Methods of Functional Analysis and Topology
Vol. 15 (2009), no. 1, pp. 15–30

ORIGINATION OF THE SINGULAR CONTINUOUS SPECTRUM IN THE CONFLICT DYNAMICAL SYSTEMS

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Dedicated to the memory of wise Ukrainian mathematician A. Ya. Povzner

Abstract. We study the spectral properties of the limiting measures in the conflict dynamical systems modeling the alternative interaction between opponents. It has been established that typical trajectories of such systems converge to the invariant mutually singular measures. We show that "almost always" the limiting measures are purely singular continuous. Besides we find the conditions under which the limiting measures belong to one of the spectral type: pure singular continuous, pure point, or pure absolutely continuous.

1. Introduction

We study the spectral properties of the limiting invariant measures of the dynamical systems with composition of the alternative conflict interaction. This composition generates the evolution of occupations of the living positions (regions) for a couple of opponents. A starting state of the system is fixed by a pair of probability measures μ and ν corresponding to opponents which exist on the common space Ω (the space of the living resources). The fight between opponents for the control in regions of Ω is described by the non-commutative and non-linear operation (composition) which is defined in terms of μ and ν. This transformation we call the conflict composition and denote by $\divideontimes$ (see below Section 4).

Actually the transformation $\divideontimes$ provides a certain redistribution between opponents of the occupation probabilities in regions at various moments of the conflict. Iteration of this transformation generates the evolution of the probability redistribution in terms of the changed measures at a sequence of the discrete moments of time $t = 1, 2, \dots, N, \dots$ So, the dynamical system of conflict appears

$$\{\mu^{N-1}, \nu^{N-1}\} \xrightarrow{\divideontimes} \{\mu^N, \nu^N\}, \quad \mu^0 = \mu, \quad \nu^0 = \nu, \quad N = 1, 2, \dots$$

In papers [15, 16] it has been proved (for finite or countable spaces Ω) that the limiting states

$$\mu^\infty = \lim_{N\to\infty} \mu^N, \quad \nu^\infty = \lim_{N\to\infty} \nu^N$$

exist and are invariant with respect to the conflict composition $\divideontimes$.

In the present paper we extend the above result for the case of measures having the similar structure on the segment $[0, 1]$ (c.f. with [17, 1]).

The significant fact is that an every similar structure measure (for the exact definition see Section 3) belongs to the pure spectral type is the following sense. Namely, it means

2000 *Mathematics Subject Classification.* 91A05, 91A10, 90A15, 90D05, 37L30, 28A80.
Key words and phrases. Conflict dynamical system, similar structure probability measure, conflict composition, local priority, directed priority, pure point, pure absolutely continuous, pure singular continuous measures.

that each such measure has a single component in the Lebesgue decomposition: either purely point, or purely absolutely continuous, or purely singular continuous.

A key problem is to find the conditions which ensure that one or both limiting measures μ^∞, ν^∞ belong to the before chosen spectral type. Some results in this direction have been already obtained in [17, 1].

In this paper we stress that the limiting measures μ^∞, ν^∞ are singular continuous almost always. Only in exceptional cases they might become point or absolutely continuous. This fact exposes the mostly distinctive property of the considered conflict dynamical systems. In particular, starting with absolutely continuous measures $\mu, \nu \in \mathcal{M}_{\rm ac}$ and applying an infinite sequence of conflict composition $\divideontimes$, we usually obtain in the limit the singular continuous measures $\mu^\infty, \nu^\infty \in \mathcal{M}_{\rm sc}$ of the Cantor type. The supports of $\mu^\infty, \nu^\infty \in \mathcal{M}_{\rm sc}$ are nowhere dense sets of zero Lebesgue measure. We will produce the conditions ensuring this result.

At the same time in the exceptional cases we obtain the criterion guaranteeing that one of limiting measures will be purely point or purely absolutely continuous. Of course, the appearance of the such type of limiting measures is rather exotic. We illustrate it by simple examples.

Finally we remark that the growing interest in study of singular continuous measures and their supports (=spectra) takes place not only in the fractal geometry but in mathematical physics too. In particular, in the connection with the problem of physical interpretation of the fine effects from presence of the singular spectrum (see, for example, [3, 8, 18, 24]). We also refer reader to the papers [6, 23] where conflict composition $\divideontimes$ has been used in the applications for construction of the migration and population models .

2. The similar structure measures

In the present paper we shall use a specific class of measures, the similar structure measures, having a certain structural similarity of its supports. This class of measures are exactly appropriate for construction of the conflict dynamical systems. The set of similar structure measures are considerably wider than the well-known class of the self-similar measures introduced by Hutchinson [13] (see also [12, 24]).

Let us describe shortly the notion of probability similar structure measure which are supported on the segment $\Delta_0 = [0, 1]$.

Let $T = \{T_{ik}\}_{i=1}^n$, $k = 1, 2, \dots$, $2 \leq n < \infty$ be a family of contractive similarities on $\mathbb{R}^1$ having the following properties. For all k

$$(a) \quad 0 < c \leq c_{ik} < 1,$$

where c_{ik} is the contraction coefficient for T_{ik},

$$(b) \quad \Delta_0 = \bigcup_{i=1}^{n} T_{ik}\Delta_0,$$

$$(c) \quad \lambda(T_{ik}\Delta_0 \bigcap T_{i'k}\Delta_0) = 0, \quad i \neq i',$$

where λ denotes Lebesgue measure.

Put

$$U_{i_1 \cdots i_k, i'_1 \cdots i'_k} := T_{i_1 1} \cdots T_{i_k k}(T_{i'_1 1} \cdots T_{i'_k k})^{-1}, \quad 1 \leq i_k, \quad i'_k \leq n.$$

We recall that each $T_{i_k k}$ is a bijection and hence the inverse transformation $T_{i_k k}^{-1}$ is well defined.

We say that set $S \subseteq \Delta_0$ has a *structural similarity*, if for some fixed family of contractive similarities T with above properties, S admits the representation

$$S = \bigcup_{i_1=1}^{n} s_{i_1}, \quad s_{i_1} = \bigcup_{i_2=1}^{n} s_{i_1 i_2}, \cdots, s_{i_1\cdots i_{k-1}} = \bigcup_{i_k=1}^{n} s_{i_1 i_2\cdots i_k}, \cdots, \tag{2.1}$$

where $s_{i_1\cdots i_k} \subset T_{i_1 1}\cdots T_{i_k k}\Delta_0$ and all non-empty $s_{i_1\cdots i_k}$ of every fixed rank $k = 1, 2, \ldots$ are similar one to other

$$s_{i_1\cdots i_k} = U_{i_1\cdots i_k, i'_1\cdots i'_k} s_{i'_1\cdots i'_k}. \tag{2.2}$$

We observe that

$$\mathrm{diam}(s_{i_1\cdots i_k}) \to 0, \quad k \to \infty, \tag{2.3}$$

and

$$\lambda\big(s^{\mathrm{cl}}_{i_1\cdots i_k} \bigcap s^{\mathrm{cl}}_{i'_1\cdots i'_k}\big) = 0, \quad \text{if} \quad i_l \neq i'_l \tag{2.4}$$

at least for single $1 \leq l \leq k$, where cl stands for closure.

Emphasize that in contrast to the definition of a self-similar set (see [13, 24]), subsets of different ranks $s_{i_1}, s_{i_1 i_2}, \cdots, s_{i_1\cdots i_k}, \cdots$ are in general not similar. In particular, they are not similar in general to the whole set S. Roughly speaking, each similar structure set on any ε-level ($\varepsilon > 0$) admits the decomposition into a finite amount of "cells" with diameters smaller or equal to ε, which are similar one to other but are not necessarily similar to whole set and to "cells" of another level of decomposition.

A Borel measure μ on Δ_0 is called the *similar structure* measure if its support $S = \mathrm{supp}\mu \equiv S_\mu$ has the structural similarity (see (2.1)–(2.4)) and besides,

$$\mu\big(s_{i_1\cdots i_k} \bigcap s_{i'_1\cdots i'_k}\big) = 0, \quad \text{if} \quad i_l \neq i'_l \tag{2.5}$$

at least for single $1 \leq l \leq k$, and the defined for non-empty sets $s_{i_1\cdots i_k}$ and $s_{i_1\cdots i_{k-1}}$ the ratios

$$\frac{\mu(s_{i_1\cdots i_{k-1} i_k})}{\mu(s_{i_1\cdots i_{k-1}})} =: p_{i_k k} > 0 \quad (s_{i_0} \equiv \Delta_0) \tag{2.6}$$

are independent on $i_1, \ldots, i_{k-1}$. We put $p_{i_k k} = 0$ if $s_{i_1\cdots i_k}$ is empty. The family of such measures will be denoted by $\mathcal{M}^{\mathrm{ss}}(\Delta_0) \equiv \mathcal{M}^{\mathrm{ss}}$ (ss stands for similar structure).

