Methods of Functional Analysis and Topology
Vol. 14 (2008), no. 4, pp. 386–396

SUFFICIENT CONDITIONS FOR SUPERSTABILITY OF MANY-BODY INTERACTIONS

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Abstract. A detailed analysis of sufficient conditions on a family of many-body potentials, which ensure stability, superstability or strong superstability of a statistical system is given in present work.There has been given also an example of superstable many-body interaction.

1. Introduction

From the second half of the previous century different sufficient conditions and restrictions on 2-body potential, which imply superstable or strong superstable interaction have been studied (see [8] for survey of the results). It is obvious, that the research of systems with respect to many-body interaction requires the same conditions on potential energy of interaction of any finite number of particles to be fulfilled. In accordance with this fact, one has a similar problem to describe the sufficient conditions on a sequence of p-body ($p \geq 3$) potentials, which ensure stability, superstability or strong superstability of an infinite statistical system. We have to mention, that such conditions (which ensure an existence of correlation function in the thermodynamic limit) have been written in rather abstract form in works [2], [3] and more implicitly in works [4], [5], [6], [7], [10], [11]. There is another interesting work in this field (see [1]), in which authors consider a finite sequence of finite range many-body potentials, one of which is *stabilizing*, and ensures stability of a whole system. In the present paper we consider an infinite system of infinite range many-body potentials taking into account the traditional concept, i.e. in some sense p-body potential plays less important role in the total energy of interaction than $p-1$-one. p-body potential is a symmetric function of p variables which has positive and negative parts. The conditions on a sequence of p-body ($p > 2$) potentials, which ensure stability, superstability or strong superstability of a system, if such a behavior is enabled by 2-body (pair) potential of interaction are formulated in this article. In the next section we give necessary definitions and formulate main result. In section 4 we give an example of a many-body interaction, which yields above mentioned conditions.

2. Definitions and main result

Let $\mathbb{R}^d$ be a d-dimensional Euclidean space. Following [9] for each $r \in \mathbb{Z}^d$ and $\lambda \in \mathbb{R}_+$ we define an elementary cube with a rib λ and center r

$$\Delta_\lambda(r) = \left\{ x \in \mathbb{R}^d \mid \lambda\left(r^i - 1/2\right) \leq x^i < \lambda\left(r^i + 1/2\right)\right\}. \tag{2.1}$$

We will sometimes write Δ instead of $\Delta_\lambda(r)$, if a cube Δ is considered to be arbitrary and there is no reason to emphasize that it is centered in the particular point $r \in \mathbb{Z}^d$. We denote by $\overline{\Delta_\lambda}$ the corresponding partition of $\mathbb{R}^d$ into cubes Δ. Let us consider a general

2000 *Mathematics Subject Classification.* 82B05, 82B21.
Key words and phrases. Continuous classical system, many-body interaction, criteria of superstability.

type of many-body interaction specified by a family of p-body potentials $V_p : \left(\mathbb{R}^d\right)^p \to \mathbb{R}$, $p \geq 2$ and define also positive and negative parts of interaction potential

$$V_p^+ (x_1, \dots, x_p) := \max \{0; V_p (x_1, \dots, x_p)\},$$
$$V_p^- (x_1, \dots, x_p) := \min \{0; V_p (x_1, \dots, x_p)\}.$$

We assume for the family of potentials $V := \{V_p\}_{p\geq 2}$ the following conditions:

A1. Symmetry. For any $p \geq 2$, any $(x_1, \dots, x_p) \in \left(\mathbb{R}^d\right)^p$ and any permutation π of the numbers $\{1, \dots, p\}$

$$V_p (x_1, \dots, x_p) = V_p \left(x_{\pi(1)}, \dots, x_{\pi(p)}\right).$$

A2. Translation invariance. For any $p \geq 2$, any $(x_1, \dots, x_p) \in \left(\mathbb{R}^d\right)^p$ and $a \in \mathbb{R}^d$

$$V_p (x_1, \dots, x_p) = V_p (x_1 + a, \dots, x_p + a).$$

A3. Repulsion for small distances. There exists a partition of $\mathbb{R}^d$ into cubes $\overline{\Delta_\lambda}$ (see (2.1)) such that for any $(x_1, \dots, x_p) \subset \Delta$, $p \geq 2$

$$V_p (x_1, \dots, x_p) \geq 0.$$

A4. Integrability.

$$\sup_{\{x_1,\dots,x_k\}\in(\mathbb{R}^d)^k} \int_{(\mathbb{R}^d)^{p-k}} \left|V_p^- (x_1, \dots, x_p)\right| dx_{k+1} \cdots dx_p < +\infty, \quad 1 \leq k \leq p-1. \tag{2.2}$$

