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# THE CRITERIA OF MAXIMAL DISSIPATIVITY AND SELF-ADJOINTNESS FOR A CLASS OF DIFFERENTIAL-BOUNDARY OPERATORS WITH BOUNDED OPERATOR COEFFICIENTS

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*This paper is dedicated to 100 anniversary of Mark Krein.*

Abstract. A class of the second order differential-boundary operators acting in the Hilbert space of infinite-dimensional vector-functions is investigated. The domains of considered operators are defined by nonstandard (e.g., multipoint-integral) boundary conditions.

The criteria of maximal dissipativity and the criteria of self-adjointness for investigated operators are established.

## 1. Introduction

This paper is devoted to studying of some differential-boundary operators acting in the space of functions taking values in a Hilbert space. It should be noted that the mentioned operators and their abstract models were investigated by many authors (see [4], [6] and references therein) and that our interest in the subject indicated in the headline is motivated in [13], [14], where similar problems were considered.

As in [13], [14] we use the following notation:$D(T)$, $R(T)$, $\ker T$ are, respectively, the domain, range and kernel of a (linear) operator $T$; $\mathcal{B}(X,Y)$ is the set of linear bounded operators $A : X \to Y$ such that $D(A) = X$; $\mathcal{B}(X) = \mathcal{B}(X,X)$; $\mathcal{C}(X)$ is the class of closed densely defined linear operators acting in $X$; $A|E$ is the restriction of a mapping $A$ onto a set $E$; $\overline{E}$ is the closure of $E$; $\operatorname{sp} E$ is the linear manifold generated by a set $E$; $\mathbf{1}_X$ is the identity in $X$; $A^*$ is the adjoint of operator $A$; $\oplus$ is the symbol of orthogonal sum. If $A_i : X \to Y_i$, $(i = 1, \dots, n)$ are linear operators then the notation $A = A_1 \oplus \cdots \oplus A_n$ means that $\forall x \in X \quad Ax = (A_1x, \dots, A_nx)$.

Under $H_0$ we understand a separable Hilbert space, suppose that for every $x \in [a,b]$ $(-\infty < a < b < +\infty)$ $p(x) = p(x)^* \in \mathcal{B}(H_0)$ is a positively definite operator and suppose that $p(.)$ is strongly continuous on $[a,b]$ (these assumptions may be weakened). Put

$$l[y] = -y''(x) + p(x)y \tag{1}$$

and denote by $L$ and $L_0$, respectively, maximal and minimal operators generated in the Hilbert space $H \overset{\text{def}}{=} L_2(H_0;(a,b))$ equipped with the inner product

$$\forall y,z \in H \quad (y|z) = \int_a^b (y(x)|z(x))_{H_0}\,dx$$

by the expression (1) (see [11], [16], [20] for details).

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Further, suppose that $a < c_1 < c_2 < b$, $G^{(i)}$ are closed linear subspaces of $H_0$, $P^{(i)}$ are the orthoprojections $H_0 \to G^{(i)}$ and $Q^{(i)} \overset{\text{def}}{=} \mathbf{1}_{H_0} - P^{(i)}$ $(i = 1, 2)$. Define the operators $L_{\min}$, $L_{\max}$ as follows:

$$D(L_{\min}) = \Big\{ y \in D(L_0) : P^{(i)} y(c_i) = 0, \quad i = 1, 2 \Big\}, \quad L_{\min} \subset L_0,$$

$$D(L_{\max}) = \Big\{ y \in H : y \text{ is absolutely continuous on } [a, b],$$
$$y' \text{ is absolutely continuous on } [a, c_1 - 0] \cup [c_1 + 0, c_2 - 0] \cup [c_2 + 0, b],$$
$$Q^{(i)} y'(c_i - 0) = Q^{(i)} y'(c_i + 0) \;\; (i = 1, 2), \;\; l_{\text{cl}}[y] \in H \Big\},$$

$$\forall y \in D(L_{\max}) \quad L_{\max} y = l_{\text{cl}}[y]$$

(here and below $l_{\text{cl}}[y]$ means the expression (1), in which all derivatives are interpreted in the classical sense).

