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VANISHING OF THE FIRST (σ,τ)-COHOMOLOGY GROUP OF TRIANGULAR BANACH ALGEBRAS

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Abstract. In this paper, we define the first topological (σ,τ)-cohomology group and examine vanishing of the first (σ,τ)-cohomology groups of certain triangular Banach algebras. We apply our results to study the (σ,τ)-weak amenability and (σ,τ)-amenability of triangular Banach algebras.

1. Introduction and preliminaries

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two unital algebras with units $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively. Recall that a vector space $\mathcal{M}$ is a unital $\mathcal{A}-\mathcal{B}$-bimodule whenever it is both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module satisfying

$$a(mb) = (am)b, \quad 1_{\mathcal{A}}m = m1_{\mathcal{B}} = m \quad (a, b \in \mathcal{A},\ m \in \mathcal{M}).$$

Then $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}) = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix};\ a \in \mathcal{A},\ m \in \mathcal{M},\ b \in \mathcal{B} \right\}$ equipped with the usual 2×2 matrix-like addition and matrix-like multiplication is an algebra.

An algebra $\mathcal{T}$ is called a triangular algebra if there exist algebras $\mathcal{A}$ and $\mathcal{B}$ and nonzero $\mathcal{A}-\mathcal{B}$-bimodule $\mathcal{M}$ such that $\mathcal{T}$ is (algebraically) isomorphic to $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. For example, the algebra $\mathcal{T}_n$ of $n\times n$ upper triangular matrices over the complex field $\mathbb{C}$, may be viewed as a triangular algebra when $n > 1$. In fact, if $n > k$, we have $\mathcal{T}_n = \mathrm{Tri}(\mathcal{T}_{n-k}, M_{n-k,k}(\mathbb{C}), \mathcal{T}_k)$ in which $M_{n-k,k}(\mathbb{C})$ is the space of $(n-k)\times k$ complex matrices, cf. [1].

Let $\mathcal{T}$ be a triangular algebra. If $1 = \begin{bmatrix} u & p \\ 0 & v \end{bmatrix}$, and $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is denoted by $a\oplus b$, then it can be easily verified that $e = u\oplus 0$ is an idempotent such that $(1-e)\mathcal{T}e = 0$ but $e\mathcal{T}(1-e) \neq 0$. Conversely, if there exists an idempotent $e \in \mathcal{T}$ such that $(1-e)\mathcal{T}e = 0$ but $e\mathcal{T}(1-e) \neq 0$. Then the mapping $x \mapsto \begin{bmatrix} exe & ex(1-e) \\ 0 & (1-e)x(1-e) \end{bmatrix}$ is an isomorphism between $\mathcal{T}$ and $\mathrm{Tri}(e\mathcal{T}e, e\mathcal{T}(1-e), (1-e)\mathcal{T}(1-e))$; cf. [1].

By a triangular Banach algebra we mean a Banach algebra A which is also a triangular algebra. Many algebras such as upper triangular Banach algebras [4], nest algebras [2], semi-nest algebras [3], and joins [6] are triangular algebras.

Following [1], consider a triangular Banach algebra $\mathcal{T}$ with an idempotent e satisfying $e\mathcal{T}(1-e) \neq 0$ and $(1-e)\mathcal{T}e = 0$. Put $\mathcal{A} = e\mathcal{T}e, \mathcal{B} = (1-e)\mathcal{T}(1-e)$ and $\mathcal{M} = e\mathcal{T}(1-e)$. Then $\mathcal{A}$ and $\mathcal{B}$ are closed subalgebras of $\mathcal{T}$, $\mathcal{M}$ is a Banach $\mathcal{A}-\mathcal{B}$-bimodule, and $\mathcal{T} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. Conversely, given Banach algebras $(\mathcal{A}, \|.\|_{\mathcal{A}})$ and $(\mathcal{B}, \|.\|_{\mathcal{B}})$ and an $\mathcal{A}-\mathcal{B}$-bimodule $\mathcal{M}$, then the triangular algebra $\mathcal{T} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ is a Banach algebra with respect to the norm given by $\left\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \right\|_{\mathcal{T}} = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}$. It is not hard

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to show that each norm $\|.\|$ making $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ into a triangular Banach algebra is equivalent to $\|.\|_{\mathcal{T}}$, if the natural restrictions of $\|.\|$ to $\mathcal{A},\mathcal{B}$ and $\mathcal{M}$ are equivalent to the given norms on $\mathcal{A},\mathcal{B}$ and $\mathcal{M}$, respectively. See also [12, 16]

The concept of topological cohomology arose from the problems concerning extensions by H. Kamowitz [11], derivations by R. V. Kadison and J. R. Ringrose [9, 10] and amenability by B. E. Johnson [8] and has been extensively developed by A. Ya. Helemskii and his school [7]. The reader is referred to [7, 19] for undefined notation and terminology.

