THE INTEGRATION OF DOUBLE-INFINITE TODA LATTICE BY MEANS OF INVERSE SPECTRAL PROBLEM AND RELATED QUESTIONS

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Dedicated to the fond memory of A. Ya. Povzner (1915–2008)

ABSTRACT. The solution of the Cauchy problem for differential-difference double-infinite Toda lattice by means of inverse spectral problem for semi-infinite block Jacobi matrix is given. Namely, we construct a simple linear system of three differential equations of first order whose solution gives the spectral matrix measure of the aforementioned Jacobi matrix. The solution of the Cauchy problem for the Toda lattice is given by the procedure of orthogonalization w.r.t. this spectral measure, i.e. by the solution of the inverse spectral problem for this Jacobi matrix.

1. INTRODUCTION

The classical method of integration of the Cauchy problem for the KdV equation on $(x, t) \in [0, \infty) \times [0, \infty)$ by means of the inverse spectral problem for Sturm-Liouville equation on $x \in [0, \infty)$ can be adapted to the semi-infinite Toda lattice

(1)
$$\dot{\alpha}_n(t) = \frac{1}{2}\alpha_n(t)(\beta_{n+1}(t) - \beta_n(t)),$$
$$\dot{\beta}_n(t) = \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n = 0, 1, \ldots =: \mathbb{N}_0, \quad t \in [0, \infty); \quad \alpha_{-1}(t) = 0$$

(results on the inverse problem can be found in the book [33]). Corresponding results were published in [2, 3, 5]. There instead of the Sturm-Liouville equation we used semi-infinite Jacobi matrices; the inverse spectral problem for them is classical and simple (see, e.g., [1]). Important results for the finite Toda lattice were obtained in [28, 37].

But the analogous Cauchy problem with given initial data

$$(\alpha_n(0), \beta_n(0))_{n=-\infty}^{\infty}$$

for the double-infinite Toda lattice when in (1) $n = \ldots, -1, 0, 1, \ldots =: \mathbb{Z}$ ($\alpha_{-1}(t)$ is arbitrary) is a difficult problem. In this case an application of the spectral theory for double-infinite Jacobi matrix encounters some serious difficulties. For example, if the duplication method [1] is used and the consideration is reduced to a difference equation (1) on the half-line for 2×2-matrix $\alpha_n(t), \beta_n(t)$ then one can not integrate the equation for the corresponding matrix spectral measure with respect to t [11]. A direct approach with application of Toda lattices on positive and negative half-axes gives a Riccati type equation for this measure [47].

There are quite a few papers (in both the scalar and the non-Abelian matrix case) in which one manages to overcome this difficulties in some special situations and for some special initial data by not restricting oneself to the spectral approach alone. For example, in the periodic case the equation was integrated in [32] in terms of theta functions (also

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see previous work [27]), in [34, 21, 22] by the inverse scattering problem method (for a difference problem), and for initial data tending to zero as $|n| \to \infty$, in [45, 46, 47] (the latter method is similar to the one used in [2, 3, 5]). New classes of solutions were found in [24]. See also [16, 17, 19, 18]. Along the indicated lines of investigation, there is also a number of papers of more physical nature. For some of them, see the collections of papers of the conference mentioned in [6]; [29, 30], etc. The original book of M. Toda [42] also is of a more physical nature. We note that a recently published book [41] contains a mathematically rigorous presentation of some questions related to the integration of the Cauchy problem for nonlinear differential-difference equations. New book [48] contains a number of recent results in this direction.

Let us dwell in some more detail on the integration method in [2, 3, 5] for the Cauchy problem in the case of the Toda lattice (1) on the half-line $n \in \mathbb{N}_0$ using the spectral theory for classical Jacobi matrices. The essence of the method is as follows. To the solution $u(t) = (\alpha_n(t), \beta_n(t))_{n=-\infty}^{\infty}, t \in [0, \infty)$, one assigns a selfadjoint Jacobi matrix $J(t)$ with $\beta_n(t)$ on the main diagonal and $\alpha_n(t)$ on the two neighboring diagonals. We assume that the solution $u(t)$ is bounded on every $[0, T) \ni t$, therefore $\forall t \in [0, \infty)$ $J(t)$ is a bounded selfadjoint operator on the space ℓ_2. Let $d\rho(\lambda; t)$ be its spectral measure. It is possible to prove that this measure changes in time t in a simple way: roughly speaking

$$(2) \qquad\qquad d\rho(\lambda; t) = e^{\lambda t} d\rho(\lambda; 0), \quad \lambda \in \mathbb{R}, \quad t \in [0, \infty).$$

Therefore, the spectral measure $d\rho(\lambda; 0)$ of the matrix $J(0)$ corresponding to the initial value $u(0)$ of the solution $u(t)$, makes it possible to use (2) to find the measure $d\rho(\lambda; t)$ for $t > 0$. Then we can use the classical inverse spectral problem for Jacobi matrices in terms of $d\rho(\lambda; t)$ to recover $J(t)$, i.e. to find the solution $u(t)$ of our Cauchy problem (note that the case of an unbounded solution $u(t)$ was considered in [40]).

As it was mentioned earlier, in the article [11] M. Gekhtman and I tried to apply such approach to the case of the Toda lattice (1) on $n \in \mathbb{Z}$. There we have double-infinite Jacobi matrices $J(t)$ for which the spectral measure $d\rho(\lambda; t)$ is a 2×2-matrix measure. It is possible to understand this measure as a spectral matrix of semi-infinite Jacobi matrix but with 2×2-matrix elements (duplication method [1]). This leads to some matrix differential equation in t for $d\rho(\lambda; t)$ which is impossible to solve due to noncommutativity of its coefficients (in the case of semi-axis $n \in \mathbb{N}_0$ such an equation is very simple and its solution is (2)).

In this article we propose some way of overcoming the last difficulty. Namely, in the standard duplication method we construct by sequence $(\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots)$ of complex numbers ξ_n the semi-infinite sequence $(x_0, x_1, \ldots)$ of vectors $x_n = (\xi_n, \xi_{-n-1}) \in \mathbb{C}^2$. As a result, our double-infinite Jacobi matrix $J(t)$ transforms into a semi-infinite Jacobi matrix whose elements are 2×2-matrices. Such matrix acts on the space

$$\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots;$$

corresponding spectral theory is known [31, 1]. But its application to our situation gives the noncommutativity of coefficients of the differential equation for $d\rho(\lambda; t)$ and impossibility to solve it.

In this paper we go in a different direction: we use the duplication of the form:

$$(3) \ (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots) \mapsto x_0 = \xi_0 \in \mathbb{C}^1, \ x_1 = (\xi_1, \xi_{-1}) \in \mathbb{C}^2, \ x_2 = (\xi_2, \xi_{-2}) \in \mathbb{C}^2, \ldots.$$

Corresponding Jacobi matrix $J(t)$ acts on the space

$$\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \ldots$$

and its spectral theory is not standard. The main difficulty is that the first matrix elements on side diagonals are not invertible since these elements act from $\mathbb{C}^1$ into $\mathbb{C}^2$ or vice versa.

Such a situation presents, at first, essential difficulties for the finding, with the help of recurrence relations, the corresponding orthogonal polynomials. This can be overcome by means of a certain point of view on the construction of spectral matrix which was used in [7, 26]. Note also, that the spectral theory of unitary and normal operator Jacobi matrices with elements acting between different Hilbert spaces was developed in [8, 9]. In [7] another point of view was suggested: there, we considered the matrix operator with some "boundary conditions" at the point $n = 0$.

It is necessary to say that transfer (3) really means that we choose the spectral matrix $d\rho(\lambda; t)$ of our difference operator on $\mathbb{Z}$ in a nonstandard manner. But this choice allows one to write a simple differential equation for $d\rho(\lambda; t)$ w.r.t. $t \in [0, \infty)$ (recall, that a choice of a spectral matrix in the case of a difference equation on $\mathbb{Z}$ is not unique: it depends on two chosen fundamental solutions of our difference equations).

An approach sketched above gives a possibility to find a system of three first order linear differential equations w.r.t. $t \in [0, \infty)$ for the functions $\rho_{\alpha,\beta}(\lambda; t)$, $\alpha, \beta = 0, 1$ ($\rho_{0,1}(\lambda; t) = \rho_{1,0}(\lambda; t)$).

The coefficients of these equations depend in a simple manner on

(4) $$\beta_0(t), \quad t \in [0, \infty), \quad \alpha_0(0).$$

In general, it is impossible to give exact formulas for the solution of this simple linear system (if the functions (4) are constant, it is easy to do). But we get a certain "linearization" of our Cauchy problem for (1), $n \in \mathbb{Z}$, because the construction of its required solution $u(t) = (\alpha_n(t), \beta_n(t))_{n=-\infty}^{\infty}$, i.e. the matrix $J(t)$, is an relatively standard procedure: it is achieved via a certain orthogonalization procedure.

Of course, we can get exact formulas for the solution of the Cauchy problem for (1), $n \in \mathbb{Z}$, only in some special cases, but in the general case the method of obtaining the solution is sufficiently simple and "computable".

It is worth to stress that our Cauchy problem has an essential peculiarity: to find its solution it is necessary to give the standard initial data $(\alpha_n(0), \beta_n(0))_{n=-\infty}^{\infty}$ and, in addition, the function (4). This situation reminds the so-called shock problem for Toda lattice ([38, 43, 44], see also [4, 5]); it will be investigated in another paper. Note also, that in the case of semi-infinite Toda lattice the data (4) is also implicitly present: $\alpha_0(t)$ as a coefficient at $\dot{\alpha}_n(t), \dot{\beta}_n(t)$ and $\beta_0(t)$ via the procedure of normalization of the spectral measure (see [5], pp. 27–28, 37) is connected with the constant of normalization of the spectral measure to a probability measure.

The above mentioned results concerning the double-infinite Toda lattices are contained in Sections 6–9 of the paper. Sections 2, 3 contain preliminary constructions devoted to the spectral theory of difference expressions on $\mathbb{N}_0$ with operator coefficients [1, 11] and Toda lattices [42, 11]. These constructions help to understand the results in the next Sections.

Sections 4, 5 contain a result about integration of matrix Toda lattice on the half-axis $n \in \mathbb{N}_0$, the coefficients of which are commutative matrices. These results are connected with Section 6–9 and are useful. Note that the articles [11, 23, 20] are linked with Sections 4 and 5.

Above we have used only the spectral theory of selfadjoint Jacobi matrices, with scalar or matrix elements. But in [12] an analogous theory was developed for a solution of the Cauchy problem for some types of matrix nonlinear equations connected with a normal or unitary Jacobi matrix with operator elements; article [25] gave an impetus to such an investigation.

Note also that the described above way of integration of some nonlinear differential-difference equations (with scalar or matrix coefficients) can be transferred to so-called non-isospectral equations for which the spectrum of the corresponding matrix $J(t)$, i.e.,

the support $\forall t$ of $d\rho(\lambda;t)$, changes in t (from (2) we see that the spectrum of $J(t)$ for (1) is fixed, an analogous picture takes place in other cases mentioned above: our problems are isospectral). For the case of nonisospectral equations see articles [10, 39, 14, 15, 6, 35, 36].

2. Preliminaries: some facts about the spectral theory of difference expressions with operator coefficients

Let H be a Hilbert space with a scalar product (x,y) and a norm $\|x\|$; $x,y \in H$. Denote by $L(H)$ the set of all bounded operators in H. Define a difference expression J whose coefficients belong to $L(H)$ and are collectively bounded in the norm and which act on sequences $f = (f_n)_{n=0}^{\infty}$ of vectors $f_n \in H$ as follows :

$$
(5) \quad \begin{aligned}
(Jf)_n &= a_{n-1}f_{n-1} + b_n f_n + a_n f_{n+1}, \quad a_n, b_n \in L(H), \quad a_n > 0, \quad b_n^* = b_n; \\
&n \in \mathbb{N}_0 = \{0, 1, \ldots\}; \quad a_{-1} \in L(H) \text{ is fixed.}
\end{aligned}
$$

When calculating $(Jf)_0$ we always assume that $f_{-1} = 0$; this assumption serves as a boundary condition, therefore the value of a_{-1} is not essential in Section 2.

The expression (5) induces a bounded selfadjoint operator $\boldsymbol{J}$ on the space $\mathbf{l}_2 := \ell_2(H) = H \oplus H \oplus \ldots$. Namely, for finite sequences $f \in \mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$ we put $f \mapsto Jf \in \mathbf{l}_2$; we denote by $\boldsymbol{J}$ the closure of this map (or sometimes shortly by J). This operator is generated by the Jacobi Hermitian matrix with operator elements:

$$
(6) \quad J = \begin{bmatrix}
b_0 & c_0 & 0 & 0 & 0 & \ldots \\
a_0 & b_1 & c_1 & 0 & 0 & \ldots \\
0 & a_1 & b_2 & c_2 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad a_n, b_n, c_n \in L(H), \ c_n = a_n > 0, \ b_n = b_n^*, \ n \in \mathbb{N}_0.
$$

The uniform boundedness of elements a_n, b_n ensures that $\boldsymbol{J}$ is a bounded selfadjoint operator on $\mathbf{l}_2$.

Let $\lambda \in \mathbb{R}$ be fixed, consider the sequence of operators $(P_n(\lambda))_{n=0}^{\infty} =: P(\lambda)$ where $P_n(\lambda)$ are solutions of the operator equation

$$
(7) \quad \begin{aligned}
(JP(\lambda))_n &:= a_{n-1}P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda P_n(\lambda), \quad n \in \mathbb{N}_0, \\
P_{-1}(\lambda) &= 0, \quad P_0(\lambda) = 1
\end{aligned}
$$

(1 is the identity operator on H). The equation (7) is solvable: one can find $P_1(\lambda)$, $P_2(\lambda), \ldots$ recursively; it is possible because $\forall n \in \mathbb{N}_0 \ a_n^{-1}$ exists. In particular

$$
(8) \qquad\qquad P_1(\lambda) = a_0^{-1}(\lambda 1 - b_0), \quad \lambda \in \mathbb{R}.
$$

It is easy to see that every $P_n(\lambda)$ is an operator-valued polynomial w.r.t. $\lambda \in \mathbb{R}$ (i.e. a polynomial with coefficients from $L(H)$) of degree $n \in \mathbb{N}_0$; its top coefficient is an invertible operator on H.

It is possible to prove that every generalized eigenvector of the operator $\boldsymbol{J}$ has the form $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}$, $\varphi_n(\lambda) = P_n(\lambda)\varphi_0$ where $\varphi_0 \in H$ (initial value of $\varphi(\lambda)$) is arbitrary; here λ belongs to the spectrum of $\boldsymbol{J}$. In our case, the spectral measure $d\rho(\lambda)$ of the operator $\boldsymbol{J}$ is a nonnegative operator-valued Borel measure $\mathcal{B}(\mathbb{R}) \ni \triangle \mapsto \rho(\triangle) \in L(H)$, $\rho(\triangle) \geq 0$, which is probabilistic, $\rho(\mathbb{R}) = 1$.

For an account of necessary facts of spectral theory of the operator generated by (5), (6), it is convenient to introduce the notion of a corresponding pseudo-Hilbert space. We will do this not in a general situation but only in two cases which are necessary for us.

1). Consider the set $\mathbf{l}_2(L(H))$ of all sequences $U = (U_n)_{n=0}^{\infty}$ of operators $U_n \in L(H)$ for which the series $\sum_{n=0}^{\infty} U_n^* U_n$ is weakly convergent on H. It is easy to check that such a set is a module w.r.t. the multiplication on the right by operators $\Lambda \in L(H)$ ("pseudo-scalars"). In $\mathbf{l}_2(L(H))$ it is possible to introduce the corresponding pseudo-scalar product

$\{U, V\} = \sum\limits_{n=0}^{\infty} U_n^* V_n \in L(H)$ (this series is also weakly convergent). As a result we have the following definition and properties of a pseudo-Hilbert space $\mathbf{l}_2(L(H))$:

$$\mathbf{l}_2(L(H)) = \left\{ U = (U_n)_{n=0}^{\infty}, U_n \in L(H) \big| \sum_{n=0}^{\infty} U_n^* U_n < \infty, \text{ i.e. weakly converges} \right\},$$

(9)
$$\{U, V\} = \sum_{n=0}^{\infty} U_n^* V_n \in L(H);$$

$\forall U, V \in \mathbf{l}_2(L(H)), \quad \forall \Lambda \in L(H) \quad U + V := (U_n + V_n)_{n=0}^{\infty}, \quad U\Lambda := (U_n\Lambda)_{n=0}^{\infty}.$

Then: $(U + V)\Lambda = U\Lambda + V\Lambda, \quad \{U, V\}^* = \{V, U\}, \quad \{U, V\Lambda\} = \{U, V\}\Lambda,$

$\{U\Lambda, V\} = \Lambda^* \{U, V\}, \quad \{U + V, W\} = \{U, W\} + \{V, W\}.$

Denote by δ_n the sequence $(\underbrace{0, \ldots, 0}_{n}, 1, 0, 0, \ldots) =: \delta_n \in \mathbf{l}_2(L(H))$ and consider a

matrix $A = (a_{jk})_{j,k=0}^{\infty}$ of type (6) with arbitrary $a_{jk} \in L(H)$ (the matrix can be full). The pseudo-vectors δ_n play a role of an orthonormal basis in $\mathbf{l}_2(L(H))$. Therefore if we introduce a "pseudo-linear" operator $\mathbf{A}$ on $\mathbf{l}_2(L(H))$ in a natural way: $\forall j \in \mathbb{N}_0$

(10)
$$(\mathbf{A}U)_j = \sum_{k=0}^{\infty} a_{jk} U_k, U = (U_n)_{n=0}^{\infty} \in \mathbf{l}_{\text{fin}}(L(H));$$

then it is easy to calculate the following formula for a_{jk} :

(11)
$$a_{jk} = \{\delta_j, \mathbf{A}\delta_k\}, \quad j, k \in \mathbb{N}_0.$$

Conversely, every pseudo-linear operator $\mathbf{A} : \mathbf{l}_2(L(H)) \to \mathbf{l}_2(L(H))$ which is continuous in some natural topology in $\mathbf{l}_2(L(H))$ has a matrix representation (10), (11).

Let $U = (U_n)_{n=0}^{\infty} \in \mathbf{l}_2(L(H))$ and $x \in H$, then $Ux := (U_n x)_{n=0}^{\infty}$ is a vector from the space $\mathbf{l}_2$ and $\forall x, y \in H$

(12)
$$(\{U, V\}x, y) = (Vx, Uy)_{\mathbf{l}_2}, \quad U, V \in \mathbf{l}_2(L(H)).$$

Every linear continuous operator $\boldsymbol{A} : \mathbf{l}_2 \to \mathbf{l}_2$ gives rise to some pseudo-linear operator $\mathbf{A} : \mathbf{l}_2(L(H)) \to \mathbf{l}_2(L(H))$. Namely, $\forall U \in \mathbf{l}_2(L(H)), \forall x \in H$ we put: $(\mathbf{A}U)x = \boldsymbol{A}(Ux)$ (it is clear that $\boldsymbol{A}$ cannot be always reconstructed by $\mathbf{A}$). The matrix representation (10), (11) for $\mathbf{A}$ gives the corresponding matrix representation $(a_{jk})_{j,k=0}^{\infty}, a_{jk} \in L(H)$, for $\boldsymbol{A}$: $\forall j \in \mathbb{N}_0$

(13)
$$(\boldsymbol{A}f)_j = \sum_{k=0}^{\infty} a_{jk} f_k, \quad f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_{\text{fin}};$$
$$a_{jk} = \{\delta_j, \mathbf{A}\delta_k\}, \quad (\mathbf{A}\delta_k)x = \boldsymbol{A}(\delta_k x), \quad \delta_k x = (\underbrace{0, \ldots, 0}_{k}, x, 0, 0, \ldots), \quad x \in H.$$

The multiplication of operators on $\mathbf{l}_2$ and their matrices are connected in a usual way.

2). The second example of a pseudo-Hilbert space. Consider some nonnegative operator-valued Borel probability measure $d\rho(\lambda)$ on $\mathbb{R}$ introduced above. Let $\mathbb{R} \ni \lambda \mapsto f(\lambda) \in H$ be a continuous bounded vector-function, $g(\lambda)$ is analogical one. In an ordinary way (starting from a linear combination of vector-valued characteristic functions) it is possible to introduce the integral

(14)
$$\int_{\mathbb{R}} (d\rho(\lambda)f(\lambda), g(\lambda)) =: (f, g)_{\mathbf{L}^2}.$$

This integral defines a (quasi) scalar product f and g. Let $\mathbf{L}^2 = L^2(H, \mathbb{R}, d\rho(\lambda))$ be a completion of the set of such functions f. This Hilbert space is similar in some sense to the space $\mathbf{l}_2$.

Now we can introduce a corresponding pseudo-Hilbert space,

$$\mathbf{L}^2(L(H)) = L^2(L(H), \mathbb{R}, d\rho(\lambda)).$$

For an operator-valued continuous bounded function $\mathbb{R} \ni \lambda \mapsto U(\lambda) \in L(H)$ and a similar function $V(\lambda)$ define a pseudo-scalar product by the formula

$$(15) \qquad \{U, V\} = \int_{\mathbb{R}} U^*(\lambda) d\rho(\lambda) V(\lambda)$$

(here the integral is introduced similarly to (14), starting from operator-valued characteristic functions). Such set of functions $U(\lambda)$ and pseudo-scalars $\Lambda \in L(H)$ form a module w.r.t. ordinary sum of functions and multiplication on Λ on the right. Expression (15) determines a pseudo-scalar product on this module. As a result, we get some pseudo-Hilbert space $\mathbf{L}^2(L(H))$ with properties analogous to (9), (12).

Let's return to the spectral theory of the operator $\boldsymbol{J}$ generated by Jacobi matrix (6) on the space $\mathbf{l}_2$. As was said above, every generalized eigenvector $\varphi(\lambda)$ of this operator has the form

$$(16) \qquad \varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^\infty, \quad \varphi_n(\lambda) = P_n(\lambda)\varphi_0, \quad \varphi_0 \in H, \quad \lambda \in \text{ spectrum } \boldsymbol{J},$$

where $P_n(\lambda)$ are operator-valued polynomials which are solutions of equation (7).

The corresponding Fourier transform $\widehat{}$ is

$$(17) \qquad \mathbf{l}_2 \ni f = (f_n)_{n=0}^\infty \mapsto \widehat{f}(\lambda) = \sum_{n=0}^\infty P_n^*(\lambda) f_n \in \mathbf{L}^2$$

($\mathbf{L}^2 = L^2(H, \mathbb{R}, d\rho(\lambda))$) is constructed from the spectral measure $d\rho(\lambda)$ of $\boldsymbol{J}$). The unitary mapping (17) between the spaces $\mathbf{l}_2$ and $\mathbf{L}^2$ at first is defined on $f \in \mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$ and then is extended by continuity to the whole $\mathbf{l}_2$.

