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ONE REMARK ABOUT THE UNCONDITIONAL EXPONENTIAL BASES AND COSINE BASES, CONNECTED WITH THEM

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Abstract. In the paper we consider examples of basis families $\{\cos\lambda_k t\}_1^\infty$, $\lambda_k > 0$, in the space $L_2(0,\sigma)$, such that systems $\{e^{i\lambda_k t}, e^{-i\lambda_k t}\}_1^\infty$ don't form an unconditional basis in space $L_2(-\sigma,\sigma)$.

Let Λ be a sequence of complex numbers belonging to the some strip $|\mathrm{Im} z| \leqslant h$. It is centrally symmetric, i.e. if $\lambda_k \in \Lambda$, then $-\lambda_k \in \Lambda$. Let us assume that the corresponding exponential family

$$\mathcal{E}(\Lambda) := \{e^{i\lambda_k t} : \lambda_k \in \Lambda\} \tag{1}$$

forms an unconditional basis in $L_2(-\sigma,\sigma)$ space (we assume $0 \notin \Lambda$). Let us recall that the a family of vectors, $\{u_k\}$, in a Hilbert space $\mathfrak{H}$ forms an unconditional basis if it is complete in $\mathfrak{H}$ and there exists a constant $m > 0$ such that

$$m^{-1}\sum_k |c_k|^2\|u_k\|^2 \leqslant \left\|\sum_k c_k u_k\right\|^2 \leqslant m\sum |c_k|^2\|u_k\|^2 \tag{2}$$

for every finite complex-valued sequence $\{c_k\}$. In what follows, the notation

$$\sum_k |c_k|^2\|u_k\|^2 \asymp \|\sum_k c_k u_k\|^2$$

will mean that the two-sided estimates (2) take place.

If the family (1) forms an unconditional basis in $L_2(-\sigma,\sigma)$ then the cosine family

$$C(\Lambda_+) := \{\cos\lambda_k t : \lambda_k \in \Lambda_+\}, \quad \Lambda_+ := \{\lambda_k \in \Lambda : \mathrm{Re}\lambda_k > 0\} \tag{3}$$

forms an unconditional basis in $L_2(0,\sigma)$ space.

Indeed, if

$$\|e^{i\lambda_k t}\|^2_{L_2(-\sigma,\sigma)} \asymp 1, \quad \|\cos\lambda_k t\|^2_{L_2(0,\sigma)} \asymp 1, \quad \lambda_k \in \Lambda,$$

then it follows from the family (1) basis property that

$$\begin{aligned}
\left\|\sum a_k \cos\lambda_k t\right\|^2_{L_2(0,\sigma)} &= \frac{1}{4}\left\|\sum a_k e^{i\lambda_k t} + \sum a_k e^{-i\lambda_k t}\right\|^2_{L_2(0,\sigma)} \\
&= \frac{1}{8}\left\|\sum a_k e^{i\lambda_k t} + \sum a_k e^{-i\lambda_k t}\right\|^2_{L_2(-\sigma,\sigma)} \\
&\asymp \sum |a_k|^2\|e^{i\lambda_k t}\|^2 + \sum \|a_k\|^2\|e^{-i\lambda_k t}\|^2 \\
&\asymp \sum |a_k|^2\|\cos\lambda_k t\|^2_{L_2(0,\sigma)}, \quad \lambda_k \in \Lambda_+
\end{aligned}$$

for every finite sequence of complex numbers $\{a_k\}$. Then, for every function $f \in L_2(0,\sigma)$ we have

$$2\int_0^\sigma \cos\lambda_k t f(t)dt = \int_{-\sigma}^\sigma e^{i\lambda_k t}\varphi(t)dt, \quad \lambda_k \in \Lambda,$$

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where $\varphi \in L_2(-\sigma, \sigma)$ and is defined by the formula

$$\varphi(t) = f(t), \quad 0 \leqslant t \leqslant \sigma, \qquad \varphi(t) = f(-t), \quad -\sigma \leqslant t \leqslant 0.$$

So the cosine family is complete in $L_2(0, \sigma)$ and system (3) forms an unconditional basis in the space $L_2(0, \sigma)$.