Let $\mathcal{A}$ be a Banach algebra and σ,τ be continuous homomorphisms on $\mathcal{A}$. Suppose that $\mathcal{E}$ is a Banach $\mathcal{A}$-bimodule. A linear mapping $d:\mathcal{A}\to\mathcal{E}$ is called a (σ,τ)-derivation if

$$d(ab)=d(a)\sigma(b)+\tau(a)d(b)\quad (a,b\in\mathcal{A}).$$

We mean by a σ-derivation, a (σ,σ)-derivation. For example (i) Every ordinary derivation of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule is an $id_{\mathcal{A}}$-derivation, where $id_{\mathcal{A}}$ is the identity mapping on the algebra $\mathcal{A}$. (ii) Every point derivation $d:\mathcal{A}\to\mathbb{C}$ at the character θ on $\mathcal{A}$ is a θ-derivation.

A linear mapping $d:\mathcal{A}\longrightarrow\mathcal{E}$ is called (σ,τ)-inner derivation if there exists $x\in\mathcal{E}$ such that $d(a)=\tau(a)x-x\sigma(a)\quad(a\in\mathcal{A})$. See also [13, 14, 17, 18] and references therein.

We denote the set of continuous (σ,τ)-derivations from $\mathcal{A}$ into $\mathcal{E}$ by $Z^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{E})$ and the set of inner (σ,τ)-derivations by $B^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{E})$. we define the space $H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{E})$ as the quotient space $Z^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{E})/B^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{E})$. The space $H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{E})$ is called the first $(\sigma-\tau)$-cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{E}$.

From now on, $\mathcal{A}$ and $\mathcal{B}$ denote unital Banach algebras with units $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, $\mathcal{M}$ denotes a unital Banach $\mathcal{A}-\mathcal{B}$-bimodule and $\mathcal{T}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ is the triangular matrix algebra. In addition, $\mathcal{X}$ is a unital Banach $\mathcal{T}$-bimodule, $\mathcal{X}_{\mathcal{A}\mathcal{A}}=1_{\mathcal{A}}\mathcal{X}1_{\mathcal{A}},\mathcal{X}_{\mathcal{B}\mathcal{B}}=1_{\mathcal{B}}\mathcal{X}1_{\mathcal{B}},\mathcal{X}_{\mathcal{A}\mathcal{B}}=1_{\mathcal{A}}\mathcal{X}1_{\mathcal{B}}$ and $\mathcal{X}_{\mathcal{B}\mathcal{A}}=1_{\mathcal{B}}\mathcal{X}1_{\mathcal{A}}$. For instance, with $\mathcal{X}=\mathcal{T}$ we have $\mathcal{X}_{\mathcal{A}\mathcal{A}}=\mathcal{A},\mathcal{X}_{\mathcal{B}\mathcal{B}}=\mathcal{B},\mathcal{X}_{\mathcal{A}\mathcal{B}}=\mathcal{M}$ and $\mathcal{X}_{\mathcal{B}\mathcal{A}}=0$.

In this paper, we examine vanishing of the first (σ,τ)-cohomology groups of certain triangular Banach algebras. We apply our results to investigate the (σ,τ)-weak amenability and (σ,τ)-amenability of triangular Banach algebras.

2. Vanishing of the first (σ,τ)-cohomology group

In this section, using some ideas of [5], we investigate the relation between the first (σ,τ)-cohomology of $\mathcal{T}$ with coefficients in $\mathcal{X}$ and those of $\mathcal{A}$ and $\mathcal{B}$ with coefficients in $\mathcal{X}_{\mathcal{A}\mathcal{A}}$ and $\mathcal{X}_{\mathcal{B}\mathcal{B}}$, respectively, whenever $\mathcal{X}_{\mathcal{A}\mathcal{B}}=0$ in a direct method.

We start our work by investigating the structure of bounded (σ,τ)-derivations from a triangular Banach algebra into bimodules.

Let σ and τ be two homomorphisms on $\mathcal{T}$ with the following properties:

$$\tau(1\oplus 0)=1\oplus 0,\quad \tau(0\oplus 1)=0\oplus 1; \tag{2.1}$$

$$\sigma(1\oplus 0)=1\oplus 0,\quad \sigma(0\oplus 1)=0\oplus 1. \tag{2.2}$$

The above relation implies easily that $\sigma(\mathcal{A})\subseteq\mathcal{A}$ and $\sigma(\mathcal{B})\subseteq\mathcal{B}$ if we identify $a\in\mathcal{A}$ with $\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix}$ and $b\in\mathcal{B}$ with $\begin{bmatrix} 0 & 0\\ 0 & b\end{bmatrix}$. So with no ambiguity, we can consider σ and τ as homomorphisms on $\mathcal{A}$ or $\mathcal{B}$, when it is necessary.