The Fourier transform $\widehat{}$ (17) can be rewritten in another convenient form:

$$(18) \qquad \mathbf{l}_{\text{fin}}(L(H)) \ni U = (U_n)_{n=0}^\infty \mapsto \widehat{U}(\lambda) = \sum_{n=0}^\infty P_n^*(\lambda) U(\lambda) \in \mathbf{L}^2(L(H));$$

$$\forall x \in H \quad (Ux)\widehat{}(\lambda) = (\widehat{U}(\lambda))x.$$

As it was mentioned, the mapping (17), (18) is unitary between $\mathbf{l}_2$ and $\mathbf{L}^2$. The corresponding Parseval equality has the form (see (9), (15))

$$(19) \qquad \forall f, g \in \mathbf{l}_2 \quad (f, g)_{\mathbf{l}_2} = (\widehat{f}, \widehat{g})_{\mathbf{L}^2} = \int_{\mathbb{R}} (\widehat{f}(\lambda) d\rho(\lambda), \widehat{g}(\lambda)) \quad \text{or}$$

$$\forall U, V \in \mathbf{l}_2(L(H)) \quad \{U, V\} = \{\widehat{U}, \widehat{V}\} = \int_{\mathbb{R}} (\widehat{U}(\lambda))^* d\rho(\lambda) \widehat{V}(\lambda).$$

From this Parseval equality and (17), (18) it follows that the polynomials $P_n(\lambda)$ are orthonormal in the following sense:

$$(20) \qquad \forall x, y \in H \quad \int_{\mathbb{R}} (P_j^*(\lambda) x d\rho(\lambda), P_k^*(\lambda) y) = \delta_{j,k}(x, y), \quad j, k \in \mathbb{N}_0,$$

$$\text{or} \quad \int_{\mathbb{R}} P_j(\lambda) d\rho(\lambda) P_k^*(\lambda) = \delta_{j,k} 1, \quad j, k \in \mathbb{N}_0.$$

Using the second equality in (20) and (7) we conclude that the following representations take place:

$$(21) \qquad a_n = \int_{\mathbb{R}} \lambda P_n(\lambda) d\rho(\lambda) P_{n+1}^*(\lambda), \quad b_n = \int_{\mathbb{R}} \lambda P_n(\lambda) d\rho(\lambda) P_n^*(\lambda), \quad n \in \mathbb{N}_0.$$

The second equality in (20) is more convenient. It gives a possibility to introduce a generalization of the classical Schmidt procedure of orthogonalization of a sequence of vectors in a Hilbert space to the case of pseudo-Hilbert space. We will introduce such a procedure in the essential for us case of operator-valued polynomials $P(\lambda)$ on $\lambda \in \mathbb{R}$.

So, we assume that our measure $d\rho(\lambda)$ with higher invertible coefficient has the property that the following integral exists and is a positive invertible operator in H:

$$(22) \qquad \{P, P\} = \int_{\mathbb{R}} P^*(\lambda) d\rho(\lambda) P(\lambda).$$

Consider the following sequence of polynomials from $\mathbf{L}^2(L(H)) = L^2(L(H), \mathbb{R}, d\rho(\lambda))$:

$$(23) \qquad 1, \lambda 1, \lambda^2 1, \ldots$$

Put $P_0(\lambda) = 1$ and consider the polynomial of the form $P_1'(\lambda) = \lambda 1 - P_0(\lambda) K_1$ where $K_1 \in L(H)$. Pick out K_1 in such a manner that $\{P_0(\lambda), P_1'(\lambda)\} = 0$, for this it is necessary to put (see (9)) $K_1 = \{P_0(\lambda), \lambda 1\}$. Using our assumption we can assert that $\{P_1'(\lambda), P_1'(\lambda)\}$ is a positive invertible operator. Therefore the operator $P_1(\lambda) = P_1'(\lambda)\{P_1'(\lambda), P_1'(\lambda)\}^{-1/2}$ is such that

$$(24) \qquad \{P_0(\lambda), P_1(\lambda)\} = 0, \quad \{P_1(\lambda), P_1(\lambda)\} = 1.$$

Assume that we construct the polynomials $P_0(\lambda), \ldots, P_n(\lambda)$ $(\forall j = 0, \ldots, n \ P_j(\lambda)$ as a linear combination with operator coefficients of $1, \ldots, \lambda^j 1)$ with the properties (24):

$$(25) \qquad \{P_j(\lambda), P_k(\lambda)\} = \delta_{j,k} 1, \quad j, k = 0, \ldots, n.$$

Construct $P_{n+1}(\lambda)$. For this we put

$$P_{n+1}'(\lambda) = \lambda^{n+1} 1 - P_0(\lambda) K_1 - \cdots - P_n(\lambda) K_n$$

with some coefficients $K_1, \ldots, K_n \in L(H)$ of the form $K_j = \{P_j(\lambda), \lambda^{n+1} 1\}$. Using (9) we get $\forall j = 0, \ldots, n$

$$\{P_j(\lambda), P_{n+1}'(\lambda)\} = \{P_j(\lambda), \lambda^{n+1} 1\} - \{P_j(\lambda), P_0(\lambda) K_1\} - \cdots - \{P_j(\lambda), P_n(\lambda) K_n\}$$
$$= \{P_j(\lambda), \lambda^{n+1} 1\} - \{P_j(\lambda), P_j(\lambda)\} K_j = 0.$$

Now it is necessary only to normalize $P_{n+1}'(\lambda)$: we put $P_{n+1}(\lambda) = P_{n+1}'(\lambda)\{P_{n+1}'(\lambda), P_{n+1}'(\lambda)\}^{-1/2}$.

So, we construct the system $P_0(\lambda), \ldots, P_{n+1}(\lambda)$ with the property (25) for $j, k = 0, \ldots, n+1$. The procedure of orthogonalization of sequence (24) is complete.

Such a construction allows us to solve the inverse spectral problem for Jacobi matrix (6): *assume we know some nonnegative operator-valued Borel probability measure $d\rho(\lambda)$ for which all operator-valued polynomials are integrable and condition of invertibility of operator (22) is fulfilled. Then such a measure is a spectral measure of the operator $\boldsymbol{J}$ generated by the matrix (6) and elements of this matrix are reconstructed by formulas (21) where $P_n(\lambda)$ are obtained via the procedure of orthogonalization of sequence (23). If we start from a spectral measure $d\rho(\lambda)$ of the operator $\boldsymbol{J}$ generated by (6) we get as a result the initial matrix (6).*

It is possible to give some conditions on the measure $d\rho(\lambda)$ which guarantees invertibility of the operator (22). We consider here only the simplest case when H is

two-dimensional: $H = \mathbb{C}^2$ (such a case will be useful in what follows). An analogous situation takes place for an arbitrary finite-dimensional H, the proof is similar.

Lemma 1. *Let $H = \mathbb{C}^2$ and a nonnegative operator-valued Borel probability measure $d\rho(\lambda)$ be such that all the integrals*

$$(26) \qquad \int_{\mathbb{R}} \lambda^n 1 d\rho(\lambda) \lambda^n 1, \quad n \in \mathbb{N}_0,$$

exist. If the support $d\rho(\lambda)$ has an infinite set of points, then every operator (22) *is invertible.*

Proof. Since H is finite-dimensional the invertibility of the operator $\{P, P\}$ (23) is equivalent to the property: if for $x \in \mathbb{C}^2$ $\{P, P\}x = 0$, then $x = 0$, i.e., $\left\| \{P,P\}^{1/2} x \right\|^2 = 0 \Rightarrow x = 0$. Here $P(\lambda)$ is an operator-valued polynomial of degree $n \in \mathbb{N}_0$.

For every Borel set $\triangle \subset \mathbb{R}$, the matrix $\rho(\triangle) = (\rho_{\alpha,\beta})^1_{\alpha,\beta=0}$ is nonnegative. Therefore $|\rho_{0,1}(\triangle)|^2 = |\rho_{1,0}(\triangle)|^2 \le \rho_{0,0}(\triangle)\rho_{1,1}(\triangle)$ and the complex-valued measures $\rho_{\alpha,\beta}(\lambda), \alpha, \beta = 0, 1$, are absolutely continuous w.r.t. the nonnegative measure $\triangle \mapsto \sigma(\lambda) := \rho_{0,0}(\triangle) + \rho_{1,1}(\lambda)$. Using the Radon-Nikodym theorem we can write that $d\rho(\lambda) = C(\lambda)d\sigma(\lambda), \lambda \in \mathbb{R}$, where $C(\lambda) = (C_{\alpha,\beta}(\lambda))^1_{\alpha,\beta=0}$ is a nonnegative matrix with summable functions $C_{\alpha,\beta}(\lambda)$. This matrix is positive on the set of full measure $d\sigma(\lambda)$.

Such a representation and formula (22) gives that $\forall x \in H$

$$(27)$$
$$\left\| \{P, P\}^{1/2} x \right\|^2 = (\{P, P\}x, x) = \left(\left(\int_{\mathbb{R}} P^*(\lambda)d\rho(\lambda)P(\lambda) \right)x, x \right)$$
$$= \left(\left(\int_{\mathbb{R}} P^*(\lambda)C(\lambda)P(\lambda)d\sigma(\lambda) \right)x, x \right) = \int_{\mathbb{R}} \left\| C^{1/2}(\lambda)P(\lambda)x \right\| d\sigma(\lambda).$$

If the left-hand side in (27) is equal to 0, then $\left\| C^{1/2}(\lambda)P(\lambda)x \right\| = 0$ for σ-almost all $\lambda \in \mathbb{R}$. The possibility of the matrix $C(\lambda)$ gives the same property also for $\left\| P(\lambda)x \right\|^2$. But the last expression is an ordinary polynomial of degree $2n$ with highest coefficient $\|A_n x\|^2$, where A_n is the invertible highest coefficient of $P(\lambda)$. The equality $\|P(\lambda)x\|^2 = 0$ means that this ordinary polynomial is equal to zero. Therefore $x = 0$. $\square$

3. Preliminaries: the double-infinite Toda lattice and its known reduction to semi-infinite 2×2-matrix Toda lattice

The double-infinite Toda lattice has the form

$$(28) \qquad \dot{\alpha}_n = \frac{1}{2}\alpha_n(\beta_{n+1} - \beta_n), \quad \dot{\beta}_n = \alpha_n^2 - \alpha_{n-1}^2, \quad n \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\},$$

where $\alpha_n = \alpha_n(t), \beta_n = \beta_n(t)$ are real continuously differentiable functions of $t \in [0, T), T \le \infty; \cdot = \frac{d}{dt}$. Expression (28) is a differential-difference nonlinear equation and for (28) it is possible to consider the following Cauchy problem: given the initial data $\alpha_n(0), \beta_n(0)$, $n \in \mathbb{Z}$, find a solution $\alpha_n(t), \beta_n(t), n \in \mathbb{Z}$, for $t \in (0, T)$.

The equation (28) can be rewritten in the form of a semi-infinite equation, but for the 2×2-matrices $\mathfrak{a}_n(t), \mathfrak{b}_n(t), n \in \mathbb{N}_0$, introduced by $\alpha_n(t), \beta_n(t), n \in \mathbb{Z}; t \in [0, T)$ (see, e.g. [11, 1]). We mention here a simpler procedure. Introduce the diagonal matrices:

$\forall t \in [0, T)$

$$
(29) \quad
\begin{aligned}
\mathfrak{a}_n(t) &= \begin{bmatrix} \alpha_n(t) & 0 \\ 0 & \alpha_{-n-1}(t) \end{bmatrix}, \quad \mathfrak{b}_n(t) = \begin{bmatrix} \beta_n(t) & 0 \\ 0 & -\beta_{-n}(t) \end{bmatrix}, \quad n \in \mathbb{N}_0; \\
\mathfrak{a}_{-1}(t) &= \begin{bmatrix} \alpha_{-1}(t) & 0 \\ 0 & \alpha_0(t) \end{bmatrix}.
\end{aligned}
$$

Construct for these 2×2-matrices $\mathfrak{a}_n(t) = \mathfrak{a}_n, \mathfrak{b}_n(t) = \mathfrak{b}_n, n \in \mathbb{N}_0, \mathfrak{a}_{-1}(t) = \mathfrak{a}_{-1}; t \in [0, T)$, the differential-difference equation similar to (28):

$$
(30) \quad \dot{\mathfrak{a}}_n = \frac{1}{2}\mathfrak{a}_n(\mathfrak{b}_{n+1} - \mathfrak{b}_n), \quad \dot{\mathfrak{b}}_n = \mathfrak{a}_n^2 - \mathfrak{a}_{n-1}^2, \quad n \in \mathbb{N}_0; \quad t \in [0, T).
$$

Lemma 2. *The equation* (28) *can be written in an equivalent form* (30) *where the matrices* $\mathfrak{a}_n, \mathfrak{b}_n$ *are of the form* (29).

Proof. The equalities (30) with matrices (29) have the form:

$$
(31) \quad
\begin{aligned}
\begin{bmatrix} \dot{\alpha}_n & 0 \\ 0 & \dot{\alpha}_{-n-1} \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} \alpha_n & 0 \\ 0 & \alpha_{-n-1} \end{bmatrix}\left(\begin{bmatrix} \beta_{n+1} & 0 \\ 0 & -\beta_{-n-1} \end{bmatrix} - \begin{bmatrix} \beta_n & 0 \\ 0 & -\beta_{-n} \end{bmatrix}\right) \\
&= \begin{bmatrix} \frac{1}{2}\alpha_n(\beta_{n+1} - \beta_n) & 0 \\ 0 & \frac{1}{2}\alpha_{-n-1}(\beta_{-n} - \beta_{-n-1}) \end{bmatrix}, \quad n \in \mathbb{N}_0; \\
\begin{bmatrix} \dot{\beta}_n & 0 \\ 0 & -\dot{\beta}_{-n} \end{bmatrix} &= \begin{bmatrix} \alpha_n & 0 \\ 0 & \alpha_{-n-1} \end{bmatrix}^2 - \begin{bmatrix} \alpha_{n-1} & 0 \\ 0 & \alpha_{-n} \end{bmatrix}^2 \\
&= \begin{bmatrix} \alpha_n^2 - \alpha_{n-1}^2 & 0 \\ 0 & \alpha_{-n-1}^2 - \alpha_{-n}^2 \end{bmatrix}, \quad n \in \mathbb{N}.
\end{aligned}
$$

Comparing the elements in equalities (31) and corresponding for $\dot{\mathfrak{b}}_0 = \mathfrak{a}_0 - \mathfrak{a}_{-1}$ we get:

$$
(32) \quad
\begin{aligned}
\dot{\alpha}_n &= \frac{1}{2}\alpha_n(\beta_{n+1} - \beta_n), \quad \dot{\beta}_n = \alpha_n^2 - \alpha_{n-1}^2, \quad n \in \mathbb{N}_0; \\
\dot{\alpha}_{-n-1} &= \frac{1}{2}\alpha_{-n-1}(\beta_{-n} - \beta_{-n-1}), \quad -\dot{\beta}_{-n} = \alpha_{-n-1}^2 - \alpha_{-n}^2, \quad n \in \mathbb{N}_0.
\end{aligned}
$$

The first line in (32) gives the equality (28) for $n = 0, 1, \ldots$ Its second line gives the equality (28) for $n = -1, -2, \ldots$ and equality $-\dot{\beta}_0 = \alpha_{-1}^2 - \alpha_0$, which we had in the first line. □

The equation (30) with matrices (29) is not applicable to integration of lattice (28) by means of inverse spectral problem for (6) since now $f_{-1} = 0$ and the value of $\mathfrak{a}_{-1}$ is not essential. But a certain modification of (30), (29) is applicable: see Sections 6–9.

4. SEMI-INFINITE OPERATOR TODA LATTICE AND THE CORRESPONDING LAX EQUATION

Return now to the general situation of matrices of type (6) with elements from $L(H)$. Denote by $J(t), t \in [0, T)$, the matrix of the form (6) with elements $a_{-1}(t), a_n(t), b_n(t) = b_n^*(t), c_n(t) = a_n(t), n \in \mathbb{N}_0$, from $L(H)$ for every t. We assume that they depend on t in a continuously differentiable way (in the operator norm sense) and their norms are bounded uniformly on $n \in \mathbb{N}_0$ and $t \in [0, T)$. Let $\boldsymbol{J}(t), t \in [0, T)$, be the operator constructed in Section 2 on the space $\mathbf{l}_2$.

Consider another matrix $A(t), t \in [0, T)$, of type (6); its $L(H)$-valued entries will be denoted by $\widetilde{a}_n = \widetilde{a}_n(t), \widetilde{b}_n = \widetilde{b}_n(t), \widetilde{c}_n = \widetilde{c}_n(t)$. They are $\forall t \in [0, T)$ arbitrary operators in $L(H)$ with above said properties of differentiability and boundedness. Our Lax equation has the form:

$$
(33) \quad \dot{J}(t) = [J(t), A(t)] := J(t)A(t) - A(t)J(t), \quad t \in [0, T).
$$

After simple calculations (see [5, 11, 12]) it is possible to conclude that the equality (33) is equivalent to the following system of equalities: $\forall n \in \mathbb{N}_0$ and $t \in [0, T)$

$$
\begin{aligned}
0 &= a_n \widetilde{c}_{n+1} - \widetilde{c}_n a_{n+1}, \\
\dot{a}_n &= b_n \widetilde{c}_n + a_n \widetilde{b}_{n+1} - \widetilde{b}_n a_n - \widetilde{c}_n b_{n+1}, \\
\dot{b}_n &= a_{n-1} \widetilde{c}_{n-1} + b_n \widetilde{b}_n + a_n \widetilde{a}_n - \widetilde{a}_{n-1} a_{n-1} - \widetilde{b}_n b_n - \widetilde{c}_n a_n, \\
\dot{a}_n &= a_n \widetilde{b}_n + b_{n+1} \widetilde{a}_n - \widetilde{a}_n b_n - \widetilde{b}_{n+1} a_n, \\
0 &= a_{n+1} \widetilde{a}_n - \widetilde{a}_{n+1} a_n.
\end{aligned}
$$
(34)

Above it is necessary to take into account that $a_{-1} = \widetilde{a}_{-1} = \widetilde{c}_{-1} = 0$. Because our operator $\boldsymbol{J}(t)$ does not depend on a_{-1} (by the construction), we can also assume that $a_{-1} = 0$.

From the first and the last equalities in (34) we conclude:

$$
(35) \qquad \widetilde{c}_n = a_{n-1}^{-1} \ldots a_0^{-1} \widetilde{c}_0 a_1 \ldots a_n, \quad \widetilde{a}_n = a_n \ldots a_1 \widetilde{a}_0 a_0^{-1} \ldots a_{n-1}^{-1}, \quad n \in \mathbb{N}.
$$

From the second and the fourth equations in (34) we conclude:

$$
(36) \qquad a_n \widetilde{b}_{n+1} + \widetilde{b}_{n+1} a_n = a_n \widetilde{b}_n + \widetilde{b}_n a_n + \widetilde{c}_n b_{n+1} + b_{n+1} \widetilde{a}_n - b_n \widetilde{c}_n - \widetilde{a}_n b_n, \quad n \in \mathbb{N}_0.
$$

Consecutively substituting $n = 0, 1, \ldots$ into (36) and using (35), we find $\widetilde{b}_1, \widetilde{b}_2, \ldots$ Here we use the fact that the equation for $\xi \in L(H)$ $a\xi + \xi a = b$, where $a, b \in L(H), a > 0$, is uniquely solvable.

As a result, we can assert that *the Lax equation* (33) *is equivalent to the second and the third equations in* (34) *and equalities* (35) *and* (36).

In a scalar case when $H = \mathbb{C}$ and the elements $a_n(t), b_n(t)$ of the matrix $J(t)$ are real-valued and $\widetilde{a}_n(t), \widetilde{b}_n(t), \widetilde{c}_n(t)$ are complex-valued functions, the Lax equation (33) is equivalent to the following simple generalization of Toda lattice (see [5]): $\forall t \in [0, T)$

$$
(37) \qquad \dot{a}_n = \frac{1}{2} f a_n (b_{n+1} - b_n), \quad \dot{b}_n = f(a_n^2 - a_{n-1}^2), \quad n \in \mathbb{N}_0, \quad a_{-1} = 0;
$$
$$
f = f(t) = a_0^{-1}(t)(\widetilde{a}_0(t) - \widetilde{c}_0(t)).
$$

If $\dim H > 1$ the situation is more complicated [11, 23]. We consider here only an interesting case when *all operators* $a_n(t), b_m(t), \widetilde{a}_j(t), \widetilde{c}_k(t)$ *are commuting and* $\widetilde{b}_l(t) = 0, t \in [0, T); n, m, j, k, l \in \mathbb{N}_0$.

In this case it follows from (35):

$$
(38) \qquad \widetilde{c}_n(t) = \widetilde{c}_0(t) a_0^{-1}(t) a_n(t), \quad \widetilde{a}_n(t) = \widetilde{a}_0(t) a_0^{-1}(t) a_n(t), \quad n \in \mathbb{N}, \quad t \in [0, T).
$$

With the help of (38) and the assumption $\widetilde{b}_l(t) = 0, t \in [0, T), l \in \mathbb{N}_0$, the condition (36) gives:

$$
\begin{aligned}
(39) \qquad 0 &= \widetilde{c}_n b_{n+1} + b_{n+1} \widetilde{a}_n - b_n \widetilde{c}_n - \widetilde{a}_n b_n = (\widetilde{c}_n + \widetilde{a}_n)(b_{n+1} - b_n) \\
&= (\widetilde{c}_0 + \widetilde{a}_0) a_0^{-1} a_n (b_{n+1} - b_n), \quad n \in \mathbb{N}_0, \quad t \in [0, T).
\end{aligned}
$$

Assume in addition that

$$
(40) \qquad \widetilde{a}_0(t) = \frac{1}{2} a_0(t), \quad \widetilde{c}_0(t) = -\frac{1}{2} a_0(t), \quad t \in [0, T).
$$

Then from (39) it follows that the condition (36) is fulfilled.

Our assumption about commutativity and equality to zero of all $\widetilde{b}_n, n \in \mathbb{N}_0$, and (38), (40) gives now that the second and the third equation from (34) have the form:

$\forall n \in \mathbb{N}_0, t \in [0, T)$

$$\dot{a}_n = b_n \widetilde{c}_n - \widetilde{c}_n b_{n+1} = \widetilde{c}_n (b_n - b_{n+1}) = \widetilde{c}_0 a_0^{-1} a_n (b_n - b_{n+1}) = \frac{1}{2} a_n (b_{n+1} - b_n),$$

$$\dot{b}_n = a_{n-1} \widetilde{c}_{n-1} + a_n \widetilde{a}_n - \widetilde{a}_{n-1} a_{n-1} - \widetilde{c}_n a_n = a_n (\widetilde{a}_n - \widetilde{c}_n) + a_{n-1} (\widetilde{c}_{n-1} - \widetilde{a}_{n-1})$$

$$= a_n^2 - a_{n-1}^2.$$

So, we have proved the following statement.