Every measure $\mu \in \mathcal{M}^{\mathrm{ss}}$ may be fixed in the following way.

Let us consider an infinite sequence of stochastic vectors $\mathbf{q}_k$, $k = 1, 2, \ldots$, from $\mathbb{R}^n$, $1 < n < \infty$ with strictly positive coordinates

$$\mathbf{q}_k = (q_{1k}, q_{2k}, \ldots, q_{nk}), \quad q_{1k}, \ldots, q_{nk} > 0, \quad q_{1k} + \cdots + q_{nk} = 1.$$

We assume that

$$q_{\mathrm{inf}} := \inf_{i,k} q_{ik} > 0. \tag{2.7}$$

Then

$$Q = \{\mathbf{q}_k\}_{k=1}^{\infty} = \{q_{ik}\}_{i=1,\ k=1}^{n,\ \infty} \tag{2.8}$$

denotes the stochastic matrix whose columns are formed with coordinates of vectors $\mathbf{q}_k$. Each matrix Q fixes on the segment Δ_0 the so-call (see [22, 2]) Q^*-representation, which we call here simply as the Q-representation on Δ_0. Let us shortly recall this construction.

For each $k = 1, 2, \ldots$ let us decompose the segment Δ_0 from the left to the right in the family of closed intervals of rank k

$$\Delta_0 = \bigcup_{i_1=1}^{n} \Delta_{i_1} = \bigcup_{i_1, i_2=1}^{n} \Delta_{i_1 i_2} = \cdots = \bigcup_{i_1,\ldots,i_k=1}^{n} \Delta_{i_1 i_2\cdots i_k} = \cdots$$

We suppose that different intervals of the same rank overlap at most only in the extreme points. That is, the lengths of these intervals are fixed by elements of the matrix Q

(2.9) $\lambda(\Delta_{i_1}) = q_{i_1 1}, \quad \lambda(\Delta_{i_1 i_2}) = q_{i_1 1} q_{i_2 2}, \quad \cdots, \quad \lambda(\Delta_{i_1 i_2 \cdots i_k}) = q_{i_1 1} q_{i_2 2} \cdots q_{i_k k}, \quad \cdots$

Of course, these decompositions are consistent

(2.10)

$$\Delta_0 = [0,1] = \bigcup_{i_1=1}^{n} \Delta_{i_1}, \quad \Delta_{i_1} = \bigcup_{i_2=1}^{n} \Delta_{i_1 i_2}, \quad \cdots, \quad \Delta_{i_1 i_2 \cdots i_k - 1} = \bigcup_{i_k=1}^{n} \Delta_{i_1 i_2 \cdots i_k}, \quad \cdots$$

It is not hard to see that relations

$$\Delta_{i_1 i_2 \cdots i_k} = T_{i_1 1} \cdots T_{i_k k} \Delta_0$$

define the one-to-one correspondence between the family of all Q-representations on Δ_0 and the family of above mentioned contractive similarities T.

Clearly that under a given Q-representation the σ-algebra generated by the family of subsets $\{\Delta_{i_1 \cdots i_k}\}_{k=0}^{\infty}$ coincides with the usual Borel σ-algebra on $[0,1]$. Moreover due to (2.7) for every point $x \in [0,1]$ there exists a sequence of embedded segments $\Delta_{i_1 i_2 \cdots i_k}$ containing this point and such that $x = \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 \cdots i_k}$. This fact can be written in the following form:

$$x = \Delta_{i_1 i_2 \cdots i_k \cdots}, \tag{2.11}$$

where the sequence of indexes $i_1, i_2, \ldots, i_k, \ldots$ defines the point x uniquely. Thus, $i_1, i_2, \ldots, i_k, \ldots$ are coordinates of x in the fixed Q-representation.

We remark that each point of a view (2.11) admits the representation in the terms of the corresponding family of similarities T:

$$x = \lim_{k \to \infty} y_k, \quad y_k = T_{i_1 1} \cdots T_{i_k k} y, \quad \forall y \in \mathbb{R}^1.$$

In what follows we will fix some Q-representation on Δ_0 or, that is the same, a family of contractive similarities T.

From (2.6) it follows that each measure $\mu \in \mathcal{M}^{\mathrm{ss}}$ is uniquely associated with the stochastic matrix

$$P \equiv \{\mathbf{p}_k\}_{k=1}^{\infty} = \{p_{ik}\}_{i=1,\ k=1}^{n,\ \infty}, \tag{2.12}$$

whose columns are formed by coordinates of stochastic vectors $\mathbf{p}_k \in \mathbb{R}^n,\ k = 1, 2, \ldots$

$$\mathbf{p}_k = (p_{1k}, \ldots, p_{nk}), \quad p_{1k}, \ldots, p_{nk} \geq 0, \quad p_{1k} + \cdots + p_{nk} = 1.$$

In order to point the dependence μ from P we write sometimes $\mu = \mu_P$.

The construction of the measure μ_P starting of P may be realized in the following way.

Using the first k columns of matrices Q and P we define the Borel measure μ_k on Δ_0 by the formula

$$\mu_k := \sum_{i_1, i_2, \ldots, i_k = 1}^{n} c_{i_1 i_2 \cdots i_k} \lambda_{i_1 i_2 \cdots i_k}, \quad c_{i_1 i_2 \cdots i_k} := \frac{p_{i_1 1} p_{i_2 2} \cdots p_{i_k k}}{q_{i_1 1} q_{i_2 2} \cdots q_{i_k k}}, \tag{2.13}$$

where $\lambda_{i_1 i_2 \cdots i_k} := \lambda | \Delta_{i_1 i_2 \cdots i_k}$ denotes the restriction of Lebesgue measure on segment $\Delta_{i_1 \cdots i_k}$. By (2.13) it follows that

$$\mu_1(\Delta_{i_1}) = p_{i_i 1}, \quad \ldots, \quad \mu_k(\Delta_{i_1 \ldots i_k}) = p_{i_1 1} p_{i_2 2} \cdots p_{i_k k}, \quad \cdots \tag{2.14}$$

Hence, $\mu_k, k = 1, 2, \ldots$ is a sequence of probability measures uniformly distributed on $\Delta_{i_1 \ldots i_k}$. From (2.13), (2.14) it also follows that the plot for the distribution function $f_k(x) = \mu_k\{(-\infty, x)\}$ for the measure μ_k is a piece-wise linear non-decreasing line. The values of the function $f_k(x), k > 1$ in end-points of each segment $\Delta_{i_1 \cdots i_{k-1}}$ of the rank $k-1$ are the same as for the function $f_{k-1}(x)$. Obviously for the sequence $\{f_k(x)\}_{k=1}^{\infty}$

we have $f_k(x) \to f(x), k \to \infty$ in the sense of the uniform convergence, where $f(x)$ is a left continuous non-decreasing function. Thus, $f(x)$ is the distribution function for a fixed probability measure μ on $[0, 1]$. So we can write

$$\mu = \lim_{k\to\infty} \mu_k.$$

The above constructed measure μ has the similar structure on all segments $\Delta_{i_1\cdots i_k}$ under each fixed $k \geq 1$. Besides, by condition (2.7) $\lambda(\Delta_{i_1\ldots i_k}) \to 0,\ k \to \infty$. Therefore, this measure belongs to the class $\mathcal{M}^{\rm ss}$.

Let us discuss in more details the structure of the support for μ, which will be often called the spectrum of μ. This set can be written as follows:

$$S_\mu \equiv {\rm supp}\mu = \bigcap_k S_{\mu_k}, \quad S_{\mu_k} = {\rm supp}\mu_k. \tag{2.15}$$

The set S_μ admits another representation

$$S_\mu \equiv {\rm supp}\mu = \Delta_0 \backslash \bar{S}_0(\mu), \quad S_0(\mu) = \bigcup_{\{i_k: p_{i_k k}=0\}} \Delta^{\rm int}_{i_1\cdots i_k}$$

where the union takes place along open intervals $\Delta^{\rm int}_{i_1\cdots i_k}$ (int means interior) for which $p_{i_k k} = 0$ and the bar stands for adding to $S_0(\mu)$ the isolated end-points if they appear. We note that all intervals $\Delta^{\rm int}_{i_1\cdots i_k}$ such that $p_{i_l l} = 0$ with some $l < k$ has also empty intersection with S_μ, because these intervals are included in $\Delta^{\rm int}_{i_1\cdots i_l}$.

