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INVERSE EIGENVALUE PROBLEMS FOR NONLOCAL STURM–LIOUVILLE OPERATORS

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To the memory of A. Ya. Povzner

Abstract. We solve the inverse spectral problem for a class of Sturm–Liouville operators with singular nonlocal potentials and nonlocal boundary conditions.

1. Introduction

This paper is dedicated to the memory of A. Ya. Povzner whose outstanding works have opened for me the area of spectral theory of Schrödinger operators. My personal acquaintance with A. Ya. Povzner took place in 1961 when I was presenting my PhD Thesis and he was an opponent. Talks with him in that time and the scientific discussions were very instructive and left an unforgettable trace in all my scientific life.

A one-dimensional Schrödinger operator with nonlocal potential has the form

$$Ly \equiv -\frac{d^2}{dx^2}y(x) + \int K(x,s)y(s)\,ds,$$

where $K(x,s) = \bar{K}(x,s)$ is a Hermitian-symmetric kernel. If $K(x,s) = v(x)\delta(s-x_0) + \delta(x-x_0)\bar{v}(s)$, where δ is Dirac's function, we have a Schrödinger operator with nonlocal point potential, $Ly \equiv -\frac{d^2y(x)}{dx^2} + v(x)y(x_0) + \delta(x-x_0)(y,v)_{L_2}$ [1]. When considering such an operator, one can avoid using Dirac's δ-function if the differential expression $ly(x) = \frac{d^2y(x)}{dx^2} + v(x)y(x_0)$, $x \neq x_0$, in the point $x = x_0$ is supplemented with the boundary-value conditions $y(x_0-0) = y(x_0+0) = y(x_0)$, $y(x_0-0) - y(x_0+0) = (y,v)_{L_2}$ [1]. Note that such nonlocal operators appear not only in quantum mechanics but in other areas such as the theory of diffusion processes, see the related references in [2].

In this paper, we study a one-dimensional Schrödinger operator with nonlocal point potential on a bounded interval with periodic boundary-value conditions. Because of the periodicity, we can limit the considerations to only the case where the point nonlocal potential has its support at an endpoint of the interval. Then we have the following nonlocal Sturm–Liouville eigenvalue problem:

$$\ell(y) := -y''(x) + v(x)y(1) = \lambda y(x), \qquad 0 \le x \le 1, \tag{1}$$

subject to the boundary-value condition

$$y(0) = y(1),\ y'(1) - y'(0) + (y,v)_{L_2} = 0. \tag{2}$$

Here $v \in L_2(0,1)$ is the nonlocal "potential" and $\lambda \in \mathbb{C}$ is a spectral parameter. This problem is close to the problem studied in [2] for equation (1) with the boundary-value conditions $y(0) = y'(1) + (y,v)_{L_2} = 0$. However, this unperturbed problem (1)–(2), for $v = 0$, has a simple eigenvalue, $\lambda = 0$, and double eigenvalues $\lambda = (2n\pi)^2$, $n \in N$,

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whereas the unperturbed problem considered in [2] has all simple eigenvalues $\lambda = (n\pi)^2$, $n \in N$. This influences the way for studying problem (1)–(2) and solving the inverse eigenvalue problem, that is, the problem of recovering the function v in equation (1) from a known collection of all eigenvalues of problem (1)–(2).

For solving the inverse eigenvalue problem, as in the case of the usual potential, in addition to eigenvalues it is also necessary to have some additional information [3]. The same is true for the real nonlocal potential, — we need to have signs of its even Fourier sine coefficients.

2. Direct spectral analysis

On the space $L_2(0,1)$, problem (1)–(2) is naturally related to an operator T_v that has domain

$$\operatorname{dom} T = \{y \in W_2^2(0,1) \mid y(0) = y'(1) - y'(0) + (y,v)_{L_2} = 0\}$$

and is defined by $T_v y = \ell(y)$. The operator T_v is a closed symmetric operator, since integration by part formula easily yields that the quadratic form $(T_v y, y)_{L_2}$ is real,

