REPRESENTATION OF COMMUTANTS FOR COMPOSITION OPERATORS INDUCED BY A HYPERBOLIC LINEAR FRACTIONAL AUTOMORPHISMS OF THE UNIT DISK

YU. S. LINCHUK

Abstract. We describe the commutant of the composition operator induced by a hyperbolic linear fractional transformation of the unit disk onto itself in the class of linear continuous operators which act on the space of analytic functions. Two general classes of linear continuous operators which commute with such composition operators are constructed.

The composition of functions is a fundamental operation in mathematics, and operators of the composition have an important role in complex analysis and the theory of operators. Composition operators naturally arise in a description of commutants for operators of multiplication by functions in different function spaces.

Let $G$ be an arbitrary domain in the complex plane. By $H(G)$ we denote the space of all functions analytic in $G$ and endowed with the topology of compact convergence [1]; by $L(H(G))$ we denote the set of all linear continuous operators that act on $H(G)$. For a fixed function $\varphi(z) \in H(G)$ with $\varphi(G) \subset G$ the formula $(K_\varphi f)(z) = f(\varphi(z))$ defines a composition operator $K_\varphi \in L(H(G))$ induced by $\varphi$.

Over the last thirty years the regular research of the properties of composition operators on the spaces of Hardy, Bergman and Dirichlet was carried out. These Banach spaces consist of functions analytical in the disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ with certain restrictions on their module. A main task of these researches is the investigation of properties of the composition operator $K_\varphi$ in terms of conditions on the generating function $\varphi(z)$. For the operators $K_\varphi$, the estimates on their norms are obtained, and in some cases exact formulas for evaluating the norms are found, conditions of compactness are studied, the structure of invariant subspaces of such operators is investigated, etc. Results of these researches are stated by E. A. Nordgren [2], J. H. Shapiro [3], R. K. Singh and J. S. Manhas [4], C. C. Cowen and B. D. MacCluer [5]. In the review of P. Rosenhtal [6] on monographs [3], [4] it is remarked that the problem of determining the commutant for the composition operator in these spaces is difficult enough. B. Cload in [7], [8] investigated the commutant of the composition operator $K_\varphi$ on the Hardy space $H^2$ in the case when the function $\varphi(z)$ is an automorphism of $D$. B. Cload has described the operators of multiplication on analytic functions that commute with composition operators induced by an elliptic disk automorphism. In conclusions to [8], the author remarks that the problem of a description of the commutant of $K_\varphi$ on the space $H^2$ for hyperbolic and parabolic disk automorphisms is not solved. The commutants of composition operators on $H^2$ were studied by T. Worner [9]. She gave examples of functions $\varphi(z)$ and operators from the class $L(H^2)$ that commute with $K_\varphi$ on $H^2$, but are not generated by operators of multiplication on analytical functions and operators of the composition, that is, are not represented as the sums of products of such operators.

2000 Mathematics Subject Classification. Primary 47B38.

Key words and phrases. Linear continuous operators, composition operator, hyperbolic automorphism, commutant.
In connection with this, there arises a problem on an investigation of images of the commutants for composition operators that act on the space $\mathcal{H}(G)$, where $G$ is an arbitrary simply connected domain in the complex plane and $G$ is distinct from $\mathbb{C}$. This problem is important and interesting for the composition operators induced by conformal automorphisms of the domain $G$. Since, by the Riemann Mapping Theorem, for each such domains there exists a conformal mapping of the corresponding domain to the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$, it is enough to solve this problem for composition operators which act on the space $\mathcal{H}(D)$. The general representation of a conformal map of the unit disk $D$ onto itself is defined by

$$\varphi(z) = e^{i\alpha} \frac{z - z_0}{1 - \overline{z}_0 z},$$

where $|z_0| < 1$ and $\alpha \in \mathbb{R}$.

A method for solving the problem on a description of the commutant for $K_\varphi$ and the final result essentially depend on the class of the function $\varphi(z)$, which could be elliptic, hyperbolic or parabolic. In this paper we solve this problem for composition operators induced by hyperbolic automorphisms of the unit disk $D$. The main result of this paper is announced in [10].

Recall that the linear fractional transformation $\varphi(z)$ (1) is hyperbolic if and only if the condition $|z_0| > |\sin \frac{\pi}{2}|$ is satisfied. A hyperbolic linear fractional transformation has two different fixed points that belong to the unit circle $\{ z \in \mathbb{C} : |z| = 1 \}$.

In [1], a one-to-one correspondence between the operators $T \in \mathcal{L}(\mathcal{H}(G))$ and its characteristic functions $t(\lambda, z) = T \left[ \frac{1}{\lambda - z} \right]$ is established. For solutions of the main problem of this paper it is convenient to use the special characteristic functions for operators from the class $\mathcal{L}(\mathcal{H}(G))$.

1. Suppose that $T \in \mathcal{L}(\mathcal{H}(D))$ and let $t(\lambda, z) = T \left[ \frac{1}{\lambda - z} \right]$ be a characteristic function, in the sense of Köthe, of the operator $T$. Then $t(\lambda, z)$ is a locally analytical function on the set $\mathbb{C}D \times D$. This means that there exists a sequence of positive numbers $(r_n)$ that monotonically increase to 1 and there exists a monotonically increasing sequence of numbers $(N(n))$ such that the function $t(\lambda, z)$ is analytical when $|\lambda| > r_{N(n)}$, $|z| < r_n$; $t(\infty, z) = 0$ for $|z| < r_n$. For an arbitrary function $f \in \mathcal{H}(D)$ we have

$$\frac{1}{2\pi i} \int_{\gamma_n} t(\lambda, z)f(\lambda) \, d\lambda,$$

where $\gamma_n = \{ \lambda : |\lambda| = r_{N(n)+1} \}$, $|z| < r_n$ (see [1]).

By $B_n$ we denote a Banach space of all functions $f(z)$ that are continuous on the set $|z| \leq r_n$ and analytic in the disk $|z| < r_n$ with the norm $\|f\| = \max_{|z| \leq r_n} |f(z)|$. Then, as it was shown in [1], the operator $T$ uniquely extends to an operator that is linear continuous and acts from $B_{N(n)}$ into $B_n$. From the proof of Theorem 19 [1] it follows that the formula (2) is correct for an arbitrary function $f(z)$ analytic in the disk $|z| < r$, where $r > r_{N(n)+1}$. Let $\mu$ be an arbitrary nonzero complex number for which $|\mu| < \frac{1}{r_{N(n)+1}}$.