We are able now to define

$$s_{i_1\cdots i_k} := S_\mu \bigcap \Delta^{\#}_{i_1\cdots i_k}, \tag{2.16}$$

where $\#$ denotes the possible removing from $\Delta_{i_1\cdots i_k}$ one of the end-points. Namely, we have to remove the right end-point, if $P^{\rm L}_k := \prod_{s=1}^\infty p_{1,k+s} > 0$, or the left one, if $P^{\rm R}_k := \prod_{s=1}^\infty p_{n,k+s} > 0$. The set $\Delta_{i_1\cdots i_k}$ in (2.16) remains without any changes, if $P^{\rm L}_k = P^{\rm R}_k = 0$. This procedure is caused by condition (2.5). Indeed, if one of the conditions $P^{\rm L}_k > 0$ or $P^{\rm R}_k > 0$ holds (they can not be fulfilled simultaneously!) the corresponding end-points of the segment $\Delta_{i_1\cdots i_k}$ become atoms of the point spectrum for the measure μ (see below Theorem 3.1). Thus the above describing construction of $s_{i_1\cdots i_k}$ ensures the validity of condition (2.5). Of course, it may occurs that $s_{i_1\cdots i_k}$ is empty.

It is evident now that the non-empty subsets $s_{i_1\cdots i_k}$, $s_{i'_1\cdots i'_k}$ defined by (2.16) are similar one to other for each fixed $k \geq 1$. It is true since all segments $\Delta_{i_1\cdots i_k}$, $\Delta_{i'_1\cdots i'_k}$ are similar, and on every step described above, only similar sets from these segments has been removed. In fact, by virtue of the previous observations subsets $s_{i_1\cdots i_k}$, $s_{i'_1\cdots i'_k}$ are connected by a certain kind of similarity of the form (2.2).

Further, due to (2.14) we obtain the important relations

$$\mu(s_{i_1\cdots i_k}) = \mu(\Delta^{\#}_{i_1\cdots i_k}) = \mu_k(\Delta_{i_1\cdots i_k}) = p_{i_1 1}\cdots p_{i_k k}. \tag{2.17}$$

We observe that $s_{i_1\cdots i_k}$ is non-empty iff

$$\mu(s_{i_1\cdots i_k}) = p_{i_1 1}\cdots p_{i_k k} \neq 0.$$

Thus (2.5) and (2.6) is fulfilled.

By the way, from (2.11) and (2.17) it follows that

$$\mu(x) = \prod_k p_{i_k k}, \tag{2.18}$$

where $i_k,\ k = 1, 2, \ldots$ denote coordinates of a point x in Q-representation (2.11).

Conversely, one can reconstruct a certain stochastic P-matrix by a given similar structure measure $\mu \in \mathcal{M}^{\rm ss}$ using meanings $\mu(s_{i_1\cdots i_k})$ (see (2.17)).

We shall use below the following well-known result [22].

Lemma 2.1. *The constructed above measure $\mu_P = \mu$ is singular with necessity if for infinite set of values k at least one coordinate of $\mathbf{p}_k$ equals zero.*

Proof. Without loss of generality we assume

$$0 = p_{i'_1 1} = \cdots = p_{i'_k k} = \cdots$$

By (2) we have

$$S_0(\mu) \supset \bigcup_{\{p_{i'_k k}=0\}} \Delta^{\text{int}}_{i_1 \cdots i_{k-1} i'_k}.$$

So we can calculate Lebesgue measure of the set $S_0(\mu)$

$$\begin{aligned}\lambda(S_0(\mu)) &\geq q_{i'_1 1} + \sum_{i_1 \neq i'_1} q_{i_1 1} q_{i'_2 2} + \sum_{i_1 \neq i'_1} q_{i_1 1} \sum_{i_2 \neq i'_2} q_{i_2 2} q_{i'_3 3} + \cdots \\ &+ \sum_{i_1 \neq i'_1} q_{i_1 1} \sum_{i_2 \neq i'_2} q_{i_2 2} \cdots \sum_{i_{k-1} \neq i'_{k-1}} q_{i_{k-1} k-1} q_{i'_k k} + \cdots \\ &= q_{i'_1 1} + q_{i'_2 2}(1 - q_{i'_1 1}) + q_{i'_3 3}(1 - q_{i'_1 1})(1 - q_{i'_2 2}) + \cdots \\ &+ q_{i'_k k}(1 - q_{i'_1 1}) \cdots (1 - q_{i'_{k-1} k-1}) + \cdots\end{aligned}$$

Obviously $\lambda(S_0(\mu)) = 0$ because

$$(1 - q_{i'_1 1})(1 - q_{i'_2 2}) \cdots (1 - q_{i'_k k}) \cdots = \prod_{k=1}^{\infty} (1 - q_{i'_k k}) = 0,$$

where we take into account (2.7). □

3. Spectral purity of the similar structure measures

One of the characteristic property of measures $\mu \in \mathcal{M}^{\text{ss}}(\Delta_0)$ is that each such measure has only a single component in the Lebesgue decomposition: either purely point, or purely absolutely continuous, or purely singular continuous (see [2]). Here we produce this fact about purity of a spectrum (=support) of the similar structure measure as a relevant version of the famous Jessen-Wintner theorem for probability distributions (see [10, 14, 22]). The proof in essence belongs to G. Torbin and actually is the same as in [2].

Let us introduce some notations. For a measure $\mu = \mu_P$ of class $\mathcal{M}^{\text{ss}}(\Delta_0)$ we write $\mu \in \mathcal{M}^{\text{ss}}_{\text{pp}}, \mathcal{M}^{\text{ss}}_{\text{ac}}, \mathcal{M}^{\text{ss}}_{\text{sc}}$, if this measure is purely point ($\mu = \mu_{\text{pp}}$), purely absolutely continuous ($\mu = \mu_{\text{ac}}$), or purely singular continuous ($\mu = \mu_{\text{sc}}$), respectively. Given stochastic matrices P and Q, we define two values

$$P_{\max}(\mu) := \prod_{k=1}^{\infty} p_{\max,k} \quad \text{and} \quad \rho(\mu, \lambda) := \prod_{k=1}^{\infty} \rho_k,$$

where $p_{\max,k} := \max_{i=1}^{n} \{p_{ik}\}$, $\rho_k := \Sigma_{i=1}^{n} \sqrt{p_{ik} q_{ik}}$.

Theorem 3.1. *Each similar structure measure $\mu = \mu_P \in \mathcal{M}^{\text{ss}}(\Delta_0)$ associated with the stochastic matrix P under the fixed Q-representation on $\Delta_0 = [0, 1]$ has a pure spectral type. Namely,*

(a) $\mu \in \mathcal{M}^{\text{ss}}_{pp}$, *if and only if* $P_{\max}(\mu) > 0$,
(b) $\mu \in \mathcal{M}^{\text{ss}}_{ac}$, *if and only if* $\rho(\mu, \lambda) > 0$,
(c) $\mu \in \mathcal{M}^{\text{ss}}_{sc}$, *if and only if* $P_{\max}(\mu) = 0$ *and* $\rho(\mu, \lambda) = 0$.

Proof. Let us denote by

$$(\Omega, \mathcal{A}, \mu^*) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, m_k)$$

the infinite direct product of discrete probability spaces

$$(\Omega_k, \mathcal{A}_k, m_k), \quad \Omega_k = \{\omega_{i_k}\}_{i_k=1}^{n}, \quad m_k(\omega_{i_k}) = p_{i_k k},$$

where space Ω_k and σ-algebra $\mathcal{A}_k$ depend on k only formally (in fact they are the same for all k). But $p_{i_k k}$ is changed with i_k and k in a correspondence with the matrix P. Thus the measure μ^* is uniquely connected with the matrix P. In particular, for the cylindrical sets $\Omega_{i_1 \cdots i_k} := \omega_{i_1} \times \cdots \times \omega_{i_k} \times \prod_{l=1}^{\infty} \Omega_{k+l} \in \mathcal{A}$,

$$\mu^*(\Omega_{i_1 \cdots i_k}) = \prod_{s=1}^{k} p_{i_s s}. \tag{3.1}$$

In what follows we use the measurable mapping from $\Omega = \Omega_1 \times \Omega_2 \cdots \times \Omega_k \cdots$ to $[0,1]$,

$$\pi : \Omega \ni \omega^* = \{\omega_{i_1} \times \omega_{i_2} \times \cdots \times \omega_{i_k} \times \cdots\} \to x = \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 \cdots i_k \cdots} \in \Delta_0.$$

