

## EXPANSION IN EIGENFUNCTIONS OF RELATIONS GENERATED BY PAIR OF OPERATOR DIFFERENTIAL EXPRESSIONS

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*Dedicated to blessed memory of Professor Alexander Povzner*

ABSTRACT. For relations generated by a pair of operator symmetric differential expressions, a class of generalized resolvents is found. These resolvents are integro-differential operators. The expansion in eigenfunctions of these relations is obtained.

### 1. The operator differential equation

$$(1) \quad l[y] - \lambda m[y] = m[f] \quad t \in \bar{\mathcal{I}}, \quad \mathcal{I} = (a, b) \subseteq R^1$$

is considered on finite or infinite intervals in the space of vector-functions with values in a separable Hilbert space  $\mathcal{H}$ , where  $l[y]$  and  $m[y]$  are symmetric operator differential expressions of order  $r$  and  $s$  respectively, where  $r + s > 0$ ,  $s$  is even. Expression  $m[y]$  is non-negative and such that an operator first-order system obtained from the homogeneous equation (1) by using quasi-derivatives contains a spectral parameter  $\lambda$  in Nevanlinna's manner.

In the paper, the equation (1) reduces in a special way to a symmetric first order system containing the spectral parameter either in a linear way ( $r > s$ ) or in a nonlinear way ( $r \leq s$ ). Using this reduction and the characteristic operator of this system (see [19], [20]) we construct a class of the generalized resolvents of the minimal relation corresponding to (1). These resolvents are integro-differential operators. From this the inversion formulas and Parseval's equality are obtained. For their proof we modify Strauss's method [27] concerning the case of the generalized resolvents as  $s = 0$  and  $m[y] \equiv y$  which are integral operators (but not integro-differential operators) depending on  $\lambda$  in a more simple way (see [1], [4], [5], [6], [16], [17], [18], [27]) comparing with the case  $s > 0$ .

The expansion formulas in the solutions of the homogeneous equation (1) were obtained in various particular situations in a number of papers. For  $\dim \mathcal{H} = 1$  in the regular case,  $r > s$ , and for special  $l$  and  $m$  this was done in [7, 14]. For  $\dim \mathcal{H} < \infty$ ,  $m[y] \equiv w(t)y$ ,  $0 \leq w(t) \in B(\mathcal{H})$ , the expansion formulas were obtained in [1] for  $r = 1$  and, for the general case, in [16]–[18] (see also [26] for the case  $r = 1$ ). Then for  $\dim \mathcal{H} < \infty$  the existence of the expansion formulas was proved in [11] under the assumption that the leading coefficient of the expression  $m[y]$  is nondegenerate, and the minimal differential operator that corresponds to this expression is uniformly positive on any finite interval (i.e. under these assumptions even the case  $m[y] \equiv w(t)y$  with degenerate weight  $w(t)$  is not covered). However in [11], as it was mentioned by the authors, the Titchmarsh-Kodaira's formula for an explicit calculation of the spectral matrix is not obtained. Also [11] does not contain an explicit expression for the resolvent and the case  $r = s$  is not considered.

Further in the paper, the boundary value problems for the equation (1) with boundary conditions depending on the spectral parameter are considered. We show that for some

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2000 *Mathematics Subject Classification.* Primary 34B05, 34B07, 34L10.

*Key words and phrases.* Relations generated by pair of operator differential expressions, characteristic operator, generalized resolvent, eigenfunction expansion.

boundary conditions, solutions of these problems are generated by a generalized resolvent if, in contrast to the case  $s = 0$ , the boundary conditions contain the derivatives of vector-function  $f(t)$  that are taken on the ends of interval.

In the general case where  $\dim \mathcal{H} < \infty$  we find the absolutely continuous part of the spectral matrix on the axis when the coefficients of the equation (1) are periodic on the semi-axes and also we find the spectral matrix on the semi-axis when the coefficients are periodic. These formulas are obtained using the results that are obtained in [18], [19] for  $s = 0$ .

Notice that it is not supposed in the paper that for  $r \geq s$  the leading coefficient of the expression  $m[y]$  (which set the metric) has an inverse in  $B(\mathcal{H})$ .

Many questions that concern differential operators and relations in the space of vector-functions are considered in the monographs [1, 2, 3, 13, 23, 24, 25, 26] containing an extensive literature. The method of studying these operators and relations based on a use of the abstract Weyl function was proposed in [9].

We denote by  $(\cdot)$  and  $\|\cdot\|$  the scalar product and the norm in various spaces with special indexes if it is necessary.

Let an interval  $\Delta \subseteq R$ ,  $f(t)$  ( $t \in \Delta$ ) be a function with values in some Banach space  $B$ . The notation  $f(t) \in C^l(\Delta)$ ,  $l = 0, 1, \dots$  (we omit the index  $l$  if  $l = 0$ ) means that, in any point of  $\Delta$ ,  $f(t)$  has a norm  $\|\cdot\|_B$  continuous derivatives of order up to and including  $l$  that are taken in the norm  $\|\cdot\|_B$ ; if  $\Delta$  is either semi-open or closed interval then on its ends belonging to  $\Delta$  the one-side continuous derivatives exist. The notation  $f(t) \in C_0^l(\Delta)$  means that  $f(t) \in C^l(\Delta)$  and  $f(t) = 0$  in the neighbourhoods of the ends of  $\Delta$ .

2. We consider an operator differential equation in a separable Hilbert space  $\mathcal{H}_1$ ,

$$(2) \quad \frac{i}{2} \left( (Q(t)x(t))' + Q^*(t)x'(t) \right) - H_\lambda(t)x(t) = W_\lambda(t)F(t), \quad t \in \bar{\mathcal{I}},$$

where  $Q(t)$ ,  $[\Re Q(t)]^{-1}$ ,  $H_\lambda(t) \in B(\mathcal{H}_1)$ ,  $Q(t) \in C^1(\bar{\mathcal{I}})$ ; the operator function  $H_\lambda(t) = H_\lambda^*(t)$  is continuous in  $t$  and is Nevanlinna's in  $\lambda$ . Namely, the following condition holds:

The set  $\mathcal{A} \supseteq C \setminus R^1$  exists, any of its points has a neighbourhood independent of  $t \in \bar{\mathcal{I}}$ , in this neighbourhood  $H_\lambda(t)$  is analytic  $\forall t \in \bar{\mathcal{I}}$ ;  $\forall \lambda \in \mathcal{A}$   $H_\lambda(t) \in C(\bar{\mathcal{I}})$ ; the weight  $W_\lambda(t) = \Im H_\lambda(t) / \Im \lambda \geq 0$  ( $\Im \lambda \neq 0$ ).