Then, the function $f_\mu(z) = \frac{z - \mu}{1 - \mu z}$ is analytic for $|z| < \frac{1}{|\mu|}$ and

$$\frac{1}{2\pi i} \int_{\gamma_n} t(\lambda, z)\frac{\lambda - \mu}{1 - \mu \lambda} \, d\lambda,$$

when $|z| < r_n$. By $t_1(\mu, z)$ we denote $(Tf_\mu)(z)$. Calculating the integral (3) we obtain

$$t_1(\mu, z) = -\text{res}_{\lambda = \frac{1}{\mu}} t(\lambda, z)\frac{\lambda - \mu}{1 - \mu \lambda} - \text{res}_{\lambda = \infty} t(\lambda, z)\frac{\lambda - \mu}{1 - \mu \lambda} = \frac{1 - \mu^2}{\mu^2} t \left( \frac{1}{\mu}, z \right) - \frac{1}{\mu} \varphi_0(z),$$
where \( \varphi_0(z) = T1 \). If \( \mu = 0 \), then \( t_1(0,z) = \varphi_1(z) \), where \( \varphi_1(z) = Tz \). Therefore, the function \( t_1(\mu,z) \) is analytic when \( |\mu| < \frac{1}{r_{N(n)+1}} \), \(|z| < r_n \), and

\[
 t_1(\mu,z) = \frac{1 - \mu^2}{\mu^2} t \left( \frac{1}{\mu}, z \right) - \frac{1}{\mu} \varphi_0(z). \tag{4}
\]

Hence, the function \( t_1(\mu,z) \) is locally analytic on the set \( \overline{D} \times D \), that is, \( t_1(\mu,z) \) is analytic on a set \( \mathcal{K} = \bigcup_{n=1}^N \mathcal{K}_{\rho_n} \times K_{r_n} \), where \( \mathcal{K}_{r_n} = \{ z \in \mathbb{C} : |z| < r_n \} \), \( \mathcal{K}_{\rho_n} = \{ \mu \in \mathbb{C} : |\mu| < \rho_n \} \), \( (r_n) \) and \( (\rho_n) \) are sequences of positive numbers each of them is monotonically increases and monotonically decreases to 1 respectively, and \( (N(n)) \) is a monotonically increasing sequence of natural numbers (in ours case \( \rho_{N(n)} = \frac{1}{r_{N(n)+1}} \), \( n = 1, 2, \ldots \)). From (4) it follows that

\[
 t_1(1,z) + t_1(-1,z) = 0, \quad z \in D. \tag{5}
\]

Conversely, let \( t_1(\mu,z) \) be a locally analytic function on \( \overline{D} \times D \) and satisfy (5). Substituting \( \mu = \frac{1}{\lambda} \) in (4), and using \( \varphi_0(z) = t_1(-1,z) \), we see that the formula

\[
 t(\lambda,z) = \frac{1}{\lambda^2 - 1} t_1 \left( \frac{1}{\lambda}, z \right) + \frac{\lambda}{\lambda^2 - 1} t_1(-1,z)
\]

defines a locally analytic function on the set \( \mathbb{C}D \times D \). According to (2) for an arbitrary function \( f \in \mathcal{H}(D) \) we have

\[
 (Tf)(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\lambda^2 - 1} t_1 \left( \frac{1}{\lambda}, z \right) f(\lambda) d\lambda,
\]

where \( \Gamma_n = \{ \lambda \in \mathbb{C} : |\lambda| = \frac{1}{\rho_{N(n)+1}} \}, \quad |z| < r_n \). The number \( N(n) \) is found for the number \( n \) by the definition of a function \( t_1(\mu,z) \) locally analytic on \( \overline{D} \times D \). Since \( T \left[ \frac{z - \mu}{1 - \mu z} \right] = t_1(\mu,z) \), we have that \( t_1(\mu,z) \) is the characteristic function of the operator \( T \). Thus, the following statement is valid.

**Theorem 1.** There exists a one-to-one correspondence between operators \( T \in \mathcal{L}(\mathcal{H}(D)) \) and its characteristic functions \( t_1(\lambda,z) = T \left[ \frac{z - \lambda}{1 - \lambda z} \right] \) that are locally analytic on the set \( \overline{D} \times D \) and satisfy the condition (5).

**Remark 1.** If \( g_1(\mu,z) \) is an arbitrary locally analytic function on the set \( \overline{D} \times D \), then the formula

\[
 t_1(\mu,z) = g_1(\mu,z) - \frac{1}{2} g_1(1,z) - \frac{1}{2} g_1(-1,z)
\]

defines a function locally analytic on \( \overline{D} \times D \) that satisfies (5). From Theorem 1 it follows that \( t_1(\mu,z) \) is the characteristic function for the same operator \( T \in \mathcal{L}(\mathcal{H}(D)) \).

2. The function \( \varphi(z) = \frac{z - z_0}{1 - z_0 z} \), where \( z_0 \in \mathbb{R} \) and \( 0 < |z_0| < 1 \) is an important partial case of a hyperbolic automorphism of the unit disk \( D \). This map plays an important role in the definition of a distance in the model of Lobachevsky geometry, which was proposed by A. Poincare [11].

Let us describe the commutant of the operator \( K_\varphi \) in the class \( \mathcal{L}(\mathcal{H}(D)) \). Suppose that \( T \in \mathcal{L}(\mathcal{H}(D)) \) commutes with \( K_\varphi \). Then the equality

\[
 TK_\varphi = K_\varphi T
\]

holds. By \( t_1(\lambda,z) = T \left[ \frac{z - \lambda}{1 - \lambda z} \right] \) we denote the characteristic function of the operator \( T \). Let \( \tilde{t}_1(\lambda,z) \) and \( \tilde{t}_1(\lambda,z) \) be characteristic functions of the operators \( TK_\varphi \) and \( K_\varphi T \) respectively. Fix an arbitrary natural number \( n \) and \( N(n) \) is found by the definition of
the function \( t_1(\lambda, z) \) locally analytic on the set \( \overline{D} \times D \). Then,

\[
(Tg)(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\tau^2 - 1} t_1 \left( \frac{1}{\tau}, z \right) g(\tau) d\tau,
\]

where \( \Gamma_n = \{ \tau \in \mathbb{C} : |\tau| = \frac{1}{\rho_{N(n)+1}} \} \), \( |z| < r_n \). Formula (7) is correct for an arbitrary function \( g(z) \) analytic in the disk \( |z| < r \), where \( r > \frac{1}{\rho_{N(n)+1}} \) (see the proof of Theorem 1).