We note that Ω possibly is replaced by $\Omega \diagdown \Omega_0$, where the set Ω_0 is not empty only if there exists k_0 such that one of the following inequalities occurs:

$$P_{k_0}^{L} = \prod_{s \geq k_0} p_{1s} > 0, \quad P_{k_0}^{R} = \prod_{s \geq k_0} p_{ns} > 0.$$

Namely,

$$\Omega_0 = \{\omega^* \in \Omega \mid \omega_{i_k} = \omega_{1_k},\ \forall k \geq k_0\},$$

or

$$\Omega_0 = \{\omega^* \in \Omega \mid \omega_{i_k} = \omega_{n_k},\ \ \forall k \geq k_0\},$$

respectively to the first or to the second case. In any case the mapping π preserves the meanings of the measure:

$$\mu^*(\Omega_{i_1 \cdots i_k}) = \mu(\Delta^{\#}_{i_1 \cdots i_k}) = p_{i_1 1} \cdots p_{i_k k},$$

where $\Delta_{i_1 \cdots i_k} = \pi(\Omega_{i_1 \cdots i_k}))$ (see (2.16), (2.17) (3.1)). By this reason μ is often called the image-measure for μ^* with respect to the mapping π. We will use this property of π to preserve the meanings of measure without reminding.

Let us prove (**a**). If $P_{\max}(\mu) = 0$, then $\mu^*(\omega^*) = 0$ for each point $\omega^* \in \Omega$. Indeed,

$$\mu^*(\omega^*) = \prod_{k=1}^{\infty} m_k(\omega_{i_k}) = \prod_{k=1}^{\infty} p_{i_k k} \leq \prod_{k=1}^{\infty} p_{\max,k} = 0.$$

In this case μ^* is continuous and so is μ. Thus, we proved the necessity of the condition $P_{\max}(\mu) > 0$ in the statement (**a**).

To prove the sufficiency let us introduce the set

$$A_+ := \{\omega^* \in \Omega \mid \mu^*(\omega^*) > 0\}.$$

A_+ is not empty, if $P_{\max}(\mu) > 0$, because this set contains at least the point $\omega^*_{\max} = \omega_{i'_1} \times \cdots \times \omega_{i'_k} \times \cdots$ with coordinates for which $p_{i'_k k} = p_{\max,k}$. Besides, A_+ contains also all points $\omega^* \in \Omega$ such that $\omega_{i_k} \neq \omega_{i'_k}$ for a finite amount of coordinates under condition $p_{i_k k} > 0$. The set A_+ does not contain another points. It follows from the condition $P_{\max}(\mu) > 0$. Indeed, since vectors $\mathbf{p}_k$ are stochastic the sequence $p_{\max,k}$ is an unique one (up to changing of a finite amount of coordinates) which converges to 1. In other words, $\omega^* \in A_+$ means that coordinates ω^* differ from coordinates of point $\omega^*_{\max}$ only in finite number of places. Thus, the set A_+ is countable. Applying now Kolmogorov's principle of zero and unit we conclude: either $\mu^*(A_+) = 0$ or $\mu^*(A_+) = 1$. However

$\mu^*(A_+) \geq \mu^*(\omega^*_{\max}) > 0$. Therefore the statement $\mu^*(A_+) = 1$ is true. This means that the measure μ^* is concentrated on a not more than countable number of atoms. Thus $\mu^* = \mu^*_{\mathrm{pp}}$ and hence for its image-measure under the mapping π, the equality $\mu = \mu_{\mathrm{pp}}$ is established too.

The statements (**b**), (**c**) are direct consequences of the below presented Kakutani's theorem (see [14, 7] and also [1, 2]). □

Theorem. ([14]). *Let*

$$(\Omega, \mathcal{A}, \mu^*) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, m_k) \quad \text{and} \quad (\Omega, \mathcal{A}, \lambda^*) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, \lambda_k)$$

be a pair of the infinite direct products of abstract probability spaces. Suppose that each measure m_k is equivalent to λ_k (notation, $m_k \sim \lambda_k$). Define

$$\rho(\mu^*, \lambda^*) := \prod_{k=1}^{\infty} \rho_k, \quad \rho_k = \int_{\Omega_k} \sqrt{\varphi_k(\omega)} d\lambda_k(\omega),$$

where $\varphi_k(\omega) = \frac{dm_k(\omega)}{d\lambda_k(\omega)}$ is the Radon-Nikodim derivative. Then

(**a**) *$\mu^* \sim \lambda^*$, only if $\rho(\mu^*, \lambda^*) > 0$;*

(**b**) *$\mu^* \perp \lambda^*$, only if $\rho(\mu^*, \lambda^*) = 0$, where $\perp$ denotes the mutual singularity.*

For our consideration we may put $\Omega_k = \{\omega_1, \dots, \omega_n\}$, $n \geq 2$ for all $k \geq 1$, assign $\lambda_k(\omega_{i_k}) = q_{i_k k}$, $m_k(\omega_{i_k}) = p_{i_k k}$, and take into account that μ_P coincides with Lebesgue measure on Δ_0, if $P = Q$.

Finally we remark that $\mu = \mu_{\mathrm{pp}}$ has the discrete spectrum which is concentrated in a finite set of atoms only if there exists some $k_0 \geq 1$ such that $p_{\max,k} = 1$ for all $k \geq k_0$. In this case the number of points in A_+ is the same as the amount of atoms in the spectrum of μ_{pp}. Otherwise, each point of the spectrum is accumulating.

4. The conflict dynamical system

Let us consider for a fixed $k \geq 1$ a pair of discrete probability measures m_k, v_k defined on a common space $\Omega_k = \{\omega_{1_k}, \dots, \omega_{n_k}\}$, $n_k > 1$. These measures are naturally associated with the stochastic vectors $\mathbf{p}_k = (p_{1k}, \dots, p_{nk})$, $\mathbf{r}_k = (r_{1k}, \dots, r_{nk})$

$$m_k(\omega_{i_k}) := p_{ik}, \quad v_k(\omega_{i_k}) := r_{ik}, \quad i_k = 1, \dots, n.$$

At first we define the conflict dynamical system for a couple of measures m_k, v_k

$$\{m_k^{N-1}, v_k^{N-1}\} \xrightarrow{*} \{m_k^N, v_k^N\}, \quad N = 1, 2, \dots, \quad m_k^0 = m_k, \quad v_k^0 = v_k \tag{4.1}$$

using the non-commutative and non-linear transformation (the conflict composition $*$) in terms of the stochastic vectors $\mathbf{p}_k$, $\mathbf{r}_k$ (for more details see [15, 16])

$$\mathbf{p}_k^N = \mathbf{p}_k^{n-1} * \mathbf{r}_k^{N-1}, \quad \mathbf{r}_k^N = \mathbf{r}_k^{N-1} * \mathbf{p}_k^{N-1}, \quad \mathbf{p}_k^0 = \mathbf{p}_k, \quad \mathbf{r}_k^0 = \mathbf{r}_k.$$

Here the coordinates of vectors $\mathbf{p}_k^N$, $\mathbf{r}_k^N$ are calculated by the formulae

$$p_{ik}^N := \frac{p_{ik}^{N-1}(1 - r_{ik}^{N-1})}{z_k^{N-1}}, \quad r_{ik}^N := \frac{r_{ik}^{N-1}(1 - p_{ik}^{N-1})}{z_k^{N-1}}, \tag{4.2}$$

with $z_k^{N-1} = 1 - (\mathbf{p}_k^{N-1}, \mathbf{r}_k^{N-1})$, where $(\cdot, \cdot)$ stands for the inner product in $\mathbb{R}^n$. The formulae (4.2) are well-defined if we suppose the requirement that $(\mathbf{p}_k, \mathbf{r}_k) \neq 1$. Then the condition $(\mathbf{p}_k^{N-1}, \mathbf{r}_k^{N-1}) \neq 1$ automatically occurs for all N. For coming to the measures in (4.1) we put $m_k^N(\omega_{i_k}) =: p_{ik}^N$, $v_k^N(\omega_{i_k}) =: r_{ik}^N$, $i = 1, \dots, n$.

One can interpret the formulas (4.2) as follows (see [16] and cf. with [11, 20]). Let the measures m_k,v_k correspond to some opponents A, B. Then the value p_{ik}^N (r_{ik}^N) means the probability for A (B) to occupy the position ω_{i_k} after N steps of the conflict actions. This value equals to the conditional probability to occupy the position ω_{i_k} by the opponent A (B) under the assumption that A could not meet B at any of n positions.

