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INTEGRAL REPRESENTATIONS FOR SPECTRAL FUNCTIONS OF SOME NONSELF-ADJOINT JACOBI MATRICES

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ABSTRACT. We study a Jacobi matrix J with complex numbers $a_n,\ n \in \mathbb{Z}_+$, in the main diagonal such that $r_0 \leq \operatorname{Im} a_n \leq r_1,\ r_0, r_1 \in \mathbb{R}$. We obtain an integral representation for the (generalized) spectral function of the matrix J. The method of our study is similar to Marchenko's method for nonself-adjoint differential operators.

1. Introduction

The main object of our present investigation will be a three-diagonal semi-infinite complex number matrix of the following form:

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \dots \\ b_0 & a_1 & b_1 & 0 & \dots \\ 0 & b_1 & a_2 & b_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

where $b_n > 0$, and

$$a_n \in \mathbb{C} : \ r_0 \leq \operatorname{Im} a_n \leq r_1, \tag{2}$$

for some $r_0, r_1 \in \mathbb{R}$, $n \in \mathbb{Z}_+$.

Thus, in the case $r_0 = r_1 = 0$ we obtain the classical Jacobi matrix. The spectral theory of Jacobi matrices is classic, see [1], [2], [3]. For the Jacobi matrix J, there is a corresponding non-decreasing function $\sigma(x),\ x \in \mathbb{R}$, which is called a *spectral function*. The procedure of a construction of $\sigma(x)$ provides a solution of the *direct spectral problem* for J. The *inverse spectral problem* is to reconstruct J from σ. The corresponding procedure is well-known and simple.

Recently, we have introduced a notion of a spectral function for some nonself-adjoint semi-infinite banded matrices, see [4], [5]. The spectral function is a bilinear (that means linear with respect to the both arguments) functional $\sigma(u, v),\ u, v \in \mathbb{P}$, defined on a set of complex polynomials $\mathbb{P}$. We will use methods which were applied by Marchenko to some nonself-adjoint Sturm-Liouville operators (see [6]) and obtain an integral representation for the spectral function $\sigma(u, v)$ of the matrix J from (1).

Notations. As usual, we denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$ the sets of real, complex, positive integer, integer, non-negative integer numbers, respectively. By $\mathbb{P}$ we denote the set of all polynomials with complex coefficients. By l^2 we denote a space of vectors $x = (x_0, x_1, x_2, \dots),\ x_n \in \mathbb{C},\ n \in \mathbb{Z}_+$, such that $\|x\| := (\sum_{n=0}^{\infty} |x_n|^2)^{\frac{1}{2}} < \infty$. By l^2_{fin} we denote a subset of l^2 which consists of finite vectors, i.e., vectors $x = (x_0, x_1, x_2, \dots)$, $x_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$, with only a finite number of nonzero elements x_n.

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2. Polynomials of the first and of the second kinds

Let J be the semi-infinite matrix from (1), (2). Consider the following difference equations:

$$a_0 y_0 + b_0 y_1 = \lambda y_0, \tag{3}$$

$$b_{n-1} y_{n-1} + a_n y_n + b_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \tag{4}$$

where y_n are unknowns and λ is a complex parameter.

By $P_n(\lambda)$, $n \in \mathbb{Z}_+$, we denote a solution of (3), (4) with the initial condition $P_0 = 1$. Polynomials $P_n(\lambda)$ we will call *polynomials of the first kind*. Denote by $Q_n(\lambda)$, $n \in \mathbb{Z}_+$, a solution of (4) with the initial conditions $Q_0 = 0$, $Q_1 = \frac{1}{b_0}$. Polynomials $Q_n(\lambda)$ we will call *polynomials of the second kind*.

If we write relation (4) for P_n and then multiply it by Q_n, we will get

$$b_{n-1} P_{n-1} Q_n + a_n P_n Q_n + b_n P_{n+1} Q_n = \lambda P_n Q_n, \quad n \in \mathbb{N}. \tag{5}$$

In a similar manner we will get

$$b_{n-1} Q_{n-1} P_n + a_n Q_n P_n + b_n Q_{n+1} P_n = \lambda Q_n P_n, \quad n \in \mathbb{N}. \tag{6}$$

Subtract (6) from (5) to get

$$b_{n-1}(P_{n-1} Q_n - P_n Q_{n-1}) = b_n (P_n Q_{n+1} - P_{n+1} Q_n), \quad n \in \mathbb{N}.$$

Using the initial conditions we obtain

$$P_{n-1}(\lambda) Q_n(\lambda) - P_n(\lambda) Q_{n-1}(\lambda) = \frac{1}{b_{n-1}}, \quad n \in \mathbb{N}. \tag{7}$$

We will use relation (7) in the sequel.