Furthermore, assume that there are given the operators $\Phi^{(1)}, \Phi^{(2)} \in \mathcal{B}(H, H_0)$ and $\alpha_{ij} \in \mathcal{B}(H_0)$ $(i, j = 1, \dots, 4)$ such that the operator matrix $(\alpha_{ij})_{i,j=1}^{4}$ is invertible in $\mathcal{B}(H_0^4)$. Put

$$u_i(y) = \alpha_{i1} y'(a) - \alpha_{i2} y'(b) + \alpha_{i3} y(a) + \alpha_{i4} y(b) \quad (i = 1, \dots, 4)$$

and define the main object of our investigation, operator $T$, by the relations

$$D(T) = \Big\{ y \in D(L_{\max}) : u_i(y) = P^{(i)} y(c_i) + \Phi^{(i)} y, \qquad (2)$$
$$P^{(i)} u_{i+2}(y) = y'(c_i + 0) - y'(c_i - 0), \; i = 1, 2 \Big\},$$

$$\forall y \in D(T) \quad Ty = l_{\text{cl}}[y] + (\Phi^{(1)})^* u_3(y) + (\Phi^{(2)})^* u_4(y). \qquad (3)$$

The purpose of this paper is to establish the criterion of maximal dissipativity and the criterion of self-adjointness for the operator (2)–(3) which (criteria) were established earlier in some special cases (see [10], [19]).

Recall that according to [2], [17] a linear operator $T : H \to H$ is called dissipative (accumulative) if for each $y \in D(T)$ $\operatorname{Im}(Ty|y) \geq 0 \, (\leq 0)$ and maximal dissipative (maximal accumulative) if, in addition, it has no proper dissipative (accumulative) extensions.

## 2. Preliminaries

Introduce the necessary notations by setting $\mathcal{H} = H_0 \oplus H_0$,

$$\forall y \in D(L) \quad \Gamma_1 y = (y'(a), -y'(b)), \quad \Gamma_2 y = (y(a), y(b)),$$

$$H^1 = \Big\{ y \in H : y \text{ is absolutely continuous on } [a, b], \; y' \in H \Big\},$$

$$H_e = \Big\{ y \in H^1 : y(a) = y(b) = 0 \Big\}.$$

It is well known [3], [9] that $H_e$ with the inner product

$$\forall u, v \in H_e \quad (u|v)_e = \int_a^b [(u'(x)|v'(x))_{H_0} + (p(x)u(x)|v(x))_{H_0}] \, dx$$

is the energetic space of $L_0$, and $L_F \overset{\text{def}}{=} L|\ker \Gamma_2$ is the Friedrichs, or according to M. Krein [7] hard, extension of $L_0$.

Further, assume that $\mathbf{H}$ is an (auxiliary) Hilbert space, $\Lambda : H^1 \to \mathbf{H}$ is a linear operator such that $\Lambda|H_e \in \mathcal{B}(H_e, \mathbf{H})$ and $W \in \mathcal{B}(D[L], \mathbf{H})$ (under $D[L]$ we understand the variety $D(L)$ interpreted as a Hilbert space with the inner product

$$\forall y, z \in D(L) \quad (y|z)_L = (y|z) + (Ly|Lz)).$$

Define the operators $\Lambda^\bullet \in \mathcal{B}(\mathbf{H}, H_e)$ and $W' \in \mathcal{B}(\mathbf{H}, D[L])$ as follows:

$$\forall u \in H_e, \quad \forall h \in \mathbf{H} \quad (\Lambda u|h)_{\mathbf{H}} = (u|\Lambda^\bullet h)_e,$$
$$\forall y \in D(L), \quad \forall h \in \mathbf{H} \quad (Wy|h)_{\mathbf{H}} = (y|W'h)_L.$$

Furthermore, put

$$\forall u \in H^1 \quad \Psi^{(i)}u = P^{(i)}u(c_i), \quad \chi^{(i)}u = \Psi^{(i)}u + \Phi^{(i)}u \quad (i = 1, 2);$$
$$\Phi = \Phi^{(1)} \oplus \Phi^{(2)}, \quad \Psi = \Psi^{(1)} \oplus \Psi^{(2)}, \quad \chi = \chi^{(1)} \oplus \chi^{(2)} \ \ (= \Phi + \Psi);$$
$$G = G^{(1)} \oplus G^{(2)}, \quad P_G = P^{(1)} \oplus P^{(2)};$$
$$\forall y \in D(L_{\max}) \quad \Gamma_1^{[\Psi]}y = (y'(a), -y'(b)), \quad \Gamma_2^{[\Psi]}y = (y(a), y(b)),$$
$$\Gamma_3^{[\Psi]}y = (P^{(1)}(y'(c_1+0) - y'(c_1-0)), P^{(2)}(y'(c_2+0) - y'(c_2-0)))$$