$$(T_v y, y) = (y', y')_{L_2} + 2\mathrm{Re}\,[y(1)(v,y)_{L_2}]. \tag{3}$$

For $v = 0$, the operator T_0 is self-adjoint on the space $L_2(0,1)$, with domain consisting of functions periodic on $(0,1)$ and belonging to the Sobolev space $W_2^2(0,1)$. Eigenfunctions of the operator T_0 are the functions 1, $\cos 2n\pi x$, $\sin 2n\pi x$, $n \in N$, and eigenvalues are $\lambda_0 = 0$, $\lambda_n = (2n\pi)^2$. The operators T_v and T_0 can be considered as extensions of the symmetric operator T_{min} that has domain

$$\operatorname{dom} T_{min} = \{y \in W_2^2(0,1) \mid y(0) = y(1) = 0,\ y'(0) = y'(1),\ (y,v)_{L_2} = 0\} \tag{4}$$

and is defined by $T_{min} y = T_v y = T_0 y = -y''$. Since $\dim(\operatorname{dom} T_v / \operatorname{dom} T_{min}) \le 2$, the self-adjoint operator T_v is a rank $r \le 2$ perturbation of the self-adjoint operator T_0. Since the operator T_0 has discrete spectrum, which consists of the numbers $\lambda_n = (2n\pi)^2$, the operator T_v has discrete spectrum consisting of real numbers $\lambda_n \to +\infty$ for $n \to \infty$.

Theorem 1. 1. *All eigenvalues of problem (1), (2), distinct from $(n\pi)^2$, $n \in N$, are simple.*

2. *The number $\lambda = (2n\pi)^2$ is an eigenvalue of problem (1), (2) if and only if*

$$v_n = \frac{1}{2}\int_0^1 v(x)\sin n\pi x dx = \begin{cases} 0, & \text{if } n \text{ even}, \\ 4n\pi, & \text{if } n \text{ odd}. \end{cases} \tag{5}$$

3. *The number $\lambda_n = (2n\pi)^2$ is a double eigenvalue of problem (1), (2) if and only if, in addition to (5), we have*

$$\sum_{k \ne n} \frac{[1 + (-1)^{k+1}]k\pi[v_k + \bar{v}_k] - \frac{1}{2}|v_k|^2}{(k\pi)^2 - (n\pi)^2} = 0. \tag{6}$$

Problem (1), (2) has no eigenvalues with multiplicity exceeding 2.

4. *The number $\lambda = 0$ is an eigenvalue of problem (1), (2) if and only if (6) holds for $n = 0$.*

Proof. The eigenvalues of problem (1), (2) coincide with the eigenvalues of the operator T_v. Let $y(x;z)$ be an eigenfunction of the operator T_v corresponding to an eigenvalue $\lambda = z^2$. If $y(0;z) = y(1;z) = 0$, then it follows from equation $-y'' = z^2$ that $y = C\sin \pi x$ and $z^2 = (n\pi)^2$, $n \in N$. Hence, for $\lambda \ne (n\pi)^2$, eigenfunctions take nonzero equal values at the endpoints of the interval $(0,1)$. This leads to simplicity of the eigenvalues $\lambda \ne (n\pi)^2$, $n \in N$. Indeed, if an eigenvalue were multiple, there would exist at least two linearly independent eigenfunctions $y_i(x;z)$, $i = 1,2$, corresponding to the same eigenvalue. But then $y(x;z) = y_1(0;z)y_2(x;z) - y_2(0;z)y_1(x;z)$ would be a nontrivial eigenfunction which vanishes at the endpoints of the interval $(0,1)$. This is impossible for $\lambda = z^2 \neq (n\pi)^2$. Claim 1 of the theorem is proved.