In view of [20]  $\forall \mu \in \mathcal{A} \cap R$ :  $W_\mu(t) = \partial H_\mu(t) / \partial \mu$  is Bochner locally integrable in the uniform operator topology.

For convenience of statements we suppose that  $0 \in \bar{\mathcal{I}}$  and we denote  $\Re Q(0) = G$ .

Let  $X_\lambda(t)$  be the operator solution of the homogeneous equation (2) satisfying the initial condition  $X_\lambda(0) = I$ , where  $I$  is an identity operator in  $\mathcal{H}_1$ .

For any  $\alpha, \beta \in \bar{\mathcal{I}}$ ,  $\alpha \leq \beta$  we denote

$$\Delta_\lambda(\alpha, \beta) = \int_\alpha^\beta X_\lambda^*(t)W_\lambda(t)X_\lambda(t)dt,$$

$$N = \{h \in \mathcal{H}_1 \mid h \in \text{Ker } \Delta_\lambda(\alpha, \beta) \forall \alpha, \beta\},$$

$P$  is an orthogonal projection onto  $N^\perp$ .  $N$  is independent of  $\lambda \in \mathcal{A}$  [20].

For  $x(t) \in \mathcal{H}_1$  or  $x(t) \in B(\mathcal{H}_1)$  we denote

$$U[x(t)] = ([\Re Q(t)]x(t), x(t)) \quad \text{or} \quad U[x(t)] = x^*(t)[\Re Q(t)]x(t),$$

respectively.

As in [20] we introduce the following.

**Definition 1.** An analytic operator-function  $M(\lambda) = M^*(\bar{\lambda}) \in B(\mathcal{H}_1)$  of non-real  $\lambda$  is called a characteristic operator (c.o.) of the equation (2) on  $\mathcal{I}$  (or, simply, c.o.), if for  $\Im \lambda \neq 0$  and for any  $\mathcal{H}_1$ -valued vector-function  $F(t) \in L_{W_\lambda}^2(\mathcal{I})$  with compact support the

corresponding solution  $x_\lambda(t)$  of the equation (3) of the form

$$(3) \quad \begin{aligned} x_\lambda(t, F) &= \mathcal{R}_\lambda F \\ &= \int_{\mathcal{I}} X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} \operatorname{sgn}(s-t) (iG)^{-1} \right\} X_\lambda^*(s) W_\lambda(s) F(s) ds \end{aligned}$$

satisfies the condition

$$(4) \quad (\Im \lambda) \lim_{(\alpha, \beta) \uparrow \mathcal{I}} (U[x_\lambda(\beta, F)] - U[x_\lambda(\alpha, F)]) \leq 0 \quad (\Im \lambda \neq 0).$$

The properties of c.o. and sufficient condition (that are close to necessary condition) of the c.o.'s existence are obtained in [19, 20].

We consider in the separable Hilbert space  $\mathcal{H}$  the equation (1), where  $l[y]$  and  $m[y]$  are symmetric differential expressions of orders  $r$  and  $s$  correspondingly (one of these orders can be equal to zero), where  $s$  is even, with sufficiently smooth coefficients from  $B(\mathcal{H})$ .

Namely,  $l[y] = \sum_{k=0}^r i^k l_k[y]$ , where  $l_{2j} = D^j p_j(t) D^j$ ,  $l_{2j-1} = \frac{1}{2} D^{j-1} \{ D q_j(t) + q_j^*(t) D \} D^{j-1}$ ,  $p_j(t) = p_j^*(t)$ ,  $q_j(t) \in B(\mathcal{H})$ ,  $p_j(t), q_j(t) \in C^j(\bar{\mathcal{I}})$ ,  $D = d/dt$ ;  $m[y]$  is defined in a similar way with  $s$  instead of  $r$  and  $\tilde{p}_j(t) = \tilde{p}_j^*(t)$ ,  $\tilde{q}_j(t) \in B(\mathcal{H})$  instead of  $p_j(t), q_j(t)$ .

We denote by  $p(t, \lambda)$  the coefficient at the highest-order derivative in the homogeneous equation (1), i.e.

$$p(t, \lambda) = \begin{cases} p_n(t), & r = 2n > s, \\ p_n(t) - \lambda \tilde{p}_n(t), & r = s = 2n, \\ -\lambda \tilde{p}_n(t), & s = 2n > r, \\ i \Re q_{n+1}(t), & r = 2n + 1 > s. \end{cases}$$

It is supposed in the paper that for non-real  $\lambda$ ,  $p^{-1}(t, \lambda) \in B(\mathcal{H})$  for any  $t \in \bar{\mathcal{I}}$ . We note that in that case where  $r = s$  the leading coefficients of both expressions  $l[y]$  and  $m[y]$  may not have inverses in  $B(\mathcal{H})$  (in particular simultaneously) for any  $t \in \bar{\mathcal{I}}$ .

Denote  $p = \max\{r, s\}$  and by  $y^{[k]}(t|L)$  we denote the quasi-derivatives [21] of the vector-function  $y(t)$  that corresponds to the differential expression  $L$ .

Using the substitution

$$(5) \quad \begin{aligned} x(t) &= x(t, \lambda) \\ &= \begin{cases} \left( \sum_{j=0}^{n-1} \oplus y^{(j)}(t) \right) \oplus \sum_{j=1}^n \oplus y^{[p-j]}(t, |l - \lambda m), & \text{if } p = 2n \text{ is even,} \\ \left( \sum_{j=0}^{n-1} \oplus y^{(j)}(t) \right) \oplus \left( \sum_{j=1}^n \oplus y^{[p-j]}(t, |l - \lambda m) \right) \oplus (-iy^{(n)}(t)), & \text{if } p = 2n + 1 > 1 \text{ is odd,} \\ y(t), & \text{if } p = 1, \end{cases} \end{aligned}$$

for  $t$  and  $\lambda$  such that  $p^{-1}(t, \lambda) \in B(\mathcal{H})$  the equation

$$(6) \quad l[y] - \lambda m[y] = 0, \quad t \in \bar{\mathcal{I}}$$

is reduced to a homogenous equation of type (2) in  $\mathcal{H}_1 = \mathcal{H}^p$ . Under this substitution for odd  $p = r > s$  we formally consider that  $s = r - 1$  and if it is necessary we set some leading coefficients in the expression  $m[y]$  to be equal to zero. Analogously for even  $p$  we formally consider that  $r = s$ .