Suppose \( C = \max\{|\varphi(z)| : |z| \leq \frac{1}{\rho_{N(n)+1}}\} \) and \( C_1 = \min\{|\varphi(\lambda)| : |\lambda| = \rho_{N(n)+1}\} \); then \( C < 1 < C_1 \). We choose \( N_1(n) \) so that \( \rho_{N_1(n)} < \min\{C_1, \frac{1}{|\lambda|}, \frac{1}{|z|}\} \). Fix an arbitrary \( \lambda \in \mathbb{C}, |\lambda| < \rho_{N_1(n)} \); then \( \frac{1}{|\lambda|} > C \). Suppose \( r(\lambda) = 1 \) for \( \frac{1}{|\lambda|} \geq 1 \), and \( r(\lambda) = \min\{|\varphi^{-1}(t) : |t| = \frac{1}{|\lambda|}, t \in \mathbb{C} \} \) for \( \frac{1}{|\lambda|} < 1 \); then from the definition of the number \( C \) it follows that \( r(\lambda) > \frac{1}{\rho_{N(n)+1}} \) and the function \( g_\lambda(z) = \frac{\varphi(z) - \lambda}{1 - \lambda \varphi(z)} \) is analytic in the disk \( |z| < r(\lambda) \). Using (7), we get

\[
\tilde{t}_1(\lambda, z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\tau^2 - 1} t_1 \left( \frac{1}{\tau}, z \right) \frac{\varphi(\tau) - \lambda}{1 - \lambda \varphi(\tau)} d\tau,
\]

when \( |z| < r_n \) and \( |\lambda| < \rho_{N_1(n)} \). The inequality \(|\varphi(\tau)| \cdot |\lambda| \leq C \rho_{N_1(n)} < 1 \) holds for \( |\tau| = \frac{1}{\rho_{N(n)+1}} \) and \( |\lambda| < \rho_{N_1(n)} \). Therefore,

\[
\frac{\varphi(\tau) - \lambda}{1 - \lambda \varphi(\tau)} = \frac{\tau - \varphi^{-1}(\lambda)}{1 - \tau \varphi^{-1}(\lambda)}
\]

for \( |\tau| = \frac{1}{\rho_{N(n)+1}} \) and \( |\lambda| < \rho_{N_1(n)} \). Indeed, we obtain

\[
\tilde{t}_1(\lambda, z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\tau^2 - 1} t_1 \left( \frac{1}{\tau}, z \right) \frac{\tau - \varphi^{-1}(\lambda)}{1 - \tau \varphi^{-1}(\lambda)} d\tau,
\]

when \( |z| < r_n, |\lambda| < \rho_{N_1(n)} \). If \( |\lambda| < \rho_{N_1(n)} \), then \( |\lambda| < C_1 |\varphi^{-1}(\lambda)| < \rho_{N(n)+1} \). From (8) it follows that \( t_1(\lambda, z) = t_1(\varphi^{-1}(\lambda), z) \) for \( |z| < r_n, |\lambda| < \rho_{N_1(n)} \). Since \( \tilde{t}_1(\lambda, z) \) is the characteristic function of the operator \( K_\varphi T \), we see that for the chosen number \( n \) there exists a number \( N_2(n) \) for which \( \tilde{t}_1(\lambda, z) = t_1(\lambda, \varphi(z)) \), where \( |z| < r_n, |\lambda| < \rho_{N_2(n)} \). The equality (6) implies that \( \tilde{t}_1(\lambda, z) = \tilde{t}_1(\lambda, z) \) on \( \mathcal{K} \). Therefore, we have

\[
t_1(\varphi^{-1}(\lambda), z) = t_1(\lambda, \varphi(z)),
\]

when \( |z| < r_n, |\lambda| < \rho_{N_3(n)} \), where \( N_3(n) = \max\{N_1(n), N_2(n)\} \). Let us make the substitution \( \lambda = \varphi(\mu) \) in (9). Similarly, there exists \( N_4(n) \) such that \( |\varphi(\mu)| < \rho_{N_3(n)} \) for \( |\mu| < \rho_{N_3(n)} \). Then, from (9), it follows that for \( |z| < r_n, |\mu| < \rho_{N_4(n)} \), and the equality \( t_1(\mu, z) = t_1(\varphi(\mu), \varphi(z)) \) holds.

Thus, the characteristic function of the operator \( TK_\varphi \) coincides with the characteristic function of the operator \( K_\varphi T \) if and only if the last equality is correct on the set \( \mathcal{K} \). The following statement holds.

**Theorem 2.** Let \( \varphi(z) = \frac{z - z_0}{1 - z z_0}, z_0 \in \mathbb{R}, 0 < |z_0| < 1 \), be a hyperbolic automorphism of the unit disk \( D \). An operator \( T \in \mathcal{L}(\mathcal{H}(D)) \) commutes with the operator \( K_\varphi \) if and only if its characteristic function \( t_1(\lambda, z) \) satisfies the equality

\[
t_1(\varphi(\lambda), \varphi(z)) = t_1(\lambda, z)
\]
on the set \( \mathcal{K} \).

**Remark 2.** In the proof of Theorem 2 we essentially use that the function \( k(z, \lambda) = \frac{z - \lambda}{1 - \lambda \overline{z}} \) satisfies the equality \( k(\varphi(z), \varphi(\lambda)) = k(\lambda, z) \) when \( |\lambda z| < 1 \).

We find the solutions of the equation (10) in the class of functions \( t_1(\lambda, z) \) that are locally analytic on \( \overline{D} \times D \).
Let \( t_1(\lambda, z) \) be a locally analytic function on \( \overline{D} \times D \) and satisfy (10) on the set \( \mathcal{K} \).