Let us consider now a pair of similar structure measures $\mu = \mu_P, \nu = \nu_R \in \mathcal{M}^{\rm ss}(\Delta_0)$ associated with stochastic matrices P and R respectively. The matrix R has a form similar to (2.12) and it is composed with a sequence of the stochastic vectors $\mathbf{r}_k \in \mathbb{R}^n_+,\ k = 1, 2, \ldots$ We suppose that the Q-representation of the interval $\Delta_0 = [0, 1]$ (or that is the same, the family of contractive similarities T on $\mathbb{R}^1$) is fixed.

By a pair of similar structure measures $\mu_P,\ \nu_R$ the conflict dynamical system is defined as follows:

$$\{\mu^{N-1}, \nu^{N-1}\} \xrightarrow{\ast} \{\mu^N, \nu^N\}, \quad N = 1, 2, \ldots, \quad \mu^0 = \mu_P, \nu^0 = \nu_R, \tag{4.3}$$

where the measures $\mu^N, \nu^N,\ N = 1, 2, \ldots$ are associated with the stochastic matrices $P^N,\ R^N$ respectively. In turn, the matrices $P^N,\ R^N$ is constructed using the stochastic vectors $\mathbf{p}^N_k$ and $\mathbf{r}^N_k$ defined by (4.2).

The next theorem follows from the so-called conflict Theorem (see [15, 16]) and Theorem 2 and Theorem 3 from [4].

Theorem 4.1. *Let $\mu = \mu_P, \nu = \nu_R \in \mathcal{M}^{\rm ss}(\Delta_0)$ be a pair of similar structure measures. Assume that stochastic matrices P, R satisfy the following condition,*

$$(\mathbf{p}_k, \mathbf{r}_k) \neq 1, \quad \forall k. \tag{4.4}$$

Then there exist two limiting measures

$$\mu^\infty = \lim_{N\to\infty} \mu^N, \quad \nu^\infty = \lim_{N\to\infty} \nu^N,$$

in the sense of the uniform convergence, which are invariant with respect to the composition $\ast$ defined in (4.3).

Further, the elements of the stochastic matrices $P^\infty,\ R^\infty$ associated with the limiting measures $\mu^\infty,\ \nu^\infty$ have the following explicit form:

if $\mathbf{p}_k \neq \mathbf{r}_k$*, then*

$$p^\infty_{ik} = \begin{cases} d_{ik}/D_k, & i \in \mathbb{N}_{+,k} \\ 0, & i \notin \mathbb{N}_{+,k} \end{cases}, \quad r^\infty_{ik} = \begin{cases} -d_{ik}/D_k, & i \in \mathbb{N}_{-,k} \\ 0, & i \notin \mathbb{N}_{-,k} \end{cases} \tag{4.5}$$

where

$$d_{ik} = p_{ik} - r_{ik}, \quad \mathbb{N}_{+,k} := \{i : d_{ik} > 0\}, \quad \mathbb{N}_{-,k} := \{i : d_{ik} < 0\},$$
$$D_k = \sum_{i\in\mathbb{N}_{+,k}} d_{ik} = \sum_{i\in\mathbb{N}_{-,k}} -d_{ik};$$

if $\mathbf{p}_k = \mathbf{r}_k$ *, then the non-zero coordinates of vectors $\mathbf{p}^\infty_k = \mathbf{r}^\infty_k$ have a form*

$$p^\infty_{ik} = r^\infty_{ik} = 1/c_k, \tag{4.6}$$

where c_k denotes an amount of non-zero initial coordinates $p_{ik} = r_{ik} \neq 0$.

Thus, one can easy calculate the values of measures μ^∞, μ^∞ on subsets $\Delta^{\#}_{i_1\cdots i_k}$:

$$\mu^\infty(\Delta^{\#}_{i_1\cdots i_k}) = \prod_{s\le k} d_{i_s s}/D_s,$$

under assumption that $i_s \in \mathbb{N}_{+,s}$ for all $s \le k$. In particular, if a coordinate i_s of the vector $\mathbf{p}_s$ has some priority with respect to the corresponding coordinate of the vector $\mathbf{r}_s$, $s \le k$: $p_{i_s s} > r_{i_s s}$, then due to (4.5) we obtain that $p^\infty_{i_s s} > 0$, but $r^\infty_{i_s s} = 0$, and therefore $\mu^\infty(\Delta^{\#}_{i_1\cdots i_k}) > 0$, but $\nu^\infty(\Delta^{\#}_{i_1\cdots i_k}) = 0$. Nevertheless, it is possible that $\mu^\infty(\Delta^{\#}_{i_1\cdots i_k}) = \nu^\infty(\Delta^{\#}_{i_1\cdots i_k}) = 0$ even in the case $\mathbf{p}_s \neq \mathbf{r}_s,\ s \le k$. It happens due to (4.5) for all coordinates such that $p_{i_s s} = r_{i_s s} > 0$ since then $d_{i_s s} = p^\infty_{i_s s} = r^\infty_{i_s s} = 0$.

5. Origination of the singular continuous spectrum

In this main section of the paper we show that origination of the singular continuous spectrum is generic for the limiting measures μ^∞ and ν^∞.

We say that similar structure measures $\mu = \mu_P, \nu = \nu_R$ are essentially different, write simply $\mu \neq \nu$, if $\mathbf{p}_k \neq \mathbf{r}_k$ almost for all k (this means that $\mathbf{p}_k = \mathbf{r}_k$ at most for a finite amount of indices k). We say that the measure μ has a *local priority* with respect to the measure ν at a position i_k, if $p_{i_k k} > r_{i_k k}$. If the measure μ has a local priority with respect to the measure ν almost for all k from some sequence (= a direction) $i_1, i_2, \dots, i_k, \dots$, then we say that μ has the directed priority with respect to ν, write $(\mu > \nu)_{i_1 i_2 \cdots i_k \cdots}$ Finally, we say that μ has the *shock directed priority* with respect to the essentially different measure ν, if the the total value of local priorities equals to infinity

$$\sum_k d_{i_k k} = \infty, \quad d_{i_k k} = p_{i_k k} - r_{i_k k} \tag{5.1}$$

and moreover, the normalized meanings of the local priorities

$$\mathbf{d}_{i_k} := \frac{d_{i_k k}}{D_k}, \quad D_k = 1/2 \sum_{i_k=1}^{\infty} |d_{i_k k}|$$

converge to 1 so quickly that

$$\mathbf{d}(\mu, \nu) := \prod_k \mathbf{d}_{i_k} > 0. \tag{5.2}$$

Clearly that condition (5.2) is highly specific and in general does not fulfilled.

The main result of the paper is formulated in the next theorem. It assert that for any couple of starting essentially different measures μ, ν, the generic type of the limiting measures μ^∞ and ν^∞ in the conflict dynamic system (4.3) is singular continuous.

Theorem 5.1. *Let $\mu = \mu_P, \nu = \nu_R \in \mathcal{M}^{\rm ss}$ be a couple of similar structure measures on $[0,1]$ associated with stochastic matrices $P = \{\mathbf{p}_k\}_{k=1}^\infty$, $R = \{\mathbf{r}_k\}_{k=1}^\infty$ respectively. Assume μ, ν are essentially different and neither μ nor ν has a shock directed priority one respect to other, in particular, $\mathbf{d}(\mu, \nu) = \mathbf{d}(\nu, \mu) = 0$ for any sequence $i_1, i_2, \dots, i_k, \dots$ Then both $\mu^\infty, \nu^\infty \in \mathcal{M}_{\rm sc}$.*

Proof. If $\mu \neq \nu$ then due to (4.5) all limiting vectors $\mathbf{p}_k^\infty, \mathbf{r}_k^\infty$ for almost all k contain at least one non-zero coordinate. Therefore, the measures μ^∞, ν^∞ are singular due to Lemma 2.1. We have only to check that these measures are continuous, i.e., that $\mu^\infty(x) = \nu^\infty(x) = 0$, $\forall x \in \Delta_0$. Indeed, by (2.11) and (2.18) a value of the measure μ^∞ at a point x with coordinates i_k is defined by the formula $\mu^\infty(x) = \prod_k p_{i_k k}^\infty$. The latter product is possible strictly positive, only if x belongs to the point spectrum of μ^∞ (see Theorem 3.1), i.e., if the following condition

$$P_{\max}(\mu^\infty) = \prod_k p_{\max,k}^\infty > 0, \quad p_{\max,k}^\infty := \max_i \{p_{ik}^\infty\} = \max_{i_k} \frac{d_{i_k k}}{D_k} \tag{5.3}$$

is carried (see Theorem 4.1). Condition (5.3) means that the convergence $\mathbf{d}_{i'_k} \to 1$, $k \to \infty$ is so quick that $\prod_k \mathbf{d}_{i'_k} > 0$, where $\mathbf{d}_{i'_k} := \max_i d_{ik}/D_k$ (see (5.2)). But then the sequence i'_k (on this sequence the maximal values $p_{\max,k}^\infty$ are realized) constitutes the shock directed priority of the measure μ with respect to ν (see (5.2)). Thus we get the contradiction with assumptions of the theorem. Thus $\mu^\infty(x) = 0$ for each point $x \in [0,1]$ and therefore $\mu^\infty \in \mathcal{M}_{\rm sc}$. The similar arguments is true also for ν. □

