GENERALIZED STOCHASTIC DERIVATIVES ON PARAMETRIZED SPACES OF REGULAR GENERALIZED FUNCTIONS OF MEIXNER WHITE NOISE

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ABSTRACT. We introduce and study Hida-type stochastic derivatives and stochastic differential operators on the parametrized Kondratiev-type spaces of regular generalized functions of Meixner white noise. In particular, we study the interconnection between the stochastic integration and differentiation. Our researches are based on the general approach that covers the Gaussian, Poissonian, Gamma, Pascal and Meixner cases.

0. INTRODUCTION

In the paper [3] Fred E. Benth introduced and studied a generalization of stochastic differential operators on the so-called Kondratiev generalized functions space in the Gaussian analysis. This generalization turns out to be useful for applications (for example, for study of properties of solutions of stochastic equations with Wick-type nonlinearities). Therefore there exists a motivation to generalize the results of [3] to generalized functions spaces in a non-Gaussian analysis.

In the papers [10, 11] the author generalized the results of [3] to the Kondratiev-type spaces of generalized functions in the so-called Gamma white noise analysis ([20, 21]). Since the Gamma measure has no the Chaotic Representation Property and has some another peculiarities, the corresponding spaces have a more complicated than in the Gaussian analysis structure; nevertheless a natural and rich in content analog of the Gaussian theory is possible.

A next natural step consists in the construction of a theory of stochastic differentiation on the generalized functions spaces in the so-called Meixner analysis. In fact, the (introduced in [26]) generalized Meixner measure μ on the Schwartz distributions space D' (the base measure of the Meixner analysis) is a direct generalization of "classical" measures on D', such as the Gaussian, Poissonian and Gamma measures. This measure is very general, but still has important "classical" properties (for example, the orthogonal polynomials in $L^2(D', \mu)$ are Schefer (generalized Appell in another terminology) ones), therefore a constructive theory is still possible.

In the papers [16, 15] the author constructed and studied generalized stochastic derivatives and differential operators on the "classical" Kondratiev-type (finite order) spaces of nonregular and regular generalized functions of Meixner white noise. But in the regular case one can consider the so-called *parametrized Kondratiev-type spaces* (see [12] and Preliminaries for details) that are much more "flexible" and more convenient for applications than the "classical" Kondratiev-type spaces. Therefore it is useful to transfer (so

far as it is possible) the results of [15] to the case of parametrized spaces. The realization of this idea is the main aim of the present paper.

The paper is organized in the following manner. In the first section we give a necessary information about the generalized Meixner measure, the parametrized spaces of test and generalized functions, a Wick calculus and a stochastic integration. In the second section we introduce and study Hida-type stochastic derivatives $\partial.$ on the parametrized generalized functions spaces $(L^2)^{-\beta}_{-q}$, $(L^2)^{-\beta}$ (in particular, we consider the interconnection of $\partial.$ with the stochastic integral). The third section is devoted to study of some another stochastic differential operators on $(L^2)^{-\beta}_{-q}$, $(L^2)^{-\beta}$ that are closely connected with $\partial.$ and coordinated with the Wick calculus and the extended stochastic integral.

1. Preliminaries

Let σ be a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ (here $\mathcal{B}$ denotes the Borel σ-algebra) satisfying the following assumptions:

- σ is absolutely continuous with respect to the Lebesgue measure and the density is an infinite differentiable function on $\mathbb{R}_+$;
- σ is a nondegenerate measure, i.e., for each nonempty open set $O \subset \mathbb{R}_+$ $\sigma(O) > 0$.

Remark 1.1. Note that these assumptions are the "simplest *sufficient* ones" for our considerations; actually it is possible to consider a much more general σ.

By D denote the set of all real-valued infinite differentiable functions on $\mathbb{R}_+$ with compact supports. This set can be naturally endowed with a (projective limit) topology of a nuclear space (by analogy with, e.g., [6]): $D = \mathrm{pr}\lim_{\tau \in T} \mathcal{H}_\tau$, where T is the set of all pairs $\tau = (\tau_1, \tau_2)$, $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that $\tau_2(t) \geq 1$ $\forall t \in \mathbb{R}_+$; $\mathcal{H}_\tau = \mathcal{H}_{(\tau_1, \tau_2)}$ is the Sobolev space of order τ_1 weighted by the function τ_2 (the integration in the scalar product in $\mathcal{H}_\tau$ with respect to σ). Hence in what follows, we understand D as the corresponding *topological space*.

Let us consider the (nuclear) chain (the rigging of $L^2(\mathbb{R}_+, \sigma)$–the space of square integrable with respect to σ real-valued functions on $\mathbb{R}_+$)

$$D' = \mathrm{ind}\lim_{\tau' \in T} \mathcal{H}_{-\tau'} \supset \mathcal{H}_{-\tau} \supset L^2(\mathbb{R}_+, \sigma) =: \mathcal{H} \supset \mathcal{H}_\tau \supset D,$$

where $\mathcal{H}_{-\tau}$, D' are the dual to $\mathcal{H}_\tau$, D with respect to $\mathcal{H}$ spaces correspondingly. Let $\langle \cdot, \cdot \rangle$ be the (generated by the scalar product in $\mathcal{H}$) dual pairing between elements of D' and D (and also $\mathcal{H}_{-\tau}$ and $\mathcal{H}_\tau$), this notation will be preserved for tensor powers and complexifications of spaces.

Remark 1.2. Note that *all scalar products and pairings in this paper are real*.

Let us fix arbitrary functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{C}$ that are smooth and satisfy

$$\theta := -\alpha - \beta \in \mathbb{R}, \quad \eta := \alpha\beta \in \mathbb{R}_+,$$

θ and η are bounded on $\mathbb{R}_+$. Further, let $\forall t \in \mathbb{R}_+$ $\widetilde{v}(\alpha(t), \beta(t), ds)$ be a probability measure on $\mathbb{R}$ that is defined by its Fourier transform

$$\int_{\mathbb{R}} e^{ius} \widetilde{v}(\alpha, \beta, ds)$$

$$= \exp\left\{ -iu(\alpha + \beta) + 2\sum_{m=1}^{\infty} \frac{(\alpha\beta)^m}{m} \left[\sum_{n=2}^{\infty} \frac{(-iu)^n}{n!}(\beta^{n-2} + \beta^{n-3}\alpha + \cdots + \alpha^{n-2}) \right]^m \right\},$$

$v(\alpha, \beta, ds) := \frac{1}{s^2}\widetilde{v}(\alpha, \beta, ds)$.

Definition 1.1. We say that a probability measure μ on the measurable space $(D', \mathcal{F})$ (here $\mathcal{F}$ is the generated by cylinder sets σ-algebra on D') with a Fourier transform

$$\int_{D'} e^{i\langle x, \xi\rangle} \mu(dx) = \exp\left\{ \int_{\mathbb{R}_+} \sigma(dt) \int_{\mathbb{R}} \upsilon(\alpha(t), \beta(t), ds)(e^{is\xi(t)} - 1 - is\xi(t)) \right\}$$

(here $\xi \in D$) is called the *generalized Meixner measure*.

Let us denote by a subindex $\mathbb{C}$ complexifications of spaces.

Theorem 1.1. ([26]). *The generalized Meixner measure μ is a generalized stochastic process with independent values in the sense of [9]. The Laplace transform of μ is given in a neighborhood of zero $\mathcal{U}_0 \subset D_\mathbb{C}$ by the following formula:*

$$l_\mu(\lambda) = \int_{D'} e^{\langle x, \lambda\rangle} \mu(dx)$$

$$= \exp\left\{ \int_{\mathbb{R}_+} \sum_{m=1}^{\infty} \frac{(\alpha(t)\beta(t))^{m-1}}{m} \right.$$

$$\times \left(\sum_{n=2}^{\infty} \frac{(-\lambda)^n}{n!}(\beta(t)^{n-2} + \beta(t)^{n-3}\alpha(t) + \cdots + \alpha(t)^{n-2}) \right)^m \sigma(dt) \right\}, \quad \lambda \in \mathcal{U}_0.$$

Remark 1.3. Accordingly to the classical classification [25] (see also [24, 23, 26]) for $\alpha = \beta = 0$ (here and below we understand all such equalities σ-a.e.) μ is the Gaussian measure; for $\alpha \neq 0$ (here and below $a(\cdot) \neq b(\cdot)$ means that $a - b \neq 0$ on some measurable set G such that $\sigma(G) > 0$), $\beta = 0$ μ is the centered Poissonian measure; for $\alpha = \beta \neq 0$ μ is the centered Gamma measure; for $\alpha \neq \beta$, $\alpha\beta \neq 0$, $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ μ is the centered Pascal measure; for $\alpha = \overline{\beta}$, $\mathrm{Im}(\alpha) \neq 0$ μ is the centered Meixner measure.

It was established in [17] that there exists $\widetilde{\tau} \in T$ such that the generalized Meixner measure is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$.

Now by $(L^2) = L^2(D', \mu)$ denote the space of square integrable with respect to μ complex-valued functions on D'. Let us construct orthogonal polynomials in (L^2).