Lemma 3. *For $t \in [0, T)$, consider the operator Hermitian Jacobi matrix $J(t)$ of type (6) and an analogous matrix $A(t)$ with elements $\widetilde{a}_n(t), \widetilde{b}_n(t), \widetilde{c}_n(t) \in L(H)$ (conditions for smoothness and boundedness of all these operators are formulated above, in the beginning of Section 4). Assume that all elements of these matrices $\forall t$ are commuting operators, $\widetilde{b}_n(t) = 0, n \in \mathbb{N}_0, t \in [0, T)$, and the elements $\widetilde{a}_0(t), \widetilde{c}_0(t)$ have the form (40).*

Then the corresponding Lax equation (33) can be written in the equivalent form:

$$(41) \qquad \begin{aligned} \dot{a}_n(t) &= \frac{1}{2} a_n(t)(b_{n+1}(t) - b_n(t)), \\ \dot{b}_n(t) &= a_n^2(t) - a_{n-1}^2(t), \quad n \in \mathbb{N}_0, \quad t \in [0, T); \quad a_{-1}(t) = 0. \end{aligned}$$

In one case, this result is applicable to the situation in Section 3: assume that the solution of Toda equation (28) partially consist of positive functions: $\alpha_n(t) > 0$ and uniformly bounded, i.e. $\exists c \in [0, \infty) : \alpha_n(t), |\beta_n(t)| \leq c, n \in \mathbb{N}_0, t \in [0, T)$. Then all unknowns $\mathfrak{a}_n(t), \mathfrak{b}_n(t) \in L(\mathbb{C}^2)$ of equations (30) are uniformly bounded and $\mathfrak{a}_n^{-1}(t)$ exist. Using Lemma 3 we can assert that the equations (30) *with the condition* $\mathfrak{a}_{-1}(t) = 0, t \in [0, T)$, are equivalent to the Lax equation (33) with the corresponding matrix $A(t)$. But this condition is very essential: we cannot consider the general solution of Toda lattice (28). We will overcome this difficulty in Sections 6–9. But now our immediate goal is to construct in Section 5 a solution of the Cauchy problem for (41) in the case when all unknowns are commuting operators.

We will now make an observation connected with Section 2 and 3.

Remark 1. With the Cauchy problem for double-infinite Toda lattices (28) one can connect in a more natural way the operator semi-infinite Jacobi matrix acting in the same space $\mathbf{l}_2 = \ell_2(\mathbb{C}^2)$.

Namely, the ordinary space ℓ_2 on the whole $\mathbb{Z}$, $\ell_2(\mathbb{C}, \mathbb{Z}) = \{\xi = (\xi_n)_{n=-\infty}^{\infty}, \xi_n \in \mathbb{C} | \sum_{n=-\infty}^{\infty} |\xi_n|^2 < \infty \}$, and the space $\ell_2(\mathbb{C}^2)$ on $\mathbb{N}_0$ introduced above are unitary isomorphic. This isomorphism I can be constructed, for example, as follows :

$$(42) \quad \ell_2(\mathbb{C}, \mathbb{Z}) \ni \xi = (\xi_n)_{n=-\infty}^{\infty} \mapsto I\xi = ((I\xi)_n)_{n=0}^{\infty} \in \ell_2(\mathbb{C}^2), \quad (I\xi)_n = (\xi_{-n-1}, \xi_n) \in \mathbb{C}^2.$$

The difference expression $\mathfrak{L}$ on sequences $\xi = (\xi_n)_{n=-\infty}^{\infty}, \xi_n \in \mathbb{C}$, connected with (28) is natural to present in the form of a double-infinite Toda scalar matrix of the form (6), [45, 46, 47], i.e.

$$(43) \qquad (\mathfrak{L}\xi)_n = \alpha_{n-1} \xi_{n-1} + \beta_n \xi_n + \alpha_n \xi_{n+1}, \quad n \in \mathbb{Z}.$$

The isomorphism (42) transforms the expression (43) into the expression $I\mathfrak{L}I^{-1}$ generated by a matrix of type (6) (with $L(\mathbb{C}^2)$-valued elements) on vector-valued sequences $f = (f_n)_{n=0}^{\infty}, f_n \in \mathbb{C}^2$ (for more details see [1, 11]).

The matrix of expression $I\mathfrak{L}I^{-1}$ is an operator Jacobi matrix with elements from $L(\mathbb{C}^2)$ but these elements are somewhat different from (29). In this way it is possible to develop a spectral theory of double-infinite Hermitian Jacobi matrices [1]. But potential applications to the integration of the double-infinite Toda lattice meet with essential difficulties: the corresponding differential equations w.r.t. $t \in [0, T)$ for the spectral measure are not possible to integrate and find a simple solution [47, 11].

5. EVOLUTION OF THE SPECTRAL MEASURE OF THE SEMI-INFINITE OPERATOR TODA
LATTICE IN THE COMMUTATIVE CASE. SOLUTION OF THE CORRESPONDING
CAUCHY PROBLEM

Consider operator matrices $J(t)$ and (in a general situation) $A(t), t \in [0, T)$, introduced in Section 4. Let $\boldsymbol{J}(t)$ be the operator on $\mathbf{l}_2$ constructed according to the rule (5).

Denote by $\boldsymbol{R}_z(t)$ the resolvent of $\boldsymbol{J}(t) : \boldsymbol{R}_z(t) = (\boldsymbol{J}(t) - z1)^{-1}, z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$ and consider its derivative $\dot{\boldsymbol{R}}_z(t)$ (which, of course, exists as a derivative in $L(H)$).

Assume that $J(t)$ satisfies the Lax equation (33), then from (33) $\forall t \in [0, T)$ we get:

(44) $\quad (\boldsymbol{J}(t) - z1)^{\cdot} = [\boldsymbol{J}(t) - z1, A(t)]; \quad \dot{\boldsymbol{R}}_z(t) = -\boldsymbol{R}_z(t)(\boldsymbol{J}(t) - z1)^{\cdot}\boldsymbol{R}_z(t) = [\boldsymbol{R}_z(t), A(t)]$

(the equation (44) can be understood as matrix equalities on $f \in \mathbf{l}_{\text{fin}}$).

According to (13), (12) we can introduce the matrix $(R_{z;jk}(t))_{j,k=0}^{\infty}$ of the operator $\boldsymbol{R}_z(t) : \mathbf{l}_2 \to \mathbf{l}_2 : \forall z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T), j, k \in \mathbb{N}_0$

$$R_{z;jk}(t) = \{\delta_j, \boldsymbol{R}_z(t)\delta_k\} \in L(H);$$
$$(R_{z;jk}(z)x, y) = (\{\delta_j, \boldsymbol{R}_z(t)\delta_k\}x, y) = (\boldsymbol{R}_z(t)(\delta_k x), \delta_j y)_{\mathbf{l}_2}.$$

Therefore,

(45) $$R_{z;jk}(t) = \delta_j^* \boldsymbol{R}_z(t)\delta_k, \quad j, k \in \mathbb{N}_0.$$

Here we understand δ_k as an operator from H into $\mathbf{l}_2$:
$H \ni x \mapsto \delta_k x := (\underbrace{0, \ldots, 0}_{k}, x, 0, 0, \ldots) \in \mathbf{l}_2$; such an understanding of δ_k will be of use also in what follows.

Introduce the (operator) Weyl function $m(z)$ for a given operator $\boldsymbol{J}(t) : \forall z \in \mathbb{C} \setminus \mathbb{R}$

(46)
$$m(z;t) = R_{z;0,0}(t) = \delta_0^* \boldsymbol{R}_z(t)\delta_0 = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda; t) \in L(H),$$
$$\mathcal{B}(R) \ni \triangle \mapsto \rho(\triangle; t) = \delta_0^* E(\triangle; t)\delta_0 \in L(H).$$

Here $d\rho(\lambda; t)$ is the spectral measure of operator $\boldsymbol{J}(t)$ on the space $\mathbf{l}_2; t \in [0, T)$ is fixed.

From the second equality in (44) we can find how $m(z;t)$ evolves in t. Namely, using (46) and (44), (45), (13), (11) we get:

$$\dot{m}(z;t) = \delta_0^* \dot{\boldsymbol{R}}_z(t)\delta_0 = \delta_0^*[\boldsymbol{R}_z(t), A(t)]\delta_0 = \delta_0^*\boldsymbol{R}_z(t)A(t)\delta_0 - \delta_0^*A(t)\boldsymbol{R}_z(t)\delta_0$$

(47)
$$= \{\delta_0, \boldsymbol{R}_z(t)A(t)\delta_0\} - \{\delta_0, A(t)\boldsymbol{R}_z(t)\delta_0\} = R_{z;00}(t)\widetilde{b}_0(t) + R_{z;01}(t)\widetilde{a}_0(t)$$
$$- \widetilde{b}_0(t)R_{z;00}(t) - \widetilde{c}_0(t)R_{z;10}(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T).$$

Let us pass now to the essential for us case when all the operators

$$a_n(t), b_n(t), \widetilde{a}_n(t), \widetilde{c}_n(t) \in L(H)$$

are commuting and $\widetilde{b}_n(t) = 0, n \in \mathbb{N}_0, t \in [0, T)$. We will use the following simple fact.

Lemma 4. *Let $\boldsymbol{A} : \mathbf{l}_2 \to \mathbf{l}_2$ be a linear continuous operator, $(a_{jk})_{j,k=0}^{\infty}$ its matrix (13). Assume that the bounded inverse operator $\boldsymbol{A}^{-1}$ exists, denote by $(b_{jk})_{j,k=0}^{\infty}$ its matrix. We assert that if some operator $c \in L(H)$ commutes with every a_{jk} then it also commutes with every b_{jk}.*

Proof. Consider the operator $\boldsymbol{C} = c1$ where 1 is the identity operator on $\mathbf{l}_2$. Since $\forall j, k \in \mathbb{N}_0 \quad a_{jk}c = ca_{jk}$, the operator $\boldsymbol{C}$ commutes with $\boldsymbol{A}$. Therefore it commutes also with $\boldsymbol{A}^{-1}$, i.e. $b_{jk}c = cb_{jk}, j, k \in \mathbb{N}_0$. $\square$

We can now deduce the difference equation for the Weyl function $m(z;t)$.

Lemma 5. *Assume as before that all the operators* $a_n(t), b_m(t), \widetilde{a}_j(t), \widetilde{c}_k(t)$ *are commuting and* $\widetilde{b}_l(t) = 0, n, m, j, k, l \in \mathbb{N}_0, t \in [0, T]$. *Then* $m(z;t)$ *satisfies the following operator differential equation (all operators below are commuting):*

$$(48) \quad \dot{m}(z;t) = a_0^{-1}(t)(\widetilde{a}_0(t) - \widetilde{c}_0(t))(1 + (z1 - b_0(t))m(z;t)), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T].$$

Proof. Since $\forall z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$, $1 = (\boldsymbol{J}(t) - z1)\boldsymbol{R}_z(t)$, therefore using (45), (11) and the calculations as in (47) we get:

$$1 = \delta_0^*(\boldsymbol{J}(t) - z1)\boldsymbol{R}_z(t)\delta_0 = \{\delta_0, (\boldsymbol{J}(t) - z1)\boldsymbol{R}_z(t)\delta_0\}$$
$$= R_{z;00}(t)(b_0(t) - z1) + R_{z;01}(t)a_0(t).$$

From this equality we find:

$$(49) \qquad R_{z;01}(t) = (1 - m(z;t)(b_0(t) - z1))a_0^{-1}(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T].$$

Using (45), (11) and (49) we get for the same z, t:

$$(50) \qquad \begin{aligned} R_{z;10}(t) &= \delta_1^*\boldsymbol{R}_z(t)\delta_0 = (\delta_0^*\boldsymbol{R}_z^*(t)\delta_1)^* = (\delta_0^*\boldsymbol{R}_{\overline{z}}(t)\delta_1)^* \\ &= (R_{\overline{z};01}(t))^* = a_0^{-1}(t)(1 - (b_0(t) - z1)m(z;t)). \end{aligned}$$

Note that here we took advantage of the equality following from (46):

$$(51) \qquad m^*(z;t) = m(\overline{z};t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T].$$

The representation (47), (49) and (50) was obtained without our assumption about commutativity. With this assumptions Lemma 4 and formulas above give (48):

$$\dot{m}(z;t) = R_{z;01}(t)\widetilde{a}_0(t) - \widetilde{c}_0(t)R_{z;10}(t) = a_0^{-1}(t)(\widetilde{a}_0(t) - \widetilde{c}_0(t))(1 - (b_0(t) - z1)m(z;t)).$$

$\square$

As in the scalar case, the operator-valued Weyl function

$$(52) \qquad m(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda) \in L(H), \quad z \in \mathbb{C} \setminus \mathbb{R}$$

$(\mathcal{B}(\mathbb{R}) \ni \triangle \mapsto \rho(\triangle) \in L(H)$ is a finite nonnegative operator-valued Borel measure) defines uniquely the measure $d\rho(\lambda)$. For the proof it suffices to apply this equality to $\forall x \in H$ and then scalar multiply it by $\forall y \in H$; thus the equation is reduced to the scalar case.

From this fact it follows that the spectral measure $d\rho(\lambda; t)$, as well as $m(z;t)$, commutes with $a_n(t), b_n(t), n \in \mathbb{N}_0, t \in [0, T]$.

Therefore the equation (48) is reduced to the equivalent equation for the measure $d\rho(\lambda; t)$. Namely, take $\widetilde{a}_0(t), \widetilde{c}_0(t)$ of the form (40). Then (48) has the form:

$$(53) \qquad \dot{m}(z;t) = 1 + (z1 - b_0(t))m(z;t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T].$$

Transform this equality in the following manner:

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} d\dot{\rho}(\lambda; t) = \left(\int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda; t) \right)^{\cdot} = \dot{m}(z;t) = 1 + (z1 - b_0(t))m(z;t)$$

$$= \int_{\mathbb{R}} \frac{\lambda - b_0(t)}{\lambda - z} d\rho(\lambda; t).$$

This equality is equivalent to the following one for the measure $d\rho(\lambda; t)$:

$$(54) \qquad d\dot{\rho}(\lambda; t) = (\lambda - b_0(t))d\rho(\lambda; t), \quad \lambda \in \mathbb{R}, \quad t \in [0, T].$$

For every t, the operators from $L(H)$, $\rho(\triangle; t)$ and $b_0(t)$, commute, therefore a solution of equation (54) is easy to write:

$$(55) \qquad d\rho(\lambda; t) = c(t)e^{\lambda t}d\rho(\lambda; 0), \quad \lambda \in \mathbb{R}, \quad t \in [0, T].$$

Here the operator-valued function $c(t)$ is equal to

$$(56) \qquad c(t) = e^{-\int\limits_0^t b_0(\tau)d\tau} \in L(H), \quad t \in [0, T].$$

To find this function it is not necessary to know $b_0(t)$: the measure (55) is probability, i.e. $\forall t \ \rho(\mathbb{R}; t) = 1$. Therefore from (55) we get:

$$(57) \qquad c(t) = \left(\int\limits_{\mathbb{R}} e^{\lambda t} d\rho(\lambda; 0) \right)^{-1}, \quad t \in [0, T].$$

As a result, we can formulate the following intermediate theorem whose proof follows from above stated facts (see also the proof of the **Theorem 5**).

Theorem 1. *Consider the semi-infinite operator Toda lattice* (41) *with continuously differentiable w.r.t. t commuting operators $a_n(t), b_n(t) \in L(H), n \in \mathbb{N}_0, t \in [0, T]$. For* (41), *consider the following Cauchy problem: for initial data $(a_0(0), a_1(0), \ldots; b_0(0), b_1(0), \ldots)$ find the solution $(a_0(t), a_1(t), \ldots; b_0(t), b_1(t), \ldots), t \in [0, T]$.*

Assume that such a solution exists with the additional properties: $a_n(t) > 0, b_n(t) = b_n^*(t), \exists C \in [0, \infty) : \|a_n(t)\|, \|b_n(t)\| \le C, n \in \mathbb{N}_0, t \in [0, T]$.

Construct from this solution a Jacobi matrix $J(t)$ (6), let $\boldsymbol{J}(t)$ be the bounded selfadjoint operator on $\mathbf{l}_2$ generated by $J(t)$. Denote by $d\rho(\lambda; t)$ the operator spectral measure of $\boldsymbol{J}(t)$. Then this measure is recovered from the initial measure $d\rho(\lambda; 0)$ with the help of formulas (55), (57).

A solution of our Cauchy problem is given by formulas (21) *in applying them to the spectral measure $d\rho(\lambda; t)$ and the corresponding orthogonal polynomials $P_n(\lambda; t)$ that can be obtained via the orthogonalization procedure.*

Remark 2. It is possible to investigate, in a similar fashion, a lattice that is a little more general than (41):

$$(58) \qquad \begin{aligned} \dot{a}_n(t) &= \frac{1}{2} f(t) a_n(t)(b_{n+1}(t) - b_n(t)), \\ \dot{b}_n(t) &= f(t)(a_n^2(t) - a_{n-1}^2(t)), \quad n \in \mathbb{N}_0, \quad t \in [0, T); \quad a_{-1}(t) = 0, \end{aligned}$$

where $[0, T) \ni t \mapsto f(t) \in L(H)$ is a given operator-valued function whose values commute with $a_n(t), b_n(t), n \in \mathbb{N}_0$. Note that to study (58) it is necessary to take $\widetilde{a}_0(t) = \frac{1}{2} a_0(t) f(t), \widetilde{c}_0(t) = -\frac{1}{2} a_0(t) f(t)$ instead of (40).

Remark 3. It is possible to check by direct calculations that the formulas for solutions of the Cauchy problem for (41), (58), given in **Theorem 1** are in fact solutions of these equations. It is possible to do this in a way similar to that in the works [5, 15, 12].

6. Another reduction of the double-infinite Toda lattice to semi-infinite matrix Toda lattice

In Section 3 one method of reduction was given, but in this case the matrix $\mathfrak{a}_{-1}$ in system (30) is not equal to zero. Therefore the constructions of Sections 4, 5 are not applicable to an investigation of the Toda lattice in a general situation. In this Section we will propose another approach to such a reduction.

Instead of the space $\mathbf{l}_2 = \ell_2(\mathbb{C}^2) = \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots$, we consider its subspace,

$$(59) \ \mathbf{l}_{2,0} = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_0 = \mathbb{C}^1, \quad \mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathbb{C}^2,$$

with elements $x = (x_n)_{n=0}^{\infty}$, where $x_0 \in \mathbb{C}$ and $x_n = (x_{n;0}, x_{n;1}) \in \mathbb{C}^2, n \in \mathbb{N}$. As in **Remark 1** we consider the space $\ell_2(\mathbb{C}, \mathbb{Z})$ of sequences $\xi = (\xi_n)_{n=-\infty}^{\infty}$ and introduce

(like in (42)) a mapping K :

(60)
$$\ell_2(\mathbb{C}, \mathbb{Z}) \ni \xi = (\xi_n)_{n=-\infty}^{\infty} \mapsto K\xi = ((K\xi)_n)_{n=0}^{\infty} \in \mathbf{l}_{2,0},$$
$$(K\xi)_0 = \xi_0 \in \mathbb{C}, (K\xi)_n = (\xi_n, \xi_{-n}) \in \mathbb{C}^2, \quad n \in \mathbb{N}.$$

This mapping is an unitary isomorphism between the spaces $\ell_2(\mathbb{C}, \mathbb{Z})$ and $\mathbf{l}_{2,0}$.

The inverse mapping K^{-1} has the form

(61)
$$\mathbf{l}_{2,0} \ni x = (x_n)_{n=0}^{\infty} \mapsto K^{-1}x = ((K^{-1}x)_n)_{n=-\infty}^{\infty} \in \ell_2(\mathbb{C}, \mathbb{Z}),$$
$$(K^{-1}x)_n = x_{n;0}, (K^{-1}x)_0 = x_0, (K^{-1}x)_{-n} = x_{n;1}, \quad n \in \mathbb{N}.$$

Consider the difference expression $\mathfrak{L}$ of the three-diagonal form, more general than (43), and acting on sequences $\xi = (\xi_n)_{n=-\infty}^{\infty}, \xi_n \in \mathbb{C}$:

(62)
$$(\mathfrak{L}\xi)_n = \alpha_{n-1}\xi_{n-1} + \beta_n\xi_n + \gamma_n\xi_{n+1}, \quad n \in \mathbb{Z},$$

where $\alpha_n, \beta_n, \gamma_n$ are real coefficients. Calculate its image under isomorphism (60), namely $L = K\mathfrak{L}K^{-1}$. This image act on sequences $x = (x_n)_{n=0}^{\infty}$, $x_0 \in \mathbb{C}$, $x_n = (x_{n;0}, x_{n;1}) \in \mathbb{C}^2$, $n \in \mathbb{N}$, and has the following form.

Let $n = 2, 3, \dots$. Then using (60), (62), (61) we get:

(63)
$$(Lx)_n = (K\mathfrak{L}K^{-1}x)_n = ((\mathfrak{L}K^{-1}x)_n, (\mathfrak{L}K^{-1}x)_{-n})$$
$$= (\alpha_{n-1}(K^{-1}x)_{n-1} + \beta_n(K^{-1}x)_n + \gamma_n(K^{-1}x)_{n+1},$$
$$\alpha_{-n-1}(K^{-1}x)_{-n-1} + \beta_{-n}(K^{-1}x)_{-n} + \gamma_{-n}(K^{-1}x)_{-n+1})$$
$$= (\alpha_{n-1}x_{n-1;0} + \beta_n x_{n;0} + \gamma_n x_{n+1;0}, \alpha_{-n-1}x_{n+1;1} + \beta_{-n}x_{n;1} + \gamma_{-n}x_{n-1;1}),$$

i.e.

(64)
$$(Lx)_n = a_{n-1}x_{n-1} + b_n x_n + c_n x_{n+1}, \quad n = 2, 3, \dots;$$

(65) $\quad a_n = \begin{bmatrix} \alpha_n & 0 \\ 0 & \gamma_{-n-1} \end{bmatrix}, \quad b_n = \begin{bmatrix} \beta_n & 0 \\ 0 & \beta_{-n} \end{bmatrix}, \quad c_n = \begin{bmatrix} \gamma_n & 0 \\ 0 & \alpha_{-n-1} \end{bmatrix}, \quad n = 2, 3, \dots.$

For the case $n = 1$ we get from (63) and (61) :

$$(Lx)_1 = (K\mathfrak{L}K^{-1}x)_1 = ((\mathfrak{L}K^{-1}x)_1, (\mathfrak{L}K^{-1}x)_{-1})$$
$$= (\alpha_0 x_0 + \beta_1 x_{1;0} + \gamma_1 x_{2;0}, \alpha_{-2}x_{2;1} + \beta_{-1}x_{1;1} + \gamma_{-1}x_0),$$

i.e. we have the expression (64) for $n = 1$ with b_1, c_1, given by (65) for $n = 1$ and a_0 of the form $a_0 = \begin{bmatrix} \alpha_0 \\ \gamma_{-1} \end{bmatrix}$.