Now let $m\in\mathcal{M}$. If $\sigma\left(\begin{bmatrix} 0 & m\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix} a' & m'\\ 0 & b'\end{bmatrix}$, then

$$\begin{aligned}\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix} &= \sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = \sigma\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \sigma\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Hence $\sigma\left(\begin{bmatrix} 0 & \mathcal{M} \\ 0 & 0 \end{bmatrix}\right) \subseteq \begin{bmatrix} 0 & \mathcal{M} \\ 0 & 0 \end{bmatrix}$. Thus one can define $\sigma_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ by $m \mapsto m'$. To simplify the notation we denote $\sigma_{\mathcal{M}}$ by σ. Thus $\sigma\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right)$ can be written as $\begin{bmatrix} \sigma(a) & \sigma(m) \\ 0 & \sigma(b) \end{bmatrix}$.

If $\sigma_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ and $\sigma_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ are homomorphisms, then $\sigma_{\mathcal{A}} \oplus \sigma_{\mathcal{B}} : \mathcal{A} \oplus \mathcal{B} \to \mathcal{A} \oplus \mathcal{B}$ defined by $(\sigma_{\mathcal{A}} \oplus \sigma_{\mathcal{B}})(a,b) = (\sigma_{\mathcal{A}}(a), \sigma_{\mathcal{B}}(b))$ is a homomorphism. Conversely every homomorphism on $\mathcal{A} \oplus \mathcal{B}$ is of the form $\sigma_{\mathcal{A}} \oplus \sigma_{\mathcal{B}}$ for some homomorphisms $\sigma_{\mathcal{A}}$ and $\sigma_{\mathcal{B}}$ on $\mathcal{A}$ and $\mathcal{B}$, respectively.

Applying our notation, let $\delta : \mathcal{T} \to \mathcal{X}$ be a bounded (σ,τ)-derivation. Then $\delta_{\mathcal{A}} : \mathcal{A} \to 1_{\mathcal{A}}\mathcal{X}1_{\mathcal{A}}$ defined by

$$\delta_{\mathcal{A}}(a) = 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}},$$

and $\delta_{\mathcal{B}} : \mathcal{B} \to 1_{\mathcal{B}}\mathcal{X}1_{\mathcal{B}}$ defined by

$$\delta\mathcal{B}(b) = 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)1_{\mathcal{B}}$$

are bounded (σ,τ)-derivations.

Moreover, the mapping $\theta : \mathcal{M} \to 1_{\mathcal{A}}\mathcal{X}1_{\mathcal{B}}$ given by

$$\theta(m) = 1_{\mathcal{A}}\delta\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}}$$

satisfies

$$\begin{aligned}\theta(am) &= 1_{\mathcal{A}}\delta\left(\begin{bmatrix} 0 & am \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}} = 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}} \\ &= 1_{\mathcal{A}}\tau(a)\delta\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}} + 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}} \\ &= \tau(a)1_{\mathcal{A}}\delta\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}} + 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}\sigma\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) \\ &= \tau(a)\theta(m) + \delta_{\mathcal{A}}(a)\sigma(m)\end{aligned} \tag{2.3}$$

and

$$\theta(mb) = \theta(m)\sigma(b) + \tau(m)\delta_{\mathcal{B}}(b). \tag{2.4}$$

Conversely, if δ_1 and δ_2 are bounded (σ,τ)-derivations of $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{X}_{\mathcal{A}\mathcal{A}}$ and $\mathcal{X}_{\mathcal{B}\mathcal{B}}$, respectively, and $\theta : \mathcal{M} \to \mathcal{X}_{\mathcal{A}\mathcal{B}}$ is any continuous linear mapping satisfies (2.3) and (2.4), then the mapping $D\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \delta_1(a) + \delta_2(b) + \theta(m)$ defines a bounded

(σ,τ)-derivation of $\mathcal{T}$ into X, since

$$\begin{aligned}
&\tau\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)D\Big(\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big)+D\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)\sigma\Big(\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big)\\
&=\tau\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)\big(\delta_1(a')+\delta_2(b')+\theta(m')\big)\\
&+\big(\delta_1(a)+\delta_2(b)+\theta(m)\big)\sigma\Big(\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big)\\
&=\tau\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)\tau(1_{\mathcal{A}})\delta_1(a')+\delta_1(a)\sigma(1_{\mathcal{A}})\sigma\Big(\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big)\\
&+\tau\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)\tau(1_{\mathcal{B}})\delta_2(b')+\delta_2(b)\sigma(1_{\mathcal{B}})\sigma\Big(\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big)\\
&+\tau\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)\tau(1_{\mathcal{A}})\theta(m')+\theta(m)\sigma(1_{\mathcal{B}})\sigma\Big(\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big)\\
&=\tau(a)\delta_1(a')+\delta_1(a)\sigma(a')+\delta_1(a)\sigma(m')+\tau(b)\delta_2(b')\\
&+\delta_2(b)\sigma(b')+\tau(m)\delta_2(b')+\tau(a)\theta(m')+\theta(m)\sigma(b')\\
&=\delta_1(aa')+\delta_2(bb')+\theta(am')+\theta(mb')\\
&=D\Big(\begin{bmatrix} aa' & am'+mb' \\ 0 & bb' \end{bmatrix}\Big)=D\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big).
\end{aligned}$$

If $\mathcal{X}_{\mathcal{AB}}=0$, then we may assume that the linear mapping θ defined above is zero. Notice that, in this case, $\delta_{\mathcal{A}}(a)\sigma(m)=\tau(m)\delta_{\mathcal{B}}(b)=0$ for every $a\in\mathcal{A}, b\in\mathcal{B}, m\in\mathcal{M}$.