Definition 1.2. We define the so-called *Wick exponential* (a generating function of the orthogonal polynomials) by setting

$$: \exp(x; \lambda) : \overset{\text{def}}{=} \exp\left\{ -\int_{\mathbb{R}_+} \left(\frac{\lambda(t)^2}{2} + \sum_{n=3}^{\infty} \frac{\lambda(t)^n}{n}(\alpha(t)^{n-2} \right. \right.$$

(1.1)
$$+ \alpha(t)^{n-3}\beta(t) + \cdots + \beta(t)^{n-2} \Big) \sigma(dt)$$

$$+ \left\langle x, \lambda + \sum_{n=2}^{\infty} \frac{\lambda^n}{n}(\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}) \right\rangle \right\},$$

where $\lambda \in \mathcal{U}_0 \subset D_\mathbb{C}$, $x \in D'$, $\mathcal{U}_0$ is some depending on x neighborhood of $0 \in D_\mathbb{C}$.

Remark 1.4. It was proved in [26] that

$$: \exp(x; \lambda) := \frac{e^{\langle x, \Psi(\lambda)\rangle}}{l_\mu(\Psi(\lambda))}$$

with $\Psi(\lambda) = \lambda + \sum_{n=2}^{\infty} \frac{\lambda^n}{n}(\alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1})$, therefore $: \exp(x; \cdot):$ is a generating function of so-called *Schefer polynomials* (or *generalized Appell polynomials* in another terminology). This fact gives us the possibility to use in our considerations well-known results of the so-called biorthogonal analysis (see, e.g., [2, 1, 22, 13, 19, 14, 4, 7] and references therein).

It is clear (see also [26]) that $: \exp(x; \cdot) :$ is a holomorphic at $0 \in D_{\mathbb{C}}$ function for each $x \in D'$. Therefore using the Cauchy inequality (e.g., [8]) and the kernel theorem (e.g., [6]) one can obtain the representation

$$: \exp(x; \lambda) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n(x), \lambda^{\otimes n} \rangle, \quad P_n(x) \in D_{\mathbb{C}}'^{\widehat{\otimes} n}, \quad x \in D', \quad \lambda \in D_{\mathbb{C}}.$$

Here (and below) $\widehat{\otimes}$ denotes a symmetric tensor product, $\lambda^{\otimes 0} = 1$ even for $\lambda \equiv 0$.

Remark 1.5. It follows from the recurrence formula for $P_n(x)$ ([26]) that actually $P_n(x) \in D'^{\widehat{\otimes} n}$ for $x \in D'$, and, moreover, if $\tau \in T$ is such that the Dirac delta-function $\delta_0 \in \mathcal{H}_{-\tau}$ then *for $x \in \mathcal{H}_{-\tau}$ we have $P_n(x) \in \mathcal{H}_{-\tau}^{\widehat{\otimes} n}$.*

Definition 1.3. We say that the polynomials $\langle P_n, f^{(n)} \rangle$, $f^{(n)} \in D_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$ are called the *generalized Meixner polynomials.*

Remark 1.6. Depending on α and β in (1.1) the generalized Meixner polynomials can be the generalized Hermite polynomials ($\alpha = \beta = 0$); the generalized Charlier polynomials ($\alpha \neq 0, \beta = 0$); the generalized Laguerre polynomials ($\alpha = \beta \neq 0$); the Meixner polynomials ($\alpha \neq \beta$, $\alpha\beta \neq 0$, $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$); the Meixner-Pollaczek polynomials ($\alpha = \bar{\beta}$, $\mathrm{Im}(\alpha) \neq 0$) (see also Remark 1.3).

In order to formulate a statement on an orthogonality of the generalized Meixner polynomials we need the following

Definition 1.4. We define the scalar product $\langle \cdot, \cdot \rangle_{\text{ext}}$ on $D_{\mathbb{C}}^{\widehat{\otimes} n}$ ($n \in \mathbb{N}$) by the formula

$$\langle f^{(n)}, g^{(n)} \rangle_{\text{ext}} := \sum_{\substack{k, l_j, s_j \in \mathbb{N}: \ j=1,\ldots,k, \ l_1 > l_2 > \cdots > l_k, \\ l_1 s_1 + \cdots + l_k s_k = n}} \frac{n!}{l_1^{s_1} \ldots l_k^{s_k} s_1! \ldots s_k!}$$

$$\times \int_{\mathbb{R}_+^{s_1 + \cdots + s_k}} f^{(n)} (\underbrace{t_1, \ldots, t_1}_{l_1}, \ldots, \underbrace{t_{s_1}, \ldots, t_{s_1}}_{l_1}, \ldots, \underbrace{t_{s_1 + \cdots + s_k}, \ldots, t_{s_1 + \cdots + s_k}}_{l_k})$$

$$\times g^{(n)} (\underbrace{t_1, \ldots, t_1}_{l_1}, \ldots, \underbrace{t_{s_1}, \ldots, t_{s_1}}_{l_1}, \ldots, \underbrace{t_{s_1 + \cdots + s_k}, \ldots, t_{s_1 + \cdots + s_k}}_{l_k}) \eta(t_1)^{l_1 - 1} \ldots \eta(t_{s_1})^{l_1 - 1}$$

$$\times \eta(t_{s_1 + 1})^{l_2 - 1} \ldots \eta(t_{s_1 + s_2})^{l_2 - 1} \ldots \eta(t_{s_1 + \cdots + s_{k-1} + 1})^{l_k - 1} \ldots \eta(t_{s_1 + \cdots + s_k})^{l_k - 1}$$

$$\times \sigma(dt_1) \ldots \sigma(dt_{s_1 + \cdots + s_k}).$$

Denote by $| \cdot |_{\text{ext}}$ the corresponding norm, i.e., $|f^{(n)}|_{\text{ext}}^2 = \langle f^{(n)}, \overline{f^{(n)}} \rangle_{\text{ext}}$. For $n = 0$ $\langle f^{(0)}, g^{(0)} \rangle_{\text{ext}} := f^{(0)} g^{(0)} \in \mathbb{C}$, $|f^{(0)}|_{\text{ext}} = |f^{(0)}|$.

Example 1.1. It is easy to see that for $n = 1$

$$\langle f^{(1)}, g^{(1)} \rangle_{\text{ext}} = \langle f^{(1)}, g^{(1)} \rangle = \int_{\mathbb{R}_+} f^{(1)}(t) g^{(1)}(t) \sigma(dt).$$

Further, for $n = 2$

$$\langle f^{(2)}, g^{(2)} \rangle_{\text{ext}} = \langle f^{(2)}, g^{(2)} \rangle + \int_{\mathbb{R}_+} f^{(2)}(t, t) g^{(2)}(t, t) \eta(t) \sigma(dt).$$

If $\eta = 0$ (this means that μ is the Gaussian or Poissonian measure, see Remark 1.3) then $\langle f^{(n)}, g^{(n)} \rangle_{\text{ext}} = \langle f^{(n)}, g^{(n)} \rangle$ for all $n \in \mathbb{Z}_+$; in general, $\langle f^{(n)}, g^{(n)} \rangle_{\text{ext}} = \langle f^{(n)}, g^{(n)} \rangle + \cdots$.

Theorem 1.2. ([26]). *The generalized Meixner polynomials are orthogonal in (L^2) in the sense that*

$$(1.2) \qquad \int_{D'} \langle P_n(x), f^{(n)} \rangle \langle P_m(x), g^{(m)} \rangle \mu(dx) = \delta_{mn} n! \langle f^{(n)}, g^{(n)} \rangle_{\text{ext}}.$$

By $\mathcal{H}_{\text{ext}}^{(n)}$ ($n \in \mathbb{N}$) denote the closure of $D_{\mathbb{C}}^{\widehat{\otimes}n}$ with respect to the norm $|\cdot|_{\text{ext}}$, $\mathcal{H}_{\text{ext}}^{(0)} := \mathbb{C}$. Of course, $\mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{Z}_+$ are Hilbert spaces; for the scalar products in these spaces it is natural to preserve the notation $\langle \cdot, \cdot \rangle_{\text{ext}}$.

Remark 1.7. It is not difficult to prove by analogy with [5] that the space $\mathcal{H}_{\text{ext}}^{(n)}$ is, generally speaking, the orthogonal sum of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n} \equiv L^2(\mathbb{R}_+, \sigma)_{\mathbb{C}}^{\widehat{\otimes}n}$ and some another Hilbert spaces (as a "limit case" one can consider $\eta = 0$, in this case $\mathcal{H}_{\text{ext}}^{(n)} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$). In this sense $\mathcal{H}_{\text{ext}}^{(n)}$ *is an extension of* $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$.

One can give another explanation of the fact that $\mathcal{H}_{\text{ext}}^{(n)}$ is a more wide space than $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$. Namely, let $F^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ ($F^{(n)}$ is an equivalence class in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$). We select a representative (a function) $\dot{F}^{(n)} \in F^{(n)}$ with the "zero diagonal", i.e., $\dot{F}^{(n)}(t_1, \ldots, t_n) = 0$ if there exist $i, j \in \{1, \ldots, n\}$, $i \neq j$ such that $t_i = t_j$. This function generates the equivalence class $\widehat{F}^{(n)}$ in $\mathcal{H}_{\text{ext}}^{(n)}$ that can be identified with $F^{(n)}$ (see [17] for details).