Let now $\lambda = (n\pi)^2$, $n \in N$, be an eigenvalue of the operator T_v, that is, there exists a nontrivial solution $y(x;n\pi)$ of problem (1), (2) for $\lambda = (n\pi)^2$. Since the nonlocal potential satisfies $v \in L_2$, this function can be represented by a Fourier sine series,

$$v(x) = \sum_{k=1}^{\infty} v_k \sin k\pi x dx, \qquad v_k = 2\int_0^1 v(x)\sin k\pi x dx. \tag{7}$$

By substituting (7) into equation (1) and solving it, we get

$$y(x;n\pi) = A\cos n\pi x + B\sin n\pi x - \frac{A}{2n\pi} v_n x\cos n\pi x - A\sum_{k\neq n} \frac{v_k \sin k\pi x}{(k\pi)^2 - (n\pi)^2}. \tag{8}$$

The boundary-value condition $y(0) = y(1)$ leads to the identity

$$A[1 - (-1)^n + \frac{v_n}{2n\pi}(-1)^n] = 0. \tag{9}$$

Hence, if $A \neq 0$, then (5) holds. If $A = 0$ and the solution y is nontrivial, then it has the form $y = B\sin n\pi x$ and $B \neq 0$. Substituting this solution into the boundary-value condition $y'(1) - y'(0) + (y,v) = 0$ we are again led to (5). Hence, problem (1)–(2) has a nontrivial solution for $\lambda = (n\pi)^2$ only if condition (5) is satisfied. Also, condition (5) is sufficient for the function $y = \sin n\pi x$ to be an eigenfunction. Claim 2 is proved.

To prove Claim 3, note that $\lambda = (n\pi)^2$ is a double eigenvalue if and only if condition (5) is satisfied and the function in (8), with $A = 1$ and $B = 0$, satisfies the boundary-value condition $y'(1) - y'(0) + (y,v) = 0$. Substituting (8) into this boundary-value condition leads to identity (6). Since problem (1)–(2) deals with a second order equation on the interval $(0,1)$, it can not have more than two linearly independent solutions and, consequently, there are no eigenvalues of multiplicity exceeding 2.

Consider Claim 4. It follows from (1) for $\lambda = 0$ that

$$y(x;0) = A + Bx - y(1)\sum_{k=1}^{\infty} \frac{v_k \sin k\pi x}{(k\pi)^2}. \tag{10}$$

The boundary-value condition $y(1) = y(0)$, since the solution $y(x;0)$ is nontrivial, leads to the condition $B = 0$, $A \neq 0$, $y(1) = A$. Substituting (10) into the boundary-value condition (2) we get identity (6) with $n = 0$. $\square$

To give an exact description of the distribution of eigenvalues of problem (1), (2), it is convenient to show that the eigenvalues are connected with zeros of an analytic function, which is a characteristic function of problem (1), (2).

To this end, consider a special solution of equation (1) with $\lambda = z^2$, satisfying the condition $y(0) = y(1)$,

$$\varphi(x;z) = \frac{\sin zx + \sin z(1-x)}{z} - \frac{\sin zx}{z}\sum_{k=1}^{\infty} \frac{v_k \sin k\pi x}{(k\pi)^2 - z^2}. \tag{11}$$

The function φ is an eigenfunction of problem (1), (2) if it satisfies the boundary-value condition (2). This gives the characteristic equation $\chi(z) = 0$, where the characteristic function $\chi(z)$ is defined by $\chi(z) = \varphi'(1) - \varphi'(0) + (\varphi, v)$ and has the form

$$\chi(z) = 2(\cos z - 1) + \frac{\sin z}{z}\sum_{k=1}^{\infty} \frac{(-1)^k k\pi\alpha_k}{(k\pi)^2 - z^2}, \tag{12}$$

where

$$(-1)^k\alpha_k = [1 + (-1)^{k+1}](v_k + \bar{v}_k) - \frac{1}{2k\pi}|v_k|^2. \tag{13}$$

Lemma 1. *The characteristic function $\chi(z)$ of the form (12) is an entire function of z and, for $z = n\pi$, $n \in N$, takes the values*

$$\chi(2m\pi) = \frac{|v_{2m}|^2}{(4m\pi)^2}, \qquad \chi((2m-1)\pi) = -|2 - \frac{v_{2m-1}}{2(2m-1)\pi)}|^2. \tag{14}$$

Proof. The proof is carried out by direct computations using the explicit form (12) of the characteristic function. □

Theorem 2. *The number $\lambda = z^2$ is an eigenvalue of problem (1), (2) if and only if z is a zero of the characteristic function $\chi(z)$. The number $\lambda = z^2 \neq 0$ is a double eigenvalue of the problem (1), (2) if and only if z is a double zero of the characteristic function. All zeros $z \neq n\pi$, $n \in N$, of the characteristic function are simple. The characteristic function does not have zeros of multiplicities greater than* 2.