Then the quasi-derivatives in (5) are equal to

$$(7) \quad y^{[j]}(t|l - \lambda m) = y^{(j)}(t), \quad j = 0, \dots, \quad [p/2] - 1,$$

$$(8) \quad y^{[n]}(t|l - \lambda m) = \begin{cases} p(t, \lambda) y^{(n)} - \frac{i}{2}(q_n - \lambda \tilde{q}_n) y^{(n-1)}, & p = 2n, \\ -\frac{i}{2}(q_{n+1} - \lambda \tilde{q}_{n+1}) y^{(n)}, & p = 2n + 1, \end{cases}$$

$$(9) \quad \begin{aligned} y^{[p-j]}(t|l - \lambda m) &= -Dy^{[p-j-1]}(t|l - \lambda m) + (p_j - \lambda \tilde{p}_j) y^{(j)} \\ &+ \frac{i}{2} \left[ (q_{j+1}^* - \lambda \tilde{q}_{j+1}^*) y^{(j+1)} - (q_j - \lambda \tilde{q}_j) y^{(j-1)} \right], \\ &j = 0, \dots, \left[ \frac{p-1}{2} \right], \quad q_0 \equiv \tilde{q}_0 \equiv 0. \end{aligned}$$

With this,  $l[y] - \lambda m[y] = y^{[p]}(t|l - \lambda m)$ .

In the homogeneous equation (2) obtained from equation (6) using substitution (5) for even  $p = 2n$ ,

$$(10) \quad \begin{aligned} Q(t) &= iJ = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix}, \\ H_\lambda(t) &= \|h_{\alpha\beta}(t, \lambda)\|_{\alpha, \beta=1}^2, \quad h_{\alpha\beta} \in B(\mathcal{H}^n), \end{aligned}$$

where  $I_n$  is an identity operator in  $B(\mathcal{H}^n)$ ,  $h_{11}(t, \lambda) = h_{11}^*(t, \bar{\lambda})$  is a three-diagonal operator matrix whose elements under the main diagonal are equal to

$$\left( \frac{i}{2}(q_1 - \lambda \tilde{q}_1), \dots, \frac{i}{2}(q_{n-1} - \lambda \tilde{q}_{n-1}) \right),$$

the elements on the main diagonal are equal to

$$\begin{aligned} &\left( -(p_0 - \lambda \tilde{p}_0), \dots, \right. \\ &\left. -(p_{n-2} - \lambda \tilde{p}_{n-2}) \frac{1}{4}(q_n^* - \lambda \tilde{q}_n^*) p^{-1}(t, \lambda) (q_n - \lambda \tilde{q}_n) - (p_{n-1} - \lambda \tilde{p}_{n-1}) \right), \end{aligned}$$

the rest of the elements are equal to zero.  $h_{12}(t, \lambda) = h_{21}^*(t, \bar{\lambda})$  is the operator matrix with identical operators  $I_1$  under the diagonal, the elements on the diagonal are equal to  $(0, \dots, 0, -\frac{i}{2}(q_n^* - \lambda q_n^*) p^{-1}(t, \lambda))$ , the rest of the elements are equal to zero.  $h_{22}(t, \lambda) = \text{diag}(0, \dots, 0, p^{-1}(t, \lambda))$ .

And for odd  $p = 2n + 1$ ,

$$(11) \quad \begin{aligned} Q(t) &= \begin{cases} \begin{pmatrix} 0 & iI_n & 0 \\ -iI_n & 0 & 0 \\ 0 & 0 & q_{n+1} \end{pmatrix}, & p > 1, \\ q_1, & p = 1, \end{cases} \\ H_\lambda(t) &= \begin{cases} \|h_{\alpha\beta}(t, \lambda)\|_{\alpha, \beta=1}^2, & p > 1, \\ p_0 - \lambda \tilde{p}_0, & p = 1, \end{cases} \end{aligned}$$

where  $B(\mathcal{H}^n) \ni h_{11}(t, \lambda) = h_{11}^*(t, \bar{\lambda})$  is a three-diagonal operator matrix whose elements under the diagonal are equal  $(\frac{i}{2}(q_1 - \lambda \tilde{q}_1), \dots, \frac{i}{2}(q_{n-1} - \lambda \tilde{q}_{n-1}))$ , the elements on the diagonal are equal to  $(-(p_0 - \lambda \tilde{p}_0), \dots, -(p_{n-1} - \lambda \tilde{p}_{n-1}))$ , the rest of the elements are equal to zero.  $B(\mathcal{H}^{n+1}, \mathcal{H}^n) \ni h_{12}(t, \lambda) = h_{21}^*(t, \bar{\lambda})$  is an operator matrix whose elements with indices  $j, j-1$  are equal to  $I_1$ ,  $j = 2, \dots, n$ , the element with index  $n, n+1$  is equal  $\frac{1}{2}(q_n^* - \lambda q_n^*)$ , the rest of the elements are equal to zero.  $B(\mathcal{H}^{n+1}) \ni h_{22}(t, \lambda) = h_{22}^*(t, \bar{\lambda})$  is an operator matrix whose last row is equal to  $(0, \dots, 0, -iI_1, -(p_n - \lambda \tilde{p}_n))$ , the rest of elements are equal to zero.

Therefore in the equation (2) with coefficients (10), (11),  $H_\lambda(t)$  depend on  $\lambda$  in a nonlinear manner for  $r \leq s$ , and in a linear manner for  $r > s$ ,

$$(12) \quad H_\lambda(t) = H_0(t) + \lambda H(t), \quad H_0^*(t) = H_0(t).$$

Similarly to the general equation (2), for the equation (2) with coefficients (10),(11), the weight is

$$(13) \quad W_\lambda(t) = \begin{cases} \Im H_\lambda(t)/\Im \lambda, & \text{if } \Im \lambda \neq 0, \\ \frac{\partial H_\lambda(t)}{\partial \lambda}, & \text{if } \Im \lambda = 0, \quad p^{-1}(t, \lambda) \in B(\mathcal{H}). \end{cases}$$

Everywhere below, unless stated otherwise, we assume that in the equation (2) with coefficients (10), (11),  $W_\lambda(t) \geq 0$  ( $\Im \lambda \neq 0$ ).