Definition 1.5. ([17]). For $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ ($n \in \mathbb{Z}_+$) we define $\langle P_n, F^{(n)} \rangle \in (L^2)$ as an (L^2)-limit

$$\langle P_n, F^{(n)} \rangle := \lim_{k \to \infty} \langle P_n, f_k^{(n)} \rangle,$$

where $(f_k^{(n)} \in D_{\mathbb{C}}^{\widehat{\otimes}n})_{k=1}^{\infty}$ is a sequence of "smooth" functions such that $f_k^{(n)} \to F^{(n)}$ (as $k \to \infty$) in $\mathcal{H}_{\text{ext}}^{(n)}$.

The following statement follows from results of [26].

Theorem 1.3. *A function* $F \in (L^2)$ *if and only if there exists a sequence of kernels*

(1.3) $$\left(F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \right)_{n=0}^{\infty}$$

such that F *can be presented in the form*

(1.4) $$F = \sum_{n=0}^{\infty} \langle P_n, F^{(n)} \rangle,$$

where the series converges in (L^2), *i.e., the* (L^2)-*norm of* F

$$\|F\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{\text{ext}}^2 < \infty.$$

Furthermore, the system $\{\langle P_n, F^{(n)} \rangle, \ F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}, \ n \in \mathbb{Z}_+\}$ *plays a role of an orthogonal basis in* (L^2) *in the sense that for* $F, G \in (L^2)$

$$(F, G)_{(L^2)} = \sum_{n=0}^{\infty} n! \langle F^{(n)}, G^{(n)} \rangle_{\text{ext}},$$

where $F^{(n)}, G^{(n)}$ *are the kernels from decompositions* (1.4) *for* F, G *(in particular,* (1.2) *for* $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $g^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$ *holds true).*

Now let us describe parametrized Kondratiev-type spaces (see [12] for more details).

Let us consider the set $\mathcal{P} := \{f = \sum_{n=0}^{N_f} \langle P_n, f^{(n)} \rangle, \ f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}, \ N_f \in \mathbb{Z}_+\} \subset (L^2)$ of polynomials and for each $q \in \mathbb{Z}_+$ and $\beta \in [0, 1]$ introduce on this set a scalar product $(\cdot, \cdot)_{q,\beta}$ by setting for $f = \sum_{n=0}^{N_f} \langle P_n, f^{(n)} \rangle$ and $g = \sum_{n=0}^{N_g} \langle P_n, g^{(n)} \rangle$

(1.5) $$(f, g)_{q,\beta} := \sum_{n=0}^{\min(N_f, N_g)} (n!)^{1+\beta} 2^{qn} \langle f^{(n)}, g^{(n)} \rangle_{\text{ext}}.$$

Let $\|\cdot\|_{q,\beta}$ be the corresponding norm:

$$\|f\|_{q,\beta} := \sqrt{(f,\overline{f})_{q,\beta}} = \sqrt{\sum_{n=0}^{N_f}(n!)^{1+\beta}2^{qn}|f^{(n)}|_{\text{ext}}^2}.$$

Definition 1.6. We define the *parametrized Kondratiev-type spaces of test functions* $(L^2)_q^\beta$, $q \in \mathbb{Z}_+$, $\beta \in [0,1]$ as the closures of $\mathcal{P}$ with respect to the norms $\|\cdot\|_{q,\beta}$; $(L^2)^\beta :=$ pr $\lim_{q\in\mathbb{Z}_+}(L^2)_q^\beta$.

It is not difficult to see that $f \in (L^2)_q^\beta$ if and only if f can be presented in the form

$$(1.6) \qquad\qquad f = \sum_{n=0}^{\infty}\langle P_n, f^{(n)}\rangle$$

with

$$\|f\|_{q,\beta}^2 := \|f\|_{(L^2)_q^\beta}^2 = \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}|f^{(n)}|_{\text{ext}}^2 < \infty,$$

and for $f,g \in (L^2)_q^\beta$ $(f,g)_{(L^2)_q^\beta} = \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}\langle f^{(n)}, g^{(n)}\rangle_{\text{ext}}$, where $f^{(n)}, g^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ are the kernels from decompositions (1.6) for f and g correspondingly. Therefore the generalized Meixner polynomials play a role of an orthogonal basis in $(L^2)_q^\beta$.

Remark 1.8. Note that the term "Kondratiev-type spaces" is connected with the fact that for $\beta = 1$ spaces of such a type were first introduced and studied (in the one-dimensional Gaussian analysis) by Yu. G. Kondratiev in [18].

Proposition 1.1. ([12]). *For each $q \in \mathbb{Z}_+$ and $\beta \in [0,1]$ $(L^2)_q^\beta$ is densely and continuously embedded in (L^2).*

Using the result of this proposition, one can consider the chain

$$(1.7) \qquad (L^2)^{-\beta} = \mathop{\text{ind}\lim}_{\widetilde{q}\in\mathbb{Z}_+}(L^2)_{-\widetilde{q}}^{-\beta} \supset (L^2)_{-q}^{-\beta} \supset (L^2) \supset (L^2)_q^\beta \supset (L^2)^\beta,$$

where $(L^2)_{-q}^{-\beta}$, $(L^2)^{-\beta}$ are the dual to $(L^2)_q^\beta$, $(L^2)^\beta$ with respect to (L^2) spaces correspondingly.

Remark 1.9. If $\beta < 0$ then $(L^2)^\beta \not\subset (L^2)$ and it is impossible to construct chain (1.7). If $\beta > 1$ then $(L^2)^{-\beta}$ will be too wide for construction of a substantive theory. Hence the choice $\beta \in [0,1]$ is optimal and (just as $q \in \mathbb{Z}_+$) will be fixed in this paper.

Note also that in (1.5) one can use K^{qn} with any $K > 1$ instead of 2^{qn} (cf. [4]). Formally this leads to a more general construction; but in fact such a generalization is formal and not fundamental.

Definition 1.7. The spaces $(L^2)_{-q}^{-\beta}$, $(L^2)^{-\beta}$ are called the *parametrized Kondratiev-type spaces of generalized functions.*

Remark 1.10. We remind that the "classical" Kondratiev-type spaces are the parametrized ones with $\beta = 1$. It is obvious that $(L^2)^1 \subset (L^2)^\beta$ and $(L^2)^{-\beta} \subset (L^2)^{-1}$ if $\beta < 1$.

Theorem 1.4. ([12]). *A generalized function $F \in (L^2)_{-q}^{-\beta}$ if and only if there exists sequence of kernels (1.3) such that F can be presented in form (1.4), where the series converges in $(L^2)_{-q}^{-\beta}$, i.e., the norm*

$$(1.8) \qquad \|F\|_{-q,-\beta}^2 := \|F\|_{(L^2)_{-q}^{-\beta}}^2 = \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}|F^{(n)}|_{\text{ext}}^2 < \infty.$$

Furthermore, the system $\{\langle P_n, F^{(n)}\rangle\colon F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}, n \in \mathbb{Z}_+\}$ *plays a role of an orthogonal basis in* $(L^2)_{-q}^{-\beta}$ *in the sense that for* $F, G \in (L^2)_{-q}^{-\beta}$

$$(F, G)_{(L^2)_{-q}^{-\beta}} = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} \langle F^{(n)}, G^{(n)}\rangle_{\text{ext}},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ *are the kernels from decompositions* (1.4) *for* F *and* G *correspondingly.*

Remark 1.11. It is easy to see that $F \in (L^2)^{-\beta}$ if and only if there exists sequence (1.3) such that F can be presented in form (1.4) with finite norm (1.8) for some $q \in \mathbb{Z}_+$.

The (real) dual pairing between elements of $(L^2)_{-q}^{-\beta}$ and $(L^2)_q^{\beta}$ (just as $(L^2)^{-\beta}$ and $(L^2)^{\beta}$) that is generated by the scalar product in (L^2) will be denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. It is easy to see that for a generalized function F of form (1.4) and a test function f of form (1.6)

$$(1.9) \qquad\qquad \langle\!\langle F, f \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle F^{(n)}, f^{(n)}\rangle_{\text{ext}}.$$

Now let us remind elements of the Wick calculus on $(L^2)^{-\beta}$ (a more complete and detailed presentation is given in [12]).

First we recall necessary definitions and statements in the case $\beta = 1$.

Definition 1.8. For $F \in (L^2)^{-1}$ we define the *S-transform* SF as a *formal* series

$$(SF)(\lambda) := \sum_{n=0}^{\infty} \langle F^{(n)}, \lambda^{\otimes n}\rangle_{\text{ext}},$$

where $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $n \in \mathbb{Z}_+$ are the kernels from decomposition (1.4) for F. In particular, $(SF)(0) = F^{(0)}$, $S1 \equiv 1$.