In [21] (see also [9, 18, 19]) it has been shown that operators with singular continuous spectrum are generic. The similar fact is valid for the class of singular continuous measures in the space $\mathcal{M}^{\rm ss}$. Here we'll construct on the space of similar structure measures

$\mathcal{M}^{\rm ss}$ the non-trivial "global" probability measure $\mathbf{m}$ and prove that the sub-class $\mathcal{M}^{\rm ss}_{\rm sc}$ is a set of full $\mathbf{m}$-measure (see Theorem 5.2 below). Thus, the statement that in the dynamical conflict system limiting measures μ^∞, ν^∞ are almost always singular continuous, has a more precise sense.

To this end we firstly construct in $\mathcal{M}^{\rm ss}$ a family of cylinder subsets defined by induction as follows.

On the first step we put

$$I_{i_1}^{\beta_{i_1}} := \{\mu \in \mathcal{M}^{\rm ss} |\ \mu(\Delta_{i_1}) \in \beta_{i_1},\ \mu(\Delta_{i_1}) = \max_{j_1=1,\dots,n} \{\mu(\Delta_{j_1})\}\},$$

where β_{i_1} stands for a usual Borel set from the σ-algebra $\mathcal{B}$ of subsets from $[0,1]$.

Further, for every $k \geq 1$ we define

$$I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}} := \{\mu \in I_{i_1\cdots i_{k-1}}^{\beta_{i_1}\cdots\beta_{i_{k-1}}} | \frac{\mu(\Delta_{i_1\cdots i_k})}{\mu(\Delta_{i_1\cdots i_{k-1}})} \in \beta_{i_k},\ \mu(\Delta_{i_1\cdots i_k}) = \max_{j_k=1,\dots,n} \{\mu(\Delta_{i_1\cdots i_{k-1}j_k})\}\},$$

where $\mu(\Delta_{i_0}) \equiv \mu(\Delta_0) = 1$.

It is clear that

$$\mathcal{M}^{\rm ss} = \bigcup_{i_1,\dots,i_k=1}^{n} I_{i_1\cdots i_k},\quad I_{i_1\cdots i_k} := \bigcup_{\beta_{i_1},\dots,\beta_{i_k}\in\mathcal{B}} I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}},\quad k = 1, 2, \dots$$

Starting of the family

$$\{I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}},\ \beta_{i_k} \in \mathcal{B},\ i_k \in \{1,\dots,n\},\ k = 1, 2, \dots\}$$

and using the standard procedure (see, for example, [5]) we generate the rink of subsets which is denoted as $\mathcal{R}$.

Now we introduce on $\mathcal{R}$ the probability measure $\mathbf{m}$. At first we define it on cylinder sets:

$$\mathbf{m}(I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}}) := q_{i_1 1}\cdots q_{i_k k}\cdot \prod_{l=1}^{k} \lambda'(\beta_{i_l}),$$

where $\lambda'(\beta_{i_1}) := c_n\lambda(\beta_{i_1}),\ c_n = \frac{n}{n-1}$ taking into account that $\lambda([1/n, 1]) = \frac{n-1}{n}$. It is easy to check that

$$\sum_{i_1,\dots,i_k=1}^{n} \mathbf{m}(I_{i_1\cdots i_k}) = \mathbf{m}(\bigcup_{i_1,\dots,i_k=1}^{n} (\bigcup_{\beta_{i_1},\cdots,\beta_{i_k}\in\mathcal{B}} I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}})) = 1,$$

since $\sum_{i_1,\dots,i_k=1}^{n} q_{i_1}\cdots q_{i_k} = 1$ and $\lambda'([1/n, 1]) = 1$.

Let $\mathbf{m}^*$ be the outer measure for $\mathbf{m}$. Denote by $\mathcal{J}^{\rm ss}$ the minimal σ-algebra generated by the family of the cylinder subsets $\{I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}}\}$ in $\mathcal{M}^{\rm ss}$. The standard procedure of restriction of $\mathbf{m}^*$ on $\sigma-$algebra $\mathcal{J}^{\rm ss}$ concludes the construction of the σ-additive probability measure, which we denote by $\mathbf{m}$.

Let us recall that $\mathcal{M}^{\rm ss}_{\rm pp}, \mathcal{M}^{\rm ss}_{\rm ac}, \mathcal{M}^{\rm ss}_{\rm sc}$ denote the classes of purely point, purely absolutely continuous and purely singularly continuous measures respectively.

Theorem 5.2. *Under a fixed Q-representation of the segment* $[0,1]$ *the class of purely singular continuous similar structure measures* $\mathcal{M}^{\rm ss}_{\rm sc}([0,1])$ *is a set of full measure for* $\mathbf{m}$*:*

$$\mathbf{m}(\mathcal{M}^{\rm ss}_{\rm sc}) = 1.$$

Proof. At first we show that

$$\mathbf{m}^*(\mathcal{M}^{\rm ss}_{\rm pp}) = \mathbf{m}^*(\mathcal{M}^{\rm ss}_{\rm ac}) = 0.$$

To this end we fix a sequence $\varepsilon_k \to 0,\ k \to \infty$ such that $\sum_k \varepsilon_k = \infty$. For example one can put $\varepsilon_k = 1/k$. By ε_k we introduce the sequence of subsets

$$\mathcal{I}_k := \bigcup_{i_1,\dots,i_k=1}^{n} I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}}, \quad \beta_{i_1} = [a_1, 1], \quad \dots, \quad \beta_{i_k} = [a_k, 1],$$

where a_k are chosen in such a case that $\lambda'(\beta_{i_k}) = \varepsilon_k$. Let us denote

$$\mathcal{I}_{\mathrm{pp},k} := \{\mu \in \mathcal{M}_{\mathrm{pp}}^{\mathrm{ss}} | \mu \in \mathcal{I}_k\} = \mathcal{I}_k \bigcap \mathcal{M}_{\mathrm{pp}}^{\mathrm{ss}}.$$

It is clear that

$$\mathbf{m}^*(\mathcal{I}_{\mathrm{pp},k}) \le \mathbf{m}(\mathcal{I}_k) \le \sum_{i_1,\dots,i_k=1}^{n} \mathbf{m}(I_{i_1\cdots i_k}^{\beta_{i_1}\cdots\beta_{i_k}})$$
$$= \sum_{i_1,\dots,i_k=1}^{n} q_{i_1 1}\cdots q_{i_k k} \prod_{l=1}^{k} \lambda'(\beta_{i_l}) \le \sum_{i_1,\dots,i_k=1}^{n} q_{i_1 1}\cdots q_{i_k k}\lambda'(\beta_{i_k}) = \varepsilon_k,$$

since $\sum_{i_1,\dots,i_k=1}^{n} q_{i_1 1}\cdots q_{i_k k} = 1$.