Analogously for $n = 0$ we have:

$$(Lx)_0 = (K\mathfrak{L}K^{-1}x)_0 = (\mathfrak{L}K^{-1}x)_0 = \alpha_{-1}(K^{-1}x)_{-1} + \beta_0(K^{-1}x)_0 + \gamma_0(K^{-1}x)_1$$
$$= \alpha_{-1}x_{1;1} + \beta_0 x_0 + \gamma_0 x_{1;0}.$$

In this case expression (64) also holds with matrices:

$$a_{-1} = 0, \quad b_0 = [\beta_0], \quad c_0 = [\gamma_0 \quad \alpha_{-1}].$$

These simple calculations show that the following lemma is valid.

Lemma 6. *The expression $L = K\mathfrak{L}K^{-1}$ where $\mathfrak{L}$ has the form (62) acts on sequences* $x = (x_n)_{n=0}^{\infty}, x_0 \in \mathbb{C}, x_n = (x_{n;0}, x_{n;1}), n \in \mathbb{N}$, *by the rule:*

(66)
$$(Lx)_n = a_{n-1}x_{n-1} + b_n x_n + c_n x_{n+1}, \quad where$$

$$a_n = \begin{bmatrix} \alpha_n & 0 \\ 0 & \gamma_{-n-1} \end{bmatrix}, \quad b_n = \begin{bmatrix} \beta_n & 0 \\ 0 & \beta_{-n} \end{bmatrix}, \quad c_n = \begin{bmatrix} \gamma_n & 0 \\ 0 & \alpha_{-n-1} \end{bmatrix}, \quad n \in \mathbb{N};$$

$$a_0 = \begin{bmatrix} \alpha_0 \\ \gamma_{-1} \end{bmatrix}, \quad b_0 = [\beta_0], \quad c_0 = [\gamma_0 \quad \alpha_{-1}]; \quad a_{-1} = 0.$$

Note, that in the Hermitian case, when $\mathfrak{L}$ has the form (43) (i.e. $\gamma_n = \alpha_n$), the expression L is also Hermitian:

(67) $$c_n = a_n, n \in \mathbb{N}; \quad a_0 = \begin{bmatrix} \alpha_0 \\ \alpha_{-1} \end{bmatrix}, \quad c_0 = [\alpha_0 \quad \alpha_{-1}] = a_0^*, \quad a_{-1} = 0.$$

Return now to the double-infinite Toda lattice (28) and write it in the form of a Lax equation (of course, this fact is well known).To this end, introduce the double-infinite Jacobi type matrices with real elements acting on sequences $\xi = (\xi_n)_{n=-\infty}^\infty, \xi \in \mathbb{C}$:

(68)
$$\mathcal{J} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \dots 0 & \alpha_{-2} & \beta_{-1} & \alpha_{-1} & 0 \dots & \\ & \dots 0 & \alpha_{-1} & \beta_0 & \alpha_0 & 0 \dots \\ & & \dots 0 & \alpha_0 & \beta_1 & \alpha_1 & 0 \dots \\ & & & \ddots & \ddots & \ddots \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \dots 0 & \widetilde{\alpha}_{-2} & \widetilde{\beta}_{-1} & \widetilde{\gamma}_{-1} & 0 \dots & \\ & \dots 0 & \widetilde{\alpha}_{-1} & \widetilde{\beta}_0 & \widetilde{\gamma}_0 & 0 \dots \\ & & \dots 0 & \widetilde{\alpha}_0 & \widetilde{\beta}_1 & \widetilde{\gamma}_1 & 0 \dots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Assuming that the elements of matrices (68) depend smoothly on $t \in [0, T), T \le \infty$ (as often before, we will not designate this dependence) introduce the Lax equation:

(69) $$\dot{\mathcal{J}}(t) = [\mathcal{J}(t), \mathcal{A}(t)] := \mathcal{J}(t)\mathcal{A}(t) - \mathcal{A}(t)\mathcal{J}(t), \quad t \in [0, T).$$

This equality in the "coordinate" form can be rewritten as follows (compare with (33), (34) for operator elements; see also [5, 12]):

(70)
$$\begin{aligned} 0 &= \alpha_n \widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n \alpha_{n+1}, \\ \dot{\alpha}_n &= \beta_n \widetilde{\gamma}_n + \alpha_n \widetilde{\beta}_{n+1} - \widetilde{\beta}_n \alpha_n - \widetilde{\gamma}_n \beta_{n+1}, \\ \dot{\beta}_n &= \alpha_{n-1} \widetilde{\gamma}_{n-1} + \alpha_n \widetilde{\alpha}_n - \widetilde{\alpha}_{n-1} \alpha_{n-1} - \widetilde{\gamma}_n \alpha_n, \\ \dot{\alpha}_n &= \alpha_n \widetilde{\beta}_n + \beta_{n+1} \widetilde{\alpha}_n - \widetilde{\alpha}_n \beta_n - \widetilde{\beta}_{n+1} \alpha_n, \\ 0 &= \alpha_{n+1} \widetilde{\alpha}_n - \widetilde{\alpha}_{n+1} \alpha_n, \quad n \in \mathbb{Z}, \quad t \in [0, T). \end{aligned}$$

We will assume that $\forall t \in [0, T)$ and $\forall n \in \mathbb{Z}$ $\alpha_n(t) > 0$. Then from (70) it is easy to calculate that

(71) $$\widetilde{\alpha}_n = \alpha_n \alpha_0^{-1} \widetilde{\alpha}_0, \quad \widetilde{\gamma}_n = \alpha_n \alpha_0^{-1} \widetilde{\gamma}_0, \quad \widetilde{\beta}_n = \widetilde{\beta}_0 + (2\alpha_0)^{-1}(\widetilde{\alpha}_0 + \widetilde{\gamma}_0)(\beta_n - \beta_0),$$
$$n \in \mathbb{Z}, \quad t \in [0, T).$$

It is possible also to give a simple formula for calculation of $\widetilde{\beta}_n$ by $\widetilde{\alpha}_0, \widetilde{\gamma}_0$ and $\alpha_0, \beta_0, \beta_n$.
From the second and third equalities in (70) and (71) we conclude that $\forall t \in [0, T)$

(72) $$\dot{\alpha}_n = \frac{1}{2}f(t)\alpha_n(\beta_{n+1} - \beta_n), \quad \dot{\beta}_n = f(t)(\alpha_n^2 - \alpha_{n-1}^2)$$

where $f(t) = \alpha_0^{-1}(\widetilde{\alpha}_0 - \widetilde{\gamma}_0), n \in \mathbb{Z}$.

Let, similarly to (40),

$$\text{(73)} \qquad \widetilde{\alpha}_0(t) = \frac{1}{2}\alpha_0(t), \quad \widetilde{\gamma}_0(t) = -\frac{1}{2}\alpha_0(t), \quad t \in [0, T),$$

then the function $f(t) = 1, t \in [0, T)$, and equalities (72) have the form of the Toda lattice (28). In this case it is possible also to put: $\widetilde{\beta}_n = 0, n \in \mathbb{Z}$ (see a similar calculation in [5, 12]).

Conversely, it is easy to see that Toda lattice (28) gives the Lax equation (69). Therefore *Toda lattice* (28) *and the Lax equation* (69), (70) *(with conditions* (73)*) are equivalent (under the assumption:* $\alpha_n(t) > 0, t \in [0, T), n \in \mathbb{Z}$*).*

Our nearest aim is to rewrite Toda lattice (28) in terms of matrices (66), (67) (similar to (41)). For this we need to apply to our objects the mappings K, K^{-1} (60), (61).

Let $\mathcal{J}(t) = \mathfrak{L}(t)$ have the form (43), i.e. (68), and $\mathcal{A}(t)$ be given by (68) with conditions (73), $t \in [0, T)$. Denote $\forall t \in [0, T)$

$$\text{(74)} \qquad J(t) = K\mathcal{J}(t)K^{-1}; \quad A(t) = K\mathcal{A}(t)K^{-1} : \mathbf{l}_{\text{fin}} \to \mathbf{l}_{\text{fin}}.$$

The Lax equality (69) gives the corresponding Lax equation for $J(t), A(t)$:

$$\text{(75)} \qquad \dot{J}(t) = K\dot{\mathcal{J}}(t)K^{-1} = K(\mathcal{J}(t)\mathcal{A}(t) - \mathcal{A}(t)\mathcal{J}(t))K^{-1} = J(t)A(t) - A(t)J(t),$$
$$t \in [0, T).$$

According to **Lemma 6**, (66), (67) the matrix elements a_n, b_n and $\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_n, n \in \mathbb{N}_0$, of the three-diagonal matrices $J(t)$ and $A(t)$ are:

$$\text{(76)} \qquad
\begin{aligned}
a_n &= \begin{bmatrix} \alpha_n & 0 \\ 0 & \alpha_{-n-1} \end{bmatrix}, \quad b_n = \begin{bmatrix} \beta_n & 0 \\ 0 & \beta_{-n} \end{bmatrix}, \\
\widetilde{a}_n &= \begin{bmatrix} \widetilde{\alpha}_n & 0 \\ 0 & \widetilde{\gamma}_{-n-1} \end{bmatrix}, \quad \widetilde{b}_n = \begin{bmatrix} \widetilde{\beta}_n & 0 \\ 0 & \widetilde{\beta}_{-n} \end{bmatrix}, \quad \widetilde{c}_n = \begin{bmatrix} \widetilde{\gamma}_n & 0 \\ 0 & \widetilde{\alpha}_{-n-1} \end{bmatrix}, \quad n \in \mathbb{N}; \\
a_0 &= \begin{bmatrix} \alpha_0 \\ \alpha_{-1} \end{bmatrix}, \quad b_0 = [\beta_0], \quad c_0 = [\alpha_0 \quad \alpha_{-1}] = a_0^*, \\
\widetilde{a}_0 &= \begin{bmatrix} \widetilde{\alpha}_0 \\ \widetilde{\gamma}_{-1} \end{bmatrix}, \quad \widetilde{b}_0 = [\widetilde{\beta}_0], \quad \widetilde{c}_0 = [\widetilde{\gamma}_0 \quad \widetilde{\alpha}_{-1}].
\end{aligned}$$

Therefore the Lax equality (75) has the form (33) but the matrices $J(t), A(t)$ act on the space $\mathbf{l}_{2,0}$ (59) distinct from the space $\mathbf{l}_2 = \ell_2(H)$: in (59) the first term is not equal to the other.

Nevertheless *we can claim that equality* (75) *is equivalent to a system of type* (34) *with special elements.*

This fact follows from a simple general calculation in [12], §2, (9), (23), (25). Namely, we get for the matrices (76) using their commutativity:

$$\text{(77)} \qquad
\begin{aligned}
0 &= a_n\widetilde{c}_{n+1} - \widetilde{c}_n a_{n+1}, \\
\dot{a}_n &= b_n\widetilde{c}_n + a_n\widetilde{b}_{n+1} - \widetilde{b}_n a_n - \widetilde{c}_n b_{n+1}, \\
\dot{b}_n &= a_{n-1}\widetilde{c}_{n-1} + a_n\widetilde{a}_n - \widetilde{a}_{n-1}a_{n-1} - \widetilde{c}_n a_n, \\
\dot{a}_n &= a_n\widetilde{b}_n + b_{n+1}\widetilde{a}_n - \widetilde{a}_n b_n - \widetilde{b}_{n+1}a_n, \\
0 &= a_{n+1}\widetilde{a}_n - \widetilde{a}_{n+1}a_n, \quad n = 2, 3, \dots
\end{aligned}$$

For $n = 1$ we also have equalities (77) but instead of the third one of them we get:

$$\text{(78)} \qquad \dot{b}_1 = a_0\widetilde{c}_0 + a_1\widetilde{a}_1 - \widetilde{a}_0 a_0^* - \widetilde{c}_1 a_1.$$

For $n = 0$ the corresponding equalities have the form:

(79)
$$
\begin{aligned}
0 &= a_0^* \widetilde{c}_1 - \widetilde{c}_0 a_1, \\
\dot{a}_0^* &= b_0 \widetilde{c}_0 + a_0^* \widetilde{b}_1 - \widetilde{b}_0 a_0^* - \widetilde{c}_0 b_1, \\
\dot{b}_0 &= a_0^* \widetilde{a}_0 - \widetilde{c}_0 a_0, \\
\dot{a}_0 &= a_0 \widetilde{b}_0 + b_1 \widetilde{a}_0 - \widetilde{a}_0 b_0 - \widetilde{b}_1 a_0, \\
0 &= a_1 \widetilde{a}_0 - \widetilde{a}_1 a_0.
\end{aligned}
$$

The system (77) is a partial case of the general system (34) with $H = \mathbb{C}^2$ and $n = 2, 3, \ldots$ instead of $n = 0, 1, \ldots$; $\widetilde{a}_1, \widetilde{c}_1$ are not necessarily equal to zero. All the operators $a_n(t), b_m(t), \widetilde{a}_j(t), \widetilde{c}_k(t), n, m, j, k = 2, 3, \ldots, t \in [0, T)$, are commuting, see (76); the operators $a_n(t)$ are invertible. Also we can set $\widetilde{\beta}_n = 0, n \in \mathbb{Z}$, then $\widetilde{b}_n = 0, n \in \mathbb{N}_0$.

We will rewrite this system in a more simple form at first for the case $n = 2, 3, \ldots$. Its coefficients $\widetilde{a}_n, \widetilde{c}_n$ have the the form of (76) with the elements $\widetilde{\alpha}_n, \widetilde{\gamma}_n$ satisfying the conditions (71) and (73) (with an index 2 instead of 0). So, we have:

(80)
$$
\widetilde{\alpha}_m = \alpha_m \alpha_2^{-1} \widetilde{\alpha}_2 = \frac{1}{2} \alpha_m, \quad \widetilde{\gamma}_m = \alpha_m \alpha_2^{-1} \widetilde{\gamma}_2 = -\frac{1}{2} \alpha_m, \quad m \in \mathbb{Z};
$$

(81)
$$
\begin{aligned}
\widetilde{a}_n &= \begin{bmatrix} \widetilde{\alpha}_n & 0 \\ 0 & \widetilde{\gamma}_{-n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\alpha_n & 0 \\ 0 & -\frac{1}{2}\alpha_{-n-1} \end{bmatrix} = -\frac{1}{2} q a_n, \quad q := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \\
\widetilde{c}_n &= \begin{bmatrix} \widetilde{\gamma}_n & 0 \\ 0 & \widetilde{\alpha}_{-n-1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\alpha_n & 0 \\ 0 & \frac{1}{2}\alpha_{-n-1} \end{bmatrix} = \frac{1}{2} q a_n, \quad n \in \mathbb{N}.
\end{aligned}
$$

Since $\widetilde{b}_n = 0, n \in \mathbb{N}_0$, we get conditions equivalent to (77): $\forall n = 2, 3, \ldots$

(82)
$$
\begin{aligned}
0 &= a_n(\frac{1}{2} q a_{n+1}) - (\frac{1}{2} q a_n) a_{n+1}, \\
\dot{a}_n &= b_n(\frac{1}{2} q a_n) - (\frac{1}{2} q a_n) b_{n+1}, \\
\dot{b}_n &= a_{n-1}(\frac{1}{2} q a_{n-1}) - a_n(\frac{1}{2} q a_n) + (\frac{1}{2} q a_{n-1}) a_{n-1} - (\frac{1}{2} q a_n) a_n, \\
\dot{a}_n &= -b_{n+1}(\frac{1}{2} q a_n) + (\frac{1}{2} q a_n) a_n, \\
0 &= -a_{n+1}(\frac{1}{2} q a_n) + (\frac{1}{2} q a_{n+1}) a_n.
\end{aligned}
$$

The first and the last conditions in (82) are obviously true; the other three equalities in (82) are equivalent to the following:

(83)
$$
\dot{a}_n = \frac{1}{2} a_n q (b_n - b_{n+1}), \quad \dot{b}_n = q(a_{n-1}^2 - a_n^2), \quad n = 2, 3, \ldots
$$

Consider the case $n = 1$. As earlier, we conclude that the first and last equalities in (82) are true and the second and the forth equalities are equivalent to the first equality in (83). For the next calculations it is necessary to find $\widetilde{a}_0, \widetilde{c}_0$. Using (80) and (76) we conclude:

(84)
$$
\widetilde{a}_0 = \frac{1}{2} \begin{bmatrix} \alpha_0 \\ -\alpha_{-1} \end{bmatrix} = -\frac{1}{2} q a_0, \quad \widetilde{c}_0 = \frac{1}{2}[-\alpha_0 \quad \alpha_{-1}] = \frac{1}{2} a_0^* q.
$$

Now the equality (78) with the help of (84), (81) and (76) gives:

$$
\begin{aligned}
\dot{b}_1 &= a_0(\tfrac{1}{2}a_0^* q) - a_1(\tfrac{1}{2}qa_1) + (\tfrac{1}{2}qa_0)a_0^* - (\tfrac{1}{2}qa_1)a_1 \\
&= -qa_1^2 + \frac{1}{2}\begin{bmatrix} -\alpha_0^2 & \alpha_0\alpha_{-1} \\ -\alpha_{-1}\alpha_0 & \alpha_{-1}^2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \alpha_0^2 & \alpha_0\alpha_{-1} \\ -\alpha_{-1}\alpha_0 & -\alpha_{-1}^2 \end{bmatrix} \\
&= -qa_1^2 + \begin{bmatrix} -\alpha_0^2 & 0 \\ 0 & \alpha_{-1}^2 \end{bmatrix}, \quad \text{i.e.} \quad \dot{b}_1 = -qa_1^2 + \frac{1}{2}(a_0 a_0^* q + q a_0 a_0^*).
\end{aligned}
$$

(85)

Consider the case $n = 0$. Now the equalities of type (77) have the form (79); using the formulas (76), (81), (84) we get:

$$
\begin{aligned}
0 &= a_0^*(\tfrac{1}{2}qa_1) - (\tfrac{1}{2}a_0^* q)a_1, \\
\dot{a}_0^* &= b_0(\tfrac{1}{2}a_0^* q) - (\tfrac{1}{2}a_0^* q)b_1, \\
\dot{b}_0 &= -a_0^*(\tfrac{1}{2}qa_0) - (\tfrac{1}{2}a_0^* q)a_0, \\
\dot{a}_0 &= -b_1(\tfrac{1}{2}qa_0) + (\tfrac{1}{2}qa_0)b_0, \\
0 &= -a_1(\tfrac{1}{2}qa_0) + (\tfrac{1}{2}qa_1)a_0.
\end{aligned}
$$

(86)

The first and the last conditions in (86) are true; the other three equalities in (86) are equivalent to the following:

(87) $$ \dot{a}_0^* = \frac{1}{2}(b_0 a_0^* - a_0^* b_1)q, \quad \dot{b}_0 = -a_0^* q a_0, \quad \dot{a}_0 = \frac{1}{2}q(a_0 b_0 - b_1 a_0). $$

As a result, we can state that the Lax equation (75) is equivalent to the system (83) for $n \in \mathbb{N}$ (where for $n = 1$ it is necessary to replace the second equality by (85)) and system (87) for $n = 0$. Also it is clear that in (87) the first equality is equivalent to the third one.

Now we formulate the final result of this Section.

Theorem 2. *The Toda lattice* (28) *with an assumption* $\alpha_n(t) > 0$ *and uniform boundedness of* $\alpha_n(t), \beta_n(t), t \in [0, T], T \le \infty, n \in \mathbb{Z}$, *is equivalent to the Lax equation*

(88) $$ \dot{\boldsymbol{J}}(t) = \boldsymbol{J}(t)A(t) - A(t)\boldsymbol{J}(t), \quad t \in [0, T), $$

with bounded operators $\boldsymbol{J}(t), A(t)$ *acting on the space* $\mathbf{l}_{2,0}$ (59) *and having the form of a Jacobi matrix* $J(t)$ *of type* (6) *with elements respectively*
(89)

$$
a_n = c_n = \begin{bmatrix} \alpha_n & 0 \\ 0 & \alpha_{-n-1} \end{bmatrix}, \quad b_n = \begin{bmatrix} \beta_n & 0 \\ 0 & \beta_{-n} \end{bmatrix}, \quad n \in \mathbb{N},
$$

$$
a_0 = \begin{bmatrix} \alpha_0 \\ \alpha_{-1} \end{bmatrix}, \quad b_0 = [\beta_0], \quad c_0 = a_0^*;
$$

$$
\widetilde{a}_n = -\frac{1}{2}qa_n, \quad \widetilde{b}_n = 0, \quad n \in \mathbb{N}_0, \quad \widetilde{c}_n = \frac{1}{2}qa_n, \quad n \in \mathbb{N}, \quad \widetilde{c}_0 = \frac{1}{2}a_0^* q; \quad q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

This Toda lattice is also equivalent to the system:

(90)
$$
\dot{a}_n = \frac{1}{2}a_n q(b_n - b_{n+1}), \quad n \in \mathbb{N}, \quad \dot{b}_n = q(a_{n-1}^2 - a_n^2), \quad n = 2, 3, \ldots,
$$

$$
\dot{b}_1 = \frac{1}{2}(a_0 a_0^* q + q a_0 a_0^*) - qa_1^2, \quad \dot{a}_0 = \frac{1}{2}q(a_0 b_0 - b_1 a_0), \quad \dot{b}_0 = -a_0^* q a_0.
$$

The proof follows from the calculations above. See formulas (76), (81), (84) and (83), (85), (87). Of course it is also easy to check directly that lattice (28) is equivalent to (90).

7. Direct and inverse spectral problems for a single-valued degenerate Jacobi matrix

Our aim is to get a solution of the Cauchy problem for a double-infinite Toda lattice by using Theorem 2 and scheme of Section 5. But now our Jacobi matrix (6) acts on the space $\mathbf{l}_{2,0}$ and is single-valued degenerate: its element a_0 is not invertible. The aim of this Section is to develop the spectral theory for such matrices.

Thus we consider the operator Jacobi matrix on the space $\mathbf{l}_{2,0}$ (59)

$$(91) \quad J = \begin{bmatrix} b_0 & a_0^* & 0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \begin{aligned} & a_0 = \begin{bmatrix} \alpha_0 \\ \alpha_{-1} \end{bmatrix} : \mathbb{C}^1 \to \mathbb{C}^2, \\ & a_0^* = [\alpha_0 \quad \alpha_{-1}] : \mathbb{C}^2 \to \mathbb{C}^1, \\ & b_0 = [\beta_0] : \mathbb{C}^1 \to \mathbb{C}^1; \\ & a_n, b_n \in L(\mathbb{C}^2), \ a_n > 0, \ b_n = b_n^*, \ n \in \mathbb{N}. \end{aligned}$$

All the matrices $a_n, b_n, n \in \mathbb{N}_0$, are real. For simplicity we will assume that the matrices $a_n = \begin{bmatrix} a_{n;0} & 0 \\ 0 & a_{n;1} \end{bmatrix}$, $b_n = \begin{bmatrix} b_{n;0} & 0 \\ 0 & b_{n;1} \end{bmatrix}$, $n \in \mathbb{N}$, are diagonal and $\alpha_{-1} > 0$ (only such matrices will be used in what follows, see Theorem 2).