Make the substitution

\[
\mu = \frac{z - \lambda}{1 - \lambda z}
\]

in (10). For each number \( n \) by \( G_n \) we denote the open set \( G_n = \{ \frac{z - \lambda}{1 - \lambda z} : \lambda | < \rho_{N(n)}, |z| < r_n \} \). Let \( G = \bigcup_{n=1}^{\infty} G_n \). Clearly, \( G \) is open in \( \mathbb{C} \). Since for \( \lambda \in K_{\rho_{N(1)}} \) the points \( (\lambda,0) \in K_{\rho_{N(1)}} \times K_{r_1} \), then \( K_{\rho_{N(1)}} \subset G \). Thus,

\[
(12) \quad K_{\rho_{N(1)}} \subset G.
\]

If \( |\lambda| = 1, z \in K_{r_n} \), then \( (\lambda, z) \in \mathcal{K} \), and \( \{ \mu : |\lambda| = 1, |z| < 1 \} \subset \mathcal{K} \). Therefore, \( \{ \frac{z - \lambda}{1 - \lambda z} : \lambda | = 1, |z| < 1 \} \subset G \). We have

\[
\left\{ \frac{z - i}{1 - iz} : |z| < 1 \right\} = \{ \mu : \text{Im} \mu < 0 \}, \quad \left\{ \frac{z + i}{1 + iz} : |z| < 1 \right\} = \{ \mu : \text{Im} \mu > 0 \}
\]

for \( \lambda = \pm i \). Thus, we see that

\[
(13) \quad \{ \mu \in \mathbb{C} : \text{Im} \mu \neq 0 \} \subset G.
\]

For an arbitrary positive \( \varepsilon \) by \( G_{\varepsilon} \) we denote the set \( G_{\varepsilon} = \mathbb{C} \setminus \{ \mu \in \mathbb{R} : |\mu| \geq 1 + \varepsilon \} \). From (12) and (13) it follows that there exists \( \varepsilon > 0 \) such that \( G_{\varepsilon} \subset G \).

Substituting \( \lambda \) with (11) into (10) we obtain

\[
(14) \quad t_1 \left( \varphi \left( \frac{z - \mu}{1 - \mu z} \right), \varphi(z) \right) = t_1 \left( \frac{z - \mu}{1 - \mu z}, z \right)
\]

for \( \mu \in G_n, |z| < r_n \). We have

\[
\varphi \left( \frac{z - \mu}{1 - \mu z} \right) = \frac{\varphi(z) - \mu}{1 - \mu \varphi(z)}
\]

when \( \mu \in G_n, z \in K_{r_n} \). Therefore, (14) can be presented in the form

\[
(15) \quad t_1 \left( \frac{\varphi(z) - \mu}{1 - \mu \varphi(z)}, \varphi(z) \right) = t_1 \left( \frac{z - \mu}{1 - \mu z}, z \right),
\]

\( z \in K_{r_n}, \mu \in G_n \). By \( t_2(\mu, z) \) we denote \( t_1 \left( \frac{z - \mu}{1 - \mu z}, z \right) \). The function \( t_2(\mu, z) \) satisfies the equality

\[
(16) \quad t_2(\mu, \varphi(z)) = t_2(\mu, z)
\]

for \( \mu \in G_n, z \in K_{r_n} \). The function \( t_2(\mu, z) \) is analytic on the set \( G \times D \) since this function is analytic on the set \( G_n \times K_{r_n} \) for each number \( n \). Returning to the function \( t_1(\lambda, z) \) we obtain

\[
(17) \quad t_1(\lambda, z) = t_2 \left( \frac{z - \lambda}{1 - \lambda z}, z \right)
\]

if \( |\lambda| < \rho_{N(n)}, |z| < r_n \).

Thus, there exists \( \varepsilon > 0 \) such that \( t_2(\mu, z) \) is an analytic function on the set \( G_{\varepsilon} \times D \) and the equality (16) holds on this set, which proves the necessity.

**Theorem 3.** Let \( \varphi(z) = \frac{z - z_0}{1 - z_0 z}, \; z_0 \in \mathbb{R}, \; |z_0| < 1 \), be a hyperbolic automorphism of the unit disk \( D \). A function \( t_1(\lambda, z) \), locally analytic on the set \( \overline{D} \times D \), is a solution of the equation (10) if and only if it has the form (17), where \( t_2(\mu, z) \) is an analytic function on the same set \( G_{\varepsilon} \times D \) and satisfies (16).

**Proof.** **Sufficiency.** Assume that there exists \( \varepsilon > 0 \) such that \( t_2(\mu, z) \) is an analytic function on \( G_{\varepsilon} \times D \) and satisfies on this set (16). Let us show that for any number \( r, 0 < r < 1 \), there exists \( \rho > 1 \) for which \( G(r, \rho) \subset G_{\varepsilon} \), where \( G(r, \rho) = \left\{ \frac{z - \lambda}{1 - \lambda z} : |z| < r, |\lambda| < \rho \right\} \). Indeed, fix an arbitrary \( r \in (0, 1) \) and choose \( \rho \) such that \( 1 < \rho < \min \left\{ \frac{1}{r}, 1 + \varepsilon, \frac{1 + 1}{1 + \varepsilon} \right\} \).
We show that for this ρ the inclusion \(G(r, ρ) ⊂ G_ε\) holds. Suppose the contrary. Assume that the inclusion \(G(r, ρ) ⊂ G_ε\) does not hold. Then there are numbers \(z_0, λ_0\) for which \(|z_0| < r, |λ_0| < ρ, \) and \(\frac{−z_0−λ_0}{1−λ_0z_0} = x, \) where \(x \in \mathbb{R}, |x| > 1 + ε.\) Without lost of a generality it can be assumed that \(x > 0,\) then \(x > 1 + ε.\) We have \(z_0 = \frac{λ_0+x}{1+λ_0x}.\)

Let \(ψ(λ) = \frac{λ−z_0}{1+λz_0}\) be the auxiliary linear fractional transformation. By \(K\) denote the image of the circle \(\{λ ∈ \mathbb{C} : |λ| = ρ\}\) under the map \(ψ(λ).\) The points \(ψ(ρ) = \frac{ρ+x}{1+ρx}\) and \(ψ(−ρ) = −\frac{−ρ+x}{−ρ−x}\) are diametrically opposite for \(K.\) The disk \(\{λ ∈ \mathbb{C} : |λ| < ρ\}\) is mapped on the exterior of \(K.\) By the choice of the number \(ρ\) we have \(−\frac{−ρ+x}{−ρ−x} < −r < r < \frac{ρ+x}{1+ρx}.\)

Then, the circle \(K\) contains the disk \(\{z ∈ \mathbb{C} : |z| < r\}\) and consequently the equality \(z_0 = \frac{λ_0+x}{1+λ_0x}\) is impossible. This contradicts the assumption, so the inclusion \(G(r, ρ) ⊂ G_ε\) is proved. Then, the formula (17) defines a function \(t_1(λ, z),\) locally analytic on the set \(D × D,\) which satisfies (10). Theorem 3 is proved.

