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ON UNITARY OPERATORS IN WEIGHTED SPACES $A^2_\omega(\mathbb{C})$ OF ENTIRE FUNCTIONS

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Dedicated to the memory of M. G. Krein.

Abstract. The paper gives a complete characterization of all unitary operators acting in some wide Hilbert spaces $A^2_\omega(\mathbb{C})$ of entire functions possessing weighted square integrable modulus over the whole finite complex plane, which exhaust the set of all entire functions.

1. Introduction

Investigations on the well-known A^p_α spaces of functions holomorphic in the unit disc of the complex plane were initiated by a work of L. Biberbach [1] devoted to approximations in the space of holomorphic functions, the derivatives of which have square integrable modulus over the unit disc of the complex plane. Later, the theory of A^p_α spaces was developed by numerous authors, and other connections with different fields of mathematics were found (a detailed reference list can be found in [5] and [7], see also [6]). The new stage in the development of the theory of A^p_α spaces was provided by the general analytic apparatus of these spaces, which was created in the works of M. M. Djrbashian [3, 4], where the representations of the spaces A^p_ω were obtained as the initial stage to an improvement of a result of R. Nevanlinna [2] related to the density of zeros and poles of similar classes of meromorphic functions. Besides, the work [4] contains an investigation of some particular A^p_ω spaces of entire functions.

The general theory of Banach spaces A^p_ω of functions holomorphic in $|z| < 1$ is created in [7]. These spaces are arbitrarily wide in the sense that their sum coincides with the set of all functions holomorphic in $|z| < 1$. The same work describes also arbitrarily wide in the same sense $A^p_\omega(\mathbb{C})$ $(2 \le p < +\infty)$ spaces of entire functions. According to [7], the Hilbert space $A^2_\omega(\mathbb{C})$ is the set of those entire functions $f(z)$ for which

$$\|f\|^2_\omega = \iint_{\mathbb{C}} |f(z)|^2 d\mu_\omega(z) < +\infty, \tag{1}$$

where $d\mu_\omega(re^{i\vartheta}) = -(2\pi)^{-1} d\vartheta d\omega(r^2)$ and $\omega(x)$ is a strictly decreasing function on the whole half-axis $[0, +\infty)$, such that $\omega(0) = 1$ and

$$\Delta^\infty_n(\omega) = -\int_0^{+\infty} t^n d\omega(t) < +\infty \quad \text{for any} \quad n = 0, 1, 2 \dots$$

This paper gives a complete description of all unitary operators, which act in arbitrarily wide Hilbert spaces $A^2_\omega(\mathbb{C})$ of entire functions, and reveals that these operators are very similar to the projector $L^2_\omega(\mathbb{C}) \to A^2_\omega(\mathbb{C})$ given by the representation of $A^p_\omega(\mathbb{C})$ spaces in [7].

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2. Main result

Before coming to our main result, we recall from [7] that the inner product of $A^2_\omega(\mathbb{C})$ is defined as

$$(f,g)_\omega = \iint_{\mathbb{C}} f(z)\overline{g(z)}\,d\mu_\omega(z), \quad f,g\in A^2_\omega(\mathbb{C}), \tag{2}$$

and the representation formula is of the form

$$f(z) = \iint_{\mathbb{C}} f(\zeta)C_\omega(z\overline{\zeta})\,d\mu_\omega(\zeta), \quad f,g\in A^2_\omega(\mathbb{C}), \tag{3}$$

where the kernel is the entire function

$$C_\omega(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k^\infty(\omega)}. \tag{4}$$

The following theorem is the main result of the present paper.