(evidently, in the latter relation the operators $P^{(1)}$, $P^{(2)}$ may be omitted),

$$\Gamma_4^{[\Psi]}y = (P^{(1)}y(c_1), P^{(2)}y(c_2)) \quad (\text{ i.e. } \Gamma_4^{[\Psi]} = \Psi|D(L_{\max}));$$
$$A_{11} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} \alpha_{13} & \alpha_{14} \\ \alpha_{23} & \alpha_{24} \end{pmatrix},$$
$$A_{21} = \begin{pmatrix} \alpha_{31} & \alpha_{32} \\ \alpha_{41} & \alpha_{42} \end{pmatrix}, \quad A_{22} = \begin{pmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{43} & \alpha_{44} \end{pmatrix};$$
$$U_1 = A_{11}\Gamma_1 + A_{12}\Gamma_2, \quad U_2 = A_{21}\Gamma_1 + A_{22}\Gamma_2, \quad U = U_1 \oplus U_2;$$

(4)
$$J = \begin{pmatrix} 0 & i\mathbf{1}_{\mathcal{H}} \\ -i\mathbf{1}_{\mathcal{H}} & 0 \end{pmatrix}.$$

Define the operator $\widetilde{A} = (\widetilde{\alpha}_{ij})$ ($\widetilde{\alpha}_{ij} \in \mathcal{B}(H_0)$; $i, j = 1, \dots, 4$) from the equation

(5)
$$AJ\widetilde{A}^* = J$$

and introduce the following notations:

$$\widetilde{A}_{11} = \begin{pmatrix} \widetilde{\alpha}_{11} & \widetilde{\alpha}_{12} \\ \widetilde{\alpha}_{21} & \widetilde{\alpha}_{22} \end{pmatrix}, \quad \widetilde{A}_{12} = \begin{pmatrix} \widetilde{\alpha}_{13} & \widetilde{\alpha}_{14} \\ \widetilde{\alpha}_{23} & \widetilde{\alpha}_{24} \end{pmatrix},$$
$$\widetilde{A}_{21} = \begin{pmatrix} \widetilde{\alpha}_{31} & \widetilde{\alpha}_{32} \\ \widetilde{\alpha}_{41} & \widetilde{\alpha}_{42} \end{pmatrix}, \quad \widetilde{A}_{22} = \begin{pmatrix} \widetilde{\alpha}_{33} & \widetilde{\alpha}_{34} \\ \widetilde{\alpha}_{43} & \widetilde{\alpha}_{44} \end{pmatrix};$$
$$\widetilde{U}_i = \widetilde{A}_{i1}\Gamma_1 + \widetilde{A}_{i2}\Gamma_2 \ \ (i = 1, 2), \quad \widetilde{U} = \widetilde{U}_1 \oplus \widetilde{U}_2;$$
$$U_i^{[\Psi]} = A_{i1}\Gamma_1^{[\Psi]} + A_{i2}\Gamma_2^{[\Psi]} \ \ (i = 1, 2), \quad U^{[\Psi]} = U_1^{[\Psi]} \oplus U_2^{[\Psi]};$$
$$\widetilde{U}_i^{[\Psi]} = \widetilde{A}_{i1}\Gamma_1^{[\Psi]} + \widetilde{A}_{i2}\Gamma_2^{[\Psi]} \ \ (i = 1, 2), \quad \widetilde{U}^{[\Psi]} = \widetilde{U}_1^{[\Psi]} \oplus \widetilde{U}_2^{[\Psi]}.$$

Using the introduced notations one can readily check that

$$D(T) = \Big\{ y \in D(L_{\max}) : U_1^{[\Psi]}y - \Gamma_4^{[\Psi]}y = \Phi y, \ P_G U_2^{[\Psi]}y = \Gamma_3^{[\Psi]}y \Big\},$$
$$\forall y \in D(T) \quad Ty = L_{\max}y + \Phi^* U_2^{[\Psi]}y.$$

In other words,

$$D(T) = \Big\{ y \in D(L) + R(\chi^\bullet) : y + \chi^\bullet U_2^{[\Psi]}y \in D(L), \ U_1^{[\Psi]}y = \chi y \Big\},$$
$$\forall y \in D(T) \quad Ty = L(y + \chi^\bullet U_2^{[\Psi]}y).$$

Moreover, $T \in \mathcal{C}(H)$ and

(6) $$D(T^*) = \left\{ z \in D(L_{\max}) : \widetilde{U}_1^{[\Psi]} z - \Gamma_4^{[\Psi]} z = \Phi z, \ P_G \widetilde{U}_2^{[\Psi]} z = \Gamma_3^{[\Psi]} z \right\},$$

(7) $$\forall z \in D(T^*) \quad T^* z = L_{\max} z + \Phi^* \widetilde{U}_2^{[\Psi]} z$$

or equivalently

(8) $$D(T^*) = \left\{ z \in D(L) + R(\chi^\bullet) : z + \chi^\bullet \widetilde{U}_2^{[\Psi]} z \in D(L), \ \widetilde{U}_1^{[\Psi]} z = \chi z \right\},$$

(9) $$\forall z \in D(T) \quad T^* z = L(z + \chi^\bullet \widetilde{U}_2^{[\Psi]} z).$$

The proof of (6)–(9) is analogous to that of Theorem 2 in [13].