From (17) it follows that \(t_1(1, z) = t_2(−1, z),\) \(t_1(−1, z) = t_2(1, z).\) Therefore, for the function \(t_1(λ, z),\) which is defined by (17), relation (5) is equivalent to

\[t_2(−1, z) + t_2(1, z) = 0\]

for \(|z| < 1.\) Thus, Theorems 1–3 imply that an operator \(T ∈ \mathcal{L}(\mathcal{H}(D))\) commutes with the operator \(K_φ\) if and only if its characteristic function \(t_1(λ, z)\) has the form (17), where \(t_2(µ, z)\) is a function analytic on some set \(G_ε × D\) and satisfies (16), (18). Using the equality

\[t(λ, z) = \frac{1}{λ^2−1}t_2 \left(\frac{λ−z}{λ−z}, z\right)\]

we restore \(T ∈ \mathcal{L}(\mathcal{H}(D))\) by means of its characteristic function \(t(λ, z),\) that is, the following statement holds.

**Theorem 4.** Let \(φ(λ) = Φ(\frac{z−z_0}{1−z_0z}, z_0 ∈ \mathbb{R}, 0 < |z_0| < 1,\) be a hyperbolic automorphism of the unit disk \(D.\) An operator \(T ∈ \mathcal{L}(\mathcal{H}(D))\) commutes with the operator \(K_φ\) if and only if

\[Tf(z) = \frac{1}{2πi} \int_{γ_ε} \frac{1}{λ^2−1}t_2 \left(\frac{λ−z}{λ−z}, z\right) f(λ) dλ.\]

Here \(t_2(µ, z)\) is a function analytic on some set \(G_ε × D\) and satisfies (16), (18). The contour of integration \(γ_ε\) is chosen for \(z ∈ D\) by definition of the function \(t(λ, z),\) locally analytic on \(\mathbb{C} × D,\) that is defined by (19).

**Remark 3.** Let \(t_2(µ, z)\) be a function analytic on the some set \(G_ε × D\) and satisfy (16). Theorem 3 and Remark 1 imply that the function \(t_1(λ, z) = t_2 \left(\frac{z−λ}{1−λz}, z\right) − \frac{1}{2}t_2(1, z) − \frac{1}{2}t_2(−1, z)\) is analytic on \(G_ε × D\) also, and \(t_1(1, z) + t_1(−1, z) = 0\) when \(|z| < 1.\) Therefore, \(t_1(λ, z)\) is a characteristic function of an operator \(T ∈ \mathcal{L}(\mathcal{H}(D))\) that commutes with the operator \(K_φ.\) From Theorem 4 it follows that formula (20) defines this operator for an arbitrary function \(f ∈ \mathcal{H}(D).\)

Suppose \(t_2(µ, z) = µ^n,\) \(n\) is an arbitrary fixed natural number. Then the formula

\[T_n f(z) = \frac{1}{2πi} \int_{γ_{ε_n}} \frac{(λ−z)^n}{(λ^2−1)(λ−z)^n} f(λ) dλ,\]

where \(γ_{ε_n} = \{λ : |λ| = r\},\) and \(r\) is such that \(|z| < r < 1,\) defines an operator \(T_n ∈ \mathcal{L}(\mathcal{H}(D)).\) Remark 3 and Theorem 4 imply that \(T_n\) commutes with the operator \(K_φ.\)

Let \(t_2(µ, z)\) be the entire function on \(µ,\) analytic in \(z\) in the disk \(D\) and satisfy (16) for \(µ ∈ \mathbb{C}, z ∈ D.\) Then, from Remark 3 and Theorem 4 it follows that the formula (3)
defines an operator $T \in \mathcal{L}(\mathcal{H}(D))$ that commutes with the operator $K_{\varphi}$. In this case $\gamma_z$ is the circle $|\lambda| = r, |z| < r < 1$. We expand the function $t_2(\mu, z)$ into a series

$$t_2(\mu, z) = \sum_{n=0}^{\infty} \psi_n(z)\mu^n,$$

$\mu \in \mathbb{C}, z \in D$. Then $(\psi_n(z))$ is a sequence of analytic functions in the disk $D$. For the function $t_2(\mu, z)$ the condition (16) is equivalent to

$$\psi_n(\varphi(z)) = \psi_n(z), \quad z \in D, \quad n = 0, 1, 2, \ldots$$

Substituting (22) in (20) and integrating the corresponding series, we have

$$(T f)(z) = \sum_{n=1}^{\infty} \psi_n(z)(T_n f)(z)$$

for an arbitrary functions $f \in \mathcal{H}(D), |z| < 1$. We prove that $(\psi_n(z))$ satisfies the condition

$$\forall r < 1 \lim_{n \to \infty} r \max_{|z| \leq r} |\psi_n(z)| = 0.$$ 

Fix an arbitrary $r < 1$ and $0 < \rho < \infty$. From Cauchy’s inequalities for coefficients of the expansion of the function $t_2(\mu, z)$ entire in $\mu$ in series (22) it follows that

$$|\psi_n(z)| \leq \frac{\max |t_2(\mu, z)|}{\rho^n}, \quad n = 0, 1, \ldots,$$

when $|z| \leq r$. Therefore,

$$\max_{|z| \leq r} |\psi_n(z)| \leq \frac{C}{\rho^n}, \quad n = 0, 1, \ldots,$$

where $C = \max \{|t_2(\mu, z)| : |\mu| = \rho, |z| \leq r\}$. The last inequalities implies

$$\lim_{n \to \infty} r \max_{|z| \leq r} |\psi_n(z)| \leq \frac{1}{\rho}. \quad \text{Suppose } \rho \to \infty \text{ in this inequalities; then (25) holds.}$$

Conversely, let $(\psi_n(z))$ be a sequence functions that are analytic in the disk $D$ and satisfy (23), (25). We show that the formula (24) defines some operator $T \in \mathcal{L}(\mathcal{H}(D))$ that commutes with the operator $K_{\varphi}$.