Theorem 1. *Given a unitary operator U in $A^2_\omega(\mathbb{C})$, there is a function $K(z,\zeta)$ which is of $A^2_\omega(\mathbb{C})$ by the variables z and $\overline{\zeta}$ and satisfies the conditions*

$$K(z,\zeta) = K(\overline{\zeta},\overline{z}), \quad z,\zeta\in\mathbb{C}, \tag{5}$$

$$C_\omega(\zeta_1\overline{\zeta}_2) = \iint_{\mathbb{C}} K(z,\zeta_2)\overline{K(z,\zeta_1)}\,d\mu_\omega(z), \quad \zeta_1,\zeta_2\in\mathbb{C}. \tag{6}$$

By this kernel, the equalities $g=Uf$ and $f=U^{-1}g$ ($f,g\in A^2_\omega$) are written in the forms

$$g(z) = \iint_{\mathbb{C}} f(\zeta)\overline{K(\zeta,z)}\,d\mu_\omega(\zeta), \quad z\in\mathbb{C}, \tag{7}$$

$$f(z) = \iint_{\mathbb{C}} g(\zeta)K(z,\zeta)\,d\mu_\omega(\zeta), \quad z\in\mathbb{C}. \tag{8}$$

Conversely, if a function $K(z,\zeta)$ belongs to $A^2_\omega(\mathbb{C})$ by the variables z and $\overline{\zeta}$ and the conditions (5) *and* (6) *are fulfilled, then formulas* (7) *and* (8) *respectively represent an unitary operator U and its inversion U^{-1} in $A^2_\omega(\mathbb{C})$.*

Proof. Assume that U is a unitary operator in $A^2_\omega(\mathbb{C})$ and observe that $C_\omega(z\overline{\zeta})$ is of $A^2_\omega(\mathbb{C})$ by z and $\overline{\zeta}$. Namely, one can easily verify that $\|C_\omega(z)\|^2 = C_\omega(1) < +\infty$. Now, for any fixed $\zeta\in\mathbb{C}$ denote

$$UC_\omega(z\overline{\zeta}) = H(z,\zeta) \quad\text{and}\quad U^{-1}C_\omega(z\overline{\zeta}) = K(z,\zeta). \tag{9}$$

Then obviously $H(z,\zeta)$ and $K(z,\zeta)$ are of $A^2_\omega(\mathbb{C})$ by both variables z and $\overline{\zeta}$ and, in addition,

$$H(z,\zeta) = H(\overline{\zeta},\overline{z}) \quad\text{and}\quad K(z,\zeta) = K(\overline{\zeta},\overline{z}). \tag{10}$$

Assuming that $g=Uf$ for some functions $f,g\in A^2_\omega(\mathbb{C})$ and using the equality $U^{-1}=U^*$, where U^* is the conjugate operator of U, for any fixed $\zeta\in\mathbb{C}$ we get

$$\begin{aligned}\big(g(z),C_\omega(z\overline{\zeta})\big)_\omega &= \big(Uf(z),C_\omega(z\overline{\zeta})\big)_\omega = \big(f(z),U^*C_\omega(z\overline{\zeta})\big)_\omega \\ &= \big(f(z),U^{-1}C_\omega(z\overline{\zeta})\big) = \big(f(z),K(z,\zeta)\big)_\omega,\end{aligned}$$

i.e.

$$\iint_{\mathbb{C}} g(z)\overline{C_\omega(z\overline{\zeta})}\,d\mu_\omega(z) = \iint_{\mathbb{C}} f(z)\overline{K(z,\zeta)}\,d\mu_\omega(z).$$

Consequently, by (3)

$$g(\zeta)=\iint_{\mathbb{C}} f(z)\overline{K(z,\zeta)}\,d\mu_\omega(z),\quad \zeta\in\mathbb{C}. \tag{11}$$

Thus, formula (7) is true. Further, it is easy to see that for any fixed $\zeta\in\mathbb{C}$

$$\begin{aligned}\big(f(z),C_\omega(z\overline{\zeta})\big)_\omega &= \big(U^{-1}g(z),C_\omega(z\overline{\zeta})\big)_\omega=\big(g(z),UC_\omega(z\overline{\zeta})\big)_\omega\\ &= \big(g(z),H(z,\zeta)\big)_\omega.\end{aligned}$$

Consequently, by (3)

$$f(\zeta)=\iint_{\mathbb{C}} g(z)\overline{H(z,\zeta)}\,d\mu_\omega(z),\quad \zeta\in\mathbb{C}. \tag{12}$$