Proposition 1. *Let $y_n = y_n(\lambda)$, $n \in \mathbb{Z}_+$, be an arbitrary solution of difference equation (4). The following relation holds true:*

$$\sum_{j=1}^{n-1} (\operatorname{Im} a_j - \operatorname{Im} \lambda) |y_j(\lambda)|^2 = b_{n-1} \operatorname{Im}(y_{n-1}(\lambda) \overline{y_n(\lambda)}) - b_0 \operatorname{Im}(y_0(\lambda) \overline{y_1(\lambda)}), \tag{8}$$
$$n = 2, 3, \ldots$$

Proof. Set $\widehat{a}_n = \widehat{a}_n(\lambda) = a_n - \lambda$, $n \in \mathbb{N}$, and rewrite relation (4) in the following form:

$$b_{n-1} y_{n-1} + \widehat{a}_n y_n + b_n y_{n+1} = 0, \quad n \in \mathbb{N}. \tag{9}$$

Apply the complex conjugation to the both sides of (9) to get

$$b_{n-1} \overline{y_{n-1}} + \overline{\widehat{a}_n} \overline{y_n} + b_n \overline{y_{n+1}} = 0, \quad n \in \mathbb{N}. \tag{10}$$

Multiply relation (9) by $\overline{y_n}$, relation (10) by y_n, and then subtract to obtain

$$b_{n-1}(y_{n-1}\overline{y_n} - \overline{y_{n-1}} y_n) + (\widehat{a}_n - \overline{\widehat{a}_n}) y_n \overline{y_n} + b_n (y_{n+1} \overline{y_n} - \overline{y_{n+1}} y_n) = 0, \quad n \in \mathbb{N}. \tag{11}$$

Set

$$A_n = A_n(\lambda) = b_n (y_n \overline{y_{n+1}} - \overline{y_n} y_{n+1}) = b_n 2i \operatorname{Im}(y_n \overline{y_{n+1}}), \quad n \in \mathbb{Z}_+.$$

Then we can write

$$(\widehat{a}_n - \overline{\widehat{a}_n}) y_n \overline{y_n} = A_n - A_{n-1}, \quad n \in \mathbb{N}. \tag{12}$$

Summing up we obtain

$$\sum_{j=1}^{n-1} (\widehat{a}_j - \overline{\widehat{a}_j}) y_j \overline{y_j} = A_{n-1} - A_0 = 2i b_{n-1} \operatorname{Im}(y_{n-1} \overline{y_n}) - 2i b_0 \operatorname{Im}(y_0 \overline{y_1}), \quad n = 2, 3, \ldots \tag{13}$$

Therefore relation (8) is true. □

Corollary 1. *Let $P_n(\lambda)$ and $Q_n(\lambda)$, $n \in \mathbb{Z}_+$, be polynomials of the first and of the second kinds for difference equations (3), (4), respectively. Polynomials $P_n(\lambda)$ satisfy the following relation:*

$$\sum_{j=0}^{n-1} (\operatorname{Im} a_j - \operatorname{Im} \lambda)|P_j(\lambda)|^2 = b_{n-1} \operatorname{Im}(P_{n-1}(\lambda)\overline{P_n(\lambda)}), \quad n \in \mathbb{N}. \tag{14}$$

Choose an arbitrary $w \in \mathbb{C}$ and consider the polynomials

$$\Psi_n(\lambda, w) = wP_n(\lambda) + Q_n(\lambda), \quad n \in \mathbb{Z}_+. \tag{15}$$

The polynomials $\Psi_n(\lambda, w)$ satisfy the following relation:

$$\sum_{j=0}^{n-1} (\operatorname{Im} a_j - \operatorname{Im} \lambda)|\Psi_j(\lambda, w)|^2 = b_{n-1} \operatorname{Im}(\Psi_{n-1}(\lambda, w)\overline{\Psi_n(\lambda, w)}) - \operatorname{Im} w, \quad n \in \mathbb{N}. \tag{16}$$

Proof. To obtain relations (14), (16) for $n \geq 2$, it is sufficient to write relation (8) for the polynomials $P_n(\lambda)$ and $\Psi_n(\lambda, w)$, respectively, and to use the initial conditions. For the case $n = 1$ relations (14), (16) can be verified using the initial conditions. □

Set

$$\Pi = \Pi(r_0, r_1) = \{\lambda \in \mathbb{C} : \ r_0 \leq \operatorname{Im} \lambda \leq r_1\}. \tag{17}$$

Corollary 2. *Let $P_n(\lambda)$, $n \in \mathbb{Z}_+$, be polynomials of the first kind for difference equations (3), (4). The roots of polynomials $P_n(\lambda)$ lie in the strip $\Pi(r_0, r_1)$.*

Proof. For an arbitrary root $\lambda_0 \in \mathbb{C}$ of $P_{n-1}(\lambda)$, $n = 2, 3, \ldots$, by (14) we obtain

$$\sum_{j=0}^{n-1} (\operatorname{Im} a_j - \operatorname{Im} \lambda_0)|P_j(\lambda_0)|^2 = 0. \tag{18}$$

Suppose that $\operatorname{Im} \lambda_0 > r_1$. By (2) we obtain

$$\operatorname{Im} a_j - \operatorname{Im} \lambda_0 < 0, \quad j \in \mathbb{Z}_+.$$

Then (18) leads to a contradiction since $P_0 = 1$. If we suppose that $\operatorname{Im} \lambda_0 < r_0$, we will get

$$\operatorname{Im} a_j - \operatorname{Im} \lambda_0 > 0, \quad j \in \mathbb{Z}_+.$$