Fix numbers $r_1, r$ such that $0 < r_1 < r < 1$. From (21) it follows that

$$|(T_n f)(z)| \leq \frac{r(r_1r + 1)^n}{(1 - r^2)(r - r_1)^n} \max_{|\lambda| = r} |f(\lambda)|$$

for an arbitrary function $f \in \mathcal{H}(D), |z| \leq r_1$. Therefore,

$$\max_{|z| \leq r_1} |(T_n f)(z)| \leq \frac{r}{1 - r^2} \left(\frac{r_1r + 1}{r - r_1}\right)^n \max_{|\lambda| = r} |f(\lambda)|.$$ 

Choose a number $h, 0 < h < \frac{1}{2} \left(\frac{r - r_1}{r_1r + 1}\right)$. From (25) it follows that there exists a constant $C > 0$ such that $\max_{|z| \leq r_1} |\psi_n(z)| \leq C \rho^n$, $n = 0, 1, \ldots$ From last inequalities we have

$$\max_{|z| \leq r_1} |(T_n f)(z)| \leq \frac{C r}{1 - r^2} \left(\frac{1}{2}\right)^n \max_{|\lambda| = r} |f(\lambda)|, n = 0, 1, \ldots$$

Using (26), we see that the series in the right-hand side of (24) converges uniformly on the disk $|z| \leq r_1$. This series converges uniformly inside $D$ since $r_1 < 1$ is arbitrary. Weierstrass’ theorem on uniformly convergent series of analytic functions implies that the formula (24) defines some operator $T$ that acts on the space $\mathcal{H}(D)$. The linearity of $T$ follows from linearity of the operators $T_n$. Continuity of $T$ follows from estimates (26). Hence, indeed, the formula (24) defines an operator $T \in \mathcal{L}(\mathcal{H}(D))$. By the immediate check we see that $T$ commutes with $K_{\varphi}$. Thus, the following statement holds.
Corollary 1. Let \( \psi_n(z) \) be a sequence of functions analytic on the disk \( D \) that satisfy conditions (23), (25), and the operators \( T_n \) be defined by formula (21). Then the operator \( T \) that is defined by the formula (24) belongs to \( \mathcal{L}(\mathcal{H}(D)) \) and commutes with the operator \( K_\varphi \).

Using the residues and calculating the integral (21), we can write the operator \( T_n \) in an explicit form, that is, in the form of a differential operator of finite order with variable coefficients.

For an arbitrary function \( f \in \mathcal{H}(D) \), we have

\[
(T_n f)(z) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{(\lambda z - 1)^n}{(\lambda z - z)^n} \frac{f'(\lambda)}{\lambda^2 - 1} d\lambda = \frac{1}{(n-1)!} \left( \frac{(\lambda z - 1)^n}{\lambda^2 - 1} f(\lambda) \right)^{(n-1)}|_{\lambda=z}
\]

\[
= \sum_{k=0}^{n-1} C_{n-1}^k \frac{n}{(k+1)!} z^{n-1-k} (z^2 - 1)^{k+1} \left( \frac{f(z)}{z^2 - 1} \right)^{(k)}
\]

for \(|z| < r < 1\). But

\[
\left( \frac{f(z)}{z^2 - 1} \right)^{(k)} = \frac{1}{2} \left[ \left( f(z) \frac{1}{z-1} \right)^{(k)} - \left( f(z) \frac{1}{z+1} \right)^{(k)} \right]
\]

\[
= \frac{1}{2} \sum_{s=0}^{k} C_s^k (-1)^{k-s} (k-s)! \left[ \frac{1}{(z-1)^{k-s+1}} - \frac{1}{(z+1)^{k-s+1}} \right] f^{(s)}(z).
\]

Therefore,

\[
(T_n f)(z) = \frac{1}{2} \sum_{k=0}^{n-1} \sum_{s=0}^{k} C_s^k C_{n-1}^k \frac{n}{(k+1)!} (-1)^{k-s} z^{n-1-k} (z^2 - 1)^{k+1} \left( \frac{f(z)}{z^2 - 1} \right)^{(k)}
\]

\[
\times \left[ (z+1)^{k+1} (z-1)^s - (z-1)^{k+1} (z+1)^s \right] f^{(s)}(z).
\]

Interchanging the order of summing, we obtain

\[
(T_n f)(z) = \frac{1}{2} \sum_{s=0}^{n-1} \sum_{k=s}^{n-1} C_s^k C_{n-1}^k \frac{n}{(k+1)!} (-1)^{k-s} z^{n-1-k} (z^2 - 1)^{k+1} \left( \frac{f(z)}{z^2 - 1} \right)^{(k)}
\]

\[
\times \left[ (z+1)^{k+1} (z-1)^s - (z-1)^{k+1} (z+1)^s \right] f^{(s)}(z).
\]

This formula implies

\[
(T_1 f)(z) = f(z),
\]

\[
(T_2 f)(z) = (z^2 - 1) f'(z),
\]

\[
(T_3 f)(z) = f(z) + z (z^2 - 1) f'(z) + \frac{(z^2 - 1)^2}{2} f''(z).
\]

Each of the functions \( \psi_n(z) \) that satisfies the condition \( \psi_n(\varphi(z)) = \psi_n(z), z \in D \), is automorphic with respect to the cyclic group generated by the fractional linear transformation \( \varphi(z) \). Let us remark that in [12] classes of such automorphic functions are constructed with Blaschke products.

We give one more class of operators from \( \mathcal{L}(\mathcal{H}(D)) \) that commute with \( K_\varphi \). Fix \( \mu_0 \in \mathbb{R}, |\mu_0| > 1 \). Let \( t_2(\mu, z) \) be an analytic function satisfying the equality \( t_2(\mu, \varphi(z)) = t_2(\mu, z), \mu \in \mathbb{C} \setminus \{\mu_0\}, z \in D \). Theorem 4 and Remark 3 imply that the formula (20) defines an operator \( T \in \mathcal{L}(\mathcal{H}(D)) \) that commutes with the operator \( K_\varphi \). We represent the function \( t_2(\mu, z) \) in the form

\[
t_2(\mu, z) = \sum_{n=1}^{\infty} \frac{\psi_n(z)}{(|\mu - \mu_0|)^n}.
\]


Then, from the analyticity condition on \( t_2(\mu, z) \) and relations \( t_2(\mu, \varphi(z)) = t_2(\mu, z) \) it follows that the functions \( \psi_n(z) \) are analytic in the disk \( D \) and satisfy the equality \( \psi_n(\varphi(z)) = \psi_n(z) \), \( z \in D \), \( n = 1, 2, \ldots \). From (20) we obtain

\[
(T f)(z) = \sum_{n=1}^{\infty} \psi_n(z) (\tilde{T}_n f)(z)
\]

for an arbitrary function \( f \in \mathcal{H}(D) \), \( |z| < 1 \), where

\[
(\tilde{T}_n f)(z) = \frac{1}{(z - \mu_0)^n} \frac{1}{2\pi i} \int_{\gamma_z} \frac{(\lambda - z)^n}{(\lambda - \frac{1}{z-\mu_0})} \frac{f(\lambda)}{\lambda^2 - 1} d\lambda,
\]

\( \gamma_z = \{ \lambda : |\lambda| = r \} \) and \( r \) is chosen such that the point \( \frac{1}{z-\mu_0} = \frac{1}{r} - \mu_0 \) belongs to \( |\lambda| < r \). Theorem 4 implies that each of the operators \( \tilde{T}_n \), \( n = 1, 2, \ldots \), commutes with \( K_\varphi \).