Definition 1.9. For $F, G \in (L^2)^{-1}$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ we define the *Wick product* $F \Diamond G \in (L^2)^{-1}$ and the *Wick version of* h $h^{\Diamond}(F) \in (L^2)^{-1}$ by setting

$$F \Diamond G := S^{-1}(SF \cdot SG), \quad h^{\Diamond}(F) := S^{-1}h(SF).$$

The correctness of this definition and, moreover, the fact that the Wick multiplication is continuous in the topology of $(L^2)^{-1}$ proved in [17].

Remark 1.12. It is easy to see that the Wick multiplication $\Diamond$ is commutative, associative and distributive (over the field $\mathbb{C}$). Further, if h from Definition 1.9 is presented in the form

$$(1.10) \qquad\qquad h(u) = \sum_{m=0}^{\infty} h_m (u - (SF)(0))^m$$

then $h^{\Diamond}(F) = \sum_{m=0}^{\infty} h_m(F - (SF)(0))^{\Diamond m}$, where $F^{\Diamond m} := \underbrace{F \Diamond \cdots \Diamond F}_{m \ times}$.

Let us write out the "coordinate form" of $F \Diamond G$ and $h^{\Diamond}(F)$.

Lemma 1.1. ([17]). *Let* $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$, $G^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$, $n, m \in \mathbb{Z}_+$. *We define the element* $F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{\text{ext}}^{(n+m)}$ *as follows. Let* $\dot{F}^{(n)} \in F^{(n)}$, $\dot{G}^{(m)} \in G^{(m)}$ *be some representatives*

(functions) from the equivalence classes $F^{(n)}$, $G^{(m)}$. Set

$$(\widetilde{F^{(n)}\dot{G}^{(m)}})(t_1, \ldots, t_n; t_{n+1}, \ldots, t_{n+m})$$

$$:= \begin{cases} \dot{F}^{(n)}(t_1, \ldots, t_n)\dot{G}^{(m)}(t_{n+1}, \ldots, t_{n+m}), & \text{if} \quad \substack{\forall i \in \{1,\ldots,n\}, \\ \forall j \in \{n+1,\ldots,n+m\}} \ t_i \neq t_j \\ 0, & \text{in other cases} \end{cases},$$

$\widehat{F^{(n)}\dot{G}^{(m)}} := \Pr\widetilde{F^{(n)}\dot{G}^{(m)}}$, *where* $\Pr$ *is the symmetrization operator. Then* $F^{(n)} \diamond G^{(m)}$ *is the generated by* $\widehat{F^{(n)}\dot{G}^{(m)}}$ *equivalence class in* $\mathcal{H}_{\text{ext}}^{(n+m)}$, *this class is well-defined and does not depend on a choice of the representatives* $\dot{F}^{(n)}$, $\dot{G}^{(m)}$. *Moreover,*

$$(1.11) \qquad\qquad |F^{(n)} \diamond G^{(m)}|_{\text{ext}} \leq |F^{(n)}|_{\text{ext}} |G^{(m)}|_{\text{ext}}.$$

Remark 1.13. Note that, nonstrictly speaking, $F^{(n)} \diamond G^{(m)}$ is the symmetrization of the "function"

$$\widetilde{F^{(n)}G^{(m)}}(t_1, \ldots, t_n; t_{n+1}, \ldots, t_{n+m})$$

$$:= \begin{cases} F^{(n)}(t_1, \ldots, t_n)G^{(m)}(t_{n+1}, \ldots, t_{n+m}), & \text{if} \quad \substack{\forall i \in \{1,\ldots,n\}, \\ \forall j \in \{n+1,\ldots,n+m\}} \ t_i \neq t_j \\ 0, & \text{in other cases} \end{cases}$$

with respect to $n + m$ "variables".

It is obvious that the "multiplication" $\diamond$ is commutative, associative and distributive (over the field $\mathbb{C}$).

Remark 1.14. Note that for $\eta = 0$ (the Gaussian and Poissonian cases) $F^{(n)} \diamond G^{(m)} = F^{(n)}\widehat{\otimes}G^{(m)}$ (we recall that in this case $\mathcal{H}_{\text{ext}}^{(n)} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ for each $n \in \mathbb{Z}_+$).

Proposition 1.2. ([17]). *For* $F, G \in (L^2)^{-1}$ *and a holomorphic at* $(SF)(0)$ *function* $h : \mathbb{C} \to \mathbb{C}$

$$(1.12) \qquad\qquad F \Diamond G = \sum_{n=0}^{\infty} \Big\langle P_n, \sum_{k=0}^{n} F^{(k)} \diamond G^{(n-k)} \Big\rangle,$$

$$(1.13) \qquad h^{\Diamond}(F) = h_0 + \sum_{n=1}^{\infty} \Big\langle P_n, \sum_{m=1}^{n} h_m \sum_{\substack{k_1,\ldots,k_m \in \mathbb{N}: \\ k_1+\cdots+k_m=n}} F^{(k_1)} \diamond \cdots \diamond F^{(k_m)} \Big\rangle,$$

where $F^{(k)}, G^{(k)} \in \mathcal{H}_{\text{ext}}^{(k)}$ *are the kernels from decompositions* (1.4) *for* F *and* G *correspondingly,* $h_m \in \mathbb{C}$ ($m \in \mathbb{Z}_+$) *are the coefficients from decomposition* (1.10) *for* h.

Remark 1.15. It follows from (1.12) that, in particular,

$$\langle P_n, F^{(n)} \rangle \Diamond \langle P_m, G^{(m)} \rangle = \langle P_{n+m}, F^{(n)} \diamond G^{(m)} \rangle,$$

$$F \Diamond \langle P_m, G^{(m)} \rangle = \sum_{n=0}^{\infty} \langle P_{n+m}, F^{(n)} \diamond G^{(m)} \rangle.$$

The first formula can be used in order to *define* the Wick product and the Wick version of a holomorphic function (as a series) without the S-transform. Formulas (1.12) and (1.13) also can be used as definitions. Finally we note that for $F_1, \ldots, F_m \in (L^2)^{-1}$

$$F_1 \Diamond \cdots \Diamond F_m = \sum_{n=0}^{\infty} \Big\langle P_n, \sum_{k_1,\ldots,k_m \in \mathbb{Z}_+: k_1+\cdots+k_m=n} F_1^{(k_1)} \diamond \cdots \diamond F_m^{(k_m)} \Big\rangle,$$

where $F_j^{(k_j)} \in \mathcal{H}_{\text{ext}}^{(k_j)}$ ($j \in \{1, \ldots, m\}$, $k_j \in \mathbb{Z}_+$) are the kernels from decompositions (1.4) for F_j.

Let us pass to the case $\beta < 1$. First we consider a key property of the Wick product.

Theorem 1.5. ([12]). *Let* $F, G \in (L^2)^{-\beta}$. *Then the Wick product* $F \Diamond G \in (L^2)^{-\beta}$. *Moreover, the Wick multiplication is continuous in the topology of* $(L^2)^{-\beta}$: *for* $F_1, \ldots, F_m \in (L^2)^{-\beta}$, $m \in \mathbb{N}$ *there exist* $q, q' \in \mathbb{N}$ *and* $c > 0$ *such that*

$$\|F_1 \Diamond \cdots \Diamond F_m\|_{-q,-\beta} \leq c \|F_1\|_{-q',-\beta} \cdots \|F_m\|_{-q',-\beta}.$$

Now let us consider the Wick versions of holomorphic functions. As it follows from the previous theorem, if $F \in (L^2)^{-\beta}$ and $h : \mathbb{C} \to \mathbb{C}$ is a *polynomial* then $h^{\Diamond}(F) \in (L^2)^{-\beta}$. But, unfortunately, in contrast to the case $\beta = 1$, for $F \in (L^2)^{-\beta}$ and a holomorphic at $(SF)(0)$ *not polynomial* $h : \mathbb{C} \to \mathbb{C}$ it is possible $h^{\Diamond}(F) \notin (L^2)^{-\beta}$. More exactly, we have

Theorem 1.6. ([12]). *Let* $h : \mathbb{C} \to \mathbb{C}$ *be a holomorphic at* $u_0 \in \mathbb{C}$ *not polynomial function such that all coefficients* h_m, $m \in \mathbb{Z}_+$ *from the decomposition* $h(u) = \sum_{m=0}^{\infty} h_m (u - u_0)^m$ *are non-negative. Then for each* $\beta \in [0, 1)$ *there exists* $F \in (L^2)^{-\beta}$ *such that* $(SF)(0) = u_0$ *and* $h^{\Diamond}(F) \notin (L^2)^{-\beta}$.

It follows from this theorem that there are no estimates on the coefficients h_m from decomposition (1.10) for a not polynomial h that could guarantee $h^{\Diamond}(F) \in (L^2)^{-\beta}$ ($\beta < 1$) for each $F \in (L^2)^{-\beta}$. Nevertheless, we have

Theorem 1.7. ([12]). *Let* $F = \sum_{k=0}^{N} \langle P_k, F^{(k)} \rangle \in \mathcal{P}$ *and* $h(u) = \sum_{m=0}^{\infty} h_m (u - F^{(0)})^m$ *be such that* $\exists K > 0$: $\forall m \in \mathbb{N}$

$$|h_m| \leq \frac{K^m}{m^{m N^{\frac{1-\beta}{2}}}}.$$

Then $h^{\Diamond}(F) \in (L^2)^{-\beta}$.