Proof. It follows that the squares $z^2 \neq (n\pi)^2$ of zeros of the characteristic function $\chi(z)$ are eigenvalues from the fact that the special solution (11) is an eigenfunction and vice versa. For $z^2 = (n\pi)^2 = \lambda$, the number $\lambda = z^2$ is an eigenvalue if and only if (5) is satisfied which, by (14), is equivalent to $\chi(n\pi) = 0$. It is easy to check that condition (6) is equivalent to $\dot{\chi}(n\pi) = 0$ which, in its turn, together with $\chi(n\pi) = 0$, is equivalent to that $n\pi$ is a double eigenvalue. Condition (6), for $n = 0$, is equivalent to $\chi(0) = 0$. Hence, the eigenvalues, counting multiplicities, coincide with squares of zeros of $\chi(z)$, counting multiplicities.

If the function $\chi(z)$ had a multiple root $z_0 \neq n\pi$, this would imply that $\frac{\partial}{\partial z}\varphi(x;z)|_{z=z_0}$ were a generalized eigenfunction, which is impossible since the operator T_v is self-adjoint. In the same way, we can prove that there are no zeros of $\chi(z)$ that have multiplicities greater than 2 for $z = n\pi$. □

Theorem 3. *The increasingly ordered sequence $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$ of all eigenvalues of problem (1), (2), counting multiplicities, has the following properties:*

1. *the sequence weakly alternates with the sequence $(n\pi)^2$,*

$$\lambda_n \leq (n\pi)^2 \leq \lambda_{n+1}, \qquad n \in N; \tag{15}$$

2. *there is an asymptotic representation,*

$$\sqrt{\lambda_{2n}} = 2n\pi - \tfrac{\beta_{2n}}{n}, \qquad \sqrt{\lambda_{2n+1}} = 2n\pi + \tfrac{\beta_{2n+1}}{n}, \tag{16}$$

where $\beta_j \geq 0$ and $\sum_{j=1}^{\infty} \beta_j^2 < +\infty$.

Proof. An upper estimate for λ_n is easily obtained from the variation minimax principle [4],

$$\begin{aligned} \lambda_n &= \sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\substack{||\psi||=1,\, \psi\perp\varphi_k,\\ \psi(0)=\psi(1)}} (T_v\psi,\psi) \\ &\leq \sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\substack{||\psi||=1,\, \psi\perp\varphi_k,\\ \psi(0)=\psi(1)=0}} (\psi',\psi') = (n\pi)^2. \end{aligned}$$

For large n, we have the strict inequality

$$[(2n-1)\pi]^2 < \lambda_{2n} \leq \lambda_{2n+1} < [(2n+1)\pi]^2. \tag{17}$$

Indeed, by the Rouché theorem, the entire function $\chi(z)$ and the function $2(\cos z - 1)$ have the same number of zeros, counting multiplicities, in the strip $-(2n+1)\pi < \operatorname{Re} z < (2n+1)\pi$ for large n. Since, by Theorem 2, the eigenvalues λ_n of problem (1), (2) are squares of zeros of the function $\chi(z)$, the function $\chi(z)$ is even, and double zeros of the function $2(\cos z - 1)$ are the numbers $z = 2n\pi$, $n = 0, \pm 1, \ldots$, which shows that inequalities (17) hold.

On the other hand, if

$$v_{2m} \neq 0, \qquad v_{2m+1} \neq 4(2m+1)\pi, \tag{18}$$

then, by (14), $\chi(n\pi) \neq 0$ for any n and, moreover, $\chi(n\pi)(-1)^n > 0$. Hence, in every interval $(n\pi, (n+1)\pi)$, the characteristic function χ has at least one zero z_0 and, consequently, there is one eigenvalue $\lambda = z^2$ in the interval $I_n = ((n\pi)^2, (n+1)^2\pi^2)$. The assumption that at least one interval I_n contains more than one eigenvalue leads to a contradiction with estimate (17). Hence, if conditions (18) are satisfied, inequality (15) holds,

$$\lambda_n < (n\pi)^2 < \lambda_{n+1}. \tag{19}$$

Since condition (18) can be satisfied by an arbitrary small change of the potential, passing to the limit in (19) we get (15).