Under the assumptions **A1-A4** we introduce the energy $U(\gamma) : \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$, which corresponds to the family of potentials $V_p : \left(\mathbb{R}^d\right)^p \to \mathbb{R}$, $p \geq 2$ and which is defined by

$$U(\gamma) = \sum_{p\geq 2} \sum_{\{x_1,\dots,x_p\}\subset\gamma} V_p(x_1, \dots, x_p), \quad \gamma \in \Gamma_0, \quad |\gamma| \geq 2, \tag{2.3}$$

where Γ_0 is the space of finite configurations

$$\Gamma_0 = \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}, \Gamma^{(n)} := \left\{\gamma \subset \mathbb{R}^d \middle| |\gamma| = n\right\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad \Gamma^{(0)} = \{\emptyset\}. \tag{2.4}$$

Let us consider also the part of a total energy, defined only by p-body potential:

$$U^{(p)}(\gamma) = \sum_{\{x_1,\dots,x_p\}\subset\gamma} V_p(x_1, \dots, x_p), \quad \gamma \in \Gamma_0, \quad |\gamma| \geq 2. \tag{2.5}$$

We introduce 3 kinds of interactions, defined by the family of potentials $V := \{V_p\}_{p\geq 2}$.

Definition 1. Interaction, defined by the family of potentials $V := \{V_p\}_{p\geq 2}$ is called:

a) stable, if there exists $B > 0$ such that

$$U(\gamma) \geq -B|\gamma| \quad \text{for any} \quad \gamma \in \Gamma_0; \tag{2.6}$$

b) superstable, if there exist $A > 0$, $B \geq 0$ and partition into cubes $\overline{\Delta_\lambda}$ such that

$$U(\gamma) \geq A \sum_{\Delta\in\overline{\Delta_\lambda}} |\gamma_\Delta|^2 - B|\gamma| \quad \text{for any} \quad \gamma \in \Gamma_0; \tag{2.7}$$

c) strong superstable, if there exist $A > 0$, $B \geq 0$, $m \geq 2$ and $\lambda' > 0$ such that

$$U(\gamma) \geq A \sum_{\Delta\in\overline{\Delta_\lambda}} |\gamma_\Delta|^m - B|\gamma| \quad \text{for any} \quad \gamma \in \Gamma_0, \quad \lambda \leq \lambda'. \tag{2.8}$$

In the above conditions constants A, B can depend on $\overline{\Delta_\lambda}$ and consequently on λ. In our future estimates we will use several notations, which we introduce below.

Definition 2. Let Δ, $\Delta_i \in \overline{\Delta_\lambda}$; $n, m, k_i, k \in \mathbb{N}$, $k_1 + \dots + k_n = p$, $p \geq 2$. Then

(2.9)

$$a)\ I_p^{k_1,\dots,k_n}(\Delta_1,\dots,\Delta_n) := \sup_{\left\{x_1^{(1)},\dots,x_{k_1}^{(1)}\right\}\subset\Delta_1,\dots,\left\{x_1^{(n)},\dots,x_{k_n}^{(n)}\right\}\subset\Delta_n} \left| V_p^{-}\left(x_1^{(1)},\dots,x_{k_n}^{(n)}\right)\right|,$$

(2.10)

$$b)\ I_p^{k|m}(\Delta) := \sum_{(\Delta_1,\dots,\Delta_m)\subset\overline{\Delta_\lambda}} I_p^{k,\overbrace{1,\dots,1}^{m}}(\Delta,\Delta_1,\dots,\Delta_m), \quad k+m=p.$$

Definition 3. Under the conditions of the def. 2 let $\Delta_i \neq \Delta_j$, if $i \neq j$, $\Delta_i \neq \Delta$, $1 \leq i \leq m$. Then

$$I_p^{k|\{k_1,\dots,k_m\}}(\Delta) := \sum_{\{\Delta_1,\dots,\Delta_m\}\subset\overline{\Delta_\lambda}} {\sum_{\pi\in P_m}}' I_p^{k,k_{\pi(1)},\dots,k_{\pi(m)}}(\Delta,\Delta_1,\dots,\Delta_m), \tag{2.11}$$

where P_m is a set of all permutations of numbers $\{1,\dots,m\}$, but the sum $\sum'_{\pi\in P_m}$ means that we consider only different permutations of numbers $\{k_1,\dots,k_m\}$ (for example if $k_i = k_j$ for some i, j, permutation of numbers k_i, k_j is considered only once).

There are three useful remarks and two lemmas, which will be used in our estimates.

Remark 1. From the above definitions the following equality holds:

$$I_p^{k|\{k_1,\dots,k_i,\dots,k_m\}}(\Delta) = I_p^{k_i|\{k_1,\dots,k,\dots,k_m\}}(\Delta).$$

Remark 2. If $\lambda \to 0$ then the following is true:

(2.12)

$$\lambda^{md} I_p^{k|m}(\Delta) \to \sup_{\{x_1,\dots,x_k\}\subset\Delta} \int_{R^m} \left|V_p^{-}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+m})\right| dx_{k+1}\dots dx_{k+m}.$$

If we multiply $I_p^{k|m}(\Delta)$ by λ^{md} we obtain definition of integral sums in the r.h.s of (2.12). It allows us to write an estimate for the value of $I_p^{k|m}(\Delta)$ (see (4.21)).