We are now ready to provide our main theorem.

Theorem 2.1. *Let* $\mathcal{X}_{\mathcal{AB}}=1_{\mathcal{A}}\mathcal{X}1_{\mathcal{B}}=0$. *Then*

$$H^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{X})=H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}_{\mathcal{AA}})\oplus H^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{X}_{\mathcal{BB}}).$$

Proof. Suppose that $\mathcal{X}_{\mathcal{AB}}=0$ and consider the linear mapping

$$\rho: Z^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{X})\to H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}_{\mathcal{AA}})\oplus H^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{X}_{\mathcal{BB}})$$

defined by

$$\delta\mapsto\big(\delta_{\mathcal{A}}+N^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}_{\mathcal{AA}}),\delta_{\mathcal{B}}+N^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{X}_{\mathcal{BB}})\big).$$

If $\delta_1\in Z^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}_{\mathcal{AA}})$ and $\delta_2\in Z^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{X}_{\mathcal{BB}})$, then $D\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)=\delta_1(a)+\delta_2(b)$ is a (σ,τ)-derivation from $\mathcal{T}$ into $\mathcal{X}$ and

$$\begin{aligned}
\rho(D)&=\big(D_A+N^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}_{\mathcal{AA}}),D_{\mathcal{B}}+N^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{X}_{\mathcal{BB}})\big)\\
&=\big(\delta_1+N^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}_{\mathcal{AA}}),\delta_2+N^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{X}_{\mathcal{BB}})\big).
\end{aligned}$$

The last equation is deduced from the fact that

$$D_{\mathcal{A}}(a)=1_{\mathcal{A}}\big(\delta_1(a)+\delta_2(0)\big)1_{\mathcal{A}}=\delta_1(a),$$

and

$$\delta_{\mathcal{B}}(b)=1_{\mathcal{B}}\big(\delta_1(0)+\delta_2(b)\big)1_{\mathcal{B}}=\delta_2(b).$$

Thus ρ is surjective.

If $\delta \in \ker\rho$, then $\delta_{\mathcal{A}} \in N^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}_{\mathcal{A}\mathcal{A}})$ and $\delta_{\mathcal{B}} \in N^1_{(\sigma,\tau)}(\mathcal{B}, \mathcal{X}_{\mathcal{B}\mathcal{B}})$. Then $\delta_{\mathcal{A}}(a) = \tau(a)x - x\sigma(a)$ for some $x \in \mathcal{X}_{\mathcal{A}\mathcal{A}}$ and $\delta_{\mathcal{B}}(b) = \tau(b)y - y\sigma(b)$ for some $y \in \mathcal{X}_{\mathcal{B}\mathcal{B}}$. Then

$$\begin{aligned} D\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) &= \delta_{\mathcal{A}}(a) + \delta_{\mathcal{B}}(b) \\ &= \big(\tau(a)x - x\sigma(a)\big) + \big(\tau(b)y - y\sigma(b)\big) \\ &= \big(\tau(a) + \tau(m) + \tau(b)\big)(x+y) - (x+y)\big(\sigma(a) + \sigma(m) + \sigma(b)\big) \\ &= \tau\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right)(x+y) - (x+y)\sigma\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right). \end{aligned}$$

Thus $D \in N^1_{(\sigma,\tau)}(\mathcal{T}, \mathcal{X})$.

It is straightforward to show that

$$\begin{aligned} \delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) &= 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}} + 1_{\mathcal{B}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}} + 1_{\mathcal{B}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}} \\ &= 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}} + 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}\sigma(a). \end{aligned}$$

Similarly,

$$\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)1_{\mathcal{B}} - \tau(b)1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}},$$

and also

$$\begin{aligned} \delta\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) &= \delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) \\ &= 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}\sigma\left(\begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix}\right) \\ &\quad - \tau\left(\begin{bmatrix} a & m \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}. \end{aligned}$$

These follow that

$$\begin{aligned} &(\delta - D)\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) \\ &= \delta\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) - 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}} - 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)1_{\mathcal{B}} \\ &= \left(\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) - 1_{\mathcal{A}}\delta\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}\right) + \delta\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) \\ &+ \left(\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) - 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)1_{\mathcal{B}}\right) \\ &= 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}\sigma\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) + 1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}\sigma\left(\begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix}\right) \\ &- \tau\left(\begin{bmatrix} a & m \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}} - \tau\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}} \\ &= -\,\delta_{1_{\mathcal{B}}\delta\left(\begin{bmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right)1_{\mathcal{A}}}\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right). \end{aligned}$$

We therefore have $\delta - D \in N^1_{(\sigma,\tau)}(\mathcal{T}, \mathcal{X})$, and so $\delta \in N^1_{(\sigma,\tau)}(\mathcal{T}, \mathcal{X})$.