That contradicts relation (18) as well. □

3. Weyl's discs

Like in the classical case (see [1]), an important role in our further considerations will play the following function:

$$w_n(\lambda, \tau) = -\frac{Q_n(\lambda) - \tau Q_{n-1}(\lambda)}{P_n(\lambda) - \tau P_{n-1}(\lambda)}, \tag{19}$$

where $\lambda, \tau \in \mathbb{C}$, $n \in \mathbb{N}$ (P_n, Q_n are polynomials of the first and of the second kinds for difference equations (3), (4)). We set

$$\Pi_+ = \Pi_+(r_1) = \{\lambda \in \mathbb{C} : \ \operatorname{Im} \lambda > r_1\}, \quad \Pi_- = \Pi_-(r_0) = \{\lambda \in \mathbb{C} : \ \operatorname{Im} \lambda < r_0\}, \tag{20}$$

$$\Pi_0 = \Pi_0(r_0, r_1) = \Pi_+(r_1) \cup \Pi_-(r_0). \tag{21}$$

1) Choose an arbitrary $\lambda \in \Pi_+(r_1)$ and $n \in \mathbb{N}$. By virtue of Corollary 2, relations (14) and (2) we get

$$b_{n-1} \operatorname{Im}(P_{n-1}(\lambda)\overline{P_n(\lambda)}) = b_{n-1}|P_{n-1}(\lambda)|^2 \operatorname{Im}\left(\overline{\left(\frac{P_n(\lambda)}{P_{n-1}(\lambda)}\right)}\right) < 0.$$

Thus, we have

$$\mathrm{Im}\left(\frac{P_n(\lambda)}{P_{n-1}(\lambda)}\right) > 0. \tag{22}$$

So, a pole of the map $w_n(\lambda,\tau)$ (for the fixed $\lambda\in\Pi_+,\ n\in\mathbb{N}$) lies in the upper half-plane $\mathbb{C}'_+ = \{\lambda\in\mathbb{C}:\ \mathrm{Im}\,\lambda>0\}$. In particular, this means that the real line $\mathbb{R}$ is mapped on a circle $C_n(\lambda)$ in the w-plane (the complex plane of the variable w). The lower half-plane $\mathbb{C}_- = \{\tau\in\mathbb{C}:\ \mathrm{Im}\,\tau\le 0\}$ is mapped on a disc $D_n(\lambda)$. The inverse map for $w_n(\lambda,\tau)$ has the following form:

$$\tau_n(\lambda,w) = \frac{wP_n(\lambda)+Q_n(\lambda)}{wP_{n-1}(\lambda)+Q_{n-1}(\lambda)} = \frac{\Psi_n(\lambda,w)}{\Psi_{n-1}(\lambda,w)}. \tag{23}$$

For an arbitrary $w\in\mathbb{C}:\ \Psi_{n-1}(\lambda,w)\neq 0$ (this means that $w\neq -\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)}$) by virtue of relation (16) we can write

$$\sum_{j=0}^{n-1}(\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda)|\Psi_j(\lambda,w)|^2 = -b_{n-1}|\Psi_{n-1}(\lambda)|^2\,\mathrm{Im}(\frac{\Psi_n(\lambda,w)}{\Psi_{n-1}(\lambda)}) - \mathrm{Im}\,w. \tag{24}$$

In our case we have $\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda<0,\ j\in\mathbb{Z}_+$. Therefore

$$\sum_{j=0}^{n-1}|\,\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda||\Psi_j(\lambda,w)|^2 = \mathrm{Im}\,w + b_{n-1}|\Psi_{n-1}(\lambda)|^2\,\mathrm{Im}\,\tau_n(\lambda,w). \tag{25}$$

From the last relation and relation (16) for the case $w=-\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)}$, we see that the disc $D_n(\lambda)$ consists of $w\in\mathbb{C}$ such that

$$\sum_{j=0}^{n-1}|\,\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda||\Psi_j(\lambda,w)|^2 \le \mathrm{Im}\,w. \tag{26}$$

From relation (26) it follows that

$$D_{n+1}(\lambda)\subseteq D_n(\lambda),\quad n\in\mathbb{N},\quad \lambda\in\Pi_+. \tag{27}$$

Hence, there exists a non-empty intersection $D_\infty(\lambda)=\cap_{j\in\mathbb{N}}D_j(\lambda)$. From relation (26) it follows that $D_\infty(\lambda)$ consists of $w\in\mathbb{C}$ such that

$$\sum_{j=0}^{\infty}|\,\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda||\Psi_j(\lambda,w)|^2 \le \mathrm{Im}\,w. \tag{28}$$