*Remark* 1. It should be noted that $\chi^\bullet = \Psi^\bullet + L_F^{-1} \Phi^*$ and that operator $\Psi^\bullet$ has been constructed in [15].

## 3. The criterion of accumulativity of operator $T^*$

Since operator $T \in \mathcal{C}(H)$ is maximal dissipative iff $T^*$ is maximal accumulative (see [12], [17]), we are going to establish the conditions which are necessary and sufficient for the accumulativity of operator (6)–(7), or equivalently of operator (8)–(9), at first. For this purpose denote by $P$ the ortoprojection $\mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \overline{R(\chi)}$ and recall that operator $J$ has been defined by (4).

**Lemma 1.** *The set* $\{J\widetilde{U}^{[\Psi]} z : z \in D(T^*)\}$ *is dense in* $\mathcal{H} \oplus \overline{R(\chi)}$.

**Lemma 2.** *For each* $z \in D(T^*)$

$$2\,\mathrm{Im}(T^* z|z) = -(P[iULU' + J]Ph|h)_{\mathcal{H}\oplus\mathcal{H}},$$

*where* $h = J\widetilde{U}^{[\Psi]} z$.

**Corollary 1.** $T^*$ *is an accumulative operator iff*

(10) $$P[iULU' + J]P \geq 0.$$

*Put*

(11) $$L_1 \stackrel{\text{def}}{=} L|\ker U_1.$$

**Corollary 2.** *If* $T^*$ *is an accumulative operator, then* $L_1^*$ *is an accumulative operator. In particular, the maximal dissipativity of* $T$ *implies the maximal dissipativity of* $L_1$.

**Lemma 3.** *Suppose that*

i) *the condition (10) is fulfilled;*
ii) $L_1$ *is a maximal dissipative operator;*
iii) $\dim R(\chi) < +\infty$.

*Then* $T$ *is a maximal dissipative operator.*

The proofs of Lemmas 1–3 and Corollaries 1–2 are analogous to the proofs of corresponding statements in [14], [19].

## 4. Operators $T_n$

Let $\{\widetilde{e}_{k,i}\}$ be (at most countable) orthonormal basis in $G^{(i)}$. Put

$$G_n^{(i)} = \mathrm{sp}\{\widetilde{e}_{1,i}, \dots, \widetilde{e}_{n,i}\}, \quad Q_{G_n^{(i)}} = \mathbf{1}_{H_0} - P_{G_n^{(i)}},$$

where $P_{G_n^{(i)}}$ is the ortoprojection $H_0 \to G_n^{(i)}$ $(i = 1, 2)$, and define the operators $L_{\min}^{(n)}$, $L_{\max}^{(n)}$ by the relations

$$D(L_{\min}^{(n)}) = \Big\{ y \in D(L_0) : P_{G_n^{(i)}} y(c_i) = 0, \quad i = 1, 2 \Big\}, \quad L_{\min}^{(n)} \subset L_0,$$

$$\begin{gathered} D(L_{\max}^{(n)}) = \Big\{ y \in H : y \text{ is absolutely continuous on } [a, b], \\ y' \text{ is absolutely continuous on } [a, c_1 - 0] \cup [c_1 + 0, c_2 - 0] \cup [c_2 + 0, b], \\ Q_{G_n^{(i)}} y'(c_i - 0) = Q_{G_n^{(i)}} y'(c_i + 0) \ \ (i = 1, 2), \quad l_{\mathrm{cl}}[y] \in H \Big\}, \\ \forall y \in D(L_{\max}^{(n)}) \quad L_{\max}^{(n)} y = l_{\mathrm{cl}}[y]. \end{gathered}$$

Introduce the following notations:

$$G_n = G_n^{(1)} \oplus G_n^{(2)}, \quad P_{G_n} = P_{G_n^{(1)}} \oplus P_{G_n^{(2)}}, \quad \Psi_n = P_{G_n} \Psi.$$