Moreover, tacitly we assume that the following condition holds:

$$(14) \quad \begin{aligned} \exists \lambda_0 \in C; \quad \alpha, \beta \in \bar{\mathcal{I}}, \quad 0 \in [\alpha, \beta], \quad \text{the number } \delta > 0 : \\ p^{-1}(t, \lambda_0) \in B(\mathcal{H}), \quad \forall t \in [\alpha, \beta], \\ m[\chi_{\alpha, \beta y}(t, \lambda_0), \chi_{\alpha, \beta y}(t, \lambda_0)] \geq \delta \|x(0, \lambda_0)\|^2. \end{aligned}$$

For any solution  $y(t, \lambda_0)$  of the equation (6) as  $\lambda = \lambda_0$ , where

$$(15) \quad \begin{aligned} m[f(t), g(t)] &= \int_{\bar{\mathcal{I}}} \sum_{k=0}^s m_k[f(t), g(t)] dt, \\ m_{2j}[f(t), g(t)] &= (\tilde{p}_{2j}(t)f^{(j)}(t), g^{(j)}(t)), \\ m_{2j-1}[f(t), g(t)] &= \frac{i}{2} \{(\tilde{q}_j^*(t)f^{(j)}(t), g^{(j-1)}(t)) - (\tilde{q}_j(t)f^{(j-1)}(t), g^{(j)}(t))\}, \end{aligned}$$

$\chi_{\alpha, \beta}$  is a characteristic function of the interval  $(\alpha, \beta)$ ,  $x(t, \lambda)$  is defined by (5).

For sufficiently smooth vector-function  $f(t)$  we denote

$$(16) \quad \mathcal{H}^p \ni F_\lambda(t) = \begin{cases} \left( \sum_{j=0}^{s/2} \oplus f^{(j)}(t) \right) \oplus \mathbf{O} \oplus \dots \\ \dots \oplus \mathbf{O}, \quad r = 2n, \quad r = 2n + 1 > 1, \quad s < 2n, \\ \left( \sum_{j=0}^{n-1} \oplus f^{(j)}(t) \right) \oplus \mathbf{O} \oplus \dots \oplus \\ \oplus \mathbf{O} \oplus (-if^{(n)}(t)), \quad r = 2n + 1 > 1, \quad s = 2n, \\ f(t), \quad r = 1, \\ \text{an analog of (5) for } f(t), \quad r \leq s. \end{cases}$$

**Lemma 1.** *Let the vector-function  $f(t) \in C^s(\bar{\mathcal{I}})$ ,  $F_{\bar{\lambda}}(t)$  be defined by (16) with  $\bar{\lambda}$  instead of  $\lambda$ ,  $W_\lambda(t)$  be defined by (13), (10), (11). Then*

$$(17) \quad W_\lambda(t)F_{\bar{\lambda}}(t) = \begin{cases} \left( \sum_{j=0}^{s/2-1} \oplus \left( f^{[s-j]}(t|m) + (f^{[s-j-1]}(t|m))' \right) \right) \oplus \\ f^{[s/2]}(t|m) \oplus \mathbf{O} \oplus \dots \oplus \mathbf{O}, \\ r = 2n + 1, \quad r = 2n, \quad 0 < s < 2n, \\ \left( \sum_{j=0}^{n-1} \oplus \left( f^{[s-j]}(t|m) + (f^{[s-j-1]}(t|m))' \right) \right) \\ \oplus \mathbf{O} \oplus \dots \oplus \mathbf{O} \oplus (-if^{[n]}(t|m)), \\ r = 2n + 1 > 1, \quad s = 2n, \\ \tilde{p}_0(t)f(t) \oplus \mathbf{O} \oplus \dots \oplus \mathbf{O}, \quad s = 0, \\ \left( \sum_{j=0}^{n-1} \oplus \left( f^{[s-j]}(t|m) + (f^{[s-j-1]}(t|m))' \right) \right) \oplus \mathbf{O} \oplus \dots \\ \dots \oplus \mathbf{O} + H_\lambda(t)(\mathbf{O} \oplus \dots \oplus \mathbf{O} \oplus f^{[n]}(t|m)), \quad r \leq s = 2n, \end{cases}$$

for  $\lambda, t$  such that  $p^{-1}(t, \lambda) \in B(\mathcal{H})$ .

*Proof.* The proof for  $r > s$  follows from (7)–(11), (16).

Let  $r \leq s = 2n$ . Let  $\Im \lambda \neq 0$ . Since

$$(18) \quad W_\lambda(t)F_{\bar{\lambda}} = \frac{1}{2i\Im \lambda} ((H_\lambda(t)F_\lambda(t) - H_{\bar{\lambda}}(t)F_{\bar{\lambda}}) + H_\lambda(t)(F_{\bar{\lambda}}(t) - F_\lambda(t))),$$

using (7)–(10) and the fact that

$$(19) \quad H_\lambda(t)F_\lambda(t) = iJ(F_\lambda(t))' - \text{col} \left\{ f^{[s]}(t|l - \lambda m), 0, \dots, 0 \right\},$$

we obtain (17) since  $\Im \lambda \neq 0$ .

For  $\lambda_0 \in R$ ,  $t \in \bar{\mathcal{I}}$  which imply that  $p^{-1}(t, \lambda_0) \in B(\mathcal{H})$ , formula (17) is proved by passing to the limit for  $\lambda \rightarrow \lambda_0 + i0$ . The lemma is proved.  $\square$

As is seen from the proof, Lemma 1 remains true without assuming that  $W_\lambda(t) \geq 0$  ( $\Im \lambda \neq 0$ ) and (14).