2) Choose an arbitrary $\lambda\in\Pi_-(r_0)$ and $n\in\mathbb{N}$. Reasoning similarly, we obtain that

$$\mathrm{Im}\left(\frac{P_n(\lambda)}{P_{n-1}(\lambda)}\right) < 0. \tag{29}$$

A pole of the map $w_n(\lambda,\tau)$ lies in the lower half-plane $\mathbb{C}'_- = \{\lambda\in\mathbb{C}:\ \mathrm{Im}\,\lambda<0\}$. The real line is mapped on a circle $C_n(\lambda)$ and the upper half-plane $\mathbb{C}_+ = \{\tau\in\mathbb{C}:\ \mathrm{Im}\,\tau\ge 0\}$ is mapped on a disc $D_n(\lambda)$. For $w\in\mathbb{C}:\ \Psi_{n-1}(\lambda,w)\neq 0$, by virtue of relation (16) we can write relation (24). In our case we have $\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda>0,\ j\in\mathbb{Z}_+$, therefore

$$\sum_{j=0}^{n-1}|\,\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda||\Psi_j(\lambda,w)|^2 = -\,\mathrm{Im}\,w - b_{n-1}|\Psi_{n-1}(\lambda)|^2\,\mathrm{Im}\,\tau_n(\lambda,w). \tag{30}$$

From relation (15) and relation (16) for the case $w=-\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)}$, we see that the disc $D_n(\lambda)$ consists of $w\in\mathbb{C}$ such that

$$\sum_{j=0}^{n-1}|\,\mathrm{Im}\,a_j-\mathrm{Im}\,\lambda||\Psi_j(\lambda,w)|^2 \le -\,\mathrm{Im}\,w. \tag{31}$$

From relation (31) it follows that

$$D_{n+1}(\lambda) \subseteq D_n(\lambda), \quad n \in \mathbb{N}, \quad \lambda \in \Pi_+. \tag{32}$$

Thus, there exists a non-empty intersection $D_\infty(\lambda) = \cap_{j\in\mathbb{N}} D_j(\lambda)$. From relation (31) it follows that $D_\infty(\lambda)$ consists of $w \in \mathbb{C}$ such that

$$\sum_{j=0}^{\infty} |\operatorname{Im} a_j - \operatorname{Im}\lambda||\Psi_j(\lambda, w)|^2 \le -\operatorname{Im} w. \tag{33}$$

The radius of $C_n(\lambda)$ is denoted by $r_n(\lambda)$, $\lambda \in \Pi_0$. We will need an analytic expression for $r_n(\lambda)$.

Proposition 2. *Let $\lambda \in \Pi_0$ and $n \in \mathbb{N}$. The radius of the circle $D_n(\lambda)$ is equal to*

$$r_n(\lambda) = \frac{1}{b_{n-1}|P_n(\lambda)\overline{P_{n-1}(\lambda)} - P_{n-1}(\lambda)\overline{P_n(\lambda)}|} = \frac{1}{2\sum_{j=0}^{n-1} |\operatorname{Im} a_j - \operatorname{Im}\lambda||P_j(\lambda)|^2}. \tag{34}$$

Proof. To obtain the first equality in (34), one repeats the standard arguments from the proof of Theorem 1.2.3 in [1]. The second equality follows from relation (14). □

Consider a sequence of functions,

$$\widehat{w}_n(\lambda) := w_n(\lambda, 0) = -\frac{Q_n(\lambda)}{P_n(\lambda)}, \quad \lambda \in \Pi_0(r_0, r_1), \quad n \in \mathbb{N}.$$

Notice that $\widehat{w}_n(\lambda) \in D_n(\lambda)$, $n \in \mathbb{N}$, $\lambda \in \Pi_0$. Hence, using relations (26), (31) we can write

$$\begin{aligned}|\widehat{w}_n(\lambda)|^2 = |\widehat{w}_n(\lambda)P_0(\lambda) + Q_0(\lambda)|^2 &\le \sum_{j=0}^{n-1} \frac{|\operatorname{Im} a_j - \operatorname{Im}\lambda|}{|\operatorname{Im} a_0 - \operatorname{Im}\lambda|}|\widehat{w}_n(\lambda)P_j(\lambda) + Q_j(\lambda)|^2 \\ &\le \frac{|\operatorname{Im}\widehat{w}_n(\lambda)|}{|\operatorname{Im} a_0 - \operatorname{Im}\lambda|} \le \frac{|\widehat{w}_n(\lambda)|}{|\operatorname{Im} a_0 - \operatorname{Im}\lambda|}.\end{aligned}$$

Consequently, we obtain

$$|\widehat{w}_n(\lambda)| \le \frac{1}{|\operatorname{Im} a_0 - \operatorname{Im}\lambda|}, \quad \lambda \in \Pi_0, \quad n \in \mathbb{N}. \tag{35}$$

Thus, in any compact subset of Π_0, the sequence of functions $\widehat{w}_n(\lambda)$ is uniformly bounded. The functions $\widehat{w}_n(\lambda)$ are analytic in Π_0 as it follows from Corollary 2. By virtue of Montel's theorem (see [7]) we can assert that there exists a subsequence $\widehat{w}_{n_k}(\lambda)$, $k \in \mathbb{N}$, which is uniformly convergent to a function $m(\lambda)$ in Π_0. The function $m(\lambda)$ is analytic by Weierstrass's theorem. Passing to the limit in (35) with $n = n_k$, $k \to \infty$, we obtain

$$|m(\lambda)| \le \frac{1}{|\operatorname{Im} a_0 - \operatorname{Im}\lambda|}, \quad \lambda \in \Pi_0. \tag{36}$$