Conversely, let $\delta \in N^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{X})$. Then there exists $x \in \mathcal{X}$ such that

$$\delta\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big) = \tau\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big)x - x\sigma\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big).$$

Hence

$$\begin{aligned}\delta_{\mathcal{A}}(a) &= 1_{\mathcal{A}}\delta\Big(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big)1_{\mathcal{A}} = 1_{\mathcal{A}}\Big(\tau\Big(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big)x - x\sigma\Big(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big)\Big)1_{\mathcal{A}} \\ &= \tau\Big(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big)1_{\mathcal{A}}x1_{\mathcal{A}} - 1_{\mathcal{A}}x1_{\mathcal{A}}\sigma\Big(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big) = \delta_{1_{\mathcal{A}}x1_{\mathcal{A}}}(a).\end{aligned}$$

Similarly, $\delta_{\mathcal{B}}(b) = \delta_{1_{\mathcal{B}}x1_{\mathcal{B}}}(b)$. Hence $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ are inner and so $\delta \in \ker\rho$.

Thus $N^1_{(\sigma-\tau)}(\mathcal{T},\mathcal{X}) = \ker\rho$.

We conclude that

$$H^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{X}) = \frac{Z^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{X})}{N^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{X})} = \frac{Z^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{X})}{\ker\rho} = H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}_{\mathcal{A}\mathcal{A}}) \oplus H^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{X}_{\mathcal{B}\mathcal{B}}).$$

□

Corollary 2.2. $H^1_{(\sigma,\tau)}(\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}),\mathcal{M}) = 0$.

Proof. With $\mathcal{X} = \mathcal{M}$ we have

$$H^1_{(\sigma,\tau)}(\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}),\mathcal{M}) = H^1_{(\sigma,\tau)}(\mathcal{A},0) \oplus H^1_{(\sigma,\tau)}(\mathcal{B},0) = 0.$$

□

Example 2.3. $H^1_{(\sigma,\tau)}(\mathrm{Tri}(\mathcal{A},\mathcal{A},\mathcal{A}),\mathcal{A}) = 0$.

Example 2.4. Let $\mathcal{L}$ be a left Banach $\mathcal{A}$-module. Then $H^1_{(\sigma,\tau)}(\mathrm{Tri}(\mathcal{A},\mathcal{L},\mathbb{C}),\mathcal{L}) = 0$.

Corollary 2.5. $H^1_{(\sigma-\tau)}(\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}),\mathcal{A}) = H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})$.

Proof. With $\mathcal{X} = \mathcal{A}$, we have $\mathcal{X}_{\mathcal{A}\mathcal{B}} = 0, \mathcal{X}_{\mathcal{A}\mathcal{A}} = \mathcal{A}$ and $\mathcal{X}_{\mathcal{B}\mathcal{B}} = 0$. It then follows from Theorem 2.1, $H^1_{(\sigma,\tau)}(\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}),\mathcal{A}) = H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}) \oplus H^1_{(\sigma,\tau)}(\mathcal{B},0) = H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})$. □

Example 2.6. If $\mathcal{A}$ is a hyperfinite von Neumann algebra and $\mathcal{B}$ is an arbitrary unital Banach module, then $H^1_{(\sigma,\tau)}(\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}),\mathcal{A}) = H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}) = 0$, if σ and τ are ultra-weak automorphisms (see Corollary 3.4.6 of [20]).

3. (σ,τ)-weak amenability of triangular Banach algebras

With simple calculation we can observe that if $\mathcal{X} = \mathcal{T}^*$ considered as $\mathcal{T}$-bimodule, then $\mathcal{X}_{\mathcal{A}\mathcal{A}} = \mathcal{A}^*, \mathcal{X}_{\mathcal{B}\mathcal{B}} = \mathcal{B}^*$ and $\mathcal{X}_{\mathcal{A}\mathcal{B}} = 0$. Therefore by Theorem 2.1 we can conclude the following

Theorem 3.1. *Let $\mathcal{A},\mathcal{B}$ be unital Banach algebras, $\mathcal{M}$ be a unital Banach $\mathcal{A}-\mathcal{B}$-bimodule and $\mathcal{T} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. Then*

$$H^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{T}^*) = H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}^*) \oplus H^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{B}^*).$$