To prove the asymptotic representations (16) let us first find an asymptotic representation of zeros of the characteristic function $\chi(z)$ defined by (12), since $\sqrt{\lambda_n} = z_n$.

Write the characteristic function in (12), $\chi(z)$, as

$$\chi(z) = 2(\cos z - 1) - \frac{1}{z}\int_0^1 \sin zt\alpha(t)\,dt, \tag{20}$$

where $\alpha(t) \in L_2$, and the numbers α_k are coefficients in the Fourier sine expansion. If $z = 2m\pi + \varepsilon_m$, then (20) for $m \to \infty$ gives

$$-\varepsilon_m^2 + \frac{|v_{2m}|^2}{(4m\pi)^2} - \varepsilon_m \frac{w_{2m}}{2m\pi} + o(\varepsilon_m^2) = 0, \tag{21}$$

where $w_{2m} = \displaystyle\int_0^1 \cos 2m\pi t\alpha(t)t\,dt$ and $\sum_m w_{2m}^2 < \infty$.

Identity (21) gives two solutions for ε_m, one of which is nonpositive, $\varepsilon_m^+ = -\beta_{2m}$, $\beta_{2m} \geq 0$, and the other one is nonnegative, $\varepsilon_m^- = \beta_{2m+1}$, $\beta_{2m+1} \geq 0$. Here $\sum_j |\beta_j|^2 < +\infty$. This gives representation (16). □

3. Isospectral nonlocal potentials

Let $\Lambda(v) = \{\lambda_j\}_{j=1}^{\infty}$ be an ordered sequence of all eigenvalues, counting multiplicities, of problem (1), (2) with a nonlocal potential v. Two potentials v_1 and v_2 are called isospectral if $\Lambda(v_1) = \Lambda(v_2)$, that is, eigenvalues and their multiplicities of problems (1), (2), coincide for the potentials v_1 and v_2. It follows from Theorem 2 that potentials v_1 and v_2 are isospectral if and only if the corresponding characteristic functions coincide. Using representations (12), (13) the for characteristic functions we obtain a criterion for two potentials v_1 and v_2 to be isospectral in terms of their Fourier sine coefficients $v_k^{(j)}$, $j = 1, 2$, $k \in N$,

$$\begin{aligned} &|v_{2m}^{(1)}| = |v_{2m}^{(2)}|, \\ &\left|1 - \tfrac{v_{2m+1}^{(1)}}{4(2m+1)\pi}\right| = \left|1 - \tfrac{v_{2m+1}^{(2)}}{4(2m+1)\pi}\right|. \end{aligned} \tag{22}$$

On the space $L_2(0,1)$ of real-valued functions, define nonlinear projections Π_s that depend on a sequence of real numbers $s = \{s_1, s_2, ..\}$ by

$$\Pi_s\Big(\sum_{n=1}^{\infty} v_n \sin n\pi x\Big) = \sum_{n=1}^{\infty} v_n^{(s)} \sin n\pi x, \qquad \sum_{n=1}^{\infty} v_n^2 < +\infty, \tag{23}$$

where

$$v_n^{(s)} = \begin{cases} |v_n|, & \text{if } s_n = 0, \\ v_n, & \text{if } |v_n| \le |2s_n - v_n|,\ s_n \neq 0, \\ 2s_n - v_n, & \text{if } |2s_n - v_n| < |v_n|. \end{cases} \tag{24}$$

Theorem 4. *Two real nonlocal potentials $v_1, v_2 \in L_2(0,1)$ are isospectral if and only if the corresponding nonlinear projections coincide,*

$$\Pi_s v_1 = \Pi_s v_2. \tag{25}$$

Here $s = \{s_n\}_{n=1}^{\infty}$ consists of the numbers $s_{2m} = 0$ and $s_{2m-1} = 2(2m-1)\pi$.