Remark 3. Due to the assumption **A2** value of $I_p^{k|m}(\Delta)$ does not depend on the position of cubes Δ, so we can put

$$I_p^{k|m} = I_p^{k|m}(\Delta). \tag{2.13}$$

Lemma 1. *For any $p \geq 2$ the following inequality holds:*

$$\sum_{j=2}^{p} \sum_{\substack{k_l\geq 1, 1\leq l\leq j,\\ k_1+\dots+k_j=p,\\ k_1\leq\dots\leq k_j}} I_p^{k_1|\{k_2,\dots,k_j\}}(\Delta) \leq I_p^{1|p-1}(\Delta). \tag{2.14}$$

Proof. Using the definition (2.10) we can rewrite $I_p^{1|p-1}(\Delta)$ in the following form:

$$I_p^{1|p-1}(\Delta) = \sum_{(\Delta_2,\dots,\Delta_p)\subset\overline{\Delta_\lambda}} I_p^{\overbrace{1,\dots,1}^{p}}(\Delta,\Delta_2,\dots,\Delta_p). \tag{2.15}$$

The sum in the r.h.s of (2.15) can be rewritten in the form of sums over sets of cubes $\{\Delta_2,\dots,\Delta_j\}$, $j = \overline{2,p}$, which belong to the area $\overline{\Delta_\lambda} \setminus \{\Delta\}$. Then, neglecting some combinatorial coefficients, which are greater then 1 and using the fact that $I_p^{p}(\Delta) \equiv 0$

for sufficiently small λ (see **A3** and Eq. (2.9) at $n=1$, $k_1=p$), we can deduce from the equality (2.15) that

$$I_p^{1|p-1}(\Delta) \geq \sum_{j=2}^{p} \sum_{\substack{k_l\geq 1, 1\leq l\leq j,\\ k_1+\cdots+k_j=p,\\ k_1\leq\cdots\leq k_j}} \sum_{\{\Delta_2,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda}\setminus\{\Delta\}} {\sum_{\pi\in P_j}}' I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta,\Delta_2,\ldots,\Delta_j). \tag{2.16}$$

Let us take into account the following obvious estimate:

$$ {\sum_{\pi\in P_j}}' I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta,\Delta_2,\ldots,\Delta_j) \geq {\sum_{\pi\in P_{j\setminus\{1\}}}}' I_p^{k_1,k_{\pi(2)},\ldots,k_{\pi(j)}}(\Delta,\Delta_2,\ldots,\Delta_j), \tag{2.17}$$

where $P_{j\setminus\{1\}}$ is a set of all permutations of numbers $\{2,\ldots,j\}$. Using (2.11), (2.16), (2.17), we obtain finally

$$I_p^{1|p-1}(\Delta) \geq \sum_{j=2}^{p} \sum_{\substack{k_l\geq 1, 1\leq l\leq j,\\ k_1+\cdots+k_j=p,\\ k_1\leq\cdots\leq k_j}} I_p^{k_1|\{k_2,\ldots,k_j\}}(\Delta).$$

□

Lemma 2. *For any $p\geq 2$ the following inequality holds:*

$$\sum_{\substack{\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\geq 1, 1\leq r\leq j}} {\sum_{\pi\in P_j}}' I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)\sum_{i=1}^{j}|\gamma_{\Delta_i}|^p \leq j \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}} |\gamma_\Delta|^p I_p^{k_1|\{k_2,\ldots,k_j\}}(\Delta). \tag{2.18}$$

Proof. Taking into account the def. 3 we have

$$\begin{aligned} L &:= \sum_{\substack{\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\geq 1, 1\leq r\leq j}} {\sum_{\pi\in P_j}}' I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)\sum_{i=1}^{j}|\gamma_{\Delta_i}|^p \\ &= \frac{1}{j!} \sum_{\substack{\Delta_1\in\overline{\Delta_\lambda},\ldots,\Delta_j\in\overline{\Delta_\lambda}\\ \Delta_l\neq\Delta_k, l\neq k}} {\sum_{\pi\in P_j}}' I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)\sum_{i=1}^{j}|\gamma_{\Delta_i}|^p. \end{aligned} \tag{2.19}$$

By direct computation we can obtain that for any $\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda}$ the following estimate is true:

$$ {\sum_{\pi\in P_j}}' I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j) \leq \sum_{t=1}^{j} {\sum_{\pi\in P_j\setminus\{t\}}}' I_p^{k_t,k_{\pi(2)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j). \tag{2.20}$$

We obtain from (2.19), (2.20)

$$L \leq \frac{1}{j!}\sum_{r=1}^{j}\sum_{\Delta_r\in\overline{\Delta_\lambda}} \sum_{\substack{\Delta_1,\ldots,\Delta_{r-1}\in\overline{\Delta_\lambda},\\ \Delta_{r+1},\ldots,\Delta_j\in\overline{\Delta_\lambda},\\ \Delta_l\neq\Delta_k, l\neq k}} \sum_{t=1}^{j} {\sum_{\pi\in P_j\setminus\{t\}}}' I_p^{k_t,k_{\pi(2)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)|\gamma_{\Delta_r}|^p. \tag{2.21}$$