It is easy to see that for each measure $\mu \in \mathcal{M}_{\mathrm{pp}}^{\mathrm{ss}}$ there exists $k_0 = k_0(\mu)$ such that $\mu \in \mathcal{I}_k$ for all $k \ge k_0$ (see condition (**a**) in Theorem 3.1, and also (5.3)). Therefore $\mu \in \mathcal{I}_{\mathrm{pp},k}$ beginning with some k which is defined by μ. Hence

$$\delta_k := \mathbf{m}\big(\mathcal{I}_{\mathrm{pp},k} \backslash \bigcup_{l=1}^{k-1} \mathcal{I}_{\mathrm{pp},l}\big) \to 0, \quad k \to \infty \tag{5.4}$$

Let us denote $\mathcal{I}'_{\mathrm{pp},k} := \bigcup_{l=1}^{k} \mathcal{I}_{\mathrm{pp},l}$. Clearly $\mathcal{I}'_{\mathrm{pp},k} \subset \mathcal{I}'_{\mathrm{pp},k+1}$ and also $\mathcal{M}_{\mathrm{pp}}^{\mathrm{ss}} = \bigcup_{k=1}^{\infty} \mathcal{I}'_{\mathrm{pp},k}$. Thus, by (5.4) and due to the σ-additive property of the outer measure, $\mathbf{m}^*(\mathcal{M}_{\mathrm{pp}}^{\mathrm{ss}}) = 0$. Indeed by virtue of the theorem on continuity for a union of subsets (see. [5], Theorem 6.2) we have

$$\mathbf{m}^*(\mathcal{M}_{\mathrm{pp}}^{\mathrm{ss}}) = \lim_{k\to\infty} \mathbf{m}^*\big(\bigcup_{l=1}^{k} \mathcal{I}'_{\mathrm{pp},l}\big) = \lim_{k\to\infty} \mathbf{m}^*(\mathcal{I}_{\mathrm{pp},k}) \le \lim_{k\to\infty} \mathbf{m}(\mathcal{I}_k) = \lim_{k\to\infty} \varepsilon_k = 0.$$

The equality $\mathbf{m}(\mathcal{M}_{\mathrm{ac}}^{\mathrm{ss}}) = 0$ we prove on a similar way. With this aim we introduce another sequence of subsets with the same notations

$$\mathcal{I}_k := \{\mu \in \mathcal{M}^{\mathrm{ss}} | \mu \in I_{i_1\dots i_k}^{\beta_{i_1}\dots\beta_{i_k}},\ \beta_{i_l} = [q_{i_l l} - \varepsilon_l/2, q_{i_l l} + \varepsilon_l/2],\ l = 1, ..., k\}$$

and

$$\mathcal{I}_{\mathrm{ac},k} := \{\mu \in \mathcal{M}_{\mathrm{ac}}^{\mathrm{ss}} | \mu \in \mathcal{I}_k\} = \mathcal{I}_k \bigcap \mathcal{M}_{\mathrm{ac}}^{\mathrm{ss}}.$$

We note that in the case where $q_{i_k k} = 1/n$ one can put $\beta_{i_k} = [1/n, 1/n + \varepsilon_k]$. It is not hard to understand that for every measure $\mu \in \mathcal{M}_{\mathrm{ac}}^{\mathrm{ss}}$ there exists $k_0 = k_0(\mu)$ such that $\mu \in \mathcal{I}_{\mathrm{ac},k}$ for all $k \ge k_0$. Therefore

$$\delta_k := \mathbf{m}\big(\mathcal{I}_{\mathrm{ac},k} \backslash \bigcup_{l=1}^{k-1} \mathcal{I}_{\mathrm{ac},l}\big) \to 0, \quad k \to \infty. \tag{5.5}$$

Let us denote $\mathcal{I}'_{\mathrm{ac},k} := \bigcup_{l=1}^{k} \mathcal{I}_{\mathrm{ac},l}$. Clearly that $\mathcal{I}'_{\mathrm{ac},k} \subset \mathcal{I}'_{\mathrm{ac},k+1}$ and also $\mathcal{M}_{\mathrm{ac}}^{\mathrm{ss}} = \bigcup_{k=1}^{\infty} \mathcal{I}'_{\mathrm{ac},k}$. Hence $\mathbf{m}^*(\mathcal{M}_{\mathrm{ac}}^{\mathrm{ss}}) = 0$ since

$$\mathbf{m}^*(\mathcal{M}_{\mathrm{ac}}^{\mathrm{ss}}) = \lim_{k\to\infty} \mathbf{m}^*\big(\bigcup_{l=1}^{k} \mathcal{I}'_{\mathrm{ac},l}\big) = \lim_{k\to\infty} \mathbf{m}^*(\mathcal{I}_{\mathrm{ac},k}) \le \lim_{k\to\infty} \mathbf{m}(\mathcal{I}_k) = \lim_{k\to\infty} \varepsilon_k = 0,$$

where we used

$$\mathbf{m}(\mathcal{I}_k) = \sum_{i_1,\ldots,i_k=1}^{n} q_{i_1 1}\cdots q_{i_k k}\lambda'(\beta_{i_k}) = \varepsilon_k.$$

By $\mathbf{m}^*(\mathcal{M}^{\rm ss}_{\rm pp}) = \mathbf{m}^*(\mathcal{M}^{\rm ss}_{\rm ac}) = 0$ we conclude that $\mathcal{M}^{\rm ss}_{\rm pp}$, $\mathcal{M}^{\rm ss}_{\rm ac} \in \mathcal{J}^{\rm ss}$ (see, for instance, Theorem 6.1 from [5]). Thus $\mathbf{m}(\mathcal{M}^{\rm ss}_{\rm pp}) = \mathbf{m}(\mathcal{M}^{\rm ss}_{\rm ac}) = 0$. It means by Theorem 3.1 that $\mathbf{m}(\mathcal{M}^{\rm ss}_{\rm sc}) = 1$. □

6. When $\mu^\infty \in \mathcal{M}_{\rm pp}$?

We want to find the conditions ensuring the pure point type at least for one of the limiting measures.

The next theorem follows from assertion (a) of Theorem 3.1, formulae (4.5), and proof of Theorem 5.1 (especially the formula (5.3)).

Theorem 6.1. *The limiting measure μ^∞ is pure point, $\mu^\infty \in \mathcal{M}_{\rm pp}$, if and only if the initial measure μ possesses a directed shock priority with respect to ν, i.e., if (5.2) is valid.*

Proof. By Theorem 3.1 condition (5.3) is necessary and sufficient for $\mu^\infty \in \mathcal{M}_{\rm pp}$. This condition is obviously equivalent to (5.2). □

We note, it might happen that each of measures $\mu \neq \nu$ could have their own directed priorities one with respect to other (even not one), i.e., the relations $(\mu > \nu)_{i_1 i_2 \cdots i_k \cdots}$, $(\nu > \mu)_{i'_1 i'_2 \cdots i'_k \cdots}$ may fulfilled simultaneously. Moreover, measures μ, ν may have simultaneously shock directed priorities one with respect to other. So, this case is not excluded that both limiting measures are pure point. The next result is rather unexpected.

Theorem 6.2. *Let μ, ν be a couple of different similar structure measures. The limiting measure μ^∞ has a pure point type, $\mu^\infty \in \mathcal{M}_{\rm pp}$, if and only if in the set of all directed priorities there is unique one $(\mu > \nu)_{i'_1 i'_2 \cdots i'_k \cdots}$, up to replacing of a finite amount of indices, which satisfies the condition*

$$\sum_{k=1}^{\infty} \mathbf{d}_{i'_k} = \infty, \quad \mathbf{d}_{i'_k} = \frac{d_{i'_k k}}{D_k}. \tag{6.1}$$

Proof. Let $\mu^\infty \in \mathcal{M}_{\rm pp}$. Then due to Theorem 6.1 conditions (5.2) and (5.3) are held. This means that there exists the directed priority $(\mu > \nu)_{i'_1, i'_2, \ldots, i'_k, \ldots}$ such that for any other direction the differences $d_{i_k k} = p_{i_k k} - r_{i_k k},\ i_k \neq i'_k$ converge to zero when $k \to \infty$. Moreover, this convergence is so quick that the total sum of the above differences is finite, i.e., the following series is convergent,

$$\sum_{k=1}^{\infty} \sum_{i_k \neq i'_k} \mathbf{d}_{i_k} < \infty. \tag{6.2}$$

Now we observe that conditions (5.3) and (6.2) are equivalent since $p^\infty_{\max,k} = \mathbf{d}_{i'_k}$. Therefore the unique non-convergent series consist of $\mathbf{d}_{i'_k}$.

Conversely, let condition (6.1) is fulfilled for the unique directed priority fixed by the direction $i'_1, \ldots, i'_k, \ldots$ By Theorem 4.1 all non-zero coordinates of the matrix P^∞ associated with μ^∞ have a form $p^\infty_{ik} = \frac{d_{ik}}{D_k} = \mathbf{d}_{i_k}$. Therefore, by condition (6.1) we have that $\sum_k p^\infty_{i'_k k} = \sum_k \mathbf{d}_{i'_k} = \infty$ just for the direction $i'_1, i'_2, \ldots, i'_k, \ldots$ If we assume that $\mu^\infty \notin \mathcal{M}_{\rm pp}$, then $\prod_k p^\infty_{i_k k} = 0$ for any (other) directions $i_1, \ldots, i_k, \ldots$ And then, instead (6.2) we, in particular, have: $\sum_k(\sum_{i_k\neq i'_k} d_{i_kk}/D_k) = \infty$. Moreover, the series constituted from the maximal values among $d_{i_kk}/D_k,\ i_k \neq i'_k$ is divergent too

$$\sum_k \max_{i_k\neq i'_k}(d_{i_kk}/D_k) = \infty.$$

So we get the contradiction with the assumption about of uniqueness of the direction $i'_1,\dots,i'_k,\dots$ for which (6.1) is fulfilled. Thus, $\mu^\infty \in \mathcal{M}_{\rm pp}$. □

As a consequence we get the fruitful criterion of continuity of the similar structure measures.