In accordance with the results of paper [15] we obtain that $L_{\min}^{(n)}$, $L_{\max}^{(n)} \in \mathcal{C}(H)$ and $(L_{\min}^{(n)})^* = L_{\max}^{(n)}$. Moreover,

$$L_{\min}^{(n)} = L_0 | \ker \Psi_n, \quad D(L_{\max}^{(n)}) = D(L) \dot{+} R(\Psi_n^\bullet) \stackrel{\text{def}}{=} D_n,$$

$$\forall x \in D(L), \quad \forall h \in R(\Psi_n) \quad L_{\max}^{(n)}(x + \Psi_n^\bullet h) = Lx.$$

Further, let $\{\widehat{e}_{k,i}\}$ be (at most countable) orthonormal basis in $R(\chi^{(i)}) \ominus G^{(i)}$ and $P_n^{(i)}$ be the ortoprojection $H_0 \to G_n^{(i)} \oplus \mathrm{sp}\{\widehat{e}_{1,i}, \dots, \widehat{e}_{n,i}\}$ $(i = 1, 2)$. Put

$$P_n = P_n^{(1)} \oplus P_n^{(2)}, \quad \Phi_n = P_n \Phi, \quad \chi_n = P_n \chi.$$

Clearly, $P_n \Psi = P_{G_n} \Psi$, hence $\chi_n = \Psi_n + \Phi_n$. Define operator $T_n$ by the relations

$$\begin{aligned} D(T_n) &= \Big\{ y \in D_n : y + \chi_n^\bullet U_2^{[n]} y \in D(L), \quad U_1^{[n]} y = \chi_n y \Big\} \\ &\equiv \Big\{ y \in D_n : \Gamma_3^{[n]} y = P_{G_n} U_2^{[n]} y, \quad U_1^{[n]} y - \Gamma_4^{[n]} y = \Phi_n y \Big\}, \end{aligned}$$

$$\forall y \in D(T_n) \quad T_n y = L(y + \chi_n^\bullet U_2^{[n]} y) = L_{\max}^{(n)} y + \Phi_n^* U_2^{[n]} y,$$

where $U_i^{[n]} = U_i^{[\Psi_n]} (= U_i^{[\Psi]} | D_n)$ $(i = 1, 2)$, $\Gamma_3^{[n]} = \Gamma_3^{[\Psi_n]} (= \Gamma_3^{[\Psi]} | D_n)$, $\Gamma_4^{[n]} = \Gamma_4^{[\Psi_n]} (= \Psi_n | D_n)$.

Taking into account (6)–(9) we see that

$$\begin{aligned} D(T_n^*) &= \Big\{ z \in D_n : z + \chi_n^\bullet \widetilde{U}_2^{[n]} z \in D(L), \quad \widetilde{U}_1^{[n]} z = \chi_n z \Big\} \\ &\equiv \Big\{ z \in D_n : \Gamma_3^{[n]} z = P_{G_n} \widetilde{U}_2^{[n]} z, \quad \widetilde{U}_1^{[n]} z - \Gamma_4^{[n]} z = \Phi_n z \Big\}, \end{aligned}$$

$$\forall z \in D(T_n^*) \quad T_n^* z = L(z + \chi_n^\bullet \widetilde{U}_2^{[n]} z) = L_{\max}^{(n)} z + \Phi_n^* \widetilde{U}_2^{[n]} z,$$

where $\widetilde{U}_i^{[n]} = \widetilde{U}_i^{[\Psi_n]} (= \widetilde{U}_i^{[\Psi]} | D_n)$ $(i = 1, 2)$.

*Remark* 2. Assume that $y_0 \in D(T)$. Put $Q_{G_n} = \mathbf{1}_{\mathcal{H}} - P_{G_n}$ and consider the following system of equations:

$$\text{(12)} \qquad \begin{cases} U_1^{[n]} y_n = \chi_n y_0, & P_{G_n} U_2^{[n]} y_n = P_{G_n} \Gamma_3^{[\Psi]} y_0, \\ Q_{G_n} U_2^{[n]} y_n = Q_{G_n} U_2^{[\Psi]} y_0, & \Gamma_3^{[n]} y_n = P_{G_n} \Gamma_3^{[\Psi]} y_0, \\ \Gamma_4^{[n]} y_n = \Psi_n y_0, & \Phi_n y_n = \Phi_n y_0. \end{cases}$$

The arguments analogous to ones applied under the proving of [8], Lemma 4.8.7 show that (12) has a solution $y_n \in D_n$. Besides, it is easily to verify by calculations that the latter system is equivalent to the following one:

$$(13)\qquad \begin{cases} U_1^{[n]} y_n = \chi_n y_n, & P_{G_n} U_2^{[n]} y_n = \Gamma_3^{[n]} y_n, \\ Q_{G_n} U_2^{[n]} y_n = Q_{G_n} U_2^{[\Psi]} y_0, & \Gamma_3^{[n]} y_n = P_{G_n} \Gamma_3^{[\Psi]} y_0, \\ \Gamma_4^{[n]} y_n = \Psi_n y_0, & \Phi_n y_n = \Phi_n y_0. \end{cases}$$

It follows from (13) that $\forall n \in N \;\; y_n \in D(T_n)$.