As the number of sets $\{\Delta_1,\ldots,\Delta_{r-1},\Delta_{r+1},\ldots,\Delta_j\}\subset\overline{\Delta_\lambda}$ in the third group of sums in (2.21) is $(j-1)!$ and taking into account the def. 3 (see(2.11)) one can rewrite (2.21) in the following way:

$$L \le \frac{1}{j}\sum_{r=1}^{j}\sum_{\Delta_r\in\overline{\Delta_\lambda}}\sum_{t=1}^{j} I_p^{k_t|\{k_1,\dots,k_{t-1},k_{t+1},\dots,k_j\}}(\Delta_r)|\gamma_{\Delta_r}|^p. \tag{2.22}$$

We deduce finally from the remarks 1, 3 and (2.22)

$$L \le j \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\ge 1}} |\gamma_\Delta|^p I_p^{k_1|\{k_2,\dots,k_j\}}(\Delta).$$

□

We give the following definition for the positive part of interaction potential:

$$V_p^p(\Delta) := \inf_{\{x_1,\dots,x_p\}\subset\Delta} V_p^+(x_1,\dots,x_p). \tag{2.23}$$

The main result of the article is in the following theorem:

Theorem 2.1. *Let the family of p-body potentials $V_p : (\mathbb{R}^d)^p \to \mathbb{R}$, $p \ge 2$ satisfy assumptions* **A1**–**A4**. *Let also the part of interaction $U^{(2)}(\gamma)$ be stable (superstable, strong superstable). If there exists such partition of $\mathbb{R}^d$ into cubes $\overline{\Delta_\lambda}$, that the following holds:*

$$1)\ p^{p+1} I_p^{1|p-1}(\Delta) \le V_p^p(\Delta), \quad p > 2, \tag{2.24}$$

$$2)\ \sum_{p>2} p^{p+1} I_p^{1|p-1}(\Delta) < +\infty, \tag{2.25}$$

then interaction, corresponding to this family of potentials, is also stable (superstable, strong superstable).

3. Proof of Theorem 2.1

Proof. Let conditions of the theorem (2.1) hold and $\gamma\in\Gamma_0$. We can write $U^{(p)}(\gamma)$ in the following form:

$$\begin{aligned} U^{(p)}(\gamma) = &\sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\ge p}} \sum_{\{x_1,\dots,x_p\}\subset\gamma_\Delta} V_p(x_1,\dots,x_p) \\ &+ \sum_{j=2}^{p} \sum_{\substack{k_l\ge 1, 1\le l\le j,\\ k_1+\dots+k_j=p,\\ k_1\le\dots\le k_j}} \sum_{\substack{\{\Delta_1,\dots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\ge 1, 1\le r\le j}} \sum_{\substack{\pi: k_{\pi(n)}\le|\gamma_{\Delta_n}|,\\ 1\le n\le j}}{}' \\ &\times \sum_{\{x_1^{(1)},\dots,x_{k_{\pi(1)}}^{(1)}\}\subset\gamma_{\Delta_1},\dots,\{x_1^{(j)},\dots,x_{k_{\pi(j)}}^{(j)}\}\subset\gamma_{\Delta_j}} V_p(x_1^{(1)},\dots,x_{k_{\pi(j)}}^{(j)}). \end{aligned} \tag{3.1}$$

The first part of (3.1) includes the interaction of particles within every arbitrary cube Δ, the second one does the same with particles, which are situated in different cubes of $\overline{\Delta_\lambda}$ with $|\gamma_\Delta| \ge 1$. The 4-th group of sums in the second term of (3.1) is the sum over all different permutations (see def. 3) $\pi : (k_1,\dots,k_j) \to (k_{\pi(1)},\dots,k_{\pi(j)})$ and all values $k_1,\dots,k_j$ $(k_1\le\dots\le k_j)$, $k_l\ge 1$, $l=\overline{1,j}$, $k_1+\dots+k_j=p$ with the restrictions $1\le k_{\pi(n)}\le|\gamma_{\Delta_n}|$, $n=\overline{1,j}$.

Let us explain this notation by simple example. Let the number of cubes, where there are particles for 7-potential be $j = 4$. The set of k_i is $(1, 2, 2, 2)$. We consider a set of cubes $\{\Delta_1,\dots,\Delta_4\}$ such that $|\gamma_{\Delta_1}| = 1$, $|\gamma_{\Delta_2}| = 3$, $|\gamma_{\Delta_3}| = 2$, $|\gamma_{\Delta_4}| = 6$. As a result, all permutations π such that $\pi(1) = 2, \pi(1) = 3, \pi(1) = 4$ are not allowed, i.e. $k_2 = k_3 = k_4 = 2 > |\gamma_{\Delta_1}|$.