We assume that elements a_n, b_n are uniformly bounded therefore J gives rise to a bounded selfadjoint operator $\boldsymbol{J}$ on the space $\mathbf{l}_{2,0}$. To construct its eigenfunction expansion consider the difference equations: let $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^\infty, \varphi_0(\lambda) =: \varphi_{0,0}(\lambda) \in \mathbb{R}^1, \varphi_n(\lambda) = (\varphi_{n,0}(\lambda), \varphi_{n,1}(\lambda)) \in \mathbb{R}^2, n \in \mathbb{N}, \lambda \in \mathbb{R}$,

$$(J\varphi(\lambda))_0 = b_0\varphi_0(\lambda) + a_0^*\varphi_1(\lambda) = \lambda\varphi_0(\lambda), \quad \text{i.e.}$$
$$(J\varphi(\lambda))_{0,0} = \beta_0\varphi_{0,0}(\lambda) + \alpha_0\varphi_{1,0}(\lambda) + \alpha_{-1}\varphi_{1,1}(\lambda) = \lambda\varphi_{0,0}(\lambda),$$
$$(J\varphi(\lambda))_1 = a_0\varphi_0(\lambda) + b_1\varphi_1(\lambda) + a_1\varphi_2(\lambda) = \lambda\varphi_1(\lambda), \quad \text{i.e.}$$
$$(92) \quad ((J\varphi(\lambda))_{1,0}, (J\varphi(\lambda))_{1,1}) = (\alpha_0\varphi_{0,0}(\lambda) + b_{1,0}\varphi_{1,0}(\lambda) + a_{1,0}\varphi_{2,0}(\lambda),$$
$$\alpha_{-1}\varphi_{0,0}(\lambda) + b_{1,1}\varphi_{1,1}(\lambda) + a_{1,1}\varphi_{2,1}(\lambda)) = (\lambda\varphi_{1,0}(\lambda), \lambda\varphi_{1,1}(\lambda)),$$
$$(J\varphi(\lambda))_n = a_{n-1}\varphi_{n-1}(\lambda) + b_n\varphi_n(\lambda) + a_n\varphi_{n+1}(\lambda) = \lambda\varphi_n(\lambda), \quad n = 2, 3, \dots$$

We put $\varphi_{0,0}(\lambda) = c_{0,0}, \varphi_{1,0}(\lambda) = c_{1,0}$ where $c_{0,0}, c_{1,0} \in \mathbb{R}$ are some fixed constants. Then from recurrence relations (92) we can find recursively

$$(93) \qquad \varphi_{1,1}(\lambda), \quad \varphi_{2,0}(\lambda), \quad \varphi_{2,1}(\lambda), \quad \varphi_{3,0}(\lambda), \quad \varphi_{3,1}(\lambda), \quad \dots$$

because $\alpha_{-1} > 0$ and all the matrices $a_n, n \in \mathbb{N}$, are invertible.

As a result, we find a solution with the initial data $c_{0,0}, c_{1,0}$:

$$(94) \quad \begin{aligned} & \varphi(\lambda) = (\varphi_0(\lambda) = \varphi_{0,0}(\lambda), \quad \varphi_n = (\varphi_{n,0}(\lambda), \varphi_{n,1}(\lambda)))_{n=1}^\infty = (\varphi_{n,\nu_n}(\lambda))_{\nu_n=0,1;n=0}^\infty, \\ & \varphi_{n,\nu_n}(\lambda)_{\nu_n=0,1} := \varphi_{n,0}(\lambda), \varphi_{n,1}(\lambda); \quad \varphi_{0,0}(\lambda) = c_{0,0}, \quad \varphi_{1,0}(\lambda) = c_{1,0}; \quad \lambda \in \mathbb{R}. \end{aligned}$$

Note that in (94) we assumed that for $n = 0$ we have only one index $\nu_0 = 0$; for $n = 1, 2, \dots$ we have two indexes $\nu_n = 0, 1$. Such an agreement will be used in what follows.

The functions $\varphi_{n,\nu_n}(\lambda)$ from (93), (94) are polynomials w.r.t. λ with real coefficients. The form of these polynomials will be essential for the inverse problem, we will establish it a little later.

Now we only want to note that the recursion (92) is linear and it is easy to see that every solution (94) is a linear combination of two solutions $P(0; \lambda), P(1; \lambda)$ with a corresponding initial data: $\forall \alpha = 0, 1$

$$(95) \quad \begin{aligned} & P(\alpha; \lambda) = (P_{n,\nu_n}(\alpha; \lambda))_{\nu_n=0,1;n=0}^\infty, \quad \text{where} \\ & P_{0,0}(0; \lambda) = 1, \quad P_{1,0}(0; \lambda) = 0 \quad \text{and} \quad P_{0,0}(1; \lambda) = 0, \quad P_{1,0}(1; \lambda) = 1. \end{aligned}$$

Thus we have:

$$(96) \qquad \varphi(\lambda) = c_{0,0}P(0;\lambda) + c_{1,0}P(1;\lambda), \quad c_{0,0} = \varphi_{0,0}, \quad c_{1,0} = \varphi_{1,0}.$$

This formula gives a solution of the difference equation (92) with the initial conditions $\varphi_{0,0}$ and $\varphi_{1,0}$.

Remark 4. Instead of basic polynomial solutions $P(0;\lambda), P(1;\lambda)$ of system (92) it is possible to use two solutions of this system that satisfy initial data that differ from (95). Note that we need two such solutions since every solution of (92) has two free parameters. The choice of initial data (95) is the most convenient and is reminiscent of the initial data for classical Jacobi matrices.

To construct the eigenfunctions expansion of the operator $\boldsymbol{J}$ we apply the projection spectral theorem (see, e.g. [13], Ch. 15). Denote by $\mathbf{l}_{2,0}(p)$ the weighted space of vectors $f = (f_n)_{n=0}^{\infty}, f_n \in \mathcal{H}_n$ (recall that $\mathcal{H}_0 = \mathbb{C}^1, \mathcal{H}_1 = \mathcal{H}_2 = \ldots = \mathbb{C}^2$) for which

$$\|f\|_{\mathbf{l}_{2,0}(p)}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{H}_n}^2 \, p_n < \infty, \quad (f,g)_{\mathbf{l}_{2,0}(p)} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n} p_n.$$

Here $p = (p_n)_{n=0}^{\infty}, p_n > 0$, is a given sequence of weights. In what follows, $p_n \geq 1$ and $\sum\limits_{n=0}^{\infty} p_n^{-1} < \infty$, therefore the embedding of the positive space $\mathbf{l}_{2,0}(p) \subset \mathbf{l}_{2,0}$ is quasinuclear. The corresponding negative space is $\mathbf{l}_{2,0}(p^{-1}), p^{-1} := (p_n^{-1})_{n=0}^{\infty}$. As a result, we construct the quasinuclear rigging

$$(97) \qquad \mathbf{l}_0 = (\mathbf{l}_{0,\mathrm{fin}})' \supset \mathbf{l}_{2,0}(p^{-1}) \supset \mathbf{l}_{2,0} \supset \mathbf{l}_{2,0}(p) \supset \mathbf{l}_{0,\mathrm{fin}}$$

($\mathbf{l}_0$ denotes the space of all sequences $f = (f_n)_{n=0}^{\infty}, f_n \in \mathcal{H}_n$; $\mathbf{l}_{0,\mathrm{fin}}$ is the corresponding space of finite sequences).

Now we will use the result from [1], Ch. 5, and [13], Ch. 15, on the generalized eigenvectors expansion for a bounded selfadjoint operator connected with the chain (97) in a standard way. For the operator $\boldsymbol{J}$ we have the representation

$$(98) \qquad \boldsymbol{J}f = \int_{\mathbb{R}} \lambda \boldsymbol{\Phi}(\lambda) d\sigma(\lambda) f, \quad f \in \mathbf{l}_{2,0}(p),$$

where $\boldsymbol{\Phi}(\lambda) : \mathbf{l}_{2,0}(p) \to \mathbf{l}_{2,0}(p^{-1})$ is a generalized projection operator and $d\sigma(\lambda)$ is a spectral measure (with a bounded support). Since the matrix J is real the operators $\boldsymbol{J}$ and $\boldsymbol{\Phi}(\lambda)$ are also real.

For all $f, g \in \mathbf{l}_{0,\mathrm{fin}}$ we have the Parseval equality

$$(99) \qquad (f,g)_{\mathbf{l}_{2,0}} = \int_{\mathbb{R}} (\boldsymbol{\Phi}(\lambda)f, g)_{\mathbf{l}_{2,0}} d\sigma(\lambda).$$

Extending by continuity, the equality takes place for all $f, g \in \mathbf{l}_{2,0}$.

Let us denote by $\pi_{n,\nu_n}, n \in \mathbb{N}$, the operator of orthogonal projection in $\mathbf{l}_{2,0}$ onto the one-dimensional subspace $\mathcal{H}_{n,\nu_n}$ of $\mathcal{H}_n$ consisting of the $f_{n,\nu_n} \in \mathbb{C}^1; \pi_0 = \pi_{0,0}$ is an analogous projection on $\mathcal{H}_0 = \mathbb{C}^1$. Hence for all $f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_{2,0}$ we have $f_{n,\nu_n} = \pi_{n,\nu_n}f, f_0 = \pi_0 f$. This operator acts analogously in the spaces $\mathbf{l}_{2,0}(p)$ and $\mathbf{l}_{2,0}(p^{-1})$ but possibly with a norm which is not equal to one.

Let us consider the scalar matrix $(\Phi_{j,\nu_j;k,\nu_k}(\lambda))_{\nu_j,\nu_k=0,1;j,k=0}^{\infty}$ where

$$(100) \qquad \Phi_{j,\nu_j;k,\nu_k}(\lambda) = \pi_{j,\nu_j} \boldsymbol{\Phi}(\lambda) \pi_{k,\nu_k} : \mathbf{l}_{2,0} \to \mathcal{H}_{j,\nu_j} \subset \mathbf{l}_{2,0}.$$

As we have arranged in (94), here and further we assume that the index n, ν_n for $n = 0$ is equal to $0, 0$ (*one time*).

The generalized projection operator $\boldsymbol{\Phi}(\lambda)$ is real, therefore the elements of its matrix (100) are also real.

The Parseval equality (99) can be rewritten as follows: $\forall f, g \in \mathbf{l}_{2,0}$

$$
(f,g)_{\mathbf{l}_{2,0}} = \sum_{j,k=0;\nu_j,\nu_k=0,1}^{\infty} \int_{\mathbb{R}} (\mathbf{\Phi}(\lambda)\pi_{k,\nu_k}f, \pi_{j,\nu_j}g)_{\mathbf{l}_{2,0}} d\sigma(\lambda)
$$

(101)
$$
= \sum_{j,k=0;\nu_j,\nu_k=0,1}^{\infty} \int_{\mathbb{R}} (\pi_{j,\nu_j}\mathbf{\Phi}(\lambda)\pi_{k,\nu_k}f, g)_{\mathbf{l}_{2,0}} d\sigma(\lambda)
$$

$$
= \sum_{j,k=0;\nu_j,\nu_k=0,1}^{\infty} \int_{\mathbb{R}} \Phi_{j,\nu_j;k,\nu_k}(\lambda)f_{k,\nu_k}\overline{g_{j,\nu_j}} d\sigma(\lambda).
$$

It is essential to give a representation of elements of matrix (100) via polynomials (95).

Lemma 7. *For every fixed* $j, k \in \mathbb{N}_0$ *and* $\nu_j, \nu_k = 0, 1$, *the matrix* (100) *has the following representation:* $\forall \lambda \in \mathbb{R}$

(102)
$$
\begin{aligned}
\Phi_{j,\nu_j;k,\nu_k}(\lambda) &= \Phi_{0,0;0,0}(\lambda)P_{j,\nu_j}(0;\lambda)P_{k,\nu_k}(0;\lambda) + \Phi_{0,0;1,0}(\lambda)P_{j,\nu_j}(0;\lambda)P_{k,\nu_k}(1;\lambda) \\
&+ \Phi_{1,0;0,0}(\lambda)P_{j,\nu_j}(1;\lambda)P_{k,\nu_k}(0;\lambda) + \Phi_{1,0;1,0}(\lambda)P_{j,\nu_j}(1;\lambda)P_{k,\nu_k}(1;\lambda).
\end{aligned}
$$

Proof. For fixed $k \in \mathbb{N}_0$ and $\nu_k \in \{0,1\}$, the vector $\varphi(\lambda) = (\varphi_{j,\nu_j}(\lambda))_{\nu_j=0,1;j=0}^{\infty}$, where

(103)
$$
\varphi_{j,\nu_j}(\lambda) = \Phi_{j,\nu_j;k,\nu_k}(\lambda) = (\mathbf{\Phi}(\lambda)\pi_{k,\nu_k}f)_{j,\nu_j}
$$

and fixed $f \in \mathbf{l}_{0,\text{fin}}$ with $f_{k,\nu_k} = 1$ is a generalized solution, in $\mathbf{l}_0 = (\mathbf{l}_{0,\text{fin}})'$, of the equation $J\varphi(\lambda) = \lambda\varphi(\lambda)$, since $\mathbf{\Phi}(\lambda)$ is a projector onto a generalized eigenvector of the selfadjoint operator $\boldsymbol{J}$ with the corresponding generalized eigenvalue λ. Therefore for all $g \in \mathbf{l}_{0,\text{fin}}$ we have $(\varphi(\lambda), Jg)_{\mathbf{l}_{2,0}} = \lambda(\varphi(\lambda), g)_{\mathbf{l}_{2,0}}$. Transforming the finite difference Hermitian expression J to $\varphi(\lambda)$ we get $(J\varphi(\lambda), g)_{\mathbf{l}_{2,0}} = \lambda(\varphi(\lambda), g)_{\mathbf{l}_{2,0}}$. Hence, it follows that $\varphi(\lambda) \in \mathbf{l}_{2,0}(p^{-1})$ exists as a usual real solution of the difference equation $J\varphi = \lambda\varphi$, i.e. (92), with the initial conditions $\varphi_{0,0}(\lambda) = \Phi_{0,0;k,\nu_k}(\lambda)$ and $\varphi_{1,0}(\lambda) = \Phi_{1,0;k,\nu_k}(\lambda)$.

Using the representation (96) of solutions of equation (92) by its initial conditions we can write:

(104)
$$
\begin{aligned}
\Phi_{j,\nu_j;k,\nu_k}(\lambda) &= \Phi_{0,0;k,\nu_k}(\lambda)P_{j,\nu_j}(0;\lambda) + \Phi_{1,0;k,\nu_k}(\lambda)P_{j,\nu_j}(1;\lambda), \\
&\quad j \in \mathbb{N}_0, \quad \nu_j \in \{0,1\}.
\end{aligned}
$$

Operator $\mathbf{\Phi}(\lambda)$ is formally selfadjoint therefore its matrix (100) is selfadjoint w.r.t. indexes j, ν_j and k, ν_k and, similarly to (103), is a solution of equation (92) w.r.t. k, ν_k with arbitrary fixed j, ν_j. So, using (96) we get the representation similar to (104) for $\Phi_{0,0;k,\nu_k}(\lambda)$ and $\Phi_{1,0;k,\nu_k}(\lambda)$:

(105)
$$
\begin{aligned}
\Phi_{0,0;k,\nu_k}(\lambda) &= \Phi_{0,0;0,0}(\lambda)P_{k,\nu_k}(0;\lambda) + \Phi_{0,0;1,0}(\lambda)P_{k,\nu_k}(1;\lambda), \\
\Phi_{1,0;k,\nu_k}(\lambda) &= \Phi_{1,0;0,0}(\lambda)P_{k,\nu_k}(0;\lambda) + \Phi_{1,0;1,0}(\lambda)P_{k,\nu_k}(1;\lambda), \quad k \in \mathbb{N}_0, \ \nu_k \in \{0,1\}.
\end{aligned}
$$

Substituting (105) into (104), we get the representation (102). $\square$

Introduce the Fourier transform $\widehat{}$ (w.r.t. our eigenfunction expansion), at first for finite $f = (f_n)_{n=0}^{\infty} = (f_{n,\nu_n})_{\nu_n=0,1;n=0}^{\infty} \in \mathbf{l}_{0,\text{fin}}$. It is a vector-valued function $\widehat{f}(\lambda) = (\widehat{f}_0(\lambda), \widehat{f}_1(\lambda)) \in \mathbb{C}^2$ of $\lambda \in \mathbb{R}$ with the values

(106)
$$
\widehat{f}_\alpha(\lambda) = \sum_{n=0;\nu_n=0,1}^{\infty} P_{n,\nu_n}(\alpha;\lambda)f_{n,\nu_n}, \quad \alpha = 0, 1
$$

(recall that here the index n, ν_n for $n = 0$ is equal to $0,0$;).

Substitute the representation (102) into (101). For $f, g \in \mathbf{l}_{0,\text{fin}}$ using notation (106) we get:

$$(107) \quad (f, g)_{\mathbf{l}_{2,0}} = \int_{\mathbb{R}} (\Phi_{0,0;0,0}(\lambda) \overline{\widehat{g}_0(\lambda)} \widehat{f}_0(\lambda) + \Phi_{0,0;1,0}(\lambda) \overline{\widehat{g}_0(\lambda)} \widehat{f}_1(\lambda)$$

$$+ \Phi_{1,0;0,0}(\lambda) \overline{\widehat{g}_1(\lambda)} \widehat{f}_0(\lambda) + \Phi_{1,0;1,0}(\lambda) \overline{\widehat{g}_1(\lambda)} \widehat{f}_1(\lambda)) d\sigma(\lambda).$$

Introduce the 2×2-matrix-valued real measure by the formula

$$(108) \quad d\rho(\lambda) = \begin{bmatrix} \Phi_{0,0;0,0}(\lambda) & \Phi_{0,0;1,0}(\lambda) \\ \Phi_{1,0;0,0}(\lambda) & \Phi_{1,0;1,0}(\lambda) \end{bmatrix} d\sigma(\lambda) =: (d\rho_{\alpha,\beta}(\lambda))^1_{\alpha,\beta=0}, \quad \lambda \in \mathbb{R}.$$

Then the equality (107) can be rewritten in the form:
(109)
$$(f, g)_{\mathbf{l}_{2,0}} = \int_{\mathbb{R}} \left(\begin{bmatrix} \Phi_{0,0;0,0}(\lambda) & \Phi_{0,0;1,0}(\lambda) \\ \Phi_{1,0;0,0}(\lambda) & \Phi_{1,0;1,0}(\lambda) \end{bmatrix} \widehat{f}(\lambda), \widehat{g}(\lambda) \right)_{\mathbb{C}^2} d\sigma(\lambda) = \int_{\mathbb{R}} (d\rho(\lambda)\widehat{f}(\lambda), \widehat{g}(\lambda))_{\mathbb{C}^2}.$$

Lemma 8. *The matrix in (108) is nonnegative and real for almost all $\lambda \in \mathbb{R}$ w.r.t. the measure $d\sigma(\lambda)$.*

Proof. This fact follows from nonnegativity of the operator $\boldsymbol{\Phi}(\lambda)$ as an operator from $\mathbf{l}_{2,0}(p)$ into $\mathbf{l}_{2,0}(p^{-1})$. Indeed, definition (100) gives, for fixed λ, arbitrary $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$ and the corresponding vector $f = (f_0, f_1, 0, \ldots) = (\xi_0, (\xi_1, 0), (0,0), (0,0), \ldots) \in \mathbf{l}_{0,\text{fin}}$, the following:

$$\left(\begin{bmatrix} \Phi_{0,0;0,0}(\lambda) & \Phi_{0,0;1,0}(\lambda) \\ \Phi_{1,0;0,0}(\lambda) & \Phi_{1,0;1,0}(\lambda) \end{bmatrix} \xi, \xi \right)_{\mathbb{C}^2} = \left(\begin{bmatrix} \pi_0 \boldsymbol{\Phi}(\lambda) \pi_0 & \pi_0 \boldsymbol{\Phi}(\lambda) \pi_{1,0} \\ \pi_{1,0} \boldsymbol{\Phi}(\lambda) \pi_0 & \pi_{1,0} \boldsymbol{\Phi}(\lambda) \pi_{1,0} \end{bmatrix} \xi, \xi \right)_{\mathbb{C}^2}$$

$$= \pi_0 \boldsymbol{\Phi}(\lambda) \pi_0 \xi_0 \overline{\xi_0} + \pi_0 \boldsymbol{\Phi}(\lambda) \pi_{1,0} \xi_1 \overline{\xi_0} + \pi_{1,0} \boldsymbol{\Phi}(\lambda) \pi_0 \xi_0 \overline{\xi_1} + \pi_{1,0} \boldsymbol{\Phi}(\lambda) \pi_{1,0} \xi_1 \overline{\xi_1}$$

$$= (\boldsymbol{\Phi}(\lambda)(\pi_0 + \pi_{1,0})f, (\pi_0 + \pi_{1,0})f)_{\mathbf{l}_{2,0}} \geq 0.$$

$\square$

As a result, the measure (108) is a finite nonnegative 2×2-matrix-valued real Borel measure. This measure we will call the matrix spectral measure of our operator $\boldsymbol{J}$. We can introduce, as in Section 1, the corresponding L^2-type Hilbert space $\mathbf{L}^2 = L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$ with the product (14).

The Parseval equality (109) now has the form: $\forall f, g \in \mathbf{l}_{0,\text{fin}}$

$$(110) \quad (f, g)_{\mathbf{l}_{2,0}} = \int_{\mathbb{R}} (d\rho(\lambda)\widehat{f}(\lambda), \widehat{g}(\lambda))_{\mathbb{C}^2} = (\widehat{f}(\cdot), \widehat{g}(\cdot))_{\mathbf{L}^2}.$$

After extending by continuity, this equality also will be true for arbitrary $f, g \in \mathbf{l}_{2,0}$. Note that the Fourier transform of $f \in \mathbf{l}_{2,0}$ is defined as a corresponding limit in the space $\mathbf{L}^2$ of the Fourier transform (106) of vectors from $\mathbf{l}_{0,\text{fin}}$ approximating f. In this case in (110) we have an extension of the integral appearing there.

As in the classical theory of Jacobi matrices the polynomials $P_{n,\nu_n}(\alpha; \lambda)$ are orthogonal in some sense w.r.t. matrix spectral measure $d\rho(\lambda)$. The conditions of orthogonality are very easy to deduce from the Parseval equality (110) and formula (106).