Observe that $m(\lambda) \in D_n(\lambda)$ for any $n \in \mathbb{N}$, and therefore

$$m(\lambda) \in D_\infty(\lambda), \quad \lambda \in \Pi_0. \tag{37}$$

For an arbitrary $\varepsilon > 0$ we set

$$\Pi_{0,\varepsilon} = \Pi_{0,\varepsilon}(r_0, r_1) = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \le r_0 - \varepsilon\} \cup \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \ge r_1 + \varepsilon\}.$$

Proposition 3. *For the function $m(\lambda)$ the following relation holds true:*

$$m(\lambda) \to 0, \quad \lambda \to \infty, \quad \lambda \in \Pi_{0,\varepsilon}, \quad \varepsilon > 0. \tag{38}$$

Proof. Since $m(\lambda) \in D_n(\lambda)$, $\widehat{w}_n(\lambda) \in D_n(\lambda)$, $n \in \mathbb{N}$, $\lambda \in \Pi_0$, we can write

$$\left| m(\lambda) + \frac{Q_n(\lambda)}{P_n(\lambda)} \right| \leq 2r_n(\lambda), \quad \lambda \in \Pi_{0,\varepsilon}, \quad \varepsilon > 0, \quad n \in \mathbb{N}. \tag{39}$$

From relation (34) we see that

$$|r_n(\lambda)| \leq \frac{1}{2|\operatorname{Im} a_1 - \operatorname{Im}\lambda||P_1(\lambda)|^2} \leq \frac{b_0^2}{2\varepsilon|\lambda - a_0|^2}, \quad \lambda \in \Pi_{0,\varepsilon}, \quad n = 2, 3, \ldots$$

Thus, for any fixed n, $n = 2, 3, \ldots$, we obtain

$$r_n(\lambda) \to 0, \quad \lambda \to \infty, \quad \lambda \in \Pi_{0,\varepsilon}.$$

Passing to the limit in relation (39) we see that

$$m(\lambda) + \frac{Q_n(\lambda)}{P_n(\lambda)} \to 0, \quad \lambda \to \infty, \quad \lambda \in \Pi_{0,\varepsilon}.$$

It remains to notice that

$$\frac{Q_n(\lambda)}{P_n(\lambda)} \to 0, \quad \lambda \to \infty,$$

since $\deg Q_n = n - 1$, $\deg P_n = n$. □

The following theorem is valid.

Theorem 1. *Difference equation (4) has a solution* $y_n = m(\lambda)P_n(\lambda) + Q_n(\lambda)$, $n \in \mathbb{Z}_+$, *which belongs to* l^2 *for any* $\lambda \in \Pi_0$.

Proof. Since the function $m(\lambda)$, $\lambda \in \Pi_0$, belongs to the disc $D_\infty(\lambda)$, from relations (28), (33) it follows that

$$\sum_{j=0}^{\infty} |\operatorname{Im} a_j - \operatorname{Im}\lambda||m(\lambda)P_j(\lambda) + Q_j(\lambda)|^2 < \infty. \tag{40}$$

Since $|\operatorname{Im} a_j - \operatorname{Im}\lambda| \geq \operatorname{Im}\lambda - r_1 > 0$, $\lambda \in \Pi_+$, and $|\operatorname{Im} a_j - \operatorname{Im}\lambda| \geq r_0 - \operatorname{Im}\lambda > 0$, $\lambda \in \Pi_-$, the result follows. □

4. The spectral function

Let J be the semi-infinite matrix from (1), (2). Observe that it is a matrix that is complex symmetric (with respect to the transposition). Let $\{P_n(\lambda)\}_{n\in\mathbb{Z}_+}$, $\{Q_n(\lambda)\}_{n\in\mathbb{Z}_+}$ be the defined above solutions of the corresponding difference equations (3),(4). Recall (see [4, p. 474]) that a linear with respect to the both arguments functional $\sigma(u, v)$, $u, v \in \mathbb{P}$, is called a spectral function of difference equations (3),(4) if it satisfies relations

$$\sigma(P_n, P_m) = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+. \tag{41}$$

For the given difference equations (3), (4) it is not hard to obtain the spectral function using (41) as a definition and then extending this definition by the linearity. Namely, if $P(\lambda) = \sum_{j=0}^{\infty} \xi_j P_j(\lambda)$, $\xi_j \in \mathbb{C}$, and $R(\lambda) = \sum_{j=0}^{\infty} \nu_j P_j(\lambda)$, $\nu_j \in \mathbb{C}$, we set

$$\sigma(P, R) = \sum_{j=0}^{\infty} \xi_j \nu_j. \tag{42}$$

Here all sums are finite. However, representation (42) is not very convenient. It requires the knowledge of all coefficients of resolutions of the polynomials P, R via the polynomials $\{P_n(\lambda)\}_{n\in\mathbb{Z}_+}$. We are going to derive an analytic representation for the spectral function σ.