Using definitions (2.23) and (2.9) we can estimate (3.1) in the following way:

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} V_p^p(\Delta) C^p_{|\gamma_\Delta|} - \sum_{j=2}^{p} \sum_{\substack{k_l\geq 1, 1\leq l\leq j,\\ k_1+\cdots+k_j=p,\\ k_1\leq\cdots\leq k_j}} \sum_{\substack{\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\geq 1, 1\leq r\leq j}} \sideset{}{'}\sum_{\substack{\pi: k_{\pi(n)}\leq|\gamma_{\Delta_n}|,\\ 1\leq n\leq j}}$$

(3.2)

$$\times \left(\prod_{m=1}^{j} C^{k_{\pi(m)}}_{|\gamma_{\Delta_m}|}\right) I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j),$$

where $C_n^k = \frac{n!}{(n-k)!k!}$.

Using inequalities $\forall n \geq k \geq 1$, $\frac{n^k}{k^k} \leq C_n^k \leq \frac{n^k}{k!}$, we obtain

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} \frac{V_p^p(\Delta)}{p^p}|\gamma_\Delta|^p - \sum_{j=2}^{p} \sum_{\substack{k_l\geq 1, 1\leq l\leq j,\\ k_1+\cdots+k_j=p,\\ k_1\leq\cdots\leq k_j}} \sum_{\substack{\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\geq 1, 1\leq r\leq j}}$$

(3.3)

$$\times \sideset{}{'}\sum_{\substack{\pi: k_{\pi(n)}\leq|\gamma_{\Delta_n}|,\\ 1\leq n\leq j}} I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j) \prod_{m=1}^{j} \frac{|\gamma_{\Delta_m}|^{k_{\pi(m)}}}{k_{\pi(m)}!}.$$

Let us consider the following inequality:

$$\prod_{i=1}^{j} a_i^{m_i} \leq \frac{1}{m_1+\cdots+m_j}\sum_{i=1}^{j} m_i a_i^{m_1+\cdots+m_j} \leq \sum_{i=1}^{j} a_i^{m_1+\cdots+m_j}, \tag{3.4}$$

where $a_1,\ldots,a_j \in \mathbb{R}_+$; $m_1,\ldots,m_j \in \mathbb{N}$. R.h.s of this inequality is obvious, the l.h.s is a consequence of twice used Iensen's inequality. Let us denote $m = m_1+\cdots+m_j$. We have

$$\frac{1}{m}\sum_{i=1}^{j} m_i a_i^m = \sum_{i=1}^{j}\frac{m_i}{m}a_i^m \geq \left(\sum_{i=1}^{j}\frac{m_i a_i}{m}\right)^m,$$

from the other side

$$\prod_{i=1}^{j} a_i^{m_i} = \exp\left(\sum_{i=1}^{j} m_i \ln a_i\right) = \left(\exp\left(\sum_{i=1}^{j}\frac{m_i \ln a_i}{m}\right)\right)^m \leq \left(\sum_{i=1}^{j}\frac{m_i a_i}{m}\right)^m.$$

Using (3.4), we obtain

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} \frac{V_p^p(\Delta)}{p^p}|\gamma_\Delta|^p - \sum_{j=2}^{p} \sum_{\substack{k_l\geq 1, 1\leq l\leq j,\\ k_1+\cdots+k_j=p,\\ k_1\leq\cdots\leq k_j}} \sum_{\substack{\{\Delta_1,\ldots,\Delta_j\}\subset\overline{\Delta_\lambda},\\ |\gamma_{\Delta_r}|\geq 1, 1\leq r\leq j}}$$

(3.5)

$$\times \prod_{m=1}^{j}\frac{1}{k_m!} \sideset{}{'}\sum_{\substack{\pi: k_{\pi(n)}\leq|\gamma_{\Delta_n}|,\\ 1\leq n\leq j}} I_p^{k_{\pi(1)},\ldots,k_{\pi(j)}}(\Delta_1,\ldots,\Delta_j)\sum_{i=1}^{j}|\gamma_{\Delta_i}|^p.$$

Taking into account, that the sum w.r.t. π defined in (3.1) contains less number of terms, the the same one in (2.11), as it does not have the restrictions $k_{\pi(n)} \leq |\gamma_{\Delta_n}|$, $n = \overline{1,j}$

and using lemma 2, that is inequality (2.18), we obtain
(3.6)

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} \frac{V_p^p(\Delta)}{p^p}|\gamma_\Delta|^p - \sum_{j=2}^{p}\frac{j}{B(p;j)}\sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}}|\gamma_\Delta|^p \sum_{\substack{k_l\geq 1, 1\leq l\leq j,\\ k_1+\dots+k_j=p,\\ k_1\leq\dots\leq k_j}} I_p^{k_1|\{k_2,\dots,k_j\}}(\Delta),$$

where

$$B(p;j) = \inf_{\substack{k_{\pi(t)}\geq 1, 1\leq t\leq p,\\ k_{\pi(1)}+\dots+k_{\pi(j)}=p}} (k_{\pi(1)}!\dots k_{\pi(j)}!)$$

Since $\max\frac{j}{B(p;j)} = p$, $2\leq j\leq p$ and taking into account definitions 2, 3 and lemma 1, we deduce, that

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}} \frac{V_p^p(\Delta)}{p^p}|\gamma_\Delta|^p - pI_p^{1|p-1}\sum_{\substack{\Delta\subset\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}}|\gamma_\Delta|^p. \tag{3.7}$$