Now we recall the construction of the extended stochastic integral on $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$.

Let $F \in (L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$. It follows from Remark 1.11 that F can be presented in the form

$$(1.14) \qquad\qquad F = \sum_{n=0}^{\infty} \langle P_n, F_{\cdot}^{(n)} \rangle, \quad F_{\cdot}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$$

with

$$\|F\|_{(L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F_{\cdot}^{(n)}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}^2 < \infty$$

for some $q \in \mathbb{Z}_+$.

Denote by 1_A the indicator of a set A.

Lemma 1.2. ([17]). *For given* $F_{\cdot}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ *and* $t \in [0, +\infty]$ *we construct the element* $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ *by the following way. Let* $\dot{F}^{(n)} \in F_{\cdot}^{(n)}$ *be some representative (function) from the equivalence class* $F_{\cdot}^{(n)}$. *We set*

$$\widetilde{\dot{F}}_{[0,t)}^{(n)}(u_1, \ldots, u_n, u) := \begin{cases} \dot{F}_u^{(n)}(u_1, \ldots, u_n) 1_{[0,t)}(u), & \text{if} \quad u \neq u_1, \ldots, u \neq u_n, \\ 0, & \text{in other cases} \end{cases},$$

$\widehat{\dot{F}}_{[0,t)}^{(n)} := \mathrm{Pr} \widetilde{\dot{F}}_{[0,t)}^{(n)}$, *where* Pr *is the symmetrization operator. Let* $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n+1)}$ *be the equivalence class in* $\mathcal{H}_{\text{ext}}^{(n+1)}$ *that is generated by* $\widehat{\dot{F}}_{[0,t)}^{(n)}$. *This class is well-defined, does not depend on the representative* $\dot{F}^{(n)}$, *and the estimate*

$$|\widehat{F}_{[0,t)}^{(n)}|_{\text{ext}} \leq |F_{\cdot}^{(n)} 1_{[0,t)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}} \leq |F_{\cdot}^{(n)}|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}$$

is valid.

Remark 1.16. Note that in the case $\eta = 0$ (i.e., if μ is the Gaussian or Poissonian measure) $\widehat{F}_{[0,t)}^{(n)} \in \mathcal{H}_{\text{ext}}^{(n+1)} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n+1}$ is, roughly speaking, the symmetrization of $F_{\cdot}^{(n)} 1_{[0,t)}(\cdot)$ with respect to $n+1$ "variables".

Let $\{M_s := \langle P_1, 1_{[0,s)} \rangle\}_{s \geq 0}$ be the Meixner random process (this process is a locally square integrable normal martingale with independent increments, see [17, 26] for more details).

Definition 1.10. Let $t \in [0, +\infty]$ and $F \in (L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$. We define the extended stochastic integral with respect to the Meixner process $\int_0^t F(s) \, \widehat{d}M_s \in (L^2)^{-\beta}$ by setting (see (1.14) and Lemma 1.2)

$$(1.15) \qquad \int_0^t F(s) \, \widehat{d}M_s := \sum_{n=0}^{\infty} \langle P_{n+1}, \widehat{F}_{[0,t)}^{(n)} \rangle.$$

It was shown in [12] that the extended stochastic integral $\int_0^t \circ \widehat{d}M$ is well-defined as a linear continuous operator acting from $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^{-\beta}$; note also that $\int_0^t \circ \widehat{d}M$ is a direct generalization of the Itô stochastic integral (see [17, 12] for details).

Finally, let us consider the interconnection between the Wick calculus and the extended stochastic integration. Denote by M' the Meixner white noise (the generalized stochastic process from Theorem 1.1). *Formally* $M'_{\cdot} = \langle P_1, \delta_{\cdot} \rangle$, where $\delta_{\cdot}$ is the Dirac delta-function (see [17] for more details).

Theorem 1.8. ([12]). *For all $t \in [0, +\infty]$ and $F \in (L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ formally defined $\int_0^t F_s \Diamond M'_s \sigma(ds)$ can be considered as a linear continuous functional on $(L^2)^{\beta}$ that coincides with $\int_0^t F(s) \, \widehat{d}M_s$, i.e.,*

$$\int_0^t F(s) \Diamond M'_s \sigma(ds) = \int_0^t F(s) \, \widehat{d}M_s \in (L^2)^{-\beta}.$$

2. Hida-type stochastic derivatives on $(L^2)_{-q}^{-\beta}$

We begin from some "technical preparation". For $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ and $f^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$ ($n > m$) we define a "pairing" $\langle F^{(n)}, f^{(m)} \rangle_{\text{ext}} \in \mathcal{H}_{\text{ext}}^{(n-m)}$ by the formula

$$(2.1) \qquad \langle \langle F^{(n)}, f^{(m)} \rangle_{\text{ext}}, g^{(n-m)} \rangle_{\text{ext}} = \langle F^{(n)}, f^{(m)} \diamond g^{(n-m)} \rangle_{\text{ext}} \quad \forall g^{(n-m)} \in \mathcal{H}_{\text{ext}}^{(n-m)}.$$

Since (see (1.11))

$$|\langle F^{(n)}, f^{(m)} \diamond g^{(n-m)} \rangle_{\text{ext}}| \leq |F^{(n)}|_{\text{ext}} |f^{(m)} \diamond g^{(n-m)}|_{\text{ext}} \leq |F^{(n)}|_{\text{ext}} |f^{(m)}|_{\text{ext}} |g^{(n-m)}|_{\text{ext}},$$

this definition is correct and

$$(2.2) \qquad |\langle F^{(n)}, f^{(m)} \rangle_{\text{ext}}|_{\text{ext}} \leq |F^{(n)}|_{\text{ext}} |f^{(m)}|_{\text{ext}}.$$

In order to define a Hida-type stochastic derivative on $(L^2)_{-q}^{-\beta}$ we need the following statement.

Lemma 2.1. ([17]). *For given $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ ($n \in \mathbb{N}$) we construct the element $F^{(n)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ by the following way. Let $\dot{F}^{(n)} \in F^{(n)}$ be some representative (function) from the equivalence class $F^{(n)}$. We consider $\dot{F}^{(n)}(\cdot)$ (i.e., separate a one argument of $\dot{F}^{(n)}$). Let $F^{(n)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ be the generated by $\dot{F}^{(n)}(\cdot)$ equivalence class in $\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$. This class is well-defined, does not depend on the representative $\dot{F}^{(n)}$, and*

$$(2.3) \qquad |F^{(n)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}} \leq |F^{(n)}|_{\text{ext}}.$$

Remark 2.1. It was shown in [15] that for each $f^{(1)} \in \mathcal{H}_{\mathbb{C}}$

$$\int_{\mathbb{R}_+} F^{(n)}(s) f^{(1)}(s) \sigma(ds) = \langle F^{(n)}, f^{(1)} \rangle_{\text{ext}} \in \mathcal{H}_{\text{ext}}^{(n-1)}.$$

Definition 2.1. Let $F \in (L^2)_{-q}^{-\beta}$, $t \in [0, +\infty)$. We define the Hida-type stochastic derivative $1_{[0,t)}(\cdot)\partial.F \in (L^2)_{-q-1}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ by setting

$$(2.4) \quad 1_{[0,t)}(\cdot)\partial.F := \sum_{n=1}^{\infty} n \langle P_{n-1}, 1_{[0,t)}(\cdot)F^{(n)}(\cdot) \rangle \equiv \sum_{n=0}^{\infty}(n+1)\langle P_n, 1_{[0,t)}(\cdot)F^{(n+1)}(\cdot)\rangle,$$

where the kernels $F^{(n)}(\cdot) \in \mathcal{H}_{\text{ext}}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ are constructed in Lemma 2.1 starting from the kernels $F^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ from decomposition (1.4) for F. In the case $t = +\infty$ denote $1_{[0,\infty)}(\cdot)\partial.$ by $\partial.$.

Since (see (2.3))

$$\|1_{[0,t)}(\cdot)\partial.F\|_{(L^2)_{-q-1}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}}^2 = \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-(q+1)n}(n+1)^2|1_{[0,t)}(\cdot)F^{(n+1)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}^2$$

$$= \sum_{n=0}^{\infty}((n+1)!)^{1-\beta}2^{-qn}[(n+1)^{1+\beta}2^{-n}]|1_{[0,t)}(\cdot)F^{(n+1)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}^2$$

$$\leq 2^q c \sum_{n=0}^{\infty}((n+1)!)^{1-\beta}2^{-q(n+1)}|F^{(n+1)}|_{\text{ext}}^2 \leq 2^q c \|F\|_{-q,-\beta}^2,$$

where $c := \max_{n \in \mathbb{Z}_+}[(n+1)^{1+\beta}2^{-n}]$, $1_{[0,t)}(\cdot)\partial.$ is well-defined as a linear *continuous* operator acting from $(L^2)_{-q}^{-\beta}$ to $(L^2)_{-q-1}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$.