Note that according to Theorem 1 in [4] we have

$$\sigma(P,R) = \sigma(PR,1), \quad P,R \in \mathbb{P}. \tag{43}$$

That means that it is enough to obtain an analytic representation for $\sigma(u,1)$, $u \in \mathbb{P}$. If

$$u(\lambda) = \sum_{j=0}^{\infty} u_k P_k(\lambda), \quad u_k \in \mathbb{C}, \tag{44}$$

then by (41) we will get

$$\sigma(u,1) = u_0. \tag{45}$$

Let $\Psi_n(\lambda,w)$ and $m(\lambda)$ be defined as in the previous Section. We set

$$\Psi_n(\lambda) := m(\lambda)P_n(\lambda) + Q_n(\lambda), \quad \lambda \in \Pi_0, \tag{46}$$

and

$$\Psi_f(\lambda) := \sum_{j=0}^{\infty} \Psi_j(\lambda) f_j, \quad f = (f_0, f_1, f_2, \dots) \in l^2_{\text{fin}}, \quad \lambda \in \Pi_0. \tag{47}$$

Proposition 4. *Let $f = (f_0, f_1, f_2, \dots) \in l^2_{\text{fin}}$ and $\varepsilon > 0$. For the function $\Psi_f(\lambda)$ the following relation holds:*

$$\Psi_f(\lambda) = -\frac{1}{\lambda}(f_0 + \overline{o}(1)), \tag{48}$$

where $\overline{o}(1) \to 0$ as $\lambda \to \infty$ in a strip $\Pi_{0,\varepsilon}$.

Proof. Let f and ε be from the statement of the Proposition. Set $g = (g_0, g_1, g_2, \dots)$, where

$$g_0 = a_0 f_0 + b_0 f_1,$$

$$g_n = b_{n-1} f_{n-1} + a_n f_n + b_n f_{n+1}, \quad n \in \mathbb{N}.$$

Observe that $g \in l^2_{\text{fin}}$. We can write

$$\begin{aligned}
\Psi_g(\lambda) &= \sum_{j=0}^{\infty} \Psi_j(\lambda) g_j = \Psi_0(\lambda)(a_0 f_0 + b_0 f_1) + \sum_{j=1}^{\infty} \Psi_j(\lambda)(b_{j-1} f_{j-1} + a_j f_j + b_j f_{j+1}) \\
&= \Psi_0(\lambda)(a_0 f_0 + b_0 f_1) + \sum_{k=0}^{\infty} \Psi_{k+1}(\lambda) b_k f_k + \sum_{j=1}^{\infty} \Psi_j(\lambda) a_j f_j + \sum_{l=2}^{\infty} \Psi_{l-1}(\lambda) b_{l-1} f_l \\
&= \Psi_0(\lambda) a_0 f_0 + \Psi_1(\lambda) b_0 f_0 + \sum_{j=1}^{\infty} (b_{j-1}\Psi_{j-1}(\lambda) + a_j \Psi_j(\lambda) + b_j \Psi_{j+1}(\lambda)) f_j \\
&= \Psi_0(\lambda) a_0 f_0 + \Psi_1(\lambda) b_0 f_0 + \lambda \sum_{j=1}^{\infty} \Psi_j(\lambda) f_j, \quad \lambda \in \Pi_0,
\end{aligned}$$

where we have used the fact that $\Psi_j(\lambda)$ is a solution of difference equation (4).

Since $\Psi_0(\lambda) = m(\lambda)$, and $b_0 \Psi_1(\lambda) = \lambda m(\lambda) - a_0 m(\lambda) + 1$, we get

$$\Psi_g(\lambda) = \lambda m(\lambda) f_0 + f_0 + \lambda \sum_{j=1}^{\infty} \Psi_j(\lambda) f_j = f_0 + \lambda \sum_{j=0}^{\infty} \Psi_j(\lambda) f_j = f_0 + \lambda \Psi_f(\lambda).$$

Therefore

$$\Psi_f(\lambda) = \frac{1}{\lambda}(-f_0 + \Psi_g(\lambda)), \quad \lambda \in \Pi_0. \tag{49}$$

By virtue of the Cauchy-Buniakovskiy inequality we can write

$$|\Psi_g(\lambda)| \le \left(\sum_{j=0}^{\infty} |\Psi_j(\lambda)|^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} |g_j(\lambda)|^2\right)^{\frac{1}{2}}, \quad \lambda \in \Pi_0. \tag{50}$$

If $\lambda \in \Pi_{0,\varepsilon}$ then $|\operatorname{Im} a_j - \operatorname{Im}\lambda| > \varepsilon$. Since the function $m(\lambda)$, $\lambda \in \Pi_{0,\varepsilon}$, belongs to the disc $D_\infty(\lambda)$, by virtue of relations (28), (33) we can write

$$\varepsilon \sum_{j=0}^{\infty} |m(\lambda)P_j(\lambda) + Q_j(\lambda)|^2 \le \sum_{j=0}^{\infty} |\operatorname{Im} a_j - \operatorname{Im}\lambda||m(\lambda)P_j(\lambda) + Q_j(\lambda)|^2 \le |\operatorname{Im} m(\lambda)|. \tag{51}$$

Hence, we get

$$|\Psi_g(\lambda)| \le \frac{|m(\lambda)|}{\varepsilon} \left(\sum_{j=0}^{\infty} |g_j(\lambda)|^2\right)^{\frac{1}{2}}, \quad \lambda \in \Pi_{0,\varepsilon}. \tag{52}$$