Number of cubes with $|\gamma_\Delta| = k$ is not more than $\frac{|\gamma|}{k}$. Due to this, the following estimate holds:

$$\begin{aligned}\sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq 1}}|\gamma_\Delta|^p &= \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}}|\gamma_\Delta|^p + \sum_{k=1}^{p-1}\sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|=k}}|\gamma_\Delta|^p \\ &\leq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}}|\gamma_\Delta|^p + \sum_{k=1}^{p-1}k^{p-1}|\gamma|.\end{aligned} \tag{3.8}$$

Using (3.7) and (3.8), we obtain the final estimate of $U^{(p)}(\gamma)$

$$U^{(p)}(\gamma) \geq \sum_{\substack{\Delta\in\overline{\Delta_\lambda},\\ |\gamma_\Delta|\geq p}}|\gamma_\Delta|^p\left(\frac{V_p^p(\Delta)}{p^p} - pI_p^{1|p-1}\right) - pI_p^{1|p-1}\sum_{k=1}^{p-1}k^{p-1}|\gamma|. \tag{3.9}$$

Let us take into account the following obvious estimate:

$$\sum_{p>2} pI_p^{1|p-1}\sum_{k=1}^{p-1}k^{p-1} < \sum_{p>2}p^{p+1}I_p^{1|p-1}. \tag{3.10}$$

Using the last estimates (3.9), (3.10) the conditions of superstability (strong superstability) (2.24)–(2.25) are fulfilled with

$$B = B_2 + \sum_{p>2}p^{p+1}I_p^{1|p-1}, \quad A = A_2, \tag{3.11}$$

where A_2, B_2 are taken from the condition of superstability (strong superstability) of 2-body part of interaction. If pair potential is only stable, then many-body interaction is also stable with the constant B from (3.11). □

Remark 4. Actually, we proved modified superstability inequality (3.9) for the family of p-body potentials $V := \{V_p\}_{p\geq 2}$, which satisfies assumptions **A1**–**A4** and conditions (2.24), (2.25).

4. Example of many-body interaction

Example 1. Let V be a many-body interaction, specified by a family of p-body potentials $V_p : \left(\mathbb{R}^d\right)^p \to \mathbb{R}$, $p \geq 2$

$$V_p(x_1, \ldots, x_p) = \frac{A_p}{\left(\sum\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{m(p)}} - \frac{B_p}{\left(\sum\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{n(p)}},$$

(4.1)

$$A_p > 0, \quad B_p > 0; \quad m(p) > n(p), \quad n(p) > (p-1)d.$$

Write down the conditions on A_p, B_p, that ensure superstability of interaction.

Verification of assumptions **A1**–**A2** is obvious. Let us analyze the assumption **A3** for the family of p-body potentials (4.1). Consider the following estimates of the sum $\sum\limits_{1\leq i<j\leq p} |x_i - x_j|$ in (4.1).

Lemma 3.

$$\sum_{1\leq i<j\leq p} |x_i - x_j| \geq (p-1) \max_{1\leq i<j\leq p} |x_i - x_j|. \tag{4.2}$$

Proof. Let us put for definiteness $\max\limits_{1\leq i<j\leq p} |x_i - x_j| = |x_1 - x_2|$. Then the following range of estimates holds:

$$\sum_{1\leq i<j\leq p} |x_i - x_j| \geq |x_1 - x_2| + \sum_{j=3}^{p} (|x_1 - x_j| + |x_2 - x_j|)$$

(4.3)

$$\geq |x_1 - x_2| + \sum_{j=3}^{p} |x_1 - x_2| = (p-1)|x_1 - x_2|.$$

The minimum of $\sum\limits_{1\leq i<j\leq p} |x_i - x_j|$ is reached, if $p-2$ particles coincide. □

The sum $\sum\limits_{1\leq i<j\leq p} |x_i - x_j|$ can be estimated from above in such a way:

$$\sum_{1\leq i<j\leq p} |x_i - x_j| \leq \frac{p(p-1)}{2} \max_{1\leq i<j\leq p} |x_i - x_j|. \tag{4.4}$$

Remark 5. In 1-dimensional case the estimate (4.4) can be improved

$$\sum_{1\leq i<j\leq p} |x_i - x_j| \leq \left(p - \left[\frac{p}{2}\right]\right) \left[\frac{p}{2}\right] \max_{1\leq i<j\leq p} |x_i - x_j|.$$

The maximum of $\sum\limits_{1\leq i<j\leq p} |x_i - x_j|$ is reached, if $\left[\frac{p}{2}\right]$ particles are situated at one point and the rest of them are situated at another one.

It follows from (4.2), (4.4), that for any $(x_1, \ldots, x_p) \subset \Delta$, $p \geq 2$ the following is true:

$$V_p\left(x_1, \ldots, x_p\right) \geq \frac{A_p'}{\left(\max\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{m(p)}} - \frac{B_p'}{\left(\max\limits_{1\leq i<j\leq p} |x_i - x_j|\right)^{n(p)}}, \tag{4.5}$$

where $A_p' = \frac{A_p}{\left(\frac{p(p-1)}{2}\right)^{m(p)}}$, $B_p' = \frac{B_p}{(p-1)^{n(p)}}$.