Denote

$$q = \begin{cases} s/2, & r > s, \\ s, & r \leq s. \end{cases}$$

**Lemma 2.** *Let the vector-functions  $f(t), g(t) \in C^q(\bar{\mathcal{I}})$ ,  $W_\lambda(t)$  be defined by (13), (10), (11). Then*

$$(20) \quad \sum_{k=0}^s m_k [f(t), g(t)] = (W_\lambda(t)F_\lambda(t), G_\lambda(t))_{\mathcal{H}^P}$$

for  $\lambda, t$  such that  $p^{-1}(t, \lambda) \in B(\mathcal{H})$  ( $F_\lambda(t)$  is defined by (16),  $G_\lambda(t)$  is defined in a similar way using  $g(t)$ ) and, therefore,

$$(21) \quad m[\chi_{\alpha, \beta} f(t), \chi_{\alpha, \beta} g(t)] = (F_\lambda(t), G_\lambda(t))_{L^2_{W_\lambda}(\alpha, \beta)}$$

for  $\lambda, t$  such that  $p^{-1}(t, \lambda) \in B(\mathcal{H}) \forall t \in [\alpha, \beta] \subseteq \bar{\mathcal{I}}$ .

*Proof.* For  $r > s$ , (20) follows from (7)–(11), (16). For  $r \leq s$ , (20) can be proved using (5), (10), (16), (17) and (19). Lemma is proved.  $\square$

Note that the proof shows that the formula (20) is valid without the fulfilment of the conditions  $W_\lambda(t) \geq 0$  ( $\Im \lambda \neq 0$ ) and (14).

In view of Lemma 2, the left-hand side of (20) is nonnegative for  $g(t) = f(t)$  since  $W_\lambda(t) \geq 0$  in the equation (2), (10), (11), and the condition (14) is equivalent that for this equation

$$(22) \quad \begin{aligned} &\exists \lambda_0 \in C, \quad \alpha, \beta \in \bar{\mathcal{I}}, \quad 0 \in [\alpha, \beta], \quad \text{the number } \delta > 0 : \\ &p^{-1}(t, \lambda_0) \in B(\mathcal{H}), \quad \forall t \in [\alpha, \beta], \quad (\Delta_{\lambda_0}(\alpha, \beta)g, g) \geq \delta \|g\|^2, \quad g \in \mathcal{H}^P. \end{aligned}$$

Therefore, in view of [20], fulfilment of (14) implies its fulfilment with  $\delta(\lambda) > 0$  instead of the  $\delta$  for  $\lambda \in C$  such that  $p^{-1}(t, \lambda) \in B(\mathcal{H}), \forall t \in [\alpha, \beta]$ .

**Example 1.** Let  $l[y]$  be a symmetric  $2 \times 2$ -matrix differential operation of the second order with a leading coefficient  $\text{diag}(p(t), 0)$ , where  $p(t) \neq 0$ , and

$$m[y] = - \left( \begin{pmatrix} 0 & 0 \\ 0 & q(t) \end{pmatrix} y' \right)' + P(t)y,$$

where  $q(t) > 0$ , the operator  $P(t) = P^*(t) > 0$ . In this case,  $\det p(t, \lambda) \neq 0$  ( $\Im \lambda \neq 0$ ),  $W_\lambda(t) \geq 0$  ( $\Im \lambda \neq 0$ ) and (14) holds, although  $\det p_1(t) \equiv \det \tilde{p}_1(t) \equiv \det W_\lambda(t) \equiv 0$ .

**Definition 2.** Every characteristic operator of the equation (2), (10), (11) corresponding to the equation (1) is said to be a characteristic operator of the equation (1) on  $\mathcal{I}$  (or simply c.o.).

**Lemma 3.**  $1^0$ . *We establish a correspondence between the vector-function  $f(t) \in C^s(\bar{\mathcal{I}})$  and the vector-function  $F_{\bar{\lambda}}(t)$  that is obtained from (16) with  $\bar{\lambda}$  instead of  $\lambda$ .*

*Then equation (1) is equivalent to equation (2) with coefficients (10), (11), weight (13) and with  $F(t) = F_{\bar{\lambda}}(t)$  for such  $\lambda$  and  $t$  that  $p^{-1}(t, \lambda) \in B(\mathcal{H})$ . Namely, if  $y(t)$  is*

a solution of the equation (1), then

$$(23) \quad x(t) = x(t, \lambda, f) \\ = \begin{cases} \left( \sum_{j=0}^{n-1} \oplus y^{(j)}(t) \right) \oplus \left( \sum_{j=1}^{n-1} \oplus (y^{[r-j]}(t|l - \lambda m) - f^{[s-j]}(t|m)) \right) \oplus y^{[n]}(t|l - \lambda m), & r = 2n > s, \\ \left( \sum_{j=0}^{n-1} \oplus y^{(j)}(t) \right) \oplus \left( \sum_{j=1}^n \oplus (y^{[r-j]}(t|l - \lambda m) - f^{[s-j]}(t|m)) \right) \oplus (-iy^{(n)}(t)), & r = 2n + 1 > s, \quad r > 1, \\ y(t), & r = 1, \\ \left( \sum_{j=0}^{n-1} \oplus y^{(j)}(t) \right) \oplus \left( \sum_{j=1}^n \oplus (y^{[s-j]}(t|l - \lambda m) - f^{[s-j]}(t|m)) \right), & r \leq s = 2n \end{cases} \\ \text{(here } f^{[k]}(t|m) \equiv 0 \text{ as } k \leq 0)$$

is a solution of (2) with coefficients (10), (11), weight (13) and with  $F(t) = F_{\bar{\lambda}}(t)$ . Any solution of the equation (2) with coefficients (10), (11), weight (13) and with such  $F(t)$  is equal to (23), where  $y(t)$  is a solution of (1).

$2^0$ . Let  $M(\lambda)$  be a c.o. of the equation (1),  $\mathcal{H}^p$ -valued vector-function  $F(t) \in L_{W_{\lambda}}^2(\mathcal{I})$  (in particular, one can set  $F(t) = F_{\bar{\lambda}}(t)$ , where  $f(t) \in C^q(\bar{\mathcal{I}})$ ,  $m[f(t), f(t)] < \infty$ ). Then the integral (3) converges strongly and

$$(24) \quad \|\mathcal{R}_{\lambda}F(t)\|_{L_{W_{\lambda}}^2(\mathcal{I})}^2 \leq \Im(\mathcal{R}_{\lambda}F, F)_{L_{W_{\lambda}}^2(\mathcal{I})} / \Im\lambda \quad (\Im\lambda \neq 0).$$

If, additionally,  $F(t)$  has compact support, then the inequality (24) is valid without the requirement (14).

*Proof.*  $1^0$  is verified using direct calculations taking into account (7)–(11) and Lemma 1.