Corollary 3.2. *Let $\mathcal{A},\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be an unital Banach $\mathcal{A}-\mathcal{B}$-bimodule. The triangular Banach algebra $\mathcal{T} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ is (σ,τ)-weak amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are both (σ,τ)-weak amenable.*

By induction one can easily prove the following proposition

Lemma 3.3. *Suppose that $\mathcal{A},\mathcal{B}$ are unital Banach algebras and $\mathcal{M}$ is a unital Banach $\mathcal{A}-\mathcal{B}$-bimodule. If $\mathcal{X}=\mathcal{T}^{(2n)}$ then*

$$\mathcal{X}_{\mathcal{A}\mathcal{A}}=\mathcal{A}^{(2n)},\quad \mathcal{X}_{\mathcal{B}\mathcal{B}}=\mathcal{B}^{(2n)},\quad \mathcal{X}_{\mathcal{A}\mathcal{B}}=\mathcal{M}^{(2n)},\quad \mathcal{X}_{\mathcal{B}\mathcal{A}}=0.$$

Also if $\mathcal{X}=\mathcal{T}^{(2n-1)}$ then

$$\mathcal{X}_{\mathcal{A}\mathcal{A}}=\mathcal{A}^{(2n-1)},\quad \mathcal{X}_{\mathcal{B}\mathcal{B}}=\mathcal{B}^{(2n-1)},\quad \mathcal{X}_{\mathcal{A}\mathcal{B}}=0,\quad \mathcal{X}_{\mathcal{B}\mathcal{A}}=\mathcal{M}^{(2n-1)}.$$

Now by Lemma 3.3 and Theorem 2.1, we immediately obtain the next result.

Proposition 3.4. *Let $\mathcal{A},\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A}-\mathcal{B}$-bimodule. Then for all positive integers $n\in\mathbb{N}$,*

$$H^1_{(\sigma,\tau)}(\mathcal{T},\mathcal{T}^{(2n-1)})=H^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}^{(2n-1)})\oplus H^1_{(\sigma,\tau)}(\mathcal{B},\mathcal{B}^{(2n-1)}).$$

4. (σ,τ)-amenability of triangular Banach algebras

In this section, by using some ideas of [12] we investigate (σ,τ)-amenability of triangular Banach algebra $\mathcal{T}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. We shall assume that the homomorphisms σ,τ on $\mathcal{T}$ have properties asserted in (2.1) and (2.2). We need some general observation concerning (σ,τ)-amenability of Banach algebras. The first is an easy consequence of the definition of (σ,τ)-amenability.

Proposition 4.1. [15, Proposition 3.3] *Let $\mathcal{A},\mathcal{B}$ be Banach algebras and σ,σ' be continuous endomorphisms of $\mathcal{A}$ and τ,τ' be continuous homomorphisms of $\mathcal{B}$. If there is a continuous homomorphism $\varphi:\mathcal{A}\longrightarrow\mathcal{B}$ such that $\varphi(\mathcal{A})$ is a dense subalgebra of $\mathcal{B}$ and $\tau\varphi=\varphi\sigma$ and $\tau'\varphi=\varphi\sigma'$, then (σ,σ')-amenability of $\mathcal{A}$ implies (τ,τ')-amenability of $\mathcal{B}$.*

Now, suppose that $\mathcal{A}$ is a Banach algebra, $\tau,\sigma:\mathcal{A}\longrightarrow\mathcal{A}$ are two continuous endomorphisms, and $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ such that $\sigma(\mathcal{I})\subseteq\mathcal{I},\tau(\mathcal{I})\subseteq\mathcal{I}$. Then the map $\widehat{\tau},\widehat{\sigma}:\frac{\mathcal{A}}{\mathcal{I}}\longrightarrow\frac{\mathcal{A}}{\mathcal{I}}$ can be defined by $\widehat{\sigma}(a+\mathcal{I})=\sigma(a)+\mathcal{I},\widehat{\tau}(a+\mathcal{I})=\tau(a)+\mathcal{I}$. It is not hard to show the following propositions.

Proposition 4.2. [15, Proposition 3.1] *Let $\mathcal{I},\sigma,\tau$ be as above. If $\mathcal{A}$ is (σ,τ)-amenable then $\frac{\mathcal{A}}{\mathcal{I}}$ is $(\widehat{\sigma},\widehat{\tau})$-amenable.*

Proposition 4.3. [15, Proposition 3.2] *Let $\mathcal{I},\sigma,\tau$ be as above and let σ,τ be idempotent homomorphisms. If $\mathcal{I}$ is (σ,τ)-amenable and $\frac{\mathcal{A}}{\mathcal{I}}$ is $(\widehat{\sigma},\widehat{\tau})$-amenable, then $\mathcal{A}$ is (σ,τ)-amenable.*

We now extend Theorem 4.1 of [12] as follows

Proposition 4.4. *Let σ and τ be two continuous idempotent homomorphisms on triangular Banach algebra $\mathcal{T}=\mathrm{Tri}(\mathcal{A},0,\mathcal{B})$. The triangular Banach algebra $\mathcal{T}$ is (σ,τ)-amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are (σ,τ)-amenable.*