We deduce from (4.5), that for any fixed $p \geq 2$ there exists such a ball $B(D;R_p)$ with a center in the arbitrary point D and a radius R_p:

$$R_p = \left(\frac{A_p}{B_p}\right)^{\frac{1}{m(p)-n(p)}} \frac{1}{2(p-1)\left(\frac{p}{2}\right)^{\frac{m(p)}{m(p)-n(p)}}}, \tag{4.6}$$

that $V_p(x_1,\dots,x_p) \geq 0$ for any $(x_1,\dots,x_p) \subset B(D;R_p)$. So, we have to find such values of A_p, B_p that $R_0 = \inf_{p\geq 2} R_p > 0$ in order the assumption **A3** to be fulfilled for the family of potentials (4.1), where the length of a rib of d-dimensional cube in the partition $\overline{\Delta_{\lambda_0}}$ is equal to

$$\lambda_0 = \frac{2R_0}{\sqrt{d}}. \tag{4.7}$$

In the ball $B(0;R_0)$ with a center in the origin and a radius R_0: $V_p(x_1,\dots,x_p) \geq 0$ for any $(x_1,\dots,x_p) \subset B(0;R_0)$ and any $p \geq 2$.

It is clear that

$$\left|V_p^-(x_1,\dots,x_p)\right| \leq \frac{B'_p}{\left(\max\limits_{1\leq i<j\leq p}|x_i - x_j|\right)^{n(p)}}.$$

Prove that it satisfies the assumption **A4**. Let us put $x_1 = 0$ for definiteness. There are two cases for any $(x_1,\dots,x_p) \subset (\mathbb{R}^d)^p$

$$\begin{aligned}
&1)\ \operatorname{diam}(\{x_1,\dots,x_p\}) = \operatorname{dist}(x_i;x_j),\\
&0 \in B\left(\frac{x_i+x_j}{2}; \frac{|x_j - x_i|}{2}\right), \quad 1 < i \leq p, \quad 1 < j \leq p;\\
&2)\ \operatorname{diam}(\{x_1,\dots,x_p\}) = \operatorname{dist}(0;x_j), \quad 1 < j \leq p.
\end{aligned}$$

In accordance with these two cases one can write the following estimate:

$$\int_{(\mathbb{R}^d)^{p-1}} \left|V_p^-(0,x_2,\dots,x_j)\right| dx_2 dx_p \leq K_p = K_p^{(1)} + K_p^{(2)};$$

$$\begin{aligned}
K_p^{(1)} \leq B'_p C_{p-1}^2 &\int\limits_{\substack{\frac{|x_2+x_p|}{2} \leq \frac{|x_p-x_2|}{2},\\ |x_p-x_2|>2R_0}} \frac{dx_2 dx_p}{|x_p - x_2|^{n(p)}}\\
&\times \int\limits_{\left|\frac{x_p+x_2}{2} - x_3\right| \leq \frac{|x_p-x_2|}{2}} dx_3 \cdots \int\limits_{\left|\frac{x_p+x_2}{2} - x_{p-1}\right| \leq \frac{|x_p-x_2|}{2}} dx_{p-1},
\end{aligned} \tag{4.8}$$

$$K_p^{(2)} \leq B'_p(p-1) \int\limits_{|x_p|>2R_0} \frac{dx_p}{|x_p|^{n(p)}} \int\limits_{\left|\frac{x_p}{2} - x_2\right| \leq \frac{|x_p|}{2}} dx_2 \cdots \int\limits_{\left|\frac{x_p}{2} - x_{p-1}\right| \leq \frac{|x_p|}{2}} dx_{p-1}. \tag{4.9}$$

The first integral $K_p^{(1)}$ and the second $K_p^{(2)}$ refer to cases 1) and 2) respectively. In (4.9) $p-1$ is a number of $x_j, 1 < j \leq p$. Let us take into account that a volume of d-dimensional ball $B(a,R)$ is

$$\int_{|x-a|\leq R} dx = \frac{2\pi^{\frac{d}{2}} R^d}{d\Gamma\left(\frac{d}{2}\right)}. \tag{4.10}$$

Using (4.10) one can rewrite (4.8), (4.9) in the following form:

$$K_p^{(1)} \leq B_p' C_{p-1}^2 \left(\frac{\pi^{\frac{d}{2}}}{2^{d-1} d \Gamma\left(\frac{d}{2}\right)} \right)^{p-3} \int\limits_{\substack{\frac{|x_2+x_p|}{2} \leq \frac{|x_p-x_2|}{2}, \\ |x_p-x_2|>2R_0}} \frac{dx_2 dx_p}{|x_p - x_2|^{n(p)-(p-3)d}}, \tag{4.11}$$

$$K_p^{(2)} \leq B_p'(p-1) \left(\frac{\pi^{\frac{d}{2}}}{2^{d-1} d \Gamma\left(\frac{d}{2}\right)} \right)^{p-2} \int\limits_{|x_p|>2R_0} \frac{dx_p}{|x_p|^{n(p)-(p-2)d}}. \tag{4.12}$$