Remark 2.2. It is obvious that one can define $1_{[0,t)}(\cdot)\partial.$ by formula (2.4) as a linear continuous operator acting from $(L^2)^{-\beta}$ to $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$.

Remark 2.3. Note that Definition 2.1 is a *direct generalization* of the definition of the stochastic derivative $1_{[0,t)}(\cdot)\partial.$ on (L^2), see [17]. In the Gaussian analysis such a stochastic derivative is called the *Hida derivative*, therefore the term "Hida-type stochastic derivative" for defined here $1_{[0,t)}(\cdot)\partial.$ is natural.

Sometimes it can be convenient to consider $1_{[0,t)}(\cdot)\partial.$ as an operator acting from $(L^2)_{-q}^{-\beta}$ to $(L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ (unfortunately, in this case $1_{[0,t)}(\cdot)\partial.$ is not continuous). Now we give the corresponding definition.

Definition 2.2. For each $t \in [0, +\infty]$ we define the Hida-type stochastic derivative $1_{[0,t)}(\cdot)\partial. : (L^2)_{-q}^{-\beta} \to (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ with the domain

$$(2.5) \quad \begin{aligned} &\text{dom}\left(1_{[0,t)}(\cdot)\partial.\right) \\ &:= \left\{ F \in (L^2)_{-q}^{-\beta} : \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}(n+1)^2|F^{(n+1)}(\cdot)1_{[0,t)}(\cdot)|_{\mathcal{H}_{\text{ext}}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}^2 < \infty \right\} \end{aligned}$$

by formula (2.4).

Since for $\beta = 0$ and $q = 0$ $(L^2)_{-0}^{-0} = (L^2)$, it is natural to wait that properties of $1_{[0,t)}(\cdot)\partial.$ from Definition 2.2 are similar to the properties of $1_{[0,t)}(\cdot)\partial. : (L^2) \to (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ (see [17, 15]). In fact, now we will prove the corresponding statements. First we need the following

Definition 2.3. For each $t \in [0, +\infty]$ we define the extended stochastic integral $\int_0^t \circ \widehat{d}M$: $(L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C} \to (L^2)_q^\beta$ with

$$(2.6) \quad \mathrm{dom}\left(\int_0^t \circ \widehat{d}M\right) := \left\{ f \in (L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C} : \sum_{n=0}^\infty ((n+1)!)^{1+\beta} 2^{q(n+1)} |\widehat{f}_{[0,t)}^{(n)}|_{\mathrm{ext}}^2 < \infty \right\}$$

by the formula (cf. (1.15))

$$\int_0^t f(s)\, \widehat{d}M_s := \sum_{n=0}^\infty \langle P_{n+1}, \widehat{f}_{[0,t)}^{(n)} \rangle,$$

where $\widehat{f}_{[0,t)}^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n+1)}$, $n \in \mathbb{Z}_+$ are obtained in Lemma 1.2 from the kernels $f.^{(n)} \in \mathcal{H}_{\mathrm{ext}}^{(n)} \otimes \mathcal{H}_\mathbb{C}$ from the decomposition (cf. (1.14)) $f = \sum_{n=0}^\infty \langle P_n, f.^{(n)} \rangle$.

Theorem 2.1. *For each $t \in [0, +\infty]$ the Hida-type stochastic derivative $1_{[0,t)}(\cdot)\partial.$: $(L^2)_{-q}^{-\beta} \to (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_\mathbb{C}$ and the extended stochastic integral $\int_0^t \circ \widehat{d}M : (L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C} \to (L^2)_q^\beta$ are adjoint one to another, i.e.,*

$$(2.7) \qquad 1_{[0,t)}(\cdot)\partial. = \int_0^t \circ \widehat{d}M^*; \quad \int_0^t \circ \widehat{d}M = \partial.^*(\circ 1_{[0,t)}(\cdot)).$$

In particular,

$$\partial. = \int_0^\infty \circ \widehat{d}M^*; \quad \int_0^\infty \circ \widehat{d}M = \partial.^*.$$

Equalities (2.7) can be accepted as definitions of the Hida-type stochastic derivative and the extended stochastic integral.

Proof. First we note that $\int_0^t \circ \widehat{d}M^*$ and $\partial.^*(\circ 1_{[0,t)}(\cdot))$ are well-defined because

$$\mathrm{dom}\left(\int_0^t \circ \widehat{d}M\right)$$

(see (2.6)) and $\mathrm{dom}\left(1_{[0,t)}(\cdot)\partial.\right)$ (see (2.5)) are dense in $(L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C}$ and $(L^2)_{-q}^{-\beta}$ correspondingly.

By analogy with the proof of Theorem 3.2 in [17] one can show that for all $F \in \mathrm{dom}\left(1_{[0,t)}(\cdot)\partial.\right)$ and $f \in \mathrm{dom}\left(\int_0^t \circ \widehat{d}M\right)$

$$(2.8) \qquad \langle\!\langle F, \int_0^t f(s)\, \widehat{d}M_s \rangle\!\rangle = \int_0^t \langle\!\langle \partial_s F, f(s) \rangle\!\rangle \sigma(ds) \equiv (1_{[0,t)}(\cdot)\partial.F, f)_{(L^2)\otimes\mathcal{H}_\mathbb{C}}.$$

It remains to prove that a) $\mathrm{dom}\left(1_{[0,t)}(\cdot)\partial.\right) = \mathrm{dom}\left(\int_0^t \circ \widehat{d}M^*\right)$, and b) $\mathrm{dom}\left(\int_0^t \circ \widehat{d}M\right) = \mathrm{dom}\left(\partial.^*(\circ 1_{[0,t)}(\cdot))\right)$.

a) By definition, $F \in \mathrm{dom}\left(\int_0^t \circ \widehat{d}M^*\right) \subset (L^2)_{-q}^{-\beta}$ if and only if $(L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C} \supset \mathrm{dom}\left(\int_0^t \circ \widehat{d}M\right) \ni f \mapsto \langle\!\langle F, \int_0^t f(s)\, \widehat{d}M_s \rangle\!\rangle$ is a linear continuous functional. The last is possible if and only if $\exists H \in (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_\mathbb{C}$ such that $\langle\!\langle F, \int_0^t f(s)\, \widehat{d}M_s \rangle\!\rangle = (H, f)_{(L^2)\otimes\mathcal{H}_\mathbb{C}}$. But it follows from (2.8) that this condition is fulfilled if and only if $F \in (L^2)_{-q}^{-\beta}$ and $\|1_{[0,t)}(\cdot)\partial.F\|_{(L^2)_{-q}^{-\beta}\otimes\mathcal{H}_\mathbb{C}} < \infty$, i.e., (see (2.5)) $\mathrm{dom}\left(1_{[0,t)}(\cdot)\partial.\right) = \mathrm{dom}\left(\int_0^t \circ \widehat{d}M^*\right)$.

b) By definition, $f \in \mathrm{dom}\left(\partial.^*(\circ 1_{[0,t)}(\cdot))\right) \subset (L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C}$ if and only if $(L^2)_q^{-\beta} \supset \mathrm{dom}\left(1_{[0,t)}(\cdot)\partial.\right) \ni F \mapsto (1_{[0,t)}(\cdot)\partial.F, f)_{(L^2)\otimes\mathcal{H}_\mathbb{C}}$ is a linear continuous functional. The last is possible if and only if $\exists h \in (L^2)_q^\beta$ such that $(1_{[0,t)}(\cdot)\partial.F, f)_{(L^2)\otimes\mathcal{H}_\mathbb{C}} = \langle\!\langle F, h \rangle\!\rangle$. But it follows from (2.8) that this condition is fulfilled if and only if $f \in (L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C}$ and $\|\int_0^t f(s)\, \widehat{d}M_s\|_{q,\beta} < \infty$, i.e., (see (2.6)) $\mathrm{dom}\left(\int_0^t \circ \widehat{d}M\right) = \mathrm{dom}\left(\partial.^*(\circ 1_{[0,t)}(\cdot))\right)$. $\square$

Corollary. *The operators* $1_{[0,t)}(\cdot)\partial. : (L^2)^{-\beta}_{-q} \to (L^2)^{-\beta}_{-q} \otimes \mathcal{H}_{\mathbb{C}}$ *and* $\int_0^t \circ \widehat{d}M : (L^2)^{\beta}_q \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^{\beta}_q$ *are closed.*

Remark 2.4. The results of Theorem 2.1 hold true if we consider $1_{[0,t)}(\cdot)\partial. : (L^2)^{-\beta}_{-q} \to (L^2)^{-\beta}_{-q-1} \otimes \mathcal{H}_{\mathbb{C}}$ and $\int_0^t \circ \widehat{d}M : (L^2)^{\beta}_{q+1} \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^{\beta}_q$ (so as in the case $1_{[0,t)}(\cdot)\partial. : (L^2)^{-\beta} \to (L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ and correspondingly $\int_0^t \circ \widehat{d}M : (L^2)^{\beta} \otimes \mathcal{H}_{\mathbb{C}} \to (L^2)^{\beta}$), now all these operators are continuous.