Proof. At first suppose that $\mathcal{A},\mathcal{B}$ are (σ,τ)-amenable. It is easy to see that $\begin{bmatrix}\mathcal{A}&0\\0&0\end{bmatrix}$ is a closed ideal of $\begin{bmatrix}\mathcal{A}&0\\0&\mathcal{B}\end{bmatrix}$ and $\begin{bmatrix}\mathcal{A}&0\\0&\mathcal{B}\end{bmatrix}/\begin{bmatrix}\mathcal{A}&0\\0&0\end{bmatrix}\simeq\begin{bmatrix}0&0\\0&\mathcal{B}\end{bmatrix}$. Since $\mathcal{A}$ is (σ,τ)-amenable therefore $\begin{bmatrix}\mathcal{A}&0\\0&0\end{bmatrix}$ is (σ,τ)-amenable.

Let $\varphi:\begin{bmatrix}0&0\\0&\mathcal{B}\end{bmatrix}\to\begin{bmatrix}\mathcal{A}&0\\0&\mathcal{B}\end{bmatrix}/\begin{bmatrix}\mathcal{A}&0\\0&0\end{bmatrix}$ be the natural isomorphism. Then $\varphi\tau=\widehat{\tau}\varphi$ and $\varphi\sigma=\widehat{\sigma}\varphi$. By Proposition 4.1 (σ,τ)-amenability of $\begin{bmatrix}0&0\\0&\mathcal{B}\end{bmatrix}$ implies the $(\widehat{\sigma},\widehat{\tau})$-amenability of $\begin{bmatrix}\mathcal{A}&0\\0&\mathcal{B}\end{bmatrix}/\begin{bmatrix}\mathcal{A}&0\\0&0\end{bmatrix}$. Thus by utilizing Proposition 4.3, we deduce the (σ,τ)-amenability of the Banach algebra $\mathcal{T}$.

For the converse, suppose that $\mathcal{T}$ is (σ,τ)-amenable. It is obvious that $\begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix}$ is a closed ideal of $\mathcal{T}$. By Proposition 4.2, $\begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix}$ is $(\widehat{\sigma},\widehat{\tau})$-amenable. One can easily observe that there exists the natural isomorphism $\varphi : \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B} \end{bmatrix}$ and that $\varphi\widehat{\sigma} = \sigma\varphi$ and $\varphi\widehat{\tau} = \tau\varphi$. Therefore, by Proposition 4.1, $\begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B} \end{bmatrix}$, that is $\mathcal{B}$,is (σ,τ)-amenable. Similarly one can prove the (σ,τ)-amenability of $\mathcal{A}$. $\square$

Theorem 4.5. *Let σ and τ be two continuous idempotent homomorphisms on triangular Banach algebra $\mathcal{T} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. If the triangular Banach algebra $\mathcal{T}$ is (σ,τ)-amenable then $\mathcal{A}$ and $\mathcal{B}$ are (σ,τ)- amenable. In particular, σ-amenability of $\mathcal{T}$ implies $\sigma(\mathcal{M}) = \{0\}$*

Proof. Suppose that $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$ is (σ,τ)-amenable. Clearly, $\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & 0 \end{bmatrix}$ is a closed ideal of $\mathcal{T}$. Therefore, by Proposition 4.2, $\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & 0 \end{bmatrix}$ is $(\widehat{\sigma},\widehat{\tau})$-amenable. Also there exists the natural isomorphism $\varphi : \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} / \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B} \end{bmatrix}$ such that $\varphi\widehat{\sigma} = \sigma\varphi$ and $\varphi\widehat{\tau} = \tau\varphi$. Hence $\begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B} \end{bmatrix}$ is (σ,τ)-amenable. Similarly one can prove the (σ,τ)-amenability of $\mathcal{A}$.

Now suppose that the triangular Banach algebra $\mathcal{T}$ is σ-amenable.

Set $\mathcal{X} = \begin{bmatrix} \mathcal{A}^* & \mathcal{M}^* \\ 0 & \mathcal{B}^* \end{bmatrix}$. The vector space $\mathcal{X}$ is a Banach space under the norm $\| \begin{bmatrix} f & h \\ 0 & g \end{bmatrix} \| = \|f\|_{\mathcal{A}^*} + \|h\|_{\mathcal{M}^*} + \|g\|_{\mathcal{B}^*}$. The space $\mathcal{X}$ can be regarded as a Banach $\mathcal{T}$-bimodule under the following $\mathcal{T}$-module actions

$$\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} f & h \\ 0 & g \end{bmatrix} = \begin{bmatrix} 0 & bh \\ 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} f & h \\ 0 & g \end{bmatrix} \cdot \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & ha \\ 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}$ and $\begin{bmatrix} f & h \\ 0 & g \end{bmatrix} \in \mathcal{X}$. Therefore $\mathcal{X}^* = \begin{bmatrix} \mathcal{A}^{**} & M^{**} \\ 0 & B^{**} \end{bmatrix}$ is a dual Banach $\mathcal{T}$-bimodule.