Using Cauchy’s inequalities for the coefficients of the expansion of an analytic function into Laurent series (27), we obtain

\[
\forall r < 1 \lim_{n \to \infty} \sup_{|z| \leq r} \max \{|\psi_n(z)|\} = 0.
\]

Conversely, let a sequence of functions \( \{\psi_n(z)\} \) from \( \mathcal{H}(D) \) satisfy the condition (30) and \( \psi_n(\varphi(z)) = \psi_n(z) \) for \( |z| < 1 \), \( n = 1, 2, \ldots \). Show that the formula (28) defines the some operator \( T \in \mathcal{L}(\mathcal{H}(D)) \) that commutes with \( K_\varphi \). Fix an arbitrary \( r_1 \), \( 0 < r_1 < 1 \). Under the map \( w = \frac{z - \mu_0}{1 - \mu_0 z} \) the disk \( |z| \leq 1 \) will be mapped in some disk that contains in \( |z| < 1 \). Therefore, the number \( \rho = \max\{|\frac{z - \mu_0}{1 - \mu_0 z}| : |z| \leq r_1\} \) satisfies the inequality \( \rho < 1 \). Choose \( r, \rho < r_1 < 1 \). If we take \( \gamma_z = \{ \lambda : |\lambda| = r \} \) in (29), then this formula is correct for an arbitrary function \( f \in \mathcal{H}(D) \) and for any \( z, |z| \leq r_1 \). Estimating integrals of (29) we obtain

\[
| (\tilde{T}_n f)(z) | \leq \frac{1}{(|\mu_0| - r_1)^n} \frac{r}{(r - \rho)^n} \frac{r}{(1 - r^2)} \max |f(\lambda)|
\]

for an arbitrary function \( f \in \mathcal{H}(D) \), \( |z| \leq r_1 \), \( n = 1, 2, \ldots \). Using (30), (31) and the immediate check, we see that the formula (28) defines some operator \( T \) from the class \( \mathcal{L}(\mathcal{H}(D)) \) that \( T \) commutes with \( K_\varphi \).

**Corollary 2.** Let \( \{\psi_n(z)\} \) be a sequence of functions analytic in the disk \( D \), \( \psi_n(\varphi(z)) = \psi_n(z) \), \( z \in D \), \( n = 1, 2, \ldots \), and \( \psi_n(z) \) satisfy (30) . Then the operator \( T \) defined by (28) belongs to \( \mathcal{L}(\mathcal{H}(D)) \) and commutes with the operator \( K_\varphi \).

Calculating the integral of (29) for an arbitrary \( f \in \mathcal{H}(D) \), we have

\[
(\tilde{T}_n f)(z) = \frac{1}{2} \sum_{s=0}^{n-1} \sum_{k=s}^{n} C_n^k C_{n-1}^s \frac{n}{(k+1)!} \frac{1}{(z - \mu_0)^n} (1 + z)^k (-1)^{k-s} \frac{1}{(1 + \mu_0)^{n+s}} \times \left[ \frac{(1 - z)^s (1 + z)^{k+1}}{(1 + \mu_0)^{k+s+1}} - \frac{(1 + z)^s (1 - z)^{k+1}}{(1 - \mu_0)^{k-s+1}} \right] f^{(s)} \left( \frac{z\mu_0 - 1}{\mu_0 - z} \right), \quad n = 1, 2, \ldots,
\]

|\( z \)| < 1. For \( n = 1, 2 \) we get

\[
(T_1 f)(z) = \frac{1}{1 - \mu_0^2} f \left( \frac{z\mu_0 - 1}{\mu_0 - z} \right),
\]

\[
(T_2 f)(z) = \frac{2}{(1 - \mu_0^2)^2} f \left( \frac{z\mu_0 - 1}{\mu_0 - z} \right) + \frac{1}{1 - \mu_0^2} \frac{1 - z^2}{(z - \mu_0)^2} f' \left( \frac{z\mu_0 - 1}{\mu_0 - z} \right).
\]
Note that the operators $T_n$ and $\tilde{T}_n$, $n = 1, 2, \ldots$, are universal in the following sense: they commute with all operators of composition, $K_\varphi$, where $\varphi$ is an arbitrary function of the form $\varphi(z) = \frac{z - z_0}{1 - z_0\bar{z}}, \ z_0 \in \mathbb{R}, \ 0 < |z_0| < 1$.

3. The linear fractional transformation $\varphi(z) = \frac{z - z_0}{1 - z_0\bar{z}}, \ z_0 \in \mathbb{R}, \ |z_0| < 1$, is a particular case of a hyperbolic map of the unit disk $D$ onto itself. We shall describe a commutant of the composition operator $K_\varphi$, induced by an arbitrary hyperbolic transformation $\varphi_1(z) = e^{i\alpha_1} \frac{z - z_1}{1 - z_1\bar{z}}, \ |\sin \frac{\alpha_1}{2}| < |z_1| < 1$ of the unit disk $D$.

**Lemma 1.** Let $\varphi_1(z) = e^{i\alpha_1} \frac{z - z_1}{1 - z_1\bar{z}}, \ 0 < |z_1| < 1, \ \alpha_1 \in \mathbb{R}$, be a hyperbolic automorphism of the unit disk $D$. Then on the space $H(D)$ the operator $K_\varphi_1$ is equivalent to the operator $K_\varphi_2$, where $\varphi_2(z) = e^{i\alpha_1} \frac{z - z_1}{1 - z_1\bar{z}}, \ 0 < |z_1| < 1$. For the isomorphism $K_{\psi_1} \psi_1(z) = e^{-i\beta z}$, $\beta = \arg z_1$, the following identity holds:

$$K_{\varphi_1} K_{\psi_1} = K_{\psi_1} K_{\varphi_2}. \ (32)$$

**Proof.** Since $\psi_1 \circ \varphi_1 = \varphi_2 \circ \psi_1$, we have (32).

Consider the operator $K_{\varphi_2}$, where $\varphi_2(z) = e^{i\alpha_1} \frac{z - z_1}{1 - z_1\bar{z}}, \ |\sin \frac{\alpha_1}{2}| < |z_1| < 1$. We show that on the space $H(D)$ the operator $K_{\varphi_2}$ is equivalent to some operator $K_{\varphi_3}$, where $\varphi_3(z) = z^* \frac{z - z_2}{1 - z_2\bar{z}}, \ 0 < |z_2| < 1$. For this we prove that there exists a function $\psi_2(z) = z^* \frac{z - z_2}{1 - z_2\bar{z}}, \ |z_2| < 1$, that satisfies the equality