So, according to (106) the polynomial $P_{n,\nu_n}(\alpha; \lambda)$ for $n \in \mathbb{N}$, $\nu_n = 0, 1$, is equal to the α-coordinate $\widehat{f}_\alpha(\lambda)$ of the Fourier transform $\widehat{f}(\lambda) = (\widehat{f}_0(\lambda), \widehat{f}_1(\lambda))$ of the finite sequence $f = (0, \ldots, 0, 1, 0, 0, \ldots) =: \varepsilon_{n,\nu_n}$ with 1 at the n, ν_n place; for $P_{0,0}(\alpha; \lambda)$ $f = (1, 0, 0, \ldots) =:$

$\varepsilon_{0,0}$. Therefore the Parseval equality (110) gives: $\forall n, m \in \mathbb{N}_0, \nu_n, \nu_m \in \{0,1\}$

$$(111) \quad \int_{\mathbb{R}} \sum_{\alpha,\beta=0}^{1} d\rho_{\alpha,\beta}(\lambda) P_{n,\nu_n}(\beta;\lambda) P_{m,\nu_m}(\alpha;\lambda) = \int_{\mathbb{R}} (d\rho(\lambda)\widehat{\varepsilon}_{n,\nu_n}(\lambda), \widehat{\varepsilon}_{m,\nu_m}(\lambda))_{\mathbb{C}^2}$$

$$= (\varepsilon_{n,\nu_n}, \varepsilon_{m,\nu_m})_{\mathbf{l}_{2,0}} = \delta_{n,m}\delta_{\nu_n,\nu_m}.$$

It is convenient to present the Fourier transform (106) and orthogonality (111) in a form which is a partial case of general definitions (17), (20) in the case $H = \mathbb{C}^2$. For this, introduce $\forall n \in \mathbb{N}_0$ by $P_{n,\nu_n}(\alpha;\lambda)$ the matrices

$$P_n(\lambda) = (P_{n;\alpha,\beta}(\lambda))_{\alpha,\beta=0}^{1}, \quad \text{where} \quad P_{n;\alpha,\beta}(\lambda) = P_{n,\alpha}(\beta;\lambda), \quad n \in \mathbb{N};$$

$$(112) \qquad P_0(\lambda) = [P_{0;0,0}(\lambda) \quad P_{0;0,1}(\lambda)], \quad \text{where} \quad P_{0;0,0}(\lambda) = P_{0,0}(0;\lambda) = 1,$$

$$P_{0;0,1}(\lambda) = P_{0,0}(1;\lambda) = 0, \quad \text{i.e.} \quad P_0(\lambda) = [1 \quad 0].$$

Then according to (106) $\forall f \in \mathbf{l}_{0,fin}$ we have:

$$\widehat{f}(\lambda) = (\widehat{f}_0(\lambda), \widehat{f}_1(\lambda))$$

$$(113) \qquad = \left(\sum_{n=0;\nu_n=0,1}^{\infty} P_{n,\nu_n}(0;\lambda) f_{n,\nu_n}, \sum_{n=0;\nu_n=0,1}^{\infty} P_{n,\nu_n}(1;\lambda) f_{n,\nu_n} \right) = \sum_{n=0}^{\infty} P_n^*(\lambda) f_n.$$

Therefore, we get the formula (17) but only changing $P_0^*(\lambda)$ from 1 to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The conditions (111) of orthogonality also have the form (20) for $P_n(\lambda)$ of the kind (112). Indeed, according to (112) and (111) we get $\forall j, k \in \mathbf{N}$:

$$\left(\int_{\mathbb{R}} P_j(\lambda) d\rho(\lambda) P_k^*(\lambda) \right)_{\alpha,\beta} = \sum_{\gamma,\nu=0}^{1} \int_{\mathbb{R}} P_{j;\alpha,\gamma}(\lambda) d\rho_{\gamma,\nu}(\lambda) P_{k;\beta,\nu}(\lambda)$$

$$= \sum_{\gamma,\nu=0}^{1} \int_{\mathbb{R}} P_{j,\alpha}(\gamma;\lambda) d\rho_{\gamma,\nu}(\lambda) P_{k,\beta}(\nu;\lambda) = (\varepsilon_{k,\beta}, \varepsilon_{j,\alpha})_{\mathbf{l}_{2,0}} = \delta_{k,j}\delta_{\beta,\alpha}.$$

Therefore $\forall j, k \in \mathbb{N}$

$$(114) \qquad \int_{\mathbb{R}} P_j(\lambda) d\rho(\lambda) P_k^*(\lambda) = \delta_{j,k} 1.$$

Analogous calculation gives (114) for $j = 0, k \in \mathbb{N}$ and $j \in \mathbb{N}, k = 0$. For $j = k = 0$ integral (114) is equal to $\int_{\mathbb{R}} d\rho_{0,0}(\lambda) = 1$.

We can conclude that the *Fourier transform* (106) *has the standard form* (17) *where* $H = \mathbb{C}^2$ *and the operator-valued polynomials* $P_n(\lambda), n \in \mathbb{N}_0$, *are defined by* (112). *They are orthonormal in the sense of* (114) *(for* $j = k = 0$ *unity* 1 *is the usual number)*.

Formulas of type (21) are also true in our case. Namely, *we have for elements of the matrix* J (91):

$$(115) \qquad a_n^* = \int_{\mathbb{R}} \lambda P_n(\lambda) d\rho(\lambda) P_{n+1}^*(\lambda), \quad b_n = \int_{\mathbb{R}} \lambda P_n(\lambda) d\rho(\lambda) P_n^*(\lambda),$$

$$n \in \mathbb{N}_0; \quad a_n = a_n^*, \quad n \in \mathbb{N}.$$

Indeed, the Fourier transform of the operator $\boldsymbol{J}$ generated by J on the space $\mathbf{l}_{2,0}$ is the operator of multiplication by λ, this follows from (98). Using the orthogonality conditions of (114) for $j, k \in \mathbb{N}_0$ we find the formulas (115) easily.

In the general case of the space $\mathbf{l}_2 = \ell_2(H) = H \oplus H \oplus \ldots$, in Section 2 we constructed operator-valued polynomials $P_n(\lambda)$ by procedure of orthogonalization of sequence (23).

In our case of the space $\mathbf{l}_{2,0} = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \ldots$ the situation is more complicated because the first space $\mathbb{C}^1$ is different from the others $\mathbb{C}^2$ and the operator a_0 is not invertible.

Now, unlike Sections 2, 5, it is more convenient to use the ordinary orthogonalization in the Hilbert space $\mathbf{L}^2 = L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$. Consider the corresponding procedure.

First, it is necessary to establish the structure of the polynomials $P_n(\lambda), n \in \mathbb{N}_0$, constructed as solutions $P_{n,\alpha}(\beta; \lambda)$ of difference equations (92) with initial data (95).

So, from (92) for $n = 0, 1$ (the second and the fourth equalities), (95) and definition (112) we find:

$$P_0(\lambda) = [1 \quad 0],$$

$$P_1(\lambda) = \begin{bmatrix} P_{1,0}(0;\lambda) & P_{1,0}(1;\lambda) \\ P_{1,1}(0;\lambda) & P_{1,1}(1;\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{\alpha_{-1}}(\lambda - \beta_0) & -\frac{\alpha_0}{\alpha_{-1}} \end{bmatrix},$$

(116)
$$P_2(\lambda) = \begin{bmatrix} P_{2,0}(0;\lambda) & P_{2,0}(1;\lambda) \\ P_{2,1}(0;\lambda) & P_{2,1}(1;\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\alpha_0}{a_{1,0}} & \frac{1}{a_{1,0}}(-b_{1,0} + \lambda) \\ \frac{1}{a_{1,1}\alpha_{-1}}(-\alpha_{-1}^2 + (-b_{1,1} + \lambda)(-\beta_0 + \lambda)) & \frac{\alpha_0}{a_{1,1}\alpha_{-1}}(b_{1,1} - \lambda) \end{bmatrix}.$$

For $n = 2, 3, \ldots$ the last equality in (92) gives: $\forall \beta = 0, 1$

$$(a_{n-1,0}P_{n-1,0}(\beta;\lambda) + b_{n,0}P_{n,0}(\beta;\lambda) + a_{n,0}P_{n+1,0}(\beta,\lambda), a_{n-1,1}P_{n-1,1}(\beta;\lambda)$$
$$+ b_{n,1}P_{n,1}(\beta;\lambda) + a_{n,1}P_{n+1,1}(\beta;\lambda)) = (\lambda P_{n,0}(\beta;\lambda), \lambda P_{n,1}(\beta;\lambda)).$$

From this equality we conclude that

$$P_{n+1}(\lambda) = \begin{bmatrix} P_{n+1,0}(0;\lambda) & P_{n+1,0}(1;\lambda) \\ P_{n+1,1}(0;\lambda) & P_{n+1,1}(1;\lambda) \end{bmatrix}$$

(117)
$$= \begin{bmatrix} \frac{1}{a_{n,0}}(-a_{n-1,0}P_{n-1,0}(0;\lambda) - (b_{n,0} - \lambda)P_{n,0}(0;\lambda)) \\ \frac{1}{a_{n,1}}(-a_{n-1,1}P_{n-1,1}(0;\lambda) - (b_{n,1} - \lambda)P_{n,1}(0;\lambda)) \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{a_{n,0}}(-a_{n-1,0}P_{n-1,0}(1;\lambda) - (b_{n,0} - \lambda)P_{n,0}(1;\lambda)) \\ \frac{1}{a_{n,1}}(-a_{n-1,1}P_{n-1,1}(1;\lambda) - (b_{n,1} - \lambda)P_{n,1}(1;\lambda)) \end{bmatrix}, \quad n = 2, 3, \ldots.$$

Denote by $e_{n,0} = (1,0), e_{n,1} = (0,1)$ the standard basis in the space $\mathcal{H}_n = \mathbb{C}^2, n \in \mathbb{N}; e_{0,0} = e_0 = 1 \in \mathbb{C}^1 = \mathcal{H}_0$. The vectors

(118)
$$\varepsilon_{0,0} = \varepsilon_0 = (e_0, 0, \ldots), \quad \varepsilon_{n,\nu_n} = (0, \ldots, 0, \underbrace{e_{n,\nu_n}}_{n,\nu_n \text{ place}}, 0, 0, \ldots) \in \mathbf{l}_{2,0},$$

$$n \in \mathbb{N}, \quad \nu_n \in \{0, 1\},$$

comprise an orthonormal basis in the space $\mathbf{l}_{2,0}$ (these vectors are the same as in (111)). Note that the projectors $\pi_{n,\nu_n} : \mathbf{l}_{2,0} \to \mathcal{H}_n$ used earlier are the projections onto ε_{n,ν_n}.

We will use this basis here and in what follows.

The equalities (113) and (112) give a sequence of vectors ($\mathbb{C}^2$-valued functions of $\lambda \in \mathbb{R}$) from the Hilbert space $\mathbf{L}^2$:

(119)
$$(\widehat{\varepsilon}_{0,0})(\lambda) = P_0^*(\lambda)e_{0,0} = (P_{0,0}(0;\lambda), P_{0,0}(1;\lambda)) = (1, 0),$$
$$(\widehat{\varepsilon}_{n,0})(\lambda) = P_n^*(\lambda)e_{n,0} = (P_{n,0}(0;\lambda), P_{n,0}(1;\lambda)),$$
$$(\widehat{\varepsilon}_{n,1})(\lambda) = P_n^*(\lambda)e_{n,1} = (P_{n,1}(0;\lambda), P_{n,1}(1;\lambda)), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}.$$

These vectors are orthonormal in $\mathbf{L}^2$. Indeed, using (114), (14) we get: $\forall e_{j,\alpha}, e_{k,\beta}$ where $j, k \in \mathbb{N}_0, \alpha, \beta \in \{0,1\}$

$$\delta_{j,k}\delta_{\alpha,\beta} = (\delta_{j,k}1 e_{k,\beta}, e_{j,\alpha})_{\mathbb{C}^2} = \left(\int\limits_{\mathbb{R}} P_j(\lambda) d\rho(\lambda) P_k^*(\lambda) e_{k,\beta}, e_{j,\alpha} \right)_{\mathbb{C}^2}$$

$$= \int\limits_{\mathbb{R}} \left(d\rho(\lambda) P_k^*(\lambda) e_{k,\beta}, P_j^*(\lambda) e_{j,\alpha} \right)_{\mathbb{C}^2} = (P_k^*(\lambda) e_{k,\beta}, P_j^*(\lambda) e_{j,\alpha})_{\mathbf{L}^2}.$$

It is easy to see from the formulas (116), (117) that the $\mathbb{C}^2$-valued functions (119) are linear combinations of the following initial system of $\mathbb{C}^2$-valued functions:

$$(120) \qquad (1,0); (0,1), (\lambda, 0); (0, \lambda), (\lambda^2, 0); \ldots; (0, \lambda^{n-1}), (\lambda^n, 0); \ldots, \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}.$$

More precisely, $\forall n \in \mathbb{N}$ the vector-valued function $\mathbb{R} \ni \lambda \mapsto P_n^*(\lambda) e_{n,0}$ ($\mathbb{R} \in \lambda \mapsto P_n^*(\lambda) e_{n,1}$) is a linear combination of the first $2n$ $(2n+1)$ functions from the sequence (120).

It is easy to understand that, similarly to **Lemma 1**, we have the following result.

Lemma 9. *Let the support of the measure $d\rho(\lambda)$ have an infinite set of points, then the functions* (120) *are linearly independent in the space* $\mathbf{L}^2$.

Proof. As in the proof of **Lemma 1** we construct the scalar measure $d\sigma(\lambda) = d\rho_{0,0}(\lambda) + d\rho_{1,1}(\lambda)$ and a nonnegative matrix $C(\lambda)$ such that $d\rho(\lambda) = C(\lambda) d\sigma(\lambda), \lambda \in \mathbb{R}$; this matrix is positive on the set of full measure $d\sigma(\lambda)$, this measure also has infinite support.

Assume that some finite linear combination $f(\lambda)$ of $\mathbb{C}^2$-valued functions (120) is equal to zero in the space $\mathbf{L}^2$, i.e.

$$0 = \|f(\cdot)\|_{\mathbf{L}^2}^2 = \int\limits_{\mathbb{R}} (d\rho(\lambda) f(\lambda), f(\lambda))_{\mathbb{C}^2} = \int\limits_{\mathbb{R}} (C(\lambda) f(\lambda), f(\lambda))_{\mathbb{C}^2} d\sigma(\lambda).$$

Then $(C(\lambda) f(\lambda), f(\lambda))_{\mathbb{C}^2} = 0$ for σ-almost all $\lambda \in \mathbb{R}$. But $C(\lambda)$ is invertible on the set of full measure $d\sigma(\lambda)$, therefore also $f(\lambda) = (f_0(\lambda), f_1(\lambda)) = 0$ for σ-almost all λ. The functions $f_0(\lambda), f_1(\lambda)$ are some ordinary polynomials, therefore their equality to zero means that all their coefficients also equal to zero (since $d\sigma(\lambda)$ has infinite support). In other words, the functions (120) are linearly independent. $\square$

This lemma allows us to apply the procedure of orthogonalization to the sequence (120). The result is unique and gives (119).

Results obtained in this Section can be formulated as the following theorem.

Theorem 3. *Consider a Jacobi matrix J of the form* (91) *with conditions on its entries formulated at the beginning of Section 7. This matrix gives rise to a selfadjoint bounded operator $\mathbf{J}$ on the space $\mathbf{l}_{2,0}$* (59).

The direct spectral problem for $\mathbf{J}$ consists of the following. The generalized eigenvectors of operator $\mathbf{J}$ are equal to solutions $P(\alpha; \lambda) = (P_{n,\nu_n}(\alpha; \lambda))_{\nu_n=0,1;n=0}^{\infty}, \alpha = 0, 1$, (95) *of difference equations* (92) *with the initial data*

$$(121) \qquad P_{0,0}(0; \lambda) = 1, \quad P_{1,0}(0; \lambda) = 0; \quad P_{0,0}(1; \lambda) = 0, \quad P_{1,0}(1; \lambda) = 1.$$

These eigenvectors compose a full system: if we introduce $\forall f \in \mathbf{l}_{0,fin}$ the Fourier transform (106)

$$(122) \quad \widehat{f}(\lambda) = (\widehat{f}_0(\lambda), \widehat{f}_1(\lambda)) \in \mathbb{C}^2, \quad \widehat{f}_\alpha(\lambda) = \sum_{n=0;\nu_n=0,1}^{\infty} P_{n,\nu_n}(\alpha; \lambda) f_{n,\nu_n}, \quad \alpha = 0, 1,$$

then we have the Parseval equality (110)

$$(123) \qquad (f,g)_{\mathbf{l}_{2,0}} = \int_{\mathbb{R}} (d\rho(\lambda)\widehat{f}(\lambda), \widehat{g}(\lambda))_{\mathbb{C}^2}.$$

Here $d\rho(\lambda) = (d\rho_{\alpha,\beta}(\lambda))^1_{\alpha,\beta=0}$ is a 2×2-matrix nonnegative spectral measure of the operator $\boldsymbol{J}$; it is a probability measure: $\rho(\mathbb{R}) = 1$. The mapping $\mathbf{l}_{2,0} \ni f \mapsto \widehat{f}(\lambda)$ after its extension by closure is a unitary operator between the spaces $\mathbf{l}_{2,0}$ and $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$.

The inverse spectral problem for $\boldsymbol{J}$ consists of the following. Assume we are given some 2×2-matrix nonnegative Borel probability measure $d\rho(\lambda)$ on $\mathbb{R}$ for which all the functions $\mathbb{R} \ni \lambda \mapsto (\lambda^n, \lambda^n) \in \mathbb{C}^2, n \in \mathbb{N}$, are integrable and support of $d\rho(\lambda)$ has an infinite set of points.

Then such a measure is the spectral measure of some operator $\boldsymbol{J}$ generated on the space $\mathbf{l}_{2,0}$ by Jacobi matrix J (91) of the indicated kind.

The elements of this matrix J are constructed by the following procedure. Consider in the space $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda))$ a sequence of vectors (i.e. $\mathbb{C}^2$-valued functions)

$$(124) \qquad \mathbb{R} \ni \lambda \mapsto (1,0), (0, \lambda^{n-1}), (\lambda^n, 0), \quad n \in \mathbb{N}.$$

Apply to these vectors the classical Gram-Schmidt procedure of orthonormalization in the order stated in (124).

As a result, we get the sequence of $\mathbb{C}^2$-valued polynomials

$$(125) \qquad \begin{aligned} \mathbb{R} \ni \lambda \mapsto (1,0) &= (P_{0,0}(0;\lambda), P_{0,0}(1;\lambda)), (P_{n,0}(0;\lambda), P_{n,0}(1;\lambda)), \\ &\quad (P_{n,1}(0;\lambda), P_{n,1}(1;\lambda)), \quad n \in \mathbb{N}. \end{aligned}$$

Using these polynomials we construct the 2×2-matrix valued polynomials

$$(126) \qquad P_0(\lambda) = [1 \quad 0], \quad P_n(\lambda) = \begin{bmatrix} P_{n,0}(0;\lambda) & P_{n,0}(1;\lambda) \\ P_{n,1}(0;\lambda) & P_{n,1}(1;\lambda) \end{bmatrix}, \quad n \in \mathbb{N}.$$

Then the elements of the matrix J are reconstructed by formulas (115).

If we started from the Jacobi matrix J (91) and the 2×2-matrix spectral measure $d\rho(\lambda)$ of the corresponding operator $\boldsymbol{J}$ then the procedure above gives entries of our initial matrix J.

We finish this Section with some simple facts.

Let $\mathcal{B}(\mathbb{R}) \ni \triangle \mapsto E(\triangle)$ be a resolution of identity of the operator $\boldsymbol{J}$, then it is possible to write

$$(127) \qquad E(\triangle) = \int_{\triangle} \boldsymbol{\Phi}(\lambda) d\sigma(\lambda)$$

(understanding $E(\triangle)$ as an operator from $\mathbf{l}_{2,0}(p)$ into $\mathbf{l}_{2,0}(p^{-1})$). Using (118), (127) and (100) we conclude: $\forall \triangle \in \mathcal{B}(\mathbb{R})$

$$(128) \qquad \begin{aligned} (E(\triangle)\varepsilon_{k,\nu_k}, \varepsilon_{j,\nu_j})_{\mathbf{l}_{2,0}} &= (E(\triangle)\pi_{k,\nu_k}\varepsilon_{k,\nu_k}, \pi_{j,\nu_j}\varepsilon_{j,\nu_j})_{\mathbf{l}_{2,0}} \\ &= (\pi_{j,\nu_j} E(\triangle)\pi_{k,\nu_k}\varepsilon_{k,\nu_k}, \varepsilon_{j,\nu_j})_{\mathbf{l}_{2,0}} = \left(\int_{\triangle} \Phi_{j,\nu_j;k,\nu_k}(\lambda) d\sigma(\lambda)\varepsilon_{k,\nu_k}, \varepsilon_{j,\nu_j} \right)_{\mathbf{l}_{2,0}} \\ &= \int_{\triangle} \Phi_{j,\nu_j;k,\nu_k}(\lambda) d\sigma(\lambda), \quad j,k \in \mathbb{N}_0; \quad \nu_j, \nu_k \in \{0,1\}. \end{aligned}$$

In particular, from (128) for k, ν_k and $j, \nu_j = 0,0; 1,0$ and (108) we get: $\forall \triangle \in \mathcal{B}(\mathbb{R})$

$$(E(\triangle)\varepsilon_{\alpha,0}, \varepsilon_{\beta,0})_{\mathbf{l}_{2,0}} = \int_{\triangle} \Phi_{\beta,0;\alpha,0}(\lambda) d\sigma(\lambda) = \rho_{\beta,\alpha}(\triangle), \quad \alpha, \beta = 0, 1.$$

But $(E(\triangle)\varepsilon_{\alpha,0}, \varepsilon_{\beta,0})_{l_{2,0}} = \overline{(E(\triangle)\varepsilon_{\beta,0}, \varepsilon_{\alpha,0})_{l_{2,0}}}$ and the function $\Phi_{j,\nu_j;k,\nu_k}(\lambda)$ is real. *There-fore we get an essential identity for the nonnegative and real spectral measure* (108):

$$(129) \quad \rho_{\alpha,\beta}(\triangle) = (E(\triangle)\varepsilon_{\alpha,0}, \varepsilon_{\beta,0})_{l_{2,0}} = \rho_{\beta,\alpha}(\triangle), \quad \triangle \in \mathcal{B}(\mathbb{R}), \quad \alpha,\beta = 0,1; \quad \rho(\mathbb{R}) = 1.$$

Analogously to (52), we introduce for our problem the matrix Weyl function

$$(130) \quad m(z) = \int\limits_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda) = \begin{bmatrix} (\boldsymbol{R}_z\varepsilon_{0,0}, \varepsilon_{0,0})_{l_{2,0}} & (\boldsymbol{R}_z\varepsilon_{0,0}, \varepsilon_{1,0})_{l_{2,0}} \\ (\boldsymbol{R}_z\varepsilon_{1,0}, \varepsilon_{0,0})_{l_{2,0}} & (\boldsymbol{R}_z\varepsilon_{1,0}, \varepsilon_{1,0})_{l_{2,0}} \end{bmatrix} = (m_{\alpha,\beta}(z))_{\alpha,\beta=0}^{1},$$

where $\boldsymbol{R}_z$ is the resolvent of operator $\boldsymbol{J}$. The function $m(z)$ defines the measure $d\rho(\lambda)$ uniquely; from (129) it follows that the matrix $m(z)$ is symmetric.