In (4.11) we do the following substitution of variables: $\{x_2; x_p\} \to \{x_2; t\}, t = x_p - x_2$. We obtain

$$\begin{aligned} K_p^{(1)} &\leq B_p' C_{p-1}^2 \left(\frac{\pi^{\frac{d}{2}}}{2^{d-1} d \Gamma\left(\frac{d}{2}\right)} \right)^{p-3} \int\limits_{|t|>2R_0} \frac{dt}{|t|^{n(p)-(p-3)d}} \int\limits_{|x_2+\frac{t}{2}| \leq \frac{|t|}{2}} dx_2 \\ &= B_p' C_{p-1}^2 \left(\frac{\pi^{\frac{d}{2}}}{2^{d-1} d \Gamma\left(\frac{d}{2}\right)} \right)^{p-2} \int\limits_{|t|>2R_0} \frac{dt}{|t|^{n(p)-(p-2)d}}. \end{aligned} \tag{4.13}$$

Using generalized spherical coordinates, we deduce from (4.12) and (4.13) that

$$K_p^{(1)} \leq B_p' C_{p-1}^2 2^d d \left(\frac{\pi^{\frac{d}{2}}}{2^{d-1} d \Gamma\left(\frac{d}{2}\right)} \right)^{p-1} \int_{2R_0}^{+\infty} \frac{dr}{r^{n(p)+(1-p)d+1}}, \tag{4.14}$$

$$K_p^{(2)} \leq B_p'(p-1) 2^d d \left(\frac{\pi^{\frac{d}{2}}}{2^{d-1} d \Gamma\left(\frac{d}{2}\right)} \right)^{p-1} \int_{2R_0}^{+\infty} \frac{dr}{r^{n(p)+(1-p)d+1}}. \tag{4.15}$$

If $n(p) + (1-p)d > 0$ integrals (4.14) and (4.15) converge and finally

$$\begin{aligned} &\int_{(\mathbb{R}^d)^{p-1}} \left| V_p^- (0, x_2, \ldots, x_p) \right| dx_2 dx_p \\ &\quad \leq \frac{2B_p' d(p-1)^2}{(n(p)+(1-p)d)(2R_0)^{n(p)+(1-p)d}} \left(\frac{\pi^{\frac{d}{2}}}{2^{d-1} d \Gamma\left(\frac{d}{2}\right)} \right)^{p-1} \\ &\quad \leq \frac{2B_p d}{(p-1)^{n(p)-2}(n(p)+(1-p)d)(2R_0)^{n(p)+(1-p)d}} = F_p. \end{aligned} \tag{4.16}$$

As a result the assumption **A4** holds. Now we will show that assumption **A3** and the conditions (2.24), (2.25) also hold, if we put for example

$$m(p) = pd+1, \quad n(p) = pd, \quad A_p = \left(\frac{p(p-1)}{2} \right)^{m(p)}, \quad B_p = \frac{1}{p^{p+2+\varepsilon} 2^{pd-1} d^{\frac{m(p)}{2}}}, \varepsilon > 0. \tag{4.17}$$

Using (4.6), (4.7) we obtain in this case

$$R_p = p^{p+2+\varepsilon}(p-1)^{pd} 2^{pd-2} d^{\frac{pd+1}{2}}, \quad R_0 = 2^{2d+2+\varepsilon} d^{d+\frac{1}{2}}, \quad \lambda_0 = 2^{2d+3+\varepsilon} d^d. \tag{4.18}$$

So, the assumption **A3** holds. We have from (4.16), (4.17) the upper bound for F_p

$$F_p < \frac{1}{p^{p+2+\varepsilon} d^{\frac{pd+1}{2}}}. \tag{4.19}$$

Let us verify that the condition (2.25) yields. It follows directly from the estimate $p^{p+1} F_p < \frac{1}{p^{1+\varepsilon}}$. Function in the r.h.s of (4.5) achieves its minimum in the cubic area Δ

with a rib $\lambda, \lambda \le \lambda_0$ if $\max_{1\le i<j\le p} |x_i - x_j| = \lambda\sqrt{d}$. Using (4.5), (4.17) we obtain the lower bound for $V_p^p(\Delta)$

$$V_p^p(\Delta) \ge \frac{1}{(\lambda\sqrt{d})^{pd+1}} - \frac{1}{(\lambda\sqrt{d})^{pd}}. \tag{4.20}$$

We deduce from the Remark 2 that

$$I^{1|p-1} \sim \frac{F_p - \delta}{\lambda^{(p-1)d}}, \quad \delta > 0, \quad \text{if} \quad \lambda \to 0. \tag{4.21}$$

This fact, inequalities (4.19), (4.20) imply fulfillment of (2.24).

Acknowledgments. The author is grateful to Prof. O. L. Rebenko for setting this problem and stimulating discussions and Dr. D. L. Finkelstein for valuable critical remarks and shortest way to prove lemma 3 and (3.4).

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Received 14/01/2008; Revised 20/08/2008