$$K_{\varphi_2} \psi_2 = K_{\psi_2} \varphi_3. \ (33)$$

If $\sin \frac{\alpha_1}{2} = 0$ then we take $\psi_2(z) = z$. Suppose $\sin \frac{\alpha_1}{2} \neq 0$; then $e^{i\alpha_1} \neq 1$. The relation (33) is equivalent to $\psi_2 \circ \varphi_2 \circ \psi_2^{-1} = \varphi_3$. But $(\psi_2 \circ \varphi_2)(z) = e^{i\alpha_1} \frac{z - z_2}{1 - z_2\bar{z}}$, where $z^* = \frac{e^{i\alpha_1}|z_1| + z_2}{e^{i\alpha_1} + z_2|z_1|}$. Since $\psi_2^{-1}(z) = \frac{z - z_2}{1 - z_2\bar{z}}$, then $((\psi_2 \circ \varphi_2) \circ \psi_2^{-1})(z) = k \frac{z - z_2}{1 - z_2\bar{z}}$, where

$$k = \frac{z_2|z_1| - |z_2|^2 + e^{i\alpha_1}(1 - \bar{z}_2|z_1|)}{1 - z_2|z_1| + e^{i\alpha_1}(\bar{z}_2|z_1| - |z_2|^2)}, \ (34)$$

$$z_3 = \frac{e^{i\alpha_1}(|z_1| - z_2) + z_2(1 - z_2|z_1|)}{e^{i\alpha_1}(1 - \bar{z}_2|z_1|) + z_2(|z_1| - \bar{z}_2)}, \ (35)$$

We prove that there exists a complex number $z_2$, $|z_2| < 1$, for which the number $k$ is equal to one, that is, the equality

$$2z_2|z_1| - 1 - |z_2|^2 + e^{i\alpha_1}(1 - 2\bar{z}_2|z_1| + |z_2|^2) = 0 \ (36)$$

holds. The solution of this equation can be found in the form $z_2 = re^{i\varphi}, \ r \in \mathbb{R}, \ \varphi \in \mathbb{R}$. Then, the equation (36) with respect to $r$ and $\varphi$ becomes

$$(e^{i\alpha_1} - 1)r^2 + 2r|z_1| \left(e^{i\varphi} - e^{i(\alpha_1 - \varphi)}\right) + e^{i\alpha_1} - 1 = 0.$$ 

After elementary transformations we have

$$r^2 \sin \frac{\alpha_1}{2} + 2r|z_1| \sin \left(\varphi - \frac{\alpha_1}{2}\right) + \sin \frac{\alpha_1}{2} = 0.$$ 

Suppose $\varphi = \frac{\alpha_1}{2} - \frac{\pi}{2}$; then this equation is reduced to

$$r^2 \sin \frac{\alpha_1}{2} - 2r|z_1| + \sin \frac{\alpha_1}{2} = 0.$$ 

One of solutions of the last equation is the number $r = \frac{|z_1| - \sqrt{|z_1|^2 - \sin^2 \frac{\alpha_1}{2}}}{\sin \frac{\alpha_1}{2}}$. By the assumption, $0 < |\sin \frac{\alpha_1}{2}| < |z_1|$, and we see that $r \in \mathbb{R}$ and $0 < |r| < 1$.

Thus,

$$z_2 = e^{i\left(\frac{\alpha_1}{2} + \frac{\pi}{2}\right)} \frac{|z_1| - \sqrt{|z_1|^2 - \sin^2 \frac{\alpha_1}{2}}}{\sin \frac{\alpha_1}{2}} \ (37).$$
Lemma 2. Let $\varphi_2(z) = e^{i\alpha} \frac{z - |z_1|}{1 - |z_1|}$, $0 < |\sin \frac{\alpha}{2}| < |z_1| < 1$, be a hyperbolic automorphism of the unit disk $D$. Then, on the space $\mathcal{H}(D)$ the operator $K_{\varphi_2}$ is equivalent to the operator $K_{\varphi_3}$ where $\varphi_3(z) = \frac{z - z_3}{1 - z_3 z}$, and the number $z_3$ is defined by (35), (37). The equality (33) with $\varphi_2(z) = \frac{z - z_5}{1 - z_5 z_2}$ also holds.

Remark 4. From (33) it follows that the number $z_3$ is not equal to zero, since otherwise, we get $\varphi_2(z) = z$.

Using Lemma 1 for the map $\varphi_3(z) = \frac{z - z_3}{1 - z_3 z}$, we see that on the space $\mathcal{H}(D)$ the operator $K_{\varphi_3}$ is equivalent to the operator $K_{\varphi_4}$, $\varphi_4(z) = \frac{z - |z_3|}{1 - z_3 z}$. Thus,

$$K_{\varphi_3}K_{\psi_3} = K_{\psi_3}K_{\varphi_4},$$

where $\psi_3(z) = e^{-i\beta_1}z$, $\beta_1 = \arg z_3$. From (32), (33) and (38) it follows that $\psi \circ \varphi_1 = \varphi_4 \circ \psi$, thus,

$$K_{\varphi_1}K_{\psi} = K_{\psi}K_{\varphi_4}.$$  

The formula (39) and Lemma 1 [13] imply the following result.

Theorem 5. Let $\varphi_1(z) = e^{i\alpha} \frac{z - z_1}{1 - \frac{z_1}{z}}$, $|\sin \frac{\alpha}{2}| < |z_1| < 1$, be an arbitrary hyperbolic automorphism of the unit disk $D$. An operator $\tilde{T} \in \mathcal{L}(\mathcal{H}(D))$ commutes with the operator $K_{\varphi_1}$ if and only if $\tilde{T} = K_{\psi}TK_{\psi}^{-1}$, where $T \in \mathcal{L}(\mathcal{H}(D))$ and commutes with $K_{\varphi_4}$.

The commutant of the operator $K_{\varphi_4}$ is described in 2. The commutant of the composition operator induced by an arbitrary elliptic automorphism of the unit disk is described in [13]. If $\varphi(z)$ is a parabolic automorphism of the unit disk, then the commutant for the operator $K_{\varphi}$ can be described in a similar way.

References


Chernivtsi State University, 2 Kotsiubyn's'kogo, Chernivtsi, Ukraine

Received 31/01/2008

is a solution of equation (36), and $0 < |z_2| < 1$. □