Applying Proposition 3 we complete the proof. □

Theorem 2. *The spectral function σ of difference equations (3),(4) has the following representation:*

$$\begin{aligned} \sigma(P,R) = \frac{1}{2\pi i} \lim_{\delta\to 0} \Bigg\{ &\int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} P(\lambda)R(\lambda)e^{-\delta\lambda^2} m(\lambda)\,d\lambda \\ &+ \int_{\infty+i(r_0-\varepsilon)}^{-\infty+i(r_0-\varepsilon)} P(\lambda)R(\lambda)e^{-\delta\lambda^2} m(\lambda)\,d\lambda \Bigg\}, \quad P,R \in \mathbb{P}, \end{aligned} \tag{53}$$

where

$$\varepsilon > 0: \ \varepsilon > -r_1, \ \varepsilon > r_0. \tag{54}$$

Proof. We first note that the function $\Psi_f(\lambda)$ from (47) is analytic in Π_0. Choose an arbitrary $\varepsilon > 0$ which satisfies (54) and consider points $a_N^+ = -N + i(r_1+\varepsilon)$, $c^+ = i(r_1+\varepsilon)$, $b_N^+ = N + i(r_1+\varepsilon)$, and $a_N^- = -N + i(r_0-\varepsilon)$, $c^- = i(r_0-\varepsilon)$, $b_N^- = N + i(r_0-\varepsilon)$ in the complex λ-plane. We also denote

$$C_N^+ = \{\lambda \in \mathbb{C}: \ |\lambda - c^+| = N, \ \operatorname{Im}\lambda \ge r_1 + \varepsilon\},$$

$$C_N^- = \{\lambda \in \mathbb{C}: \ |\lambda - c^-| = N, \ \operatorname{Im}\lambda \le r_0 - \varepsilon\}.$$

Condition (54) ensures that the points a_N^+, c^+, b_N^+ and the half of the circle, C_N^+, lie in the open upper half-plane $\mathbb{C}'_+$. The points a_N^-, c^-, b_N^- and the half of the circle, C_N^-, lie in the open lower half-plane $\mathbb{C}'_-$. Using the analyticity we can write

$$\int_{a_N^+}^{b_N^+} \Psi_f(\lambda)\,d\lambda + \int_{C_N^+} \Psi_f(\lambda)\,d\lambda = 0, \tag{55}$$

$$\int_{b_N^-}^{a_N^-} \Psi_f(\lambda)\,d\lambda + \int_{C_N^-} \Psi_f(\lambda)\,d\lambda = 0. \tag{56}$$

By virtue of Proposition 4 we can write

$$\int_{C_N^+} \Psi_f(\lambda)\,d\lambda = -f_0 \int_{C_N^+} \frac{1}{\lambda}\,d\lambda - \int_{C_N^+} \frac{1}{\lambda}\overline{o}(1)\,d\lambda, \tag{57}$$

where $\overline{o}(1) = -\Psi_g(\lambda)$ (see (49)) is an analytic function in Π_0. Since $|\lambda| \ge N - |r_1+\varepsilon|$ in C_N^+, we get

$$\left|\frac{\overline{o}(1)}{\lambda}\right| \le \frac{|\overline{o}(1)|}{N - |r_1+\varepsilon|},$$

and the second term in the right-hand side of (57) tends to zero as $N \to \infty$. For the first term in the right-hand side of (57), we can write

$$-f_0(\ln a_N^+ - \ln b_N^+) = -f_0 i(\arg a_N^+ - \arg b_N^+) \to -\pi i f_0,$$

as $N \to \infty$. Here we have used an arbitrary analytic branch of the logarithm in $\mathbb{C}\backslash[0,+\infty)$. Calculating arguments we used that points a_N^+, b_N^+ lie in $\mathbb{C}'_+$.

Passing to the limit in (55) we get

$$\lim_{N\to\infty} \int_{a_N^+}^{b_N^+} \Psi_f(\lambda)\, d\lambda = \pi i f_0. \tag{58}$$

Proceeding in an analogous manner with relation (56) we obtain

$$\lim_{N\to\infty} \int_{b_N^-}^{a_N^-} \Psi_f(\lambda)\, d\lambda = \pi i f_0. \tag{59}$$

Summing up relations (58) and (59) we get

$$\lim_{N\to\infty} \left\{ \int_{a_N^+}^{b_N^+} \Psi_f(\lambda)\, d\lambda + \int_{b_N^-}^{a_N^-} \Psi_f(\lambda)\, d\lambda \right\} = 2\pi i f_0. \tag{60}$$

Let us show that

$$\lim_{N\to\infty} \int_{a_N^+}^{b_N^+} \Psi_f(\lambda)\, d\lambda = \lim_{\delta\to 0} \int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} e^{-\delta\lambda^2} \Psi_f(\lambda)\, d\lambda. \tag{61}$$