$2^0$  is proved in [20] for the general equation (2) satisfying a condition of type (22) as  $\lambda_0 \in \mathcal{A}$ , and therefore it is proved for the equation (2) with coefficients (10), (11), weight (13). The statement  $2^0$  for  $F(t)$  with compact support is also proved in [20] for the general equation (2) without a condition of the type (22). Lemma is proved.  $\square$

We notice that one can see from the proof that item  $1^0$  of Lemma 3 is valid without the fulfilment of the condition  $W_{\lambda}(t) \geq 0$  ( $\Im\lambda \neq 0$ ) and (14).

One can deduce from Lemmas 1–3 the following.

**Corollary 1.** Let the vector-functions  $x(t), y(t) \in C^p([\alpha, \beta])$ ,  $f(t), g(t) \in C^s([\alpha, \beta])$ ,  $p^{-1}(t, \lambda), p^{-1}(t, \mu) \in B(\mathcal{H}) \forall t \in [\alpha, \beta] \subseteq \bar{\mathcal{I}}$  and

$$l[y] - \lambda m[y] = m[f], \quad l[x] - \mu m[x] = m[g].$$

Then Green's formula is valid,

$$m[\chi_{\alpha, \beta} f(t), \chi_{\alpha, \beta} x(t)] - m[\chi_{\alpha, \beta} y(t), \chi_{\alpha, \beta} g(t)] + (\lambda - \bar{\mu})m[\chi_{\alpha, \beta} y(t), \chi_{\alpha, \beta} x(t)] \\ = ([i\Re Q(t)]x(t, \lambda, f), y(t, \mu, g))|_{\alpha}^{\beta},$$

where  $x(t, \lambda, f)$  is defined by (23),  $y(t, \mu, g)$  is defined in a similar way using  $x(t)$  instead of  $y(t)$ ,  $g$  instead of  $f$  and quasi-derivatives that correspond to the expression  $l[x] - \mu m[x]$ ,  $Q(t)$  is defined by (10), (11).

We consider pre-Hilbert spaces  $\overset{\circ}{H}$  and  $H$  of vector-functions  $y(t) \in C_0^s(\bar{\mathcal{I}})$  and  $y(t) \in C^s(\bar{\mathcal{I}})$ ,  $m[y(t), y(t)] < \infty$ , correspondingly, with the scalar product

$$(25) \quad (f(t), g(t))_m = m[f(t), g(t)],$$

where  $m[f(t), g(t)]$  is defined by (15).

**Definition 3.** By  $\overset{\circ}{L}_m^2(\mathcal{I})$  and  $L_m^2(\mathcal{I})$  we denote the completions of the spaces  $\overset{\circ}{H}$  and  $H$  in the norms  $\|\cdot\|_m = \sqrt{(\cdot, \cdot)_m}$  correspondingly. By  $\overset{\circ}{P}$  we denote the orthogonal projection in  $L_m^2(\mathcal{I})$  onto  $\overset{\circ}{L}_m^2(\mathcal{I})$ .

We consider, in  $L_m^2(\mathcal{I})$ , the symmetric relation

$$(26) \quad \mathcal{L}'_0 = \left\{ \{ \tilde{y}(t), \tilde{g}(t) \} | \tilde{y}(t) \stackrel{L_m^2(\mathcal{I})}{=} y(t), \tilde{g}(t) \stackrel{L_m^2(\mathcal{I})}{=} g(t), \right. \\ \left. y(t) \in C_0^p(\bar{\mathcal{I}}), g(t) \in C_0^s(\bar{\mathcal{I}}), l[y] = m[g] \right\}.$$

Further we assume that  $\mathcal{L}'_0$  consists of pairs of the type  $\{y(t), g(t)\}$ . We denote  $\mathcal{L}_0 = \overline{\mathcal{L}'_0}$ .

In the following theorem the generalized resolvents  $R_\lambda = \int_{-\infty}^{\infty} \frac{dE_\mu}{\mu - \lambda}$  of the relation  $\mathcal{L}_0$  are constructed and corresponding generalized spectral families  $E_\mu = E_{\mu-0}$  [10] are found. In this theorem we denote  $E_{\alpha, \beta} = \frac{1}{2}(E_{\beta+0} + E_\beta - E_{\alpha+0} - E_\alpha)$ ,  $-\infty < \alpha \leq \beta < \infty$ .

**Theorem 1.**  $1^\circ$ . Let  $M(\lambda)$  be the characteristic operator of the equation (1),

$$(27) \quad x_\lambda(t, F_\lambda) = \text{col} \{y_j(t, \lambda, f)\}_{j=1}^p \quad (y_j \in \mathcal{H}),$$

be the corresponding solution (3) of the equation (2) with coefficients (10), (11), weight (13) and  $F(t) = F_\lambda(t)$ , where  $F_\lambda(t)$  is defined by (16) with  $\bar{\lambda}$  instead of  $\lambda$ ,  $f(t) \in C^s(\bar{\mathcal{I}})$ ,  $m[f(t), f(t)] < \infty$  (and therefore  $F_\lambda(t) \in L_{W_\lambda}^2(\mathcal{I})$  in view of (20)).

Then an integro-differential operator  $R_\lambda f = y_1(t, \lambda, f)$  which is densely defined in  $L_m^2(\mathcal{I})$  and given by the first vector-valued component of the solution (27) is, after closing, the generalized resolvent of the relation  $\mathcal{L}_0$ .

$2^\circ$ . Let  $M(\lambda)$  be the characteristic operator of the equation (1) (and therefore by [20]  $\Im M(\lambda) \geq 0$  as  $\Im \lambda > 0$ ) and  $\sigma(\mu) = w - \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_0^\mu \Im M(\mu + i\varepsilon) d\mu$  be the spectral operator-function that corresponds to  $M(\lambda)$ .

Let  $E_\mu$  be the generalized spectral family corresponding to the generalized resolvent  $R_\lambda$  from the item  $1^\circ$  of this theorem. Then for any  $f(t) \in C_0^s(\bar{\mathcal{I}})$  the equality

$$(28) \quad \overset{\circ}{P} E_{\alpha, \beta} f(t) = \overset{\circ}{P} \int_\alpha^\beta [X_\mu(t)]_1 d\sigma(\mu) \varphi(\mu, f),$$

is valid in  $\overset{\circ}{L}_m^2(\mathcal{I})$ , where  $[X_\lambda(t)]_1 \in B(\mathcal{H}^p, \mathcal{H})$  is the first row of the operator solution  $X_\lambda(t)$  of the homogeneous equation (2) with coefficients (10), (11) that is written in the matrix form and such that  $X_\lambda(0) = I_p$ ,

$$(29) \quad \varphi(\mu, f) = \int_{\mathcal{I}} ([X_\mu(t)]_1)^* m[f] dt,$$

if  $p^{-1}(t, \mu) \in B(\mathcal{H}) \forall t \in \bar{\mathcal{I}}, \mu \in [\alpha, \beta]$ .