Remark 2.5. In [15] elements of the so-called Clark-Ocone theory in the case $F \in (L^2)^{-1}$ were considered. One can easily see that all corresponding results from [15] hold true in the case $F \in (L^2)^{-\beta}$.

3. Stochastic differential operators on $(L^2)^{-\beta}_{-q}$

By analogy with [3, 11, 15] we consider in this section stochastic differential operators that are closely connected with the Hida-type stochastic derivatives (see Proposition 3.3 below) and convenient for construction of an analysis on generalized functions spaces.

Definition 3.1. Let $F \in (L^2)^{-\beta}_{-q}$, $n \in \mathbb{Z}_+$, $f^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$. We define

$$(3.1) \qquad (\mathcal{D}^n F)(f^{(n)}) := \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} \langle P_m, \langle F^{(m+n)}, f^{(n)} \rangle_{\text{ext}} \rangle \in (L^2)^{-\beta}_{-q-1},$$

where $F^{(m)} \in \mathcal{H}^{(m)}_{\text{ext}}$ ($m \in \mathbb{Z}_+$) are the kernels from decomposition (1.4) for F. Denote $\mathcal{D} := \mathcal{D}^1$, this operator is called the *generalized stochastic derivative*.

Since (see (1.8), (2.2))

$$\|(\mathcal{D}^n F)(f^{(n)})\|^2_{-q-1,-\beta} = \sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-(q+1)m} \left(\frac{(m+n)!}{m!}\right)^2 |\langle F^{(m+n)}, f^{(n)} \rangle_{\text{ext}}|^2_{\text{ext}}$$

$$= 2^{(q+1)n} \sum_{m=0}^{\infty} ((m+n)!)^{1-\beta} 2^{-(q+1)(m+n)} \left(\frac{(m+n)!}{m!}\right)^{1+\beta} |\langle F^{(m+n)}, f^{(n)} \rangle_{\text{ext}}|^2_{\text{ext}}$$

$$\leq 2^{(q+1)n} |f^{(n)}|^2_{\text{ext}} \sum_{m=0}^{\infty} ((m+n)!)^{1-\beta} 2^{-q(m+n)} \left[2^{-(m+n)} \left(\frac{(m+n)!}{m!}\right)^{1+\beta}\right] |F^{(m+n)}|^2_{\text{ext}}$$

$$\leq 2^{(q+1)n} |f^{(n)}|^2_{\text{ext}} c \sum_{m=0}^{\infty} ((m+n)!)^{1-\beta} 2^{-q(m+n)} |F^{(m+n)}|^2_{\text{ext}}$$

$$\leq 2^{(q+1)n} |f^{(n)}|^2_{\text{ext}} c \|F\|^2_{-q,-\beta},$$

where $c := \max_{m \in \mathbb{Z}_+} [2^{-(m+n)}(\frac{(m+n)!}{m!})^{1+\beta}]$, $(\mathcal{D}^n \circ)(f^{(n)})$ is well-defined as a linear *continuous* operator acting from $(L^2)^{-\beta}_{-q}$ to $(L^2)^{-\beta}_{-q-1}$. Moreover, for each $F \in (L^2)^{-\beta}_{-q}$ $(\mathcal{D}^n F)(\circ)$ is a linear continuous operator acting from $\mathcal{H}^{(n)}_{\text{ext}}$ to $(L^2)^{-\beta}_{-q-1}$.

Remark 3.1. It is obvious that for each $f^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$ $(\mathcal{D}^n \circ)(f^{(n)})$ can be defined by (3.1) as a linear continuous operator acting in $(L^2)^{-\beta}$; in this case for each $F \in (L^2)^{-\beta}$ $(\mathcal{D}^n F)(\circ)$ is a linear continuous operator acting from $\mathcal{H}^{(n)}_{\text{ext}}$ to $(L^2)^{-\beta}$.

Since for $\beta < 1$ $(L^2)^{-\beta}_{-q} \subset (L^2)^{-1}_{-q}$, the following statements directly from the corresponding results of [15] follow.

Proposition 3.1. *For* $g_1^{(1)}, g_2^{(1)}, \dots, g_n^{(1)} \in \mathcal{H}_{\mathbb{C}} = \mathcal{H}^{(1)}_{\text{ext}}$

$$(\underbrace{\mathcal{D}(\dots(\mathcal{D}((\mathcal{D}F)(g_1^{(1)})))(g_2^{(1)})\dots)}_{n \text{ times}})(g_n^{(1)}) = (\mathcal{D}^n F)(g_1^{(1)} \diamond g_2^{(1)} \diamond \cdots \diamond g_n^{(1)}).$$

Proposition 3.2. *For each $F \in (L^2)^{-\beta}_{-q}$ the kernels $F^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$, $n \in \mathbb{Z}_+$ from decomposition (1.4) can be presented in the form*

$$F^{(n)} = \frac{1}{n!}\mathbf{E}(\mathcal{D}^n F),$$

where $\mathbf{E}\circ := \langle\!\langle \circ, 1 \rangle\!\rangle$ is an expectation.

Proposition 3.3. *For all $F \in (L^2)^{-\beta}_{-q}$, $f^{(1)} \in \mathcal{H}_{\mathbb{C}}$*

$$\int_{\mathbb{R}_+} \partial_s F \cdot f^{(1)}(s)\sigma(ds) = (\mathcal{D}F)(f^{(1)}).$$

Remark 3.2. Note that *formally $\partial_\circ = (\mathcal{D}\circ)(\delta.)$, where δ is the Dirac delta-function.*

Proposition 3.4. *The adjoint to $\mathcal{D}^n$ operator has the form*

$$(3.2) \qquad (\mathcal{D}^n g)(f^{(n)})^* = \sum_{m=0}^{\infty} \langle P_{m+n}, g^{(m)} \diamond f^{(n)} \rangle = g \Diamond \langle P_n, f^{(n)} \rangle \in (L^2)^{\beta}_q,$$

where $g \in (L^2)^{\beta}_{q+1}$, $f^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$, and $g^{(m)} \in \mathcal{H}^{(m)}_{\text{ext}}$ $(m \in \mathbb{Z}_+)$ are the kernels from decomposition (1.6) for g.

Theorem 3.1. *(cf. (2.8)). For all $F \in (L^2)^{-\beta}_{-q}$, $g \in (L^2)^{\beta}_{q+1}$ and $f^{(1)} \in \mathcal{H}_{\mathbb{C}}$*

$$\left\langle\!\!\left\langle F, \int_0^{\infty} g \cdot f^{(1)}(s)\, \widehat{d}M_s \right\rangle\!\!\right\rangle = \langle\!\langle F, g\Diamond\langle P_1, f^{(1)}\rangle \rangle\!\rangle = \langle\!\langle F, (\mathcal{D}g)(f^{(1)})^* \rangle\!\rangle = \langle\!\langle (\mathcal{D}F)(f^{(1)}), g \rangle\!\rangle.$$

Remark 3.3. The equality $\int_0^{\infty} g \cdot f^{(1)}(s)\, \widehat{d}M_s = g\Diamond\langle P_1, f^{(1)}\rangle = (\mathcal{D}g)(f^{(1)})^*$ can be generalized in the following sense. For a general $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$, $g \in (L^2)^{\beta}_{q+1}$ one can **define** a *multiple* extended stochastic integral

$$\int_{\mathbb{R}^n_+} g \cdot f^{(n)}(u_1, \ldots, u_n)\, \widehat{d}M_{u_1} \ldots \widehat{d}M_{u_n} := g\Diamond\langle P_n, f^{(n)}\rangle = (\mathcal{D}^n g)(f^{(n)})^* \in (L^2)^{\beta}_q$$

(see (3.2)). It is easy to see that for $f^{(n)} = f^{(1)}_1 \diamond \cdots \diamond f^{(1)}_n$, $f^{(1)}_1, \ldots, f^{(1)}_n \in \mathcal{H}_{\mathbb{C}}$ this integral is a *repeated* extended stochastic one: $(\mathcal{D}^n g)(f^{(1)}_1 \diamond \cdots \diamond f^{(1)}_n)^* = \int_0^{\infty}(\cdots(\int_0^{\infty}(\int_0^{\infty} g \cdot f^{(1)}_1(u_1)\, \widehat{d}M_{u_1})f^{(1)}_2(u_2)\, \widehat{d}M_{u_2})\ldots)f^{(1)}_n(u_n)\, \widehat{d}M_{u_n}$ (cf. Proposition 3.1).