The stated considerations were used by many authors for a reduction of the cosine bases problem to the exponential basis property problem. Even the incorrect opinion that this reduction is possible for finding an arbitrary cosine system was formed. Possibly, it explains the fact that a few papers are concerned with cosine (sine) bases. The examples when families (3) form an unconditional bases in $L_2(0, \sigma)$ but the corresponding systems $\mathcal{E}(\Lambda)$ are not bases in $L_2(-\sigma, \sigma)$ are built in this article.

The criteria of cosine families unconditional basis property and the criteria of more general function systems basis property were obtained in papers [1, 2]. Let us formulate a special case of the mentioned results. Our further constructions will be based on it.

First of all we note that in the case where the cosine family

$$\{\cos \sqrt{z_k} t : z_k \in M\}, \quad 0 \notin M \tag{4}$$

has an unconditional basis property in space $L_2(0, \sigma)$, the sequence of complex numbers M is the same as the set of roots of an entire function F the growth degree $1/2$ and of a normal type. This function satisfies two conditions:

1) $\int_0^\infty |x|^{-5/2} |F(0) - F(x)|^2 dx < \infty$,
2) $h_F(-\pi) \leqslant \sigma$,

where the growth indicator is defined by the formula

$$h_F(\alpha) := \limsup_{r \to \infty} r^{1/2} \log |F(re^{i\alpha})|, \quad -2\pi < \alpha \leqslant 0. \tag{5}$$

The entire function F which has the listed properties is called a generating function of cosine family (4).

We agree to note the sequence $\sqrt{z_k}$, $z_k \in M$, by $M^{1/2}$, moreover, in the calculation of $\sqrt{z_k}$, we'll consider $-2\pi < \arg z_k \leqslant 0$. Thus $\mathrm{Im}\sqrt{z_k} \leqslant 0$ and in what follows we'll assume that there exists a number $\delta > 0$ such that

$$\inf_k \mathrm{Im}\sqrt{z_k} > -\delta. \tag{6}$$

In the next formulation the contour γ_δ is the parabola $z(t) = (t - i\delta)^2$, $t \in \mathbb{R}$, and the notation $-M^{1/2} := \{-\lambda_k : \lambda_k \in M^{1/2}\}$ is used.

Theorem [1, 2]. *Let F be a generating function of family (4) and the sequence $M^{1/2}$ satisfy the condition (6) for some $\delta > 0$. For the cosine system (4) to be an unconditional space $L_2(0, \sigma)$, the next conditions are necessary and sufficient:*

1) *the sequence $M^{1/2} \cup (-M^{1/2})$ is separable, i.e.*

$$\inf_{k \neq j} |u_k - u_j| > 0, \quad u_k \in M^{1/2} \cup (-M^{1/2});$$

2) $h_F(-\pi) = \sigma$;
3) *the weight $|W(z)| := |z|^{-1/2} |F(z)|^2$ satisfies the Muckenhoupt condition on the contour γ_δ:*

$$\sup_{z \in \gamma_\delta} \sup_{r > 0} \left\{ r^{-1} \int_{B(z,r) \cap \gamma_\delta} |W(z)| |dz| r^{-1} \int_{B(z,r) \cap \gamma_\delta} |W(z)|^{-1} |dz| \right\} < \infty,$$

where $B(z, r) := \{u \in \mathbb{C} : |u - z| < r\}$.

Now let us consider the function

$$F_\nu(z) := (z-a)E_{1/2}(-z, 1+\nu), \quad a>0, \quad 1/2<\nu<2,$$

where the Mittag-Leffler entire function is defined by the factorization

$$E_{1/2}(-z,1+\nu) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{\Gamma(1+\nu+2n)}.$$