*Remark* 3. Clearly,

i) $\forall h \in \overline{R(\chi)} \quad \lim\limits_{n\to\infty} P_n h = h$;

ii) $\forall g \in G \quad P_n g = P_{G_n} g$, therefore $\lim\limits_{n\to\infty} P_n g = g$.

**Lemma 4.** *Suppose that $y_0 \in D(T)$, and $y_n \in D(T_n)$ $(n \in N)$ satisfy the equalities (12). Then*

$$(14)\qquad \begin{cases} \lim\limits_{n\to\infty} U_i^{[n]} y_n = U_i^{[\Psi]} y_0, \quad \lim\limits_{n\to\infty} \Gamma_{i+2}^{[n]} y_n = \Gamma_{i+2}^{[\Psi]} y_0 \quad (i = 1, 2), \\ \lim\limits_{n\to\infty} \Phi_n y_n = \Phi y_0. \end{cases}$$

*Proof.* Taking into account (12) and Remark 3, we obtain

i) $\lim\limits_{n\to\infty} U_1^{[n]} y_n = \lim\limits_{n\to\infty} \chi_n y_0 = \lim\limits_{n\to\infty} P_n \chi y_0 = \chi y_0 = U_1^{[\Psi]} y_0$;

ii)

$$\begin{aligned} U_2^{[n]} y_n &= P_{G_n} U_2^{[n]} y_n + Q_{G_n} U_2^{[n]} y_n = P_{G_n} \Gamma_3^{[\Psi]} y_0 + Q_{G_n} U_2^{[\Psi]} y_0 \\ &= P_{G_n} \Gamma_3^{[\Psi]} y_0 + U_2^{[\Psi]} y_0 - P_{G_n} U_2^{[\Psi]} y_0 = U_2^{[\Psi]} y_0 + P_{G_n}((\Gamma_3^{[\Psi]} y_0 \\ &- P_\Psi U_2^{[\Psi]} y_0) - (U_2^{[\Psi]} y_0 - P_\Psi U_2^{[\Psi]} y_0)) = U_2^{[\Psi]} y_0; \end{aligned}$$

iii) $\lim\limits_{n\to\infty} \Gamma_3^{[n]} y_n = \lim\limits_{n\to\infty} P_{G_n} \Gamma_3^{[\Psi]} y_0 = \Gamma_3^{[\Psi]} y_0$;

iv) $\lim\limits_{n\to\infty} \Gamma_4^{[n]} y_n = \lim\limits_{n\to\infty} \Psi_n y_0 = \lim\limits_{n\to\infty} P_{G_n} \Psi y_0 = \Gamma_4^{[\Psi]} y_0$;

v) $\lim\limits_{n\to\infty} \Phi_n y_n = \lim\limits_{n\to\infty} \Phi_n y_0 = \lim\limits_{n\to\infty} P_n \Phi y_0 = \Phi y_0$.

The lemma is proved. □

**Corollary 3.** In the assumptions of Lemma 4

$$\lim_{n\to\infty} \widetilde{U}_i^{[n]} y_n = \widetilde{U}_i^{[\Psi]} y_0, \quad i = 1, 2.$$

*Proof.* Put $\widetilde{U}^{[n]} = \widetilde{U}_1^{[n]} \oplus \widetilde{U}_2^{[n]}$. Since both $(\mathcal{H} \oplus \mathcal{H}, U)$ and $(\mathcal{H} \oplus \mathcal{H}, \widetilde{U})$ are boundary pairs for $(L, L_0)$, there exists a bijection $B \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ such that $\widetilde{U} = BU$ (see [8], p. 156 for details), therefore $\widetilde{U}^{[\Psi]} = BU^{[\Psi]}$, $\widetilde{U}^{[n]} = BU^{[n]}$. Taking into account Lemma 4, we obtain

$$\lim_{n\to\infty} \widetilde{U}^{[n]} y_n = B \lim_{n\to\infty} U^{[n]} y_n = BU^{[\Psi]} y_0 = \widetilde{U}^{[\Psi]} y_0.$$