We first note that the integral in the right-hand side of (61) exists, since $\Psi_f(\lambda)$ is bounded (see (48)). For an arbitrary $\widehat{\varepsilon} > 0$ we can write

$$\begin{aligned} &\left| \int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} e^{-\delta\lambda^2} \Psi_f(\lambda)\, d\lambda - \lim_{N\to\infty} \int_{a_N^+}^{b_N^+} \Psi_f(\lambda)\, d\lambda \right| \\ &= \left| \lim_{N\to\infty} \int_{a_N^+}^{b_N^+} (e^{-\delta\lambda^2} - 1)\Psi_f(\lambda)\, d\lambda \right| \le \left| \int_{a_N^+}^{b_N^+} (e^{-\delta\lambda^2} - 1)\Psi_f(\lambda)\, d\lambda \right| + \frac{\widehat{\varepsilon}}{2}, \end{aligned} \tag{62}$$

for $N \ge N_0$, $N_0 \in \mathbb{N}$. On the finite segment $[a_{N_0}^+, b_{N_0}^+]$, the function $(e^{-\delta\lambda^2} - 1)\Psi_f(\lambda)$ uniformly tends to zero as $\delta \to 0$. Therefore,

$$\int_{a_{N_0}^+}^{b_{N_0}^+} (e^{-\delta\lambda^2} - 1)\Psi_f(\lambda)\, d\lambda \to 0, \quad \delta \to 0.$$

Hence, we can choose $\widehat{\delta} > 0$ such that $|\delta| < \widehat{\delta}_0$ implies

$$\left| \int_{a_{N_0}^+}^{b_{N_0}^+} (e^{-\delta\lambda^2} - 1)\Psi_f(\lambda)\, d\lambda \right| \le \frac{\widehat{\varepsilon}}{2}. \tag{63}$$

From relations (62), (63) it follows that (61) holds. In an analogous manner we obtain

$$\lim_{N\to\infty} \int_{b_N^-}^{a_N^-} \Psi_f(\lambda)\, d\lambda = \lim_{\delta\to 0} \int_{\infty+i(r_0-\varepsilon)}^{-\infty+i(r_0-\varepsilon)} e^{-\delta\lambda^2} \Psi_f(\lambda)\, d\lambda. \tag{64}$$

From (60), (61), (64) we obtain

$$\lim_{\delta\to 0} \left\{ \int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} e^{-\delta\lambda^2} \Psi_f(\lambda)\, d\lambda + \int_{\infty+i(r_0-\varepsilon)}^{-\infty+i(r_0-\varepsilon)} e^{-\delta\lambda^2} \Psi_f(\lambda)\, d\lambda \right\} = 2\pi i f_0. \tag{65}$$

Let $u(\lambda) \in \mathbb{P}$ be an arbitrary complex polynomial which has resolution (44). A vector of coefficients $u = (u_0, u_1, u_2, \dots)$ belongs to l^2_{fin}. For $\lambda \in \Pi_0$ we can write

$$\Psi_u(\lambda) = \sum_{j=0}^{\infty} \Psi_j(\lambda) u_j = \sum_{j=0}^{\infty} (m(\lambda) P_j(\lambda) + Q_j(\lambda)) u_j = m(\lambda) u(\lambda) + \sum_{j=0}^{\infty} Q_j(\lambda) u_j. \tag{66}$$

Let us show that

$$\lim_{\delta\to 0} \left\{ \int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} e^{-\delta\lambda^2} Q_j(\lambda)\, d\lambda + \int_{\infty+i(r_0-\varepsilon)}^{-\infty+i(r_0-\varepsilon)} e^{-\delta\lambda^2} Q_j(\lambda)\, d\lambda \right\} = 0, \quad j \in \mathbb{Z}_+. \tag{67}$$

Since the function $e^{-\delta\lambda^2} Q_j(\lambda)$ is analytic in $\mathbb{C}$, we have

$$\begin{aligned} &\int_{-N+i(r_1+\varepsilon)}^{N+i(r_1+\varepsilon)} e^{-\delta\lambda^2} Q_j(\lambda)\, d\lambda + \int_{N+i(r_0-\varepsilon)}^{-N+i(r_0-\varepsilon)} e^{-\delta\lambda^2} Q_j(\lambda)\, d\lambda \\ &\quad + \int_{N+i(r_1+\varepsilon)}^{N+i(r_0-\varepsilon)} e^{-\delta\lambda^2} Q_j(\lambda)\, d\lambda + \int_{-N+i(r_0-\varepsilon)}^{-N+i(r_1+\varepsilon)} e^{-\delta\lambda^2} Q_j(\lambda)\, d\lambda = 0. \end{aligned} \tag{68}$$

The last two terms in the left-hand side of (68) tend to zero as $N \to \infty$. In fact, the length of the path of integration is constant and the function under the integral tends to zero as $N \to \infty$, in the both cases. So, proceeding to the limit in (68) we obtain (67).

If we write relation (65) for the function $\Psi_u(\lambda)$ from (66) and use (67), we will get

$$\begin{aligned} &\lim_{\delta\to 0} \left\{ \int_{-\infty+i(r_1+\varepsilon)}^{\infty+i(r_1+\varepsilon)} e^{-\delta\lambda^2} m(\lambda) u(\lambda)\, d\lambda + \int_{\infty+i(r_0-\varepsilon)}^{-\infty+i(r_0-\varepsilon)} e^{-\delta\lambda^2} m(\lambda) u(\lambda)\, d\lambda \right\} \\ &\quad = 2\pi i u_0 = 2\pi i \sigma(u(\lambda), 1). \end{aligned} \tag{69}$$

If we take into account relation (43), we will obtain relation (53). The proof is complete. □

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