Proof. The proof follows from the isospectral criterion (22) and definitions (23), (24) of the nonlinear projection Π_s. □

4. The inverse spectral analysis

The main tool for describing eigenvalues of problem (1), (2) is the characteristic function $\chi(z)$ explicitly given by (12). This function can be written using the ordered set of all eigenvalues, $\Lambda = \{\lambda_j\}$.

Theorem 5. *The characteristic function $\chi(z)$ can be expressed in terms of the ordered sequence $\{\lambda_j\}$, counting multiplicities, of problem (1), (2) as*

$$\chi(z) = (\lambda_1 - z^2) \prod_{k=2}^{\infty} \frac{\lambda_k - z^2}{4[\frac{k}{2}]^2 \pi^2}, \tag{26}$$

where $[\alpha]$ denotes the entire part of α.

Proof. Since $\chi(z)$ is an even function of exponential type 1 and

$$\lim_{\xi \to \infty} \frac{\chi(i\xi)}{2(\cos i\xi - 1)} = 1,$$

it can uniquely be represented in terms of its zeros $z_k = \sqrt{\lambda_k}$ as the product in (26). □

Consider now the problem of recovering the nonlocal potential for problem (1), (2) in terms of the set $\Lambda = \{\lambda_j\}$ of all eigenvalues of the problem. To solve this problem, we can propose the following algorithm:

Step 1. Using the eigenvalues $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ construct the characteristic function $\chi(z)$ by formula (26).
Step 2. Calculate the values $\chi(n\pi)$, $n \in N$.
Step 3. Use formula (14) to find v_n that have the minimal modulus.
Step 4. Use the Fourier coefficients v_n to construct the nonlocal potential

$$v(x) = \sum_{n=1}^{\infty} v_n \sin n\pi x. \tag{27}$$

Example. Let, for problem (1), (2), we have $\lambda_1 = \pi^2$, $\lambda_{2n} = \lambda_{2n+1} = (2n\pi)^2$, $n \in N$. Then $\chi(z) = \frac{z^2 - \pi^2}{z^2} 2(\cos z - 1)$. In this case, $\chi(2n\pi) = 0$, $\chi((2n-1)\pi) = -4 + \frac{4}{(2n-1)^2}$, $n \in N$. It follows from formulas (14) that $v_{2n} = 0$, $v_{2n-1} = 4\pi \frac{1}{2n-1+\sqrt{(2n-1)^2-1}}$, $n \in N$. Hence, the nonlocal potential $v(x)$ has the following Fourier series:

$$v(x) = 4\pi \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1+\sqrt{(2n-1)^2-1}}.$$

Since $\sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1} = \frac{\pi}{4}$, we have

$$v(x) = \frac{\pi^2}{2} + 2\pi \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{(2n-1)[2n-1+\sqrt{(2n-1)^2-1}]^2}.$$

As was remarked in the previous section, the nonlocal potential v is defined by its eigenvalues $\Lambda(v)$ non-uniquely. However, it is easy to give conditions such that the nonlocal potential can be found in a unique way, as it was done in [2]. In particular, the inverse problem for (1), (2) with nonlocal potential has a unique solution if we use $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ to find a nonlocal potential with a minimal norm and an additional assignment of signs of all nonzero even Fourier coefficients.

For the inverse problem, it is important to have a description of the initial data of the problem, that is the set $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ of all eigenvalues of problem (1), (2) with nonlocal potential $v \in L_2$.

Theorem 6. *Conditions 1 and 2 in Theorem 3 are necessary and sufficient for a sequence $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ to be an ordered sequence of all eigenvalues of problem (1), (2) with a nonlocal potential $v \in L_2$.*

Proof. Let conditions 1 and 2 of Theorem 3 be satisfied. Then the function $\chi(z)$ constructed using (26) admits representation (19) and the value $\chi(n\pi)$ can be written as in (14) with some v_n, $\sum_{n=1}^{\infty} |v_n|^2 < \infty$. The nonlocal potential $v(x) = \sum v_k \sin k\pi x \in L_2$ gives rise to a characteristic function that coincides with $\chi(z)$ that is constructed using formula (26). Hence, problem (1), (2) with the potential constructed as above has eigenvalues that coincide with $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$. □

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