8. The equations for the matrix Weyl function and the spectral matrix

Every bounded operator $\boldsymbol{A} : l_{2,0} \to l_{2,0}$ can be written as a matrix in the basis (118), namely $\forall f \in l_{2,0}$ we have:

$$f = (f_n)_{n=0}^{\infty} = (f_{n,\nu_n})_{n=0,\nu_n=0,1}^{\infty}, \quad f_{n,\nu_n} = (f, \varepsilon_{n,\nu_n})_{l_{2,0}};$$

$$(131) \qquad \forall j \in \mathbb{N}_0, \quad \nu_j \in \{0,1\}, \quad (\boldsymbol{A}f)_{j,\nu_j} = \sum_{k=0,\nu_k=0,1}^{\infty} a_{j,\nu_j;k,\nu_k} f_{k,\nu_k},$$

$$a_{j,\nu_j;k,\nu_k} = (\boldsymbol{A}\varepsilon_{k,\nu_k}, \varepsilon_{j,\nu_j})_{l_{2,0}}, \quad \text{here} \quad 0, \nu_0 = 0, 0.$$

The matrices $a_{j;k} = a_{j,\nu_j;k,\nu_k}$ from (131) act in the following way:

$$(132) \qquad a_{j;k} = \begin{bmatrix} a_{j,0;k,0} & a_{j,0;k,1} \\ a_{j,1;k,0} & a_{j,1;k,1} \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2, \quad a_{0;k} = \begin{bmatrix} a_{0,0;k,0} \\ a_{0,1;k,0} \end{bmatrix} : \mathbb{C}^1 \to \mathbb{C}^2,$$

$$a_{j;0} = \begin{bmatrix} a_{j,0;0,0} & a_{j,0;0,1} \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^1, \quad j,k \in \mathbb{N}; \quad a_{0;0} = [a_{0,0;0,0}] : \mathbb{C}^1 \to \mathbb{C}^1.$$

Using these matrices, we can rewrite (131) in the form:

$$(133) \qquad\qquad (\boldsymbol{A}f)_j = \sum_{k=0}^{\infty} a_{j;k} f_k, \quad j \in \mathbb{N}_0.$$

The general Jacobi matrix B of type (6), connected with the space $l_{2,0}$ instead of $\ell_2(H)$, acts on $f = (f_n)_{n=0}^{\infty}, f_n \in \mathcal{H}_n$, by

$$(134) \quad \begin{aligned} &(Bf)_n = a_{n-1}f_{n-1} + b_n f_n + c_n f_{n+1}, \quad n \in \mathbb{N}; \quad (Bf)_0 = b_0 f_0 + c_0 f_1; \\ &a_n, b_n, c_n : \mathbb{C}^2 \to \mathbb{C}^2, \quad n \in \mathbb{N}; \quad a_0 : \mathbb{C}^1 \to \mathbb{C}^2, \quad b_0 : \mathbb{C}^1 \to \mathbb{C}^1, \quad c_0 : \mathbb{C}^2 \to \mathbb{C}^1. \end{aligned}$$

Using (134) it is easy to calculate that for vectors (118) we have:

$$(135) \quad \begin{aligned} &B\varepsilon_{n,\nu_n} = (\underbrace{0,\ldots,0}_{n-1}, c_{n-1}e_{n,\nu_n}, b_n e_{n,\nu_n}, a_n e_{n,\nu_n} 0, 0, \ldots), \quad n \in \mathbb{N}, \quad \nu_n \in \{0,1\}, \\ &B\varepsilon_{0,0} = (b_0 e_{0,0}, a_0 e_{0,0}, 0, 0, \ldots). \end{aligned}$$

Consider the Lax equation (88). As in (44) for resolvent $\boldsymbol{R}_z(t)$ of operator $\boldsymbol{J}(t)$ we get

$$(136) \qquad\qquad \dot{\boldsymbol{R}}_z(t) = \boldsymbol{R}_z(t)A(t) - A(t)\boldsymbol{R}_z(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0,T].$$

Introduce, in agreement with (131), (132), (133), a matrix of the operator $\boldsymbol{R}_z(t)$ in the basis (118): its entries are equal to

$$R_{z;j,\nu_j;k,\nu_k}(t) = (\boldsymbol{R}_z(t)\varepsilon_{k,\nu_k}, \varepsilon_{j,\nu_j})_{l_{2,0}}, \quad j,k \in \mathbb{N}_0, \quad \nu_j, \nu_k \in \{0,1\};$$

$$(137) \qquad (\boldsymbol{R}_z(t)f)_j = \sum_{k=0}^{\infty} R_{z;j;k}(t) f_k, \quad j \in \mathbb{N}_0.$$

Using these entries, we construct, according to (130), the matrix Weyl function $m(z;t) = (m_{\alpha,\beta}(z;t))_{\alpha,\beta=0}^{1}, z \in \mathbb{C} \setminus \mathbb{R}, t \in [0,T]$. Our next goal is to deduce a differential equation for $m(z;t)$ when the resolvent $\boldsymbol{R}_z(t)$ satisfies the Lax equality (136). The proof is analogous to the case (53) but more complicated since our operators act on the space $\mathbf{l}_{2,0} = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \dots$ instead of $H \oplus H \oplus \dots$ and the matrix $a_0 : \mathbb{C}^1 \to \mathbb{C}^2$ is not invertible.

Elements of the matrix $m(z;t)$ have the form (see (130))

$$
(138) \quad
\begin{aligned}
m_{0,0}(z;t) &= (\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = \overline{(\boldsymbol{R}_{\bar{z}}(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}} = \overline{m_{0,0}(\bar{z};t)}, \\
m_{0,1}(z;t) &= (\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = \overline{(\boldsymbol{R}_{\bar{z}}(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}} = \overline{m_{1,0}(\bar{z};t)} = m_{1,0}(z;t), \\
m_{1,1}(z;t) &= (\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = \overline{(\boldsymbol{R}_{\bar{z}}(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}} = \overline{m_{1,1}(\bar{z};t)}, \\
& \qquad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0,T].
\end{aligned}
$$

Consider $m_{0,0}(z;t)$. Using (136) we get:

$$
(139) \quad \dot{m}_{0,0}(z;t) = (\dot{\boldsymbol{R}}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)A(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} - (A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}.
$$

The second formula in (135), (137) and (89) gives (below a_n, b_n, c_n, where $n \in \mathbb{N}_0$, depend on t):

$$
(140) \quad
\begin{aligned}
A(t)\varepsilon_{0,0} &= (0, \widetilde{a}_0 1, 0, 0, \dots) = (0, \tfrac{1}{2}(\alpha_0, -\alpha_{-1}), 0, 0, \dots), \\
(\boldsymbol{R}_z(t)A(t)\varepsilon_{0,0})_{j=0}^{\infty} &= (R_{z;j;1}(t)\tfrac{1}{2}(\alpha_0, -\alpha_{-1}))_{j=0}^{\infty}; \quad \text{in particular} \\
(\boldsymbol{R}_z(t)A(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} &= (\boldsymbol{R}_z(t)A(t)\varepsilon_{0,0})_{0,0} = \tfrac{1}{2}R_{z;0,0;1,0}(t)\alpha_0 \\
& \quad - \tfrac{1}{2}R_{z;0,0;1,1}(t)\alpha_{-1} = \tfrac{1}{2}\alpha_0(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} - \tfrac{1}{2}\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}.
\end{aligned}
$$

Analogously using the first formula from (135) for $n = 1$ and (89) we get:

$$
(141) \quad
\begin{aligned}
\boldsymbol{R}_z(t)\varepsilon_{0,0} &= ((\boldsymbol{R}_z(t)\varepsilon_{0,0})_0, (\boldsymbol{R}_z(t)\varepsilon_{0,0})_1, \dots), \\
A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0} &= (\widetilde{c}_0(\boldsymbol{R}_z(t)\varepsilon_{0,0})_1, \widetilde{a}_0(\boldsymbol{R}_z(t)\varepsilon_{0,0})_0 + \widetilde{c}_1(\boldsymbol{R}_z(t)\varepsilon_{0,0})_2, \dots) \\
&= \left(\tfrac{1}{2}[-\alpha_0 \quad \alpha_{-1}](\boldsymbol{R}_z(t)\varepsilon_{0,0})_1, \widetilde{a}_0(\boldsymbol{R}_z(t)\varepsilon_{0,0})_0 + \widetilde{c}_1(\boldsymbol{R}_z(t)\varepsilon_{0,0})_2, \dots \right); \\
(A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} &= (A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0})_{0,0} = \tfrac{1}{2}[-\alpha_0 \quad \alpha_{-1}](\boldsymbol{R}_z(t)\varepsilon_{0,0})_1 \\
&= -\tfrac{1}{2}\alpha_0(\boldsymbol{R}_z(t)\varepsilon_{0,0})_{1,0} + \tfrac{1}{2}\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{0,0})_{1,1} = -\tfrac{1}{2}\alpha_0(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} \\
& \quad + \tfrac{1}{2}\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,1})_{\mathbf{l}_{2,0}}.
\end{aligned}
$$

From (139), (140), (141) and (138) we conclude that

$$
(142) \quad \dot{m}_{0,0}(z;t) = \alpha_0 m_{0,1}(z;t) - \tfrac{1}{2}\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,1})_{\mathbf{l}_{2,0}} - \tfrac{1}{2}\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}.
$$

Consider $m_{0,1}(z;t)$. Analogously to (139)–(141) we have:

$$
(143) \quad
\begin{aligned}
\dot{m}_{0,1}(z;t) &= (\dot{\boldsymbol{R}}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} \\
&= (\boldsymbol{R}_z(t)A(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} - (A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}, \\
(\boldsymbol{R}_z(t)A(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} &= (\boldsymbol{R}_z(t)A(t)\varepsilon_{0,0})_{1,0} = \tfrac{1}{2}R_{z;1,0;1,0}(t)\alpha_0 \\
& \quad - \tfrac{1}{2}R_{z;1,0;1,1}(t)\alpha_{-1} = \tfrac{1}{2}\alpha_0(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} - \tfrac{1}{2}\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}.
\end{aligned}
$$

Calculate
(144)
$$(A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = (A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0})_{1,0} = \pi_{1,0}(\widetilde{a}_0(\boldsymbol{R}_z(t)\varepsilon_{0,0})_0 + \widetilde{c}_1(\boldsymbol{R}_z(t)\varepsilon_{0,0})_2).$$

But according to (89) $\widetilde{a}_0 = -\frac{1}{2}qa_0 = \frac{1}{2}\begin{bmatrix} \alpha_0 \\ -\alpha_{-1} \end{bmatrix}$, $\widetilde{c}_1 = \frac{1}{2}qa_1 = \frac{1}{2}\begin{bmatrix} -\alpha_1 & 0 \\ 0 & \alpha_{-2} \end{bmatrix}$, therefore

$\pi_{1,0}\widetilde{a}_0 = \frac{1}{2}\begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix}$, $\pi_{1,0}\widetilde{c}_1 = \frac{1}{2}\begin{bmatrix} -\alpha_1 \\ 0 \end{bmatrix}$ and (144) gives

(145) $\qquad (A(t)\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = \frac{1}{2}\alpha_0(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} - \frac{1}{2}\alpha_1(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{2,0})_{\mathbf{l}_{2,0}}.$

Using (143), (144), the last equality, and (138) we find:

$$
\begin{aligned}
(146) \qquad \dot{m}_{0,1}(z;t) = & -\frac{1}{2}\alpha_0 m_{0,0}(z;t) + \frac{1}{2}\alpha_0 m_{1,1}(z;t) - \frac{1}{2}\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} \\
& + \frac{1}{2}\alpha_1(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{2,0})_{\mathbf{l}_{2,0}}.
\end{aligned}
$$

Consider $m_{1,1}(z;t)$. We have as before
(147)
$$\dot{m}_{1,1}(z;t) = (\dot{\boldsymbol{R}}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)A(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} - (A(t)\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}.$$

Now according to the first formula in (135) for $n = 1$ and (89) we have

$$
\begin{aligned}
A(t)\varepsilon_{1,0} &= (\widetilde{c}_0 e_{1,0}, 0, \widetilde{a}_1 e_{1,0}, 0, 0, \ldots) \\
&= \left(\frac{1}{2}[-\alpha_0 \quad \alpha_{-1}]e_{1,0}, 0, \frac{1}{2}\begin{bmatrix} \alpha_1 & 0 \\ 0 & -\alpha_{-2} \end{bmatrix} e_{1,0}, 0, 0, \ldots \right) \\
&= \left(-\frac{1}{2}\alpha_0, 0, \frac{1}{2}(\alpha_1, 0), 0, 0, \ldots \right).
\end{aligned}
$$

Analogously to (140) we get:

$$
(\boldsymbol{R}_z(t)A(t)\varepsilon_{1,0})_{j=0}^{\infty} = (R_{z;j;0}(t)(-\frac{1}{2}\alpha_0) + R_{z;j;2}(t)\frac{1}{2}(\alpha_1, 0))_{j=0}^{\infty};
$$

(148) $\qquad (\boldsymbol{R}_z(t)A(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = R_{z;1,0;0,0}(t)(-\frac{1}{2}\alpha_0) + R_{z;1,0;2,0}(t)\frac{1}{2}\alpha_1$

$$
= -\frac{1}{2}\alpha_0(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} + \frac{1}{2}\alpha_1(\boldsymbol{R}_z(t)\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}.
$$

For the second term in (147) we get analogously to (141):

$$
\begin{aligned}
\boldsymbol{R}_z(t)\varepsilon_{1,0} &= ((\boldsymbol{R}_z(t)\varepsilon_{1,0})_0, (\boldsymbol{R}_z(t)\varepsilon_{1,0})_1, \ldots), \\
A(t)\boldsymbol{R}_z(t)\varepsilon_{1,0} &= (\widetilde{c}_0(\boldsymbol{R}_z(t)\varepsilon_{1,0})_1, \widetilde{a}_0(\boldsymbol{R}_z(t)\varepsilon_{1,0})_0 + \widetilde{c}_1(\boldsymbol{R}_z(t)\varepsilon_{1,0})_2, \ldots) \\
&= (\frac{1}{2}[-\alpha_0 \quad \alpha_{-1}](\boldsymbol{R}_z(t)\varepsilon_{1,0})_1, \widetilde{a}_0(\boldsymbol{R}_z(t)\varepsilon_{1,0})_0 + \widetilde{c}_1(\boldsymbol{R}_z(t)\varepsilon_{1,0})_2, \ldots).
\end{aligned}
$$

Calculate

$$(A(t)\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = (A(t)\boldsymbol{R}_z(t)\varepsilon_{1,0})_{1,0} = \pi_{1,0}(\widetilde{a}_0(\boldsymbol{R}_z(t)\varepsilon_{1,0})_0 + \widetilde{c}_1(\boldsymbol{R}_z(t)\varepsilon_{1,0})_2).$$

As is the case (144), (145) we see that the last term is equal to $\frac{1}{2}\alpha_0(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} - \frac{1}{2}\alpha_1(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{2,0})_{\mathbf{l}_{2,0}}$. Thus, we calculated $(A(t)\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}$. Using (147), (148), this equality and (138) we have:

(149) $\qquad \dot{m}_{1,1}(z;t) = -\alpha_0 m_{0,1}(z;t) + \frac{1}{2}\alpha_1(\boldsymbol{R}_z\varepsilon_{1,0}, \varepsilon_{2,0})_{\mathbf{l}_{2,0}} + \frac{1}{2}\alpha_1(\boldsymbol{R}_z\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}.$

Every equality (142), (146) and (149), where $z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$, contains two last summands which do not contain the two functions from (138). We can find a system of differential equation for functions (138) in this case if we can express these summands by

means of functions (138). Such expression can be obtained similarly to (49), using the obvious equality

$$(150) \qquad 1 + z\boldsymbol{R}_z(t) = \boldsymbol{R}_z(t)\boldsymbol{J}(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T].$$

So, from (150), the second formula in (135), (89) and (137)

(151)

$$1 + zm_{0,0}(z; t) = ((1 + z\boldsymbol{R}_z(t))\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}},$$

$$\boldsymbol{J}(t)\varepsilon_{0,0} = (b_0 e_{0,0}, a_0 e_{0,0}, 0, 0, \ldots);$$

$$(\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{0,0})_{j=0}^{\infty} = (R_{z;j;0}(t)b_0 1 + R_{z;j;1}(t)(\alpha_0, \alpha_{-1}))_{j=0}^{\infty},$$

$$(\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{0,0})_{0,0} = R_{z;0,0;0,0}(t)\beta_0 + R_{z;0,0;1,0}(t)\alpha_0$$

$$+ R_{z;0,0;1,1}(t)\alpha_{-1} = \beta_0(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} + \alpha_0(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}$$

$$+ \alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = \beta_0 m_{0,0}(z; t) + \alpha_0 m_{1,0}(z; t) + \alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}.$$

From (151) and (138) we conclude: $\forall z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$

$$(152) \qquad \alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = 1 + zm_{0,0}(z; t) - \beta_0 m_{0,0}(z; t) - \alpha_0 m_{0,1}(z; t).$$

Analogously to (151) for $m_{0,1}(z; t)$:

$$zm_{0,1}(z; t) = ((1 + z\boldsymbol{R}_z(t))\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}},$$

$$(\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{0,0})_{1,0}$$

$$= R_{z;1,0;0,0}(t)\beta_0 + R_{z;1,0;1,0}(t)\alpha_0 + R_{z;1,0;1,1}(t)\alpha_{-1}$$

$$= \beta_0(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} + \alpha_0(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} + \alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}$$

$$= \beta_0 m_{0,1}(z; t) + \alpha_0 m_{1,1}(z; t) + \alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}.$$

As a result, we get $\forall z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$

$$(153) \qquad \alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{1,1}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = zm_{0,1}(z; t) - \beta_0 m_{0,1}(z; t) - \alpha_0 m_{1,1}(z; t).$$

It is convenient to perform similar calculations for $m_{1,0}(z; t)$ instead of the third equality in (138). So, we have

$$(154) \qquad zm_{1,0}(z; t) = ((1 + z\boldsymbol{R}_z(t))\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}.$$

Analogously to calculations in (151), (152) we get:

$$\boldsymbol{J}(t)\varepsilon_{1,0} = (c_0 e_{1,0}, b_1 e_{1,0}, a_1 e_{1,0}, 0, 0, \ldots) = ([\alpha_0 \quad \alpha_{-1}]e_{1,0}, b_1 e_{1,0}, \alpha_1 e_{1,0})$$

$$= (\alpha_0, (\beta_1, 0), (\alpha_1, 0), 0, 0, \ldots);$$

$$(\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{1,0})_{j=0}^{\infty} = (R_{z;j;0}(t)\alpha_0 + R_{z;j;1}(t)(\beta_1, 0) + R_{z;j;2}(t)(\alpha_1, 0))_{j=0}^{\infty};$$

$$(\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = R_{z;0,0;0,0}(t)\alpha_0 + R_{z;0,0;1,0}(t)\beta_1 + R_{z;0,0;2,0}(t)\alpha_1$$

$$= \alpha_0(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} + \beta_1(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} + \alpha_1(\boldsymbol{R}_z(t)\varepsilon_{2,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}$$

$$= \alpha_0 m_{0,0}(z; t) + \beta_1 m_{1,0}(z; t) + \alpha_1(\boldsymbol{R}_z(t)\varepsilon_{2,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}}.$$

These calculations and (154), (138) give: $\forall z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$

$$(155) \qquad \alpha_1(\boldsymbol{R}_z(t)\varepsilon_{2,0}, \varepsilon_{0,0})_{\mathbf{l}_{2,0}} = zm_{0,1}(z; t) - \alpha_0 m_{0,0}(z; t) - \beta_1 m_{0,1}(z; t).$$

Finally, for the function $m_{1,1}(z; t)$ we have analogously to (154), (155)

$$1 + zm_{1,1}(z; t) = ((1 + z\boldsymbol{R}_z(t))\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = (\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}},$$

$$(\boldsymbol{R}_z(t)\boldsymbol{J}(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = R_{z;1,0;0,0}(t)\alpha_0 + R_{z;1,0;1,0}(t)\beta_1 + R_{z;1,0;2,0}(t)\alpha_1$$

$$= \alpha_0(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} + \beta_1(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} + \alpha_1(\boldsymbol{R}_z(t)\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}$$

$$= \alpha_0 m_{0,1}(z; t) + \beta_1 m_{1,1}(z; t) + \alpha_1(\boldsymbol{R}_z(t)\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}}.$$

So, we get $\forall z \in \mathbb{C} \setminus \mathbb{R}, t \in [0, T)$

(156) $\alpha_1(\boldsymbol{R}_z(t)\varepsilon_{2,0}, \varepsilon_{1,0})_{\mathbf{l}_{2,0}} = 1 + zm_{1,1}(z; t) - \alpha_0 m_{0,1}(z; t) - \beta_1 m_{1,1}(z; t).$

It is necessary also to add to the equalities (152), (153), (155), (156) their simple modifications (see (138)):

$$\alpha_{-1}(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,1})_{\mathbf{l}_{2,0}} = \alpha_{-1}\overline{(\boldsymbol{R}_{\bar{z}}(t)\varepsilon_{1,1}, \varepsilon_{0,0})}_{\mathbf{l}_{2,0}}$$

$$= 1 + zm_{0,0}(z; t) - \alpha_0 m_{0,1}(z; t) - \beta_0 m_{0,0}(z; t),$$

(157) $$\alpha_1(\boldsymbol{R}_z(t)\varepsilon_{0,0}, \varepsilon_{2,0})_{\mathbf{l}_{2,0}} = \alpha_1\overline{(\boldsymbol{R}_{\bar{z}}(t)\varepsilon_{2,0}, \varepsilon_{0,0})}_{\mathbf{l}_{2,0}}$$

$$= zm_{0,1}(z; t) - \alpha_0 m_{0,0}(z; t) - \beta_1 m_{0,1}(z; t),$$

$$\alpha_1(\boldsymbol{R}_z(t)\varepsilon_{1,0}, \varepsilon_{2,0})_{\mathbf{l}_{2,0}} = \alpha_1\overline{(\boldsymbol{R}_{\bar{z}}(t)\varepsilon_{2,0}, \varepsilon_{1,0})}_{\mathbf{l}_{2,0}}$$

$$= 1 + zm_{1,1}(z; t) - \alpha_0 m_{0,1}(z; t) - \beta_1 m_{1,1}(z; t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T).$$

Substituting (152), (153), (156), (157) into (142), (146), (149) we get:

$$\dot{m}_{0,0}(z; t) = (\beta_0(t) - z)m_{0,0}(z; t) + 2\alpha_0(t)m_{0,1}(z; t) - 1,$$

(158) $$\dot{m}_{0,1}(z; t) = -\alpha_0(t)m_{0,0}(z; t) + \frac{1}{2}(\beta_0(t) - \beta_1(t))m_{0,1}(z; t) + \alpha_0(t)m_{1,1}(z; t),$$

$$\dot{m}_{1,1}(z; t) = -2\alpha_0(t)m_{0,1}(z; t) - (\beta_1(t) - z)m_{1,1}(z; t) + 1,$$

$$m_{1,0}(z; t) = m_{0,1}(z; t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0, T).$$

As a result, we can now formulate the following assertion.