□

## 5. Main results

**Proposition 1.** *Operator $T$ is maximal dissipative iff (10) is fulfilled and operator $L_1$ defined by (11) is maximal dissipative.*

*Proof. Sufficiency.* Suppose that (10) is fulfilled and $L_1$ is maximal dissipative operator. Multiplying (10) from left and from right by $(\mathbf{1}_{\mathcal{H}} \oplus P_n)$ and taking into account following from the inclusion $R(\chi_n) \subset \overline{R(\chi)}$ equalities

$$(\mathbf{1}_{\mathcal{H}} \oplus P_n)P = P(\mathbf{1}_{\mathcal{H}} \oplus P_n) = \mathbf{1}_{\mathcal{H}} \oplus P_n,$$

we derive

$$(\mathbf{1}_{\mathcal{H}} \oplus P_n)[iULU' + J](\mathbf{1}_{\mathcal{H}} \oplus P_n) \geq 0 \quad (n \in N).$$

Whence using Lemma 3 we conclude that $T_n$ is a (maximal) dissipative operator.

Assume that $y_0 \in D(T)$, and $y_n \in D(T_n)$ is a solution of the system (12). Applying Theorem 5 from [15] (its proof is analogous to that of Theorem 1 from [13]) we have

$$\begin{aligned} 0 \leq 2\operatorname{Im}(T_n y_n|y_n) &= -i[(T_n y_n|y_n) - (y_n|T_n y_n)] = -i\{[(L^{(n)}_{\max} y_n|y_n) \\ &- (y_n|(L^{(n)}_{\max} y_n)] + [(U_2^{[n]} y_n|\Phi_n y_n)_{\mathcal{H}} - (\Phi_n y_n|U_2^{[n]} y_n)_{\mathcal{H}}]\} \\ &= -i\{(U_1^{[n]} y_n|\widetilde{U}_2^{[n]} y_n)_{\mathcal{H}} - (U_2^{[n]} y_n|\widetilde{U}_1^{[n]} y_n)_{\mathcal{H}} + (\Gamma_3^{[n]} y_n|\Gamma_4^{[n]} y_n)_{\mathcal{H}} \\ &- (\Gamma_4^{[n]} y_n|\Gamma_3^{[n]} y_n)_{\mathcal{H}} + (U_2^{[n]} y_n|\Phi_n y_n)_{\mathcal{H}} - (\Phi_n y_n|U_2^{[n]} y_n)_{\mathcal{H}}\}. \end{aligned}$$

Taking the limit as $n \to \infty$ and applying the mentioned theorem again we derive

$$\begin{aligned} 0 &\leq -i\{(U_1^{[\Psi]} y_0|\widetilde{U}_2^{[\Psi]} y_0)_{\mathcal{H}} - (U_2^{[\Psi]} y_0|\widetilde{U}_1^{[\Psi]} y_0)_{\mathcal{H}} + (\Gamma_3^{[\Psi]} y_0|\Gamma_4^{[\Psi]} y_0)_{\mathcal{H}} \\ &- (\Gamma_4^{[\Psi]} y_0|\Gamma_3^{[\Psi]} y_0)_{\mathcal{H}} + (U_2^{[\Psi]} y_0|\Phi y_0)_{\mathcal{H}} - (\Phi y_0|U_2^{[\Psi]} y_0)_{\mathcal{H}}\} \\ &= -i\{[(L_{\max} y_0|y_0) - (y_0|(L_{\max} y_0)] + [(\Phi^* U_2^{[\Psi]} y_0|y_0)_{\mathcal{H}} \\ &- (y_0|\Phi^* U_2^{[\Psi]} y_0)_{\mathcal{H}}]\} = -i[(Ty_0|y_0) - (y_0|Ty_0)] = 2\operatorname{Im}(Ty_0|y_0). \end{aligned}$$

Since $y_0$ is an arbitrary element from $D(T)$, $T$ is dissipative operator. In view of Corollary 1 it is maximal dissipative operator (see e.g. [3], [12], [17] for details). The sufficiency is proved.

*Necessity* follows from Corollaries 1, 2. □

*Remark* 4. Similar arguments convince that $T$ is maximal accumulative operator iff

$$P[iULU' + J]P \leq 0$$

and $L_1$ is maximal accumulative operator.