Now, taking in (11) $f(z)=C_\omega(z\overline{\zeta}_1)$, where $\zeta_1\in\mathbb{C}$ is any fixed point, and observing that in this case $g(\zeta)=Uf(\zeta)=UC_\omega(\zeta\overline{\zeta}_1)=H(\zeta,\zeta_1)$, by (3) we conclude that

$$\begin{aligned}g(\zeta)= H(\zeta,\zeta_1) &= \iint_{\mathbb{C}} C_\omega(z,\overline{\zeta}_1)\overline{K(z,\zeta)}\,d\mu_\omega(z)\\ &= \overline{\iint_{\mathbb{C}} K(z,\zeta)C_\omega(\zeta_1\overline{z})\,d\mu_\omega(z)}=\overline{K(\zeta_1,\zeta)}.\end{aligned}$$

Thus, $\overline{H(\zeta,\zeta_1)}=K(\zeta_1,\zeta)$, and hence the equality (12) becomes the inversion formula (8). For proving formula (6), we insert $f(z)=K(z,\zeta_1)$ in (7). Then using the equalities

$$g(\zeta)=Uf(\zeta)=UK(\zeta,\zeta_1)=C_\omega(\zeta\zeta_1)$$

we come to the formula

$$C_\omega(\zeta\overline{\zeta}_1)=\iint_{\mathbb{C}} K(z,\zeta_1)\overline{K(z,\zeta)}\,d\mu_\omega(z),$$

which coincides with (6).

For proving the converse statement of our theorem, we define the operators

$$UK(z,\zeta)=C_\omega(z\overline{\zeta})\quad\text{and}\quad VC_\omega(z\zeta)=K(z,\zeta) \tag{13}$$

on the sets of functions

$$\big\{K(z,\zeta):\zeta\in\mathbb{C}\big\}\quad\text{and}\quad\big\{C_\omega(z\overline{\zeta}):\zeta\in\mathbb{C}\big\}.$$

Aimed at proving that the operators U and V are unitary on the mentioned sets, one can use (3) and (6) to be convinced that

$$\begin{aligned}\big(UK(z,\zeta_1),UK(z,\zeta_2)\big)_\omega &= \big(C_\omega(z\overline{\zeta}_1),C_\omega(z\overline{\zeta}_2)\big)_\omega=\iint_{\mathbb{C}} C_\omega(z\overline{\zeta}_1)C_\omega(\zeta_2\overline{z})\,d\mu_\omega(z)\\ &=\omega(\zeta_2\overline{\zeta}_1)=\iint_{\mathbb{C}} K(z,\zeta_1)\overline{K(z,\zeta_2)}\,d\mu_\omega(z)\\ &=\big(K(z,\zeta_1),K(z,\zeta_2)\big)_\omega.\end{aligned} \tag{14}$$

Besides, one can similarly prove that

$$\begin{aligned}\big(VC_\omega(z\overline{\zeta}_1),VC_\omega(z\overline{\zeta_2})\big)_\omega &= \big(K(z,\zeta_1),K(z,\zeta_2)\big)_\omega=\iint_{\mathbb{C}} K(z,\zeta_1)\overline{K(z,\zeta_2)}\,d\mu_\omega(z)\\ &=C_\omega(\overline{\zeta}_1\zeta_2)=\iint_{\mathbb{C}} C_\omega(z\overline{\zeta_1})C_\omega(\zeta_2\overline{z})\,d\mu_\omega(z)\\ &=\big(C_\omega(z\overline{\zeta}_1),C_\omega(z\overline{\zeta}_2)\big)_\omega.\end{aligned} \tag{15}$$