From the general formulae for the degree and function type it follows from its Taylor's coefficient that the type F_ν is 1 for the growth degree 1/2. The properties of the Mittag-Leffler function studied in the paper [3] with another reference. In particular the estimates

$$\begin{aligned} &|F_\nu(x)| \leqslant C|x|^{1-\nu/2}, \qquad x \geqslant 1,\\ &|F_\nu(z)| \asymp |z|^{1-\nu/2}, \qquad z \in \gamma_\delta \end{aligned} \tag{7}$$

take place for every parabola γ_δ, $\delta>0$. Further, if $1/2<\gamma<2$, then the roots of the function F_ν (for a correct choice of a) are simple and lie on the axis $\mathbb{R}_+$, and the asymptotic formulae [3] take place,

$$z_1 = a, \quad z_k = \left(\pi(k-1) + \frac{\pi}{2}(\nu-1) + 0\left(k^{\nu-2}\right)\right)^2, \quad k \geqslant 2.$$

Thus it follows from the estimates (7) that the function F_ν is a generating one for the cosine system

$$\left\{\cos\lambda_k t : \lambda_k \in M^{1/2}\right\} \tag{8}$$

in the space $L_2(0,1)$ and, moreover, the sequence $M^{1/2}$ consists of the positive numbers

$$\lambda_1 = \sqrt{a}, \quad \lambda_k = \pi(k-1) + \frac{\pi}{2}(\nu-1) + 0(k^{\nu-2}), \quad k \geqslant 2.$$

We shall note that the sequence $M^{1/2} \cup (-M^{1/2})$ is separable, $h_{F_\nu}(-\pi) = 1$, and the weight

$$|W(z)| \asymp |z|^{-1/2} \cdot |z|^{2-\nu} = |z|^{3/2-\nu}$$

satisfies the Muckenhoupt condition on every parabola γ_δ as far as $-1 < 3/2-\nu < 1$, if $1/2<\nu<2$. Thus it follows from what has been said above that the family (8) forms an unconditional basis in space $L_2(0,1)$.

Now let us consider the exponential system in the space $L_2(-1,1)$,

$$\left\{e^{i\mu_k t} : \mu_k \in M^{1/2} \cup (-M^{-1/2})\right\} \tag{9}$$

in which the centrally symmetric sequence $\{\mu_k\}_{-\infty}^{+\infty}$ is the same as the set of roots of the entire function of exponential type

$$\Phi_\nu(z) := F_\nu(z^2).$$

By B. S. Pavlov's theorem [4], *the family (9) forms an unconditional basis in the space $L_2(-1,1)$ if and only if the following conditions hold:*

1) $\limsup_{r\to\infty} r^{-1}\log|\Phi_\nu(re^{\pm i\frac{\pi}{2}})| = 1$;
2) *the sequence $M^{1/2} \cup (-M^{1/2})$ is separable;*
3) *the weight $W(x) := |\Phi_\nu(x-i\delta)|^2$ for some $\delta>0$ satisfies the Muckenhoupt condition (A_2) on the real axis [5].*

We note that the condition 1) simply follows from the equality $h_{F_\nu}(-\pi)=1$ and the separability of the sequence $M^{1/2} \cup (-M^{-1/2})$ was proved above. Further, the two-sided estimate

$$|\Phi_\nu(x-i\delta)|^2 \asymp |x-i\delta|^{4-2\nu}, \quad x \in \mathbb{R}$$

follows from formula (7). Therefore condition 3) holds if (and only if) $-1 < 4 - 2\nu < 1$. This one and the condition $1/2 < \nu < 2$ both imply the inequalities $3/2 < \nu < 2$. Thus we proved the theorem in which formulation the notations (1), (3) are used and the number a is selected so that the roots of $\Phi_\nu(z)$ are simple.

Theorem. *Denote by* Λ *the entire function*

$$\Phi_\nu(z) = (z^2 - a)E_{1/2}(-z^2, 1+\nu), \quad a > 0, \quad 1/2 < \nu < 2,$$

such that thy roots are real and simple. Let also

$$\Lambda_+ := \{\lambda_k \in \Lambda : \lambda_k > 0\}.$$

Then the cosine family $C(\Lambda_+)$ *forms an unconditional basis in space* $L_2(0,1)$*. In addition the exponential family* $\mathcal{E}(\Lambda)$ *forms an unconditional basis in space* $L_2(-1,1)$ *only if* $3/2 < \nu < 2$.

It is clear that the analogous constructions and conclusions can be made for sine families too.

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