L^2([0, 1]))) = \mathcal{D}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$$

(resp. $\mathcal{AM}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$).

Then there exists an element $x_0 \in (L^2([0, 1]))^+$ and there exists some $n \in \mathbb{N}^*$ such that

$$Q_n \circ |S|(x_0) < Q_n \circ T(x_0).$$

Consider the operator R defined from $L^2([0, 1])$ into $l^\infty(L^2([0, 1]))$ by

$$R = T - i_n \circ Q_n \circ (T - |S|).$$

We have

$$0 < Q_n \circ (T - |S|) < Q_n \circ T$$

as operators from $L^2([0, 1])$ into $L^2([0, 1])$. By applying Theorem 4.4 of Kalton-Saab [8] (resp. Theorem 2.1 of [2]) related to the domination problem for Dunford-Pettis (resp. AM-compact) operators, we conclude that $Q_n \circ (T - |S|)$ is Dunford-Pettis (resp. AM-compact). Hence, the operator R is Dunford-Pettis (resp. AM-compact) too. But $|S| \leq R < T$, this presents a contradiction.

On the other hand, it follows from ([5], Theorem 2.1) that the subspaces $\mathcal{D}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$ and $\mathcal{AM}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$ are not sublattices of $\mathcal{L}^r(L^2([0, 1]), l^\infty(L^2([0, 1])))$. This completes the proof. \square

As consequence for the linear span of positive compact operators, it follows from Theorem 1 of [11] and Theorem 2.1:

Corollary 2.5. *Let E and F be Banach lattices. Then $\mathcal{K}^r(E, F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E, F)$, if one of the following conditions holds:*

- 1) *The Banach lattice E is discrete and its norm is order continuous.*
- 2) *The vector lattice F is discrete.*
- 3) *The topological dual E' is discrete and F is reflexive.*
- 4) *the Banach lattice E' is discrete and its norm is order continuous*
- 5) *the norms of E' and F are order continuous.*

Recall that a Banach lattice E is reflexive, if and only if the norms of its topological dual E' and of its topological bidual E'' are order continuous ([10], Theorem 5.16).

The following result for the linear span of positive weakly compact operators is a consequence of Theorem 7 of [3], Theorem 5.16 of [10] and Theorem 2.1:

Corollary 2.6. *Let E and F be Banach lattices. Then $\mathcal{W}^r(E, F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E, F)$, if one of the following conditions holds:*

- 1) *The Banach lattice E is discrete and its norm is order continuous.*
- 2) *The vector lattice F is discrete.*
- 3) *the norm of E' is order continuous*
- 4) *the norm of F is order continuous.*

To give the following consequence, recall that the lattice operations in a Banach lattice E are weakly sequentially continuous if the sequence $(|x_n|)$ converges to 0 for the weak topology $\sigma(E, E')$ whenever the sequence (x_n) converges to 0 for $\sigma(E, E')$. For example,

the lattice operations of a AM-space are weakly sequentially continuous but the lattice operations of the Banach lattice L^2 are not.

Note that in ([6], Corollary 2.2), we have proved that if E is a Banach lattice such that its topological dual E' is discrete, then the lattice operations of E are weakly sequentially continuous.

The following result for the linear span of positive Dunford-Pettis operators is a consequence of Theorem 2 of [11], Corollary 2.2 of [6] and Theorem 2.1:

Corollary 2.7. *Let E and F be Banach lattices. Then $\mathcal{D}^r(E, F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E, F)$, if one of the following conditions holds:*

- 1) *The vector lattice F is discrete.*
- 2) *the lattice operations in E are weakly sequentially continuous.*
- 3) *the norm of F is order continuous.*

Finally, we have the following result for the linear span of positive AM-compact operators is a consequence of Corollary 2.14 and Theorem 2.15 of [4], Theorem 1.2 of [2] and Theorem 2.1:

Corollary 2.8. *Let E and F be Banach lattices. Then $\mathcal{AM}^r(E, F)$ cannot be a vector lattice without being a sublattice of $\mathcal{L}^r(E, F)$, if one of the following conditions holds:*

- 1) *The Banach lattice E is discrete and its norm is order continuous.*
- 2) *The vector lattice F is discrete.*
- 3) *The topological dual E' is discrete.*
- 4) *the norm of F is order continuous.*

REFERENCES

1. Y. A. Abramovich and A. W. Wickstead, *Solutions of several problems in the theory of compact positive operators*, Proc. Amer. Math. Soc. **123** (1995), no. 10, 3021–3026.
2. C. D. Aliprantis and O. Burkinshaw, *Positive compact operators on Banach lattices*, Math. Z. **174** (1980), 289–298.
3. C. D. Aliprantis and O. Burkinshaw, *On weakly compact operators on Banach lattices*, Proc. Amer. Math. Soc. **83** (1981), no. 3, 573–578.
4. B. Aqzzouz and R. Nouira, *Sur l'idéal d'ordre des opérateurs AM-compact*, Proc. Amer. Math. Soc. **134** (2006), 3515–3523.
5. B. Aqzzouz, R. Nouira, *Order σ -completeness of the linear span of positive AM-compact operators* (submitted).
6. B. Aqzzouz, R. Nouira, and L. Zraoula, *Compactness of AM-compact operators on Banach lattices* (to appear in Annales des Sciences Mathématiques du Québec).
7. Z. L. Chen and A. W. Wickstead, *Vector lattices of weakly compact operators on Banach lattices*, Trans. Amer. Math. Soc. **352** (1999), no. 1, 397–412.
8. N. J. Kalton and P. Saab, *Ideal properties of regular operators between Banach lattices*, Illinois Journal of Math. **29** (1985), no. 3, 382–400.
9. U. Krengel, *Remark on the modulus of compact operators*, Bull. Amer. Math. Soc. **72** (1966), 132–133.
10. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin—New York, 1974.
11. A. W. Wickstead, *Converses for the Dodds-Fremlin and Kalton-Saab theorems*, Math. Proc. Camb. Phil. Soc. **120** (1996), no. 1, 175–179.
12. A. C. Zaanen, *Riesz spaces II*, North Holland Publishing Company, 1983.

UNIVERSITÉ MOHAMMED V-SOUISSI, FACULTÉ DES SCIENCES ECONOMIQUES, JURIDIQUES ET SOCIALES, DÉPARTEMENT D'ECONOMIE, B.P. 5295, SALA ELJADIDA, MOROCCO
E-mail address: baqzzouz@hotmail.com

UNIVERSITÉ IBN TOFAIL, FACULTÉ DES SCIENCES, DÉPARTEMENT DE MATHÉMATIQUES, B.P. 133, KÉNITRA, MOROCCO

Received 08/01/2008