On the other hand,

$$
\begin{aligned}
\big(VC_\omega(z\zeta_1), K(z,\zeta_2)\big)_\omega &= \iint_{\mathbb{C}} K(z,\zeta_1)\overline{K(z,\zeta_2)}\, d\mu_\omega(z) = C_\omega(\zeta_2\overline{\zeta}_1) \\
&= \iint_{\mathbb{C}} C_\omega(z\overline{\zeta_1})C_\omega(\overline{z}\zeta_2)\, d\mu_\omega(z) = \big(C_\omega(z\overline{\zeta}_1), C_\omega(z\overline{\zeta}_2)\big)_\omega \\
&= \big(C_\omega(z\zeta_1), UK(z,\zeta_2)\big)_\omega .
\end{aligned} \tag{16}
$$

Formulas (14), (15) and (16) show that

$$
(Uf,Ug)_\omega = (f,g)_\omega, \quad (Vf,Vg)_\omega = (f,g)_\omega \quad \text{and} \quad (Vf,g)_\omega = (f,Ug)_\omega \tag{17}
$$

for f and g from the mentioned sets. Further, the following extension of U and V to the linear spans of the mentioned sets generated by the functions $K(z,\zeta)$ and $C_\omega(z\overline{\zeta})$ is natural:

$$
\text{if} \quad f(z) = \sum_{i=1}^{\infty} a_i K(z,\zeta_i) \quad \text{and} \quad g(z) = \sum_{i=1}^{\infty} b_i C_\omega(z\overline{\zeta}_i),
$$

then we set

$$
Uf(z) = \sum_{i=1}^{\infty} a_i C_\omega(z\overline{\zeta}_i) \quad \text{and} \quad Vg(z) = \sum_{i=1}^{\infty} b_i K(z,\zeta_i).
$$

By the previous methods, one can easily show that also the extended operators U and V satisfy (17) on the corresponding linear spans.

Suppose now that $\{\zeta_k\} \in \mathbb{C}$ is any sequence such that $\zeta_k \to a \in \mathbb{C}$ as $k \to \infty$ and ℓ is an arbitrary linear functional over the space $A^2_\omega(\mathbb{C})$, such that $\ell\big(C_\omega(z\overline{\zeta}_k)\big) = 0$ for any $k \geq 1$. Then ℓ has the form (2) and hence, using the identity (3) we conclude that there is a function $g(z) \in A^2_\omega(\mathbb{C})$ such that

$$
\begin{aligned}
\ell\big(C_\omega(z\overline{\zeta}_k)\big) &= \iint_{\mathbb{C}} C_\omega(z\overline{\zeta}_k)\overline{g(z)}\, d\mu_\omega(z) \\
&= \overline{\iint_{\mathbb{C}} g(z) C_\omega(\overline{z}\zeta_k)\, d\mu_\omega(z)} = \overline{g(\zeta_k)} = 0,
\end{aligned}
$$

i.e. $g(\zeta_k) = 0$ and hence $g(z) \equiv 0$ by the uniqueness of analytic function. Thus, $\ell \equiv 0$ and hence the set $\{C_\omega(z\overline{\zeta}) : \zeta \in \mathbb{C}\}$ is everywhere dense in $A^2_\omega(\mathbb{C})$. Similarly, one can show that also the set $\{K(z,\overline{\zeta}) : \zeta \in \mathbb{C}\}$ is everywhere dense in $A^2_\omega(\mathbb{C})$. Consequently, the operators U and V can be extended from these sets to the whole space $A^2_\omega(\mathbb{C})$, and the equalities (17) are true for any functions $f, g \in A^2_\omega(\mathbb{C})$. It is obvious that the equalities (17) are equivalent to the conditions

$$
U^*V = V^*U = I \quad \text{and} \quad V^* = U
$$

which, in their turn, mean that the operator U has left and right inversions. Consequently, U is invertible and $U^{-1} = V$. Hence, both U and $V = U^{-1}$ are unitary operators in $A^2_\omega(\mathbb{C})$ and by the already proved statements of our theorem they are of the forms (7) and (8).