Moreover, for  $f(t) \in D(\mathcal{L}'_0)$  (see (26)) and with  $r > s$  (or with  $r < s$ , if additionally  $\overset{\circ}{P}(E_{+0} - E_0)f(t) = 0$ ), the inverse formula in  $\overset{\circ}{L}_m^2(\mathcal{I})$

$$(30) \quad f(t) = \overset{\circ}{P} \int_{-\infty}^{\infty} [X_\mu(t)]_1 d\sigma(\mu) \varphi(\mu, f),$$

and Parseval's equality

$$(31) \quad m[f(t), g(t)] = (\varphi(\mu, f), \varphi(\mu, g))_{L^2(\mathbb{R}, d\sigma)},$$

are valid, where  $g(t) \in C_0^s(\bar{\mathcal{I}})$ .

Let us explain that, for  $r > s$ ,

$$\overset{\circ}{P} \int_{-\infty}^{\infty} = \lim_{\substack{\alpha \rightarrow -\infty \\ \beta \rightarrow \infty}} \overset{\circ}{P} \int_\alpha^\beta$$

in (30), and for  $r < s$ ,

$$\mathring{P} \int_{-\infty}^{\infty} = \lim_{\substack{\alpha \rightarrow -\infty \\ \beta \rightarrow -0}} \mathring{P} \int_{\alpha}^{\beta} + \lim_{\substack{\delta \rightarrow \infty \\ \gamma \rightarrow +0}} \mathring{P} \int_{\gamma}^{\delta},$$

where the limits exist in  $L_m^2(\mathcal{I})$ . Similarly,  $\int_{-\infty}^{\infty} = \int_{-\infty}^{-0} + \int_{+0}^{\infty}$  in the right-hand side of (31) for  $r < s$ .

*Proof.* Let for definiteness  $r \leq s = 2n$  (for  $r > s$  the proof becomes simpler due to (10)–(12)).

1<sup>o</sup>. Let  $\Im \lambda \neq 0$ . In view of the item 1<sup>o</sup> of Lemma 3,  $y_1(t, \lambda, f)$  is a solution of (1). Using (10) and Lemmas 1–3 one can show that

$$(32) \quad \sum_{k=0}^s m_k [y_1(t, \lambda, f), y_1(t, \lambda, f)] - \Im \left( \sum_{k=0}^s m_k [y_1(t, \lambda, f), f(t)] \right) / \Im \lambda \\ = (W_{\lambda}(t)x(t, \lambda, f), x(t, \lambda, f))_{\mathcal{H}^s} - \Im (W_{\lambda}(t)x(t, \lambda, f), F_{\lambda}^{-}(t))_{\mathcal{H}^s} / \Im \lambda,$$

although for  $r \leq s$  the corresponding items in the right- and left-hand sides of (32) do not coincide. Therefore<sup>1</sup>

$$(33) \quad \|y_1(t, \lambda, f)\|_{L_m^2(\alpha, \beta)}^2 - \Im (y_1(t, \lambda, f), f(t))_{L_m^2(\alpha, \beta)} / \Im \lambda \\ = \|x(t, \lambda, f)\|_{L_{W_{\lambda}}^2(\alpha, \beta)}^2 - \Im (x(t, \lambda, f), F_{\lambda}^{-}(t))_{L_{W_{\lambda}}^2(\alpha, \beta)} / \Im \lambda.$$

In view of the item 2<sup>o</sup> of Lemma 3 a nonnegative limit of the right-hand-side of (33) exists, when  $(\alpha, \beta) \uparrow \mathcal{I}$ . Consequently

$$(34) \quad \|y_1(t, \lambda)\|_{L_m^2(\mathcal{I})}^2 \leq \Im (y_1(t, \lambda), f(t))_{L_m^2(\mathcal{I})} / \Im \lambda.$$

Since  $M(\lambda) = M^*(\bar{\lambda})$ , then the operator  $\mathcal{R}_{\lambda}$  (3) in  $L_{W_{\lambda}}^2(\alpha, \beta)$  with finite  $(\alpha, \beta) \subseteq \mathcal{I}$  possesses the property  $\mathcal{R}_{\lambda} = \mathcal{R}_{\bar{\lambda}}^*$ . Therefore  $([\Re Q(t)]R_{\lambda}F, R_{\bar{\lambda}}G)|_{\alpha}^{\beta} = 0$ . It follows from Corollary 1 and (34) that  $\forall f(t), g(t) \in C^s(\bar{\mathcal{I}}) \cap L_m^2(\mathcal{I})$

$$m[y_1(\lambda, f), g] = m[f, y_1(\bar{\lambda}, g)].$$

Thus the closure of the operator  $R_{\lambda}f = y_1(t, \lambda, f)$  in  $L_m^2(\mathcal{I})$  possesses a property

$$(35) \quad R_{\lambda} = R_{\bar{\lambda}}^*.$$

Since in view of (34) for any  $f(t), g(t) \in C^s(\bar{\mathcal{I}}) \cap L_m^2(\mathcal{I})$  and with  $(\alpha, \beta) \uparrow \mathcal{I}$ ,

$$(y_1(\lambda, f), g)_{L_m^2(\alpha, \beta)} \rightarrow (y_1(\lambda, f), g)_{L_m^2(\mathcal{I})}$$

uniformly in  $\lambda$  from any compact set  $\in C/R$ , we see that, in view of analyticity of the operator function  $M(\lambda)$  and vector-function  $W_{\lambda}(t)F_{\lambda}^{-}(t)$ , (17), the operator  $R_{\lambda}$  depends analytically on the non-real  $\lambda$  in view of [15, p. 195].

Finally, similarly to the case  $s = 0$  [20] using Corollary 1 it is verified that

$$(36) \quad R_{\lambda}(\mathcal{L}_0 - \lambda) \subset \mathbf{I},$$

where  $\mathbf{I}$  is the graph of the identical operator in  $L_m^2(\mathcal{I})$ .