Let $D : \mathcal{T} \longrightarrow \begin{bmatrix} \mathcal{A}^{**} & M^{**} \\ 0 & B^{**} \end{bmatrix}$ be defined by $D\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & \widehat{\sigma(m)} \\ 0 & 0 \end{bmatrix}$.

Now we have

$$\begin{aligned} D\left(\begin{bmatrix} a_1 & m_1 \\ 0 & b_1 \end{bmatrix}\begin{bmatrix} a_2 & m_2 \\ 0 & b_2 \end{bmatrix}\right) &= D\left(\begin{bmatrix} a_1a_2 & a_1m_2 + m_1b_2 \\ 0 & b_1b_2 \end{bmatrix}\right) \\ = \begin{bmatrix} 0 & \sigma(a_1\widehat{m_2 + m_1}b_2) \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & \sigma(a_1)\widehat{\sigma(m_2)} + \widehat{\sigma(m_1)}\sigma(b_2) \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} 0 & \sigma(a_1)\widehat{\sigma(m_2)} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \widehat{\sigma(m_1)}\sigma(b_2) \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & \widehat{\sigma(m_1)} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \sigma(a_2) & \sigma(m_2) \\ 0 & \sigma(b_2) \end{bmatrix} \end{aligned}$$

$$+\begin{bmatrix}\sigma(a_1) & \sigma(m_1)\\ 0 & \sigma(b_1)\end{bmatrix}\begin{bmatrix}0 & \widehat{\sigma(m_2)}\\ 0 & 0\end{bmatrix} = D\Big(\begin{bmatrix}a_1 & m_1\\ 0 & b_1\end{bmatrix}\Big)\sigma\Big(\begin{bmatrix}a_2 & m_2\\ 0 & b_2\end{bmatrix}\Big)$$
$$+\sigma\Big(\begin{bmatrix}a_1 & m_1\\ 0 & b_1\end{bmatrix}\Big)D\Big(\begin{bmatrix}a_2 & m_2\\ 0 & b_2\end{bmatrix}\Big).$$

Therefore D is a σ-derivation. Hence there exists $\begin{bmatrix}F & H\\ 0 & G\end{bmatrix} \in \mathcal{X}^*$ such that

$$\begin{bmatrix}0 & \widehat{\sigma(m)}\\ 0 & 0\end{bmatrix} = D\Big(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\Big) = \sigma\Big(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\Big)\begin{bmatrix}F & H\\ 0 & G\end{bmatrix}$$
$$-\begin{bmatrix}F & H\\ 0 & G\end{bmatrix}\sigma\Big(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\Big) = \begin{bmatrix}0 & \sigma(a)H\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & H\sigma(b)\\ 0 & 0\end{bmatrix}.$$

Thus

$$\widehat{\sigma(m)} = \sigma(a)H - H\sigma(b) \tag{4.1}$$

for all $m \in \mathcal{M}, a \in \mathcal{A}$ and $b \in \mathcal{B}$. Choosing $a = 0, b = 0$ in (4.1) we conclude that $\widehat{\sigma(m)} = 0$ for all $m \in \mathcal{M}$. Thus $\sigma(\mathcal{M}) = \{0\}$. □

Corollary 4.6. *Let $\mathcal{A}, \mathcal{B}$ be two unital Banach algebra and $\sigma : \mathcal{A} \longrightarrow \mathcal{A}, \tau : \mathcal{B} \longrightarrow \mathcal{B}$ be two continuous idempotent homomorphisms. The Banach algebra $\mathcal{A}\oplus\mathcal{B}$ is $\sigma\oplus\tau$-amenable if and only if $\mathcal{A}$ is σ-amenable and $\mathcal{B}$ is τ-amenable.*

Proof. It is easy to see that $\begin{bmatrix}\mathcal{A} & 0\\ 0 & \mathcal{B}\end{bmatrix} \simeq \mathcal{A}\oplus\mathcal{B}$. Define $\varphi : \begin{bmatrix}\mathcal{A} & 0\\ 0 & \mathcal{B}\end{bmatrix} \longrightarrow \begin{bmatrix}\mathcal{A} & 0\\ 0 & \mathcal{B}\end{bmatrix}$ via $\varphi\Big(\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\Big) = \begin{bmatrix}\sigma(a) & 0\\ 0 & \tau(b)\end{bmatrix}$. Therefore $\begin{bmatrix}\mathcal{A} & 0\\ 0 & \mathcal{B}\end{bmatrix}$ is φ-amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are both φ-amenable, and this holds if and only if $\mathcal{A}$ is σ-amenable and $\mathcal{B}$ is τ-amenable. □

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