Lemma 10. *The entries* (138) *of the matrix Weyl function* $(m_{\alpha,\beta}(z; t))^1_{\alpha,\beta=0}$ *with respect to* $t \in [0, T)$ *satisfy the linear system of differential equations* (158).

Rewrite the equations (158) in the form of equations for elements of the corresponding spectral matrix $(d\rho_{\alpha,\beta}(\lambda; t))^1_{\alpha,\beta=0}$ of the operator $\boldsymbol{J}(t), t \in [0, T)$. Recall that it is given by formulas (108), (129) and connected with the matrix Weyl function by (130).

Similar to [5, 12] and Section 5 we can write using (130), (138), (158): for a fixed $t \in [0, T)$ and $z \in \mathbb{C} \setminus \mathbb{R}$

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} d\dot{\rho}_{0,0}(\lambda; t) = \dot{m}_{0,0}(z; t)$$

(159) $$= (\beta_0(t) - z)m_{0,0}(z; t) + 2\alpha_0(t)m_{0,1}(z; t) - 1$$

$$= \int_{\mathbb{R}} \frac{\beta_0(t) - \lambda}{\lambda - z} d\rho_{0,0}(\lambda; t) + \int_{\mathbb{R}} \frac{2\alpha_0(t)}{\lambda - z} d\rho_{0,1}(\lambda; t).$$

Here we have used that $\rho(\mathbb{R}; t) = 1$ (see (129)) and therefore $\rho_{0,0}(\mathbb{R}; t) = 1$.

From (159) we conclude due to arbitrariness of z that

(160) $$\dot{\rho}_{0,0}(\lambda; t) = (\beta_0(t) - \lambda)\rho_{0,0}(\lambda; t) + 2\alpha_0(t)\rho_{0,1}(\lambda; t)$$

for almost all $\lambda \in \mathbb{R}$ w.r.t. the scalar spectral measure $d\sigma(\lambda; t)$ of the operator $\boldsymbol{J}(t)$ and the functions $\rho_{0,0}(\lambda; t), \rho_{0,1}(\lambda; t)$ are continuously differentiable in t.

Analogously to (159), (160) we can write all other equations for $\rho_{\alpha,\beta}(\alpha; t)$ which are equivalent to the differential equations (158). As a result, we have the following system for three unknowns: $\rho_{0,0}(\lambda; t), \rho_{0,1}(\lambda; t), \rho_{1,1}(\lambda; t)$:

$$\dot{\rho}_{0,0}(\lambda; t) = (\beta_0(t) - \lambda)\rho_{0,0}(\lambda; t) + 2\alpha_0(t)\rho_{0,1}(\lambda; t),$$

(161) $$\dot{\rho}_{0,1}(\lambda; t) = -\alpha_0(t)\rho_{0,0}(\lambda; t) + \frac{1}{2}(\beta_0(t) - \beta_1(t))\rho_{0,1}(\lambda; t) + \alpha_0(t)\rho_{1,1}(\lambda; t),$$

$$\dot{\rho}_{1,1}(\lambda; t) = -2\alpha_0(t)\rho_{0,1}(\lambda; t) - (\beta_1(t) - \lambda)\rho_{1,1}(\lambda; t);$$

$$\rho_{1,0}(\lambda; t) = \rho_{0,1}(\lambda; t), \quad \lambda \in \mathbb{R}, \quad t \in [0, T).$$

It is evident that the system (161) gives (158). So, we get the following result.

Lemma 11. *The Lax equation* (88) *leads to the system* (161) *for elements of the spectral matrix* $(\rho_{\alpha,\beta}(\lambda;t))^1_{\alpha,\beta=0}$ *of the operator* $\boldsymbol{J}(t)$.

According to **Theorem 2**, the Lax equation (88) is equivalent to the Toda lattice (28). Therefore from the first equality in (28) we get:

$$(162) \qquad \frac{1}{2}(\beta_0(t) - \beta_1(t)) = -\frac{\dot{\alpha}_0(t)}{\alpha_0(t)}, \quad \beta_1(t) = \beta_0(t) + 2\frac{\dot{\alpha}_0(t)}{\alpha_0(t)}, \quad t \in [0,T).$$

The system (161) now has the form:

$$
\begin{aligned}
&\dot{\rho}_{0,0}(\lambda;t) = (\beta_0(t) - \lambda)\rho_{0,0}(\lambda;t) + 2\alpha_0(t)\rho_{0,1}(\lambda;t), \\
&\dot{\rho}_{0,1}(\lambda;t) = -\alpha_0(t)\rho_{0,0}(\lambda;t) - \frac{\dot{\alpha}_0(t)}{\alpha_0(t)}\rho_{0,1}(\lambda;t) + \alpha_0(t)\rho_{1,1}(\lambda;t), \\
&\dot{\rho}_{1,1}(\lambda;t) = -2\alpha_0(t)\rho_{0,1}(\lambda;t) - \left(\beta_0(t) - \lambda + 2\frac{\dot{\alpha}_0(t)}{\alpha_0(t)}\right)\rho_{1,1}(\lambda;t); \\
&\rho_{1,0}(\lambda;t) = \rho_{0,1}(\lambda;t), \quad \lambda \in \mathbb{R}, \quad t \in [0,T).
\end{aligned}
$$
(163)

As a result, we have proved the following fact.

Theorem 4. *Consider the Toda lattice* (28) *under the assumption that* $\alpha_n(t) > 0$ *and uniformly boundedness of* $\alpha_n(t), \beta_n(t), t \in [0,T), T \le \infty, n \in \mathbb{Z}$. *This lattice is equivalent to the Lax equation* (88) *where* $\boldsymbol{J}(t), A(t)$ *are given by matrices of the form* (6) *with elements* (89) *acting on the space* $\mathrm{l}_{2,0}$ (59).

Operator $\boldsymbol{J}(t)$ *is a bounded selfadjoint operator on the space* $\mathrm{l}_{2,0}$. *Its* 2×2-*matrix valued nonnegative real and the probability spectral measure* $d\rho(\lambda;t) = (d\rho_{\alpha,\beta}(\lambda;t))^1_{\alpha,\beta=0}$ *with continuously differentiable in* t *entries* $\rho_{\alpha,\beta}(\lambda;t)$ *changes in* t *according to the system* (163) *of differential equations.*

For system (163) *it is possible to consider the Cauchy problem: given the initial data* $d\rho(\lambda;0)$, *to find the solution* $d\rho(\lambda;t), t \in [0,T)$. *This problem is solvable.*

9. The investigation of system of differential equations for the spectral matrix and the main theorem

Our first goal is to obtain some properties of solutions of system (163). Unknowns of this system form a real matrix. Also important is the determinant of this matrix:

$$
(164) \qquad
\begin{aligned}
&\rho(\lambda;t) = \begin{bmatrix} \rho_{0,0}(\lambda;t) & \rho_{0,1}(\lambda;t) \\ \rho_{0,1}(\lambda;t) & \rho_{1,1}(\lambda;t) \end{bmatrix}, \quad \lambda \in \mathbb{R}, \quad t \in [0,T); \\
&r(\lambda;t) = \mathrm{Det}\,\rho(\lambda;t).
\end{aligned}
$$

The initial data $\rho(\lambda;0)$ in (163) is generated by the spectral measure $d\rho(\lambda;0)$ of the bounded selfadjoint operator $\boldsymbol{J}(0)$: its increment on $\triangle \in \mathcal{B}(\mathbb{R})$ is equal to $\rho(\triangle;0), \rho(-\infty;0) = 0$. Therefore this matrix-valued function $\mathbb{R} \ni \lambda \mapsto \rho(\alpha;0)$ is equal to zero for $\lambda \in (-\infty, a]$ and 1 for $\lambda \in (b, \infty)$ where $[a,b]$ is the minimal closed interval containing the spectrum of $\boldsymbol{J}(0)$. Its values on $[a,b)$ are nonnegative real matrices, the function $[a,b) \ni \lambda \mapsto \rho(\lambda;0)$ is nondecreasing continuous from the left. The corresponding scalar function $r(\lambda;t)$ has analogous properties and changes from 0 to 1.

It is possible to simplify the system (163), namely the following result takes place:

Lemma 12. *For the determinant* $r(\lambda;t)$ *introduced above the following relations holds:*

$$(165) \qquad r(\lambda;t) = \frac{\alpha_0^2(0)}{\alpha_0^2(t)}r(\lambda;0), \quad \lambda \in \mathbb{R}, \quad t \in [0,T).$$

Proof. Using the equation (163) we get $\forall \lambda \in \mathbb{R}$ and $\forall t \in [0, T)$

$$\dot{r}(\lambda; t) = \dot{\rho}_{0,0}(\lambda; t)\rho_{1,1}(\lambda; t) + \rho_{0,0}(\lambda; t)\dot{\rho}_{1,1}(\lambda; t) - 2\rho_{0,1}(\lambda; t)\dot{\rho}_{0,1}(\lambda; t)$$

$$= -2\frac{\dot{\alpha}_0(t)}{\alpha_0(t)}r(\lambda; t).$$

As a result, we have a simple differential equation for $r(\lambda; t)$. Its integration gives:

$$\log r(\lambda; t) = \log r(\lambda; 0) - 2\int\limits_0^t \frac{\dot{\alpha}_0(\tau)}{\alpha_0(\tau)}d\tau = \log\left(r(\lambda; 0)\frac{\alpha_0^2(0)}{\alpha_0^2(t)}\right).$$

This equality is equivalent to (165). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $\rho(\lambda; 0)$ be the spectral measure of the operator $\boldsymbol{J}(0)$, then $\rho(\lambda; 0) = 1$ for $\lambda > b$, therefore the corresponding $r(\lambda; 0)$ also is equal to 1. For an arbitrary fixed $t > 0$ and for a sufficiently large λ $\rho(\lambda; t) = 1$ (the operator $\boldsymbol{J}(t)$ is bounded) and $r(\lambda; t) = 1$ also. For such λ the equality (165) gives that $\alpha_0(t) = \alpha_0(0)$. Hence *we have proved that*

$$(166) \qquad\qquad\qquad \alpha_0(t) = \alpha_0(0), \quad t \in [0, T).$$

Now we can rewrite the system (163) in the form:

$$(167) \qquad \begin{aligned} &\dot{\rho}_{0,0}(\lambda; t) = (\beta_0(t) - \lambda)\rho_{0,0}(\lambda; t) + 2\kappa\rho_{0,1}(\lambda; t), \\ &\dot{\rho}_{0,1}(\lambda; t) = -\kappa\rho_{0,0}(\lambda; t) + \kappa\rho_{1,1}(\lambda; t), \\ &\dot{\rho}_{1,1}(\lambda; t) = -2\kappa\rho_{0,1}(\lambda; t) - (\beta_0(t) - \lambda)(\rho_{1,1}(\lambda; t)), \\ &\kappa = \alpha_0(0), \quad \lambda \in \mathbb{R}, \quad t \in [0, T). \end{aligned}$$

So, we can now formulate the following main result.

Theorem 5. *Consider the double-infinite Toda lattice (28) and pose the following problem: find a solution $\alpha_n(t), \beta_n(t), t \in [0, T), T \leq \infty, n \in \mathbb{Z}$, of (28) which satisfies the following conditions:*

$$(168) \qquad\qquad \alpha_n(0), \beta_n(0), n \in \mathbb{Z}, \quad and \quad \beta_0(t), \quad t \in [0, T),$$

are given. Here $\alpha_n(t) > 0, \beta_n(t)$ are real continuously differentiable uniformly (w.r.t. n) bounded functions. So, we have the Cauchy problem for (28) with prescribed values of $\beta_0(t)$.

This solution can be found via the following procedure:

1) Consider the Jacobi matrix $J(0)$ of type (91) with elements $a_n(0), b_n(0), n \in \mathbb{N}_0$, constructed according to (89). Find its 2×2-matrix spectral measure $d\rho(\lambda; 0)$.

2) Find the evolution $d\rho(\lambda; t) = (d\rho_{\alpha,\beta}(\lambda; t))_{\alpha,\beta=0}^1$ of this measure. For this it is necessary to find a solution of the Cauchy problem for system (167) with the initial data $d\rho(\lambda; 0)$.

3) We know $d\rho(\lambda; t)$ for every fixed $t \in [0, T)$. Consider the Hilbert space

$$L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda; t))$$

and in this space the sequence of $\mathbb{C}^2$-valued functions (124). Apply to these vectors the classical procedure of orthonormalization, as a result we get $\mathbb{C}^2$-valued polynomials (125) (that depend on t). Using these polynomials we form the 2×2-matrix valued polynomials (126) $P_n(\lambda; t), n \in \mathbb{N}_0$.

4) The required solution $\alpha_n(t)$, $\beta_n(t)$, $t \in [0, T)$, of our initial problem is given by formulas (115) with the obtained $d\rho(\lambda; t)$, $P_n(\lambda; t)$; transferring from the calculated $a_n^*(t)$,

$b_n(t)$ *to* $\alpha_n(t)$, $\beta_n(t)$ *is given by the formulas* (89):

$$\begin{bmatrix} \alpha_n(t) & 0 \\ 0 & \alpha_{-n-1}(t) \end{bmatrix} = a_n(t) = a_n^*(t), \quad \begin{bmatrix} \beta_n(t) & 0 \\ 0 & \beta_{-n}(t) \end{bmatrix} = b_n(t), \quad n \in \mathbb{N};$$

$$\begin{bmatrix} \alpha_0(t) \\ \alpha_{-1}(t) \end{bmatrix} = a_0(t), \quad [\beta_0(t)] = b_0(t), \quad t \in [0, T).$$

The proof of this theorem follows from the results of Sections 6–8. It should be emphasized that we assumed that solutions of our problem (28), (168), exists.

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References

1. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
2. Yu. M. Berezansky, *The integration of semi-infinite Toda chain by means of inverse spectral problem*, Preprint 84.79, Institute of Mathematics, Acad. Sci. Ukr. SSR, Kiev, 1984.
3. Yu. M. Berezansky, *Integration of nonlinear difference equations by the inverse spectral problem method*, Soviet Math. Dokl. **21** (1985), 264–267.
4. Yu. M. Berezanskii, *A remark about the forced Toda lattice*, Ukrainian Math. J. **37** (1985), no. 3, 276–279.
5. Yu. M. Berezanski, *The integration of semi-infinite Toda chain by means of inverse spectral problem*, Rep. Math. Phys. **24** (1986), no. 1, 21–47.
6. Yu. M. Berezansky, *Integration of nonlinear nonisospectral difference-differential equations by means of the inverse spectral problem*, Nonlinear Physics, Theory and Experiment, Proc. of the First Workshop, Galliopoli (Lecce), Italy, June 29–July 7, 1995; World Scientific, Singapore–New Jersey–London–Hong Kong, 1996, pp. 11–20.
7. Yu. M. Berezansky, *Integration of the modified double-infinite Toda lattice with the help of inverse spectral problem*, Ukrain. Mat. Zh. **60** (2008), no. 4, 453–469.
8. Yu. M. Berezansky and M. E. Dudkin, *The direct and inverse spectral problems for the block Jacobi type unitary matrices*, Methods Funct. Anal. Topology **11** (2005), no. 4, 327–345.
9. Yu. M. Berezansky and M. E. Dudkin, *The complex moment problem and direct and inverse spectral problems for the block Jacobi type bounded normal matrices*, Methods Funct. Anal. Topology **12** (2006), no. 1, 1–31.
10. Yu. M. Berezanskii, M. I. Gekhtman, and M. E. Shmoish, *Integration of some chains of nonlinear difference equations by the method of the inverse spectral problem*, Ukrainian Math. J. **38** (1986), no. 1, 74–78.
11. Yu. M. Berezanskii and M. I. Gekhtman, *Inverse problem of the spectral analysis and non-Abelian chains of nonlinear equations*, Ukrainian Math. J. **42** (1990), no. 6, 645–658.
12. Yu. M. Berezansky and A. A. Mokhon'ko, *Integration of some differential-difference nonlinear equations using the spectral theory of normal block Jacobi matrices*, Funct. Anal. Appl. **42** (2008), no. 1, 1–18.
13. Yu. M. Berezansky, Z. G. Sheftel, G. F. Us, *Functional Analysis*, Vols. 1, 2, Birkhäuser Verlag, Basel–Boston–Berlin. 1996. (Russian edition: Vyshcha Shkola, Kiev, 1990)
14. Yu. M. Berezanskii and M. E. Shmoish, *Nonisospectral nonlinear difference equations*, Ukrainian Math. J. **42** (1990), no. 4, 492–495.
15. Yu. Berezansky and M. Shmoish, *Nonisospectral flows on semi-infinite Jacobi matrices*, Nonl. Math. Phys. **1** (1994), no. 2, 116–146.
16. M. Bruschi, S. V. Manakov, O. Ragnisco, and D. Levi, *The nonabelian Toda lattice – discrete analogue of the matrix Schrödinger spectral problem*, J. Math. Phys. **21** (1980), 2749–2753.
17. M. Bruschi, O. Ragnisco, and D. Levi, *Evolution equations associated with the discrete analog of the matrix Schrödinger spectral problem solvable by the inverse spectral transform*, J. Math. Phys. **22** (1981), 2463–2471.
18. A. Yu. Daletskii and G. B. Podkolzin, *A group approach to the integration of the infinite Toda chain*, Ukrainian Math. J. **40** (1988), no. 4, 445–447.
19. P. Deift, L. C. Li, and C. Tomei, *Toda flows with infinitely many variables*, J. Funct. Anal. **64** (1985), 358–402.
20. L. Faybusovich and M. Gekhtman, *Elementary Toda orbits and integrable lattices*, J. Math. Phys. **41** (2000), 2905–2921.
21. H. Flaschka, *On the Toda lattice.* I, Phys. Rev., B, **9** (1974), 1924–1925.

22. H. Flaschka, *On the Toda lattice*. II, Progr. Theoret. Phys. **51** (1974), 703–716.
23. M. I. Gekhtman, *Integration of non-Abelian Toda type lattices*, Funktsional. Anal. i Prilozhen. **24** (1990), no. 3, 72–73. (Russian)
24. F. Gesztesy, W. Renger, *New classes of Toda soliton solutions*, Comm. Math. Phys. **184** (1997), 27–50.
25. L. B. Golinskii, *Schur flows and orthogonal polynomials on the unit circle*, Mat. Sb. **197** (2006), no. 8, 41–62. (Russian)
26. I. Ya. Ivasiuk, *Direct spectral problem for the generalized Jacobi Hermitian matrices*, Methods Funct. Anal. Topology **15** (2009), no. 1, 3–14.
27. M. Kac and P. van Moerbeke, *A complete solution of the periodic Toda problem*, Proc. Nat. Acad. Sci. USA **72** (1975), no. 8, 2879-2880.
28. M. Kac and P. van Moerbeke, *On an explicitly soluble system of nonlinear differential equations related to certain Toda lattice*, Adv. Math. **16** (1975), no. 2, 160–169.
29. Y. Kodama, J. Ye, *Toda hierarchy with indefinite metric*, Physica D **91** (1996), 321–339.
30. Y. Kodama, J. Ye, *Toda lattice with indefinite metric* II: *topology of the iso-spectral manifolds*, Physica D **1949** (1998), 1–20.
31. M. G. Krein, *Infinite J-matrices and a matrix moment problem*, Dokl. Akad. Nauk SSSR **69** (1949), no. 2, 125–128. (Russian)
32. I. M. Krichever, *Periodic non-Abelian Toda lattice and its two-dimensional generalization*, Uspekhi Mat. Nauk **36** (1981), no. 2, 72–80 (application to the article by B. A. Dubrovin, Theta-functions and nonlinear equations) . (Russian)
33. B. M. Levitan, *Inverse Sturm-Liouville Problems*, Nauka, Moscow, 1984. (Russian)
34. S. V. Manakov, *On the complete integrability and stochastization of discrete dynamical systems*, Zh. Eksp. Teor. Fiz. **67** (1974), no. 2, 543–555. (Russian)
35. O. A. Mokhon'ko, *On some solvable classes of nonlinear nonisospectral difference equations*, Ukrainian Math. J. **57** (2005), no. 3, 427–439.
36. O. A. Mokhon'ko, *Nonisospectral flows on semiinfinite unitary block Jacobi matrices*, Ukrain. Mat. Zh. **60** (2008), no. 4, 521–544.
37. J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16** (1975), 197–220.
38. L. A. Sakhovich, *On the semi-infinite Toda lattice*, Teoret. Mat. Fizika **81** (1989), no. 1, 12–23. (Russian).
39. M. E. Shmoish, *Nonisospectral deformations of Jacobi matrices and nonlinear difference equations*, Ukrainian Math. J. **41** (1989), no. 2, 254–257.
40. M. Szczepanik, *Integration of semi-infinite Toda chain in some class of unbounded solutions*, Rep. Math. Phys. **39** (1997), no. 2, 263–273.
41. G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear lattices*, Math. Surveys Monogr., Vol. 72, Amer. Math. Soc., Providence, RI, 2000.
42. M. Toda, *Theory of Nonlinear Lattices*, Springer-Verlag, Berlin–Heidelberg–New York, 1981. (Russian translation: Mir, Moscow, 1984)
43. S. Venakides, *The Toda shock problem*, Contemporary Math. **122** (1991), 119–133.
44. S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Commun. Pure Appl. Math. **44** (1991), 1171–1242.
45. N. V. Zhernakov, *Direct and inverse problems for a periodic Jacobi matrix*, Ukrainian Math. J. **38** (1986), no. 6, 665–668.
46. N. V. Zhernakov, *Integration of Toda chains in the class of Hilbert-Schmidt operators*, Ukrainian Math. J. **39** (1987), no. 5, 527–530.
47. N. V. Zhernakov, *Spectral theory of some classes of Jacobi matrices and its application to the integration of Toda lattice*, Ph.D. Thesis, Institute of Mathematics, Acad. Sci. Ukr. SSR, Kiev, 1987. (Russian)
48. F. Gesztesy, H. Holden, J. Michor, G. Teschl, *Soliton Equations and Their Algebro-Geometric Solutions,* Vol. II, Cambridge University Press, Cambridge, 2008.

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