3. Construction of some special $K(z,\zeta)$ kernels

Now we consider the kernel $K(z,\zeta)$ for some special weights ω, for which $K(z,\zeta)$ takes somehow more explicit forms. Namely, we assume that, along with satisfying (6), $K(z,\zeta)$ *depends on the product* $z\overline{\zeta}$, i.e. $K(z,\zeta) = K(z\overline{\zeta})$. Thus, in addition to (6) we assume that

$$K(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{18}$$

Then by (6)

$$\iint_{\mathbb{C}} K(z\overline{\zeta}_1)\overline{K(z\overline{\zeta}_2)}\, d\mu_\omega(z) = \sum_{m,n=0} a_m \overline{a}_n \overline{\zeta}_1 \zeta_2 \iint_{\mathbb{C}} z^m \overline{z}^n d\mu_\omega(z).$$

On the other hand, by (4)

$$C_\omega(\overline{\zeta}_1\zeta_2) = \sum_{k=0}^{\infty} \frac{\overline{\zeta}_1^k \zeta_2^k}{\Delta_k^\infty(\omega)}.$$

Hence, requiring that the last two sums are equal we obtain

$$a_n\overline{a}_m \iint_{\mathbb{C}} z^m \overline{z}^n d\mu_\omega(z) = \begin{cases} 0 & \text{if} \quad m \neq n \\ |a_n|^2 \displaystyle\iint_{\mathbb{C}} |z|^{2n} d\mu_\omega(z) = \frac{1}{\Delta_n^\infty(\omega)} & \text{if} \quad m = n \end{cases}.$$

Consequently,

$$\frac{1}{\Delta_n^\infty(\omega)} = |a_n|^2 \iint_{\mathbb{C}} |z|^{2n} d\mu_\omega(z) = |a_n|^2 \int_0^{+\infty} x^n d\mu_\omega(x) = |a_n|^2 \Delta_n \infty(\omega),$$

and $|a_n| = [\Delta_n^\infty(\omega)]^{-1}$ or

$$a_n = \frac{e^{it_n}}{\Delta_n^\infty(\omega)}, \tag{19}$$

where $\{t_n\}$ can be any sequence of real numbers. Thus, the considered kernels K, which depend on the product $z\overline{\zeta}$, are of the form

$$K(z) = \sum_{n=0}^{\infty} \frac{e^{it_n}}{\Delta_n^\infty(\omega)} z^n, \tag{20}$$

where $\{t_n\}$ is any sequence of real numbers. For instance, this is true when $\omega(t) = e^{-\sigma t^\alpha}$ $(\sigma, \alpha > 0)$, and the class $A^2_\omega(\mathbb{C})$ coincides with that considered in [7] as a particular case, where

$$\Delta_n = \sigma \int_0^{+\infty} t^{k+\alpha-1} e^{-\sigma t^\alpha} dt = \alpha^{-1}\sigma^{-n/\alpha}\Gamma\left(1 + \frac{n}{\alpha}\right),$$

and

$$K(z) = \alpha \sum_{n=0}^{\infty} e^{it_n} \frac{\sigma^{n/\alpha} z^n}{\Gamma\left(1 + \frac{n}{\alpha}\right)}.$$

References

1. L. Biberbach, *Zur theorie und praxis der konformen abbildung*, Palermo Rendiconti **38** (1914), 98–118.
2. R. Nevanlinna, *Eindeutige Analytische Funktionen*, Springer, Berlin, 1937.
3. M. M. Djrbashian, *On canonical representation of functions meromorphic in the unit disc*, Dokl. Akad. Nauk. Armenian SSR **3** (1945), no. 1, 3–9.
4. M. M. Djrbashian, *On the representability problem of analytic functions*, Soobsch. Inst. Matem. i Mekh. Akad. Nauk Armenian SSR **2** (1948), 3–40.
5. A. E. Djrbashian and F. A. Shamoian, *Topics in the theory of A^p_α spaces*, Teubner-Texte zur Math., vol. 105, Teubner-Verlag, Leipzig, 1988.
6. H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, Berlin, 2000.
7. A. M. Jerbashian, *On the theory of weighted classes of area integrable regular functions*, Complex Variables **50** (2005), 155–183.

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