Since operator $T \in \mathcal{C}(H)$ is self-adjoint iff it is maximal dissipative and maximal accumulative simultaneously (see, e.g., [3], [12], [17]) we conclude that $T = T^*$ iff

$$P[iULU' + J]P = 0, \quad L_1 = L_1^*.$$

**Theorem 1.** *Operator $T$ is maximal dissipative (maximal accumulative) iff*

$$P[AJA^* - J]P \leq 0, \quad \ker(A_{11} + iA_{12}) = \{0\},$$
$$(P[AJA^* - J]P \geq 0, \quad \ker(A_{11} - iA_{12}) = \{0\}).$$

*Proof.* In order to prove the theorem it is sufficient to take into account established in [8] the equality $iULU' = -AJA^*$ and to apply the criterion of maximal dissipativity (maximal accumulativity) for the extension of symmetric operator in the form, indicated in [18], see also [1], [3], [5]. □

**Corollary 4.** $T = T^*$ *iff* $P[AJA^* - J]P = 0$, $\ker(A_{11} \pm iA_{12}) = \{0\}$.

## References


1. V. M. Bruk, *Extensions of symmetric relations*, Mat. Zametki **22** (1977), no. 6, 825–834. (Russian)
2. I. C. Gohberg, M. G. Krein, *Introduction to the Theory of Linear Non-Selfadjoint Operators in Hilbert Space*, Nauka, Moscow, 1965. (Russian)
3. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer Acad. Publ., Dordrecht—Boston—London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984)
4. V. I. Gorbachuk, M. L. Gorbachuk, A. N. Kochubei, *Extension theory of symmetric operators and boundary value problems*, Ukranian Math. J. **41** (1989), no. 10, 1298–1313. (Russian)
5. A. N. Kochubei, *On extensions of symmetric operators and symmetric binary relations*, Mat. Zametki **17** (1975), no. 1, 41–48. (Russian)
6. A. M. Krall, *The development of general differential and differential-boundary systems*, Rocky J. Math. **5** (1975), no. 2, 493–542.
7. M. G. Krein, *The theory of self-adjoint extensions of semibounded Hermitian operators and its applications.* I, Mat. Sb. **20** (1947), no. 3, 431–495. (Russian)
8. V. E. Lyantse, O. G. Storozh, *Methods of the Theory of Unbounded Operators*, Naukova Dumka, Kiev, 1983. (Russian)
9. S. G. Mihlin, *Variational Methods in Mathematical Physics*, Nauka, Moscow, 1970. (Russian)
10. O. Ya. Milyo, O. G. Storozh, *Selfadjointness and maximal dissipativeness conditions for a class of finite-dimensional perturbations of a positively definite operator*, Mat. Studii **7** (1997), no. 1, 97–102.
11. M. A. Naimark, *Linear Differential Operators*, Nauka, Moscow, 1969. (Russian)
12. R. S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc. **90** (1959), 193–254.
13. H. M. Pipa, O. G. Storozh, *On some perturbations changing the domain of proper extension of positively definite operator*, Methods Funct. Anal. Topology **11** (2005), no. 3, 257–269.
14. H. M. Pipa, O. G. Storozh, *Maximal dissipativeness conditions for a class of closed operators in Hilbert space*, Visn. Lviv. Univ. **64** (2006), 190–200. (Ukrainian)
15. H. M. Pipa, O. G. Storozh, *On differential-boundary operator of the Sturm-Liouville type with multipoint-integral boundary conditions in a space of infinite-dimensional vector-functions*, Dopov. Nats. Acad. Nauk Ukr. (2006), no. 10, 34–39. (Ukrainian)
16. F. S. Rofe-Beketov, *On self-adjoint extensions of differential operators in a space of vector-valued functions*, Teor. Funkts., Funkts. Analiz Prilozh. **8** (1969), 3–24. (Russian)
17. A. V. Straus, *On extensions and characteristic function of symmetric operator*, Izv. Akad. Nauk SSSR, Ser. Mat. **32** (1968), no. 1, 186–207. (Russian)
18. O. G. Storozh, *The extension theory and abstract boundary operators*, Fourth Summer School "Algebra, Topology, Functional and Stochastic Analysis", (Lviv–Kozyova, July 17–29, 2006), Lviv, 2006, pp. 62–88.
19. O. G. Storozh, O. B. Shuvar, *The criteria of maximal dissipativity for almost bounded perturbations of operators, adjoint to symmetric ones*, Ukrain. Mat. Zh. **55** (2004), no. 7, 966–976. (Ukrainian)
20. F. Z. Ziatdinov, *On linear second-order differential operators in Hilbert space of vector-functions taking values in abstract Hilbert space*, Izv. Vyssh. Uchebn. Zaved., Mat. (1960), no. 4, 89–100. (Russian)



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