Theorem 3.2. *The generalized stochastic derivative $\mathcal{D}$ is a differentiation with respect to the Wick product, i.e., $\forall F, G \in (L^2)^{-\beta}$ we have*

$$\mathcal{D}(F\Diamond G) = (\mathcal{D}F)\Diamond G + F\Diamond(\mathcal{D}G) \in (L^2)^{-\beta}.$$

Corollary. *Let $n \in \mathbb{N}$, $F, F_1, \ldots, F_n \in (L^2)^{-\beta}$, and $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $(SF)(0)$ function. Then*

$$\mathcal{D}(F_1\Diamond\cdots\Diamond F_n) = \sum_{k=1}^{n} F_1\Diamond\cdots\Diamond F_{k-1}\Diamond(\mathcal{D}F_k)\Diamond F_{k+1}\Diamond\cdots\Diamond F_n \in (L^2)^{-\beta},$$

$$\mathcal{D}(F^{\Diamond n}) = nF^{\Diamond(n-1)}\Diamond(\mathcal{D}F) \in (L^2)^{-\beta},$$

$$\mathcal{D}h^{\Diamond}(F) = h'^{\Diamond}(F)\Diamond(\mathcal{D}F) \in (L^2)^{-1},$$

where h' denotes the usual derivative of h. Under some additional conditions (for example, if h is a polynomial, or $F \in \mathcal{P}$ and h satisfies the conditions of Theorem 1.7) $\mathcal{D}h^{\Diamond}(F) \in (L^2)^{-\beta}$.

Theorem 3.3. *Let $F \in (L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$. Then $\forall t \in [0, +\infty]$ and $\forall f^{(n)} \in \mathcal{H}^{(n)}_{\text{ext}}$*

$$\left(\mathcal{D}\int_0^t F_s\, \widehat{d}M_s\right)(f^{(n)}) = \int_0^t (\mathcal{D}F_s)(f^{(n)})\, \widehat{d}M_s + \int_0^t F_s f^{(n)}(s)\sigma(ds) \in (L^2)^{-\beta}.$$

Let us define a set of formal series B_β (a characterization set of $(L^2)^{-\beta}$ in terms of the S-transform) by setting $B_\beta := S\big((L^2)^{-\beta}\big) \equiv \{K | \exists F \in (L^2)^{-\beta} : K = SF\}$.

Definition 3.2. Let $g \in \mathcal{H}_\mathbb{C}$. We define a "directional derivative" $D_g^\diamond : B_\beta \to B_\beta$ by setting for $(SF)(\cdot) = \sum_{m=0}^\infty \langle F^{(m)}, \cdot^{\otimes m}\rangle_{\text{ext}} \in B_\beta$

$$(D_g^\diamond SF)(\cdot) := \sum_{m=1}^\infty m\langle F^{(m)}, \cdot^{\otimes(m-1)} \diamond g\rangle_{\text{ext}} \equiv \sum_{m=0}^\infty (m+1)\langle\langle F^{(m+1)}, g\rangle_{\text{ext}}, \cdot^{\otimes m}\rangle_{\text{ext}} \in B_\beta.$$

It is obvious that *formally* $S^{-1}(D_g^\diamond SF) = (\mathcal{D}F)(g)$, therefore this definition is correct and, moreover, we have

Theorem 3.4. *The generalized stochastic derivative* $(\mathcal{D}\circ)(g)$ *is a pre-image of the "directional derivative"* $D_g^\diamond$ *of* $S\circ$ *under the* S*-transform, i.e., for all* $F \in (L^2)^{-\beta}$ *and* $g \in \mathcal{H}_\mathbb{C}$

$$(\mathcal{D}F)(g) = S^{-1}(D_g^\diamond SF) \in (L^2)^{-\beta}.$$

Remark 3.4. Note that if $\eta = 0$ (the Gaussian and Poissonian cases) then B_β consists of holomorphic at zero functions and $D_g^\diamond$ is the usual directional derivative.

Sometimes it can be convenient to consider $(\mathcal{D}^n\circ)(f^{(n)})$, $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ as an operator acting acting in $(L^2)_{-q}^{-\beta}$ (for example, $(L^2)_{-0}^{-0} = (L^2)$), this very important particular case was considered in details in [17, 15]). We accept the following

Definition 3.3. Let $n \in \mathbb{Z}_+$, $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$. We define $(\mathcal{D}^n\circ)(f^{(n)}) : (L^2)_{-q}^{-\beta} \to (L^2)_{-q}^{-\beta}$ with the domain

(3.3)
$$\text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})\big)$$
$$= \left\{ F \in (L^2)_{-q}^{-\beta} : \sum_{m=0}^\infty (m!)^{1-\beta} 2^{-qm}\left(\frac{(m+n)!}{m!}\right)^2 |\langle F^{(m+n)}, f^{(n)}\rangle_{\text{ext}}|_{\text{ext}}^2 < \infty \right\}$$

$(F^{(m)} \in \mathcal{H}_{\text{ext}}^{(m)}$, $m \in \mathbb{Z}_+$ are the kernels from decomposition (1.4) for F) by formula (3.1).

Theorem 3.5. *The operator* $(\mathcal{D}^n\circ)(f^{(n)}) : (L^2)_{-q}^{-\beta} \to (L^2)_{-q}^{-\beta}$ $(f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)})$ *with domain* (3.3) *is closed.*

Proof. Let us prove that there exists the second adjoint to $(\mathcal{D}^n\circ)(f^{(n)})$ operator $(\mathcal{D}^n\circ)(f^{(n)})^{**}$ and $(\mathcal{D}^n\circ)(f^{(n)}) = (\mathcal{D}^n\circ)(f^{(n)})^{**}$ (as is well known, an adjoint operator is closed).

Since $\text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})\big)$ is dense in $(L^2)_{-q}^{-\beta}$, the operator $(\mathcal{D}^n\circ)(f^{(n)})^* : (L^2)_q^\beta \to (L^2)_q^\beta$ is well-defined and, obviously, is given by (3.2). Therefore $\text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})^*\big) = \{g \in (L^2)_q^\beta : \|g\diamond\langle P_n, f^{(n)}\rangle\|_{q,\beta}^2 = \sum_{m=0}^\infty ((m+n)!)^{1+\beta} 2^{q(m+n)} |g^{(m)} \diamond f^{(n)}|_{\text{ext}}^2 < \infty\}$ is dense in $(L^2)_q^\beta$, hence $(\mathcal{D}^n\circ)(f^{(n)})^{**} : (L^2)_{-q}^{-\beta} \to (L^2)_{-q}^{-\beta}$ is well-defined and it remains to show that $\text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})\big) = \text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})^{**}\big)$. By definition, $F \in \text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})^{**}\big)$ if and only if $(L^2)_q^\beta \supset \text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})^*\big) \ni g \mapsto \langle\langle F, (\mathcal{D}^n g)(f^{(n)})^*\rangle\rangle$ is a linear continuous functional. The last is possible if and only if $\exists H \in (L^2)_{-q}^{-\beta}$ such that $\langle\langle F, (\mathcal{D}^n g)(f^{(n)})^*\rangle\rangle = \langle\langle H, g\rangle\rangle$. Using the definition of $(L^2)_q^\beta$, Theorem 1.4, (1.9), (3.2) and (2.1) one can show that this H must have the form $\sum_{m=0}^\infty \frac{(m+n)!}{m!}\langle P_m, \langle F^{(m+n)}, f^{(n)}\rangle_{\text{ext}}\rangle$, hence (see (3.3) and (1.8)) $\text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})\big) = \text{dom}\big((\mathcal{D}^n\circ)(f^{(n)})^{**}\big)$. $\square$

Remark 3.5. Let $G_n := \{F \in (L^2)_{-q}^{-\beta} : \sum_{m=0}^\infty (m!)^{1-\beta} 2^{-qm}(\frac{(m+n)!}{m!})^2 |F^{(m+n)}|_{\text{ext}}^2 < \infty\}$. For each $f^{(n)} \in \mathcal{H}_{\text{ext}}^{(n)}$ we define the operator $(\mathcal{D}^n\circ)(f^{(n)}) : G_n \to (L^2)_{-q}^{-\beta}$ by formula (3.1). It follows from Theorem 3.5 that this operator (as an operator acting in the topological

space $(L^2)^{-\beta}_{-q}$ is *closable*. Moreover, for each $F \in G_n$ the operator $(\mathcal{D}^n F)(\circ) : \mathcal{H}^{(n)}_{\text{ext}} \to (L^2)^{-\beta}_{-q}$ is continuous:

$$\|(\mathcal{D}^n F)(f^{(n)})\|^2_{-q,-\beta} = \sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |\langle F^{(m+n)}, f^{(n)} \rangle_{\text{ext}}|^2_{\text{ext}}$$

$$\leq |f^{(n)}|^2_{\text{ext}} \sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-qm} \Big(\frac{(m+n)!}{m!}\Big)^2 |F^{(m+n)}|^2_{\text{ext}}$$

(see (1.8), (2.2)).

Remark 3.6. In the case $\beta = 1$ one can consider the so-called Kondratiev-type spaces of *nonregular* test and generalized functions, and introduce and study a stochastic integral, stochastic differential operators, elements of a Wick calculus etc. on these spaces, see [17] for details. If $\beta < 1$ then *formal* studying of "nonregular" spaces and the mentioned objects on these spaces (by analogy with our considerations here) is possible; but such a studying is not unreasonable in the Gaussian and Poissonian cases only because if $\eta \neq 0$ then the nonregular "test functions" spaces are not embedded in (L^2).

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