Theorem 6.3. *A measure $\mu = \mu_P \in \mathcal{M}^{\rm ss}(\Delta_0)$ associated with the stochastic matrix P is pure continuous, i.e., $\mu \notin \mathcal{M}_{pp}$, if and only if one can pick at least two sequences of elements of the matrix P, which are different, $p_{i_kk} \neq p_{i'_kk},\ k = 1,2,\dots$, and such that*

$$\sum_k p_{i_kk} = \sum_k p_{i'_kk} = \infty. \tag{6.3}$$

Proof. Let (6.3) hold. Of course, it is possible that $p_{i_kk} = p_{i'_kk}$ for finite amount of meanings of the index k. Assume in addition that $\mu \in \mathcal{M}_{\rm pp}$. Then by Theorem 3.1, $\prod_{k=1}^\infty p_{\max,k} > 0$. Clearly, that in the such case $\sum_k p_{\max,k} = \sum_k p_{i_kk} = \infty$, where we consider that $p_{\max,k}$ is reached just on i_k. We emphasize that latter series is unique (up to replacing of a finite number of coordinates) divergent one formed from elements of the matrix P. This follows from the fact that the condition $\prod_k p_{\max,k} > 0$ is equivalent to the convergent of the series $\sum_k(\sum_{j_k\neq i_k} p_{j_kk})$. However then $\sum_k p_{i'_kk} < \infty$ for any sequence $i'_k \neq i_k$, that contradicts to (6.3). So, we prove that $\mu \notin \mathcal{M}_{\rm pp}$ under condition (6.3).

Let us prove the necessity of (6.3). Let $\mu \notin \mathcal{M}_{\rm pp}$ and therefore the series

$$\sum_k \sum_{\{i_k:p_{i_kk}\neq p_{\max,k}\}} p_{i_kk}$$

converges. Define $p_{i'_kk} := \max_{\{i_k:p_{i_kk}\neq p_{\max,k}\}}\{p_{i_kk}\}$. It is clear that $\sum_k p_{i'_kk} = \infty$. But $\sum_k p_{\max,k} = \infty$ too, since $p_{\max,k} \geq 1/n$ $(n \geq 2)$ due to vectors $\mathbf{p}_k$ are stochastic. Thus we constructed two divergent series. The theorem is proved. □

We note that if $|\mathbb{N}_{+,k}| = 1$ (see Theorem 4.1) for almost all k ($|\mathbb{N}_{+,k}|$ denotes the cardinality of the set $\mathbb{N}_{+,k}$), then the measure μ^∞ has a discrete spectrum and its support consists of the finite number of points. This number is equal to the product $1 \leq \prod_k |\mathbb{N}_{+,k}| < \infty$. But if $|\mathbb{N}_{+,k}| \geq 2$ for an infinite amount of values k, then $\sigma(\mu^\infty)$ is a countable nowhere dense set including only accumulating points.

7. Examples

In the last section we build the examples of the limiting measures which are not necessarily singular continuous.

We write $q_{ik} \sim 1/n$, if $\prod_k(\sum_i \sqrt{q_{ik}/n}) > 0$.

Example 7.1. Let measures $\mu,\nu \in \mathcal{M}^{\rm ss}$ be associated with the stochastic matrices formed with vectors $\mathbf{p}_k,\mathbf{r}_k \in \mathbb{R}^n_+$ $n = 2$. Then $\mu^\infty,\nu^\infty \in \mathcal{M}_{\rm ac}$, if $q_{ik} \sim 1/2$ and $\mathbf{p}_k = \mathbf{r}_k$ almost for all k. Indeed, in this case $p^\infty_{ik} = r^\infty_{ik} = 1/2$ almost for all k. Therefore $\rho(\mu,\nu) = \rho(\nu,\mu) > 0$ due to $q_{ik} \sim 1/2$. Thus, by Theorem 3.1 both measures μ^∞,ν^∞ are pure absolutely continuous.

But if μ,ν are different, then both limiting measures are pure point, $\mu^\infty,\nu^\infty \in \mathcal{M}_{\rm pp}$ without any additional condition on q_{ik}.

Statement 7.1. *Let $n = 2$ and $\mathbf{p}_k \neq \mathbf{r}_k$ for almost all k (excluded finite set is denoted by K). Then both limiting measures μ^∞, ν^∞ are pure point.*

Proof. If $\mathbf{p}_k \neq \mathbf{r}_k, k \notin K$ and $\mathbf{p}_k, \mathbf{r}_k \in \mathbb{R}^2$ then one of coordinates of the vectors $\mathbf{p}_k^\infty, \mathbf{r}_k^\infty$ is equal to unit, and other one is equal to zero (see (4.5)). Therefore $\prod_{k \notin K} p_{\max,k} = \prod_{k \notin K} r_{\max,k} > 0$ and $\mu^\infty, \nu^\infty \in \mathcal{M}_{\rm pp}$ due to Theorem 3.1. In such the case each of considered measure is concentrated at $2^{|K|}$ atoms, where $|K|$ is an amount of point in the set K. □

Example 7.2. Let the measures $\mu, \nu \in \mathcal{M}^{\rm ss}$ be associated with matrices formed with the stochastic vectors $\mathbf{p}_k, \mathbf{r}_k \in \mathbb{R}_+^n$, $n = 3$. Then similar to Example 7.1, $\mu^\infty, \nu^\infty \in \mathcal{M}_{\rm ac}$ only if $q_{ik} \sim 1/3$ and $\mathbf{p}_k = \mathbf{r}_k$ for almost all k. Indeed, in this case almost all elements $p_{ik}^\infty = r_{ik}^\infty = 1/3$ and therefore $\rho(\mu, \nu) = \rho(\nu, \mu) > 0$. But if $\mathbf{p}_k \neq \mathbf{r}_k$ for almost all k then, in general, both limiting measures are pure singular continuous. However, if one of initial measures, say μ, has a local priority, $p_{i_k k} > r_{i_k k}$, only at a single position for almost all k, then $\mu^\infty \in \mathcal{M}_{\rm pp}$, and $\nu^\infty \in \mathcal{M}_{\rm sc}$. It is true since almost all values $p_{\max,k}^\infty = 1$ and the corresponding coordinates $r_{i_k k}^\infty = 0$.

In general case ($n > 2$) the similar result occurs.

Statement 7.2. *Let $\mathbf{p}_k, \mathbf{r}_k \in \mathbb{R}_+^n$, $n \geq 2$ Then $\mu^\infty \in \mathcal{M}_{\rm ac}$ if and only if $q_{ik} \sim 1/n$, $n \geq 3$ and $\mu = \nu$ in the sense that $p_{ik} = r_{ik}$ for almost all k. But if $\mathrm{d}_{i_k k} > 0$ only at a single position i_k for each k, then the measure μ possesses the shock directed priority with respect to the measure ν, $\mathbf{d}(\mu, \nu) > 0$, and therefore $\mu^\infty \in \mathcal{M}_{\rm pp}$.*

Proof. If $\mu = \nu$ then by Theorem 4.1 the elements of the limiting matrices P^∞, R^∞ have the following values $p_{ik}^\infty = r_{ik}^\infty = 1/n$ for almost all k. Therefore by Theorem 3.1 we have: $\rho(\mu^\infty, \lambda) > 0$. This ensures that $\mu^\infty \in M_{\rm ac}$. Let us show the necessity of the conditions. Due to Theorem 5.1, $\mu^\infty \notin \mathcal{M}_{\rm ac}$, if $\mu \neq \nu$ in the sense that $p_{ik} \neq r_{ik}$ for an infinite amount of meanings of k. Therefore the condition $p_{ik} = r_{ik}$ is necessary almost for all k. Moreover, since $p_{ik}^\infty = 1/n$ (see (4.6)), the value $\rho(\mu^\infty, \lambda)$ is strongly positive only under the condition $q_{ik} \sim 1/n$. Finally, taken into account that the measure μ has only a single priority position i_k for each k we observe that $p_{i_k k}^\infty = 1$. Thus $\mathbf{d}(\mu, \nu) > 0$ and μ^∞ is the pure point measure. □

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Received 15/02/2008