Taking into account (34)–(36) and analyticity of  $R_{\lambda}$ , we see in view of [10] that  $R_{\lambda}$  is a generalized resolvent of  $\mathcal{L}_0$ . Item 1<sup>o</sup> is proved.

<sup>1</sup>In particular this implies that

$$\|y_1(t, \lambda, f)\|_{L_m^2(\alpha, \beta)}^2 - \frac{\Im (y_1(t, \lambda, f), f(t))_{L_m^2(\alpha, \beta)}}{\Im \lambda} = \frac{U[x(\beta, \lambda, f)] - U[x(\alpha, \lambda, f)]}{2\Im \lambda}.$$

2<sup>0</sup>. Let the vector-functions  $f(t), g(t) \in C_0^s(\mathcal{I})$ ,  $\lambda = \mu + i\varepsilon$ ,  $G_\lambda(t)$  be defined by (16) with  $g(t)$  instead of  $f(t)$ . In view of the Stieltjes inversion formula,

$$\begin{aligned}
 (E_{\alpha, \beta} f, g)_m &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\alpha}^{\beta} ([y_1(\lambda, f) - y_1(\bar{\lambda}, f)], g)_m d\mu \\
 &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\alpha}^{\beta} \left[ (x(t, \lambda, f), G_\lambda(t))_{L_{W_\lambda}^2(\mathcal{I})} - (x(t, \bar{\lambda}, f), G_{\bar{\lambda}}(t))_{L_{W_\lambda}^2(\mathcal{I})} \right. \\
 &\quad \left. + 2i \int_{\mathcal{I}} \left( (\mathfrak{S}p^{-1}(t, \lambda)) f^{[n]}(t|m), g^{[n]}(t|m) \right) dt \right] d\mu \\
 (37) \quad &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\alpha}^{\beta} \left[ \left( M(\lambda) \int_{\mathcal{I}} X_{\bar{\lambda}}^*(t) W_\lambda(t) F_{\bar{\lambda}}(t) dt, \int_{\mathcal{I}} X_{\bar{\lambda}}^*(t) W_{\bar{\lambda}}(t) G_\lambda(t) dt \right) \right. \\
 &\quad \left. - \left( M^*(\lambda) \int_{\mathcal{I}} X_{\bar{\lambda}}^*(t) W_{\bar{\lambda}}(t) F_\lambda(t) dt, \int_{\mathcal{I}} X_{\bar{\lambda}}^*(t) W_\lambda(t) G_{\bar{\lambda}}(t) dt \right) \right] d\mu \\
 &= \int_{\alpha}^{\beta} \left( d\sigma(\mu) \int_{\mathcal{I}} X_\mu^*(t) W_\mu(t) F_\mu(t) dt, \int_{\mathcal{I}} X_\mu^*(t) W_\mu(t) G_\mu(t) dt \right),
 \end{aligned}$$

where the second equality is a corollary of (10), (13), (20), (23), next to last is a corollary of (3), and the last follows from the well-known generalization of the Stieltjes inversion formula [27, proposition (B), p. 803], [4, Lemma, p. 952]. But for  $\mu \in [\alpha, \beta]$

$$(38) \quad \int_{\mathcal{I}} X_\mu^*(t) W_\mu(t) F_\mu(t) dt = \int_{\mathcal{I}} ([X_\mu(t)]_1)^* m[f] dt,$$

because, in view of (20),

$$\begin{aligned}
 \forall h \in \mathcal{H}^s : \quad &\left( \int_{\mathcal{I}} X_\mu^*(t) W_\mu(t) F_\mu(t) dt, h \right) \\
 &= \int_{\mathcal{I}} (W_\mu(t) F_\mu(t), X_\mu(t) h) = \int_{\mathcal{I}} \left( ([X_\mu]_1)^* m[f], h \right) dt.
 \end{aligned}$$

Due to (37), (38),

$$(39) \quad (E_{\alpha, \beta} f, g)_m = \int_{\alpha}^{\beta} (d\sigma(\mu) \varphi(\mu, f), \varphi(\mu, g)).$$

Replacing  $\int_{\alpha}^{\beta}$  in (39) by an integral sum and using (20), (38) we obtain that

$$\begin{aligned}
 (E_{\alpha, \beta} f, g)_m &= \left( \int_{\alpha}^{\beta} [X_\mu(t)]_1 d\sigma(\mu) \varphi(\mu, f), g(t) \right)_m \\
 &= \left( \overset{\circ}{P} \int_{\alpha}^{\beta} [X_\mu(t)]_1 d\sigma(\mu) \varphi(\mu, f), g(t) \right)_m
 \end{aligned}$$

and (28) is proved.

Since  $E_\infty f(t) = f(t)$  if  $f(t) \in D(\mathcal{L}'_0)$ , passing to the limit in (28), (39) for  $\alpha \rightarrow -\infty$ ,  $\beta \rightarrow -0$  and  $\alpha \rightarrow +0$ ,  $\beta \rightarrow \infty$  we obtain (30) and (31). Item 2<sup>o</sup> and Theorem 1 are proved.  $\square$

The following remark follows from [5, 6] and from [20, formula (1.70)].

**Remark 1.** If  $m[y] = w(t)y$  and if for the equation (2) with coefficients (10), (11) that corresponds to the equation (6), the condition

$$(40) \quad \exists \lambda_0 \in C, \quad \alpha, \beta, \delta > 0 : (\Delta_{\lambda_0}(\alpha, \beta) g, g) \geq \delta \|g\|^2 \quad \forall g \in N^\perp$$

holds true, then  $R_\lambda f$  for any generalized resolvent  $R_\lambda$  of  $\mathcal{L}_0$  and any  $f(t) \in C_0(\bar{\mathcal{I}})$  have the same representation as in item 1<sup>o</sup> of Theorem 1.

An analysis of the proof of Theorem 1 shows the following.

**Remark 2.** If (14) is not assumed to hold, then we have the following: 1) item 1° of Theorem 1 is valid either for  $f(t) \in C^s(\bar{\mathcal{I}})$ , if the interval  $\mathcal{I}$  is finite or for  $f(t) \in C_0^s(\mathcal{I})$ , if  $L_m^2(\mathcal{I}) = L_m^2(\bar{\mathcal{I}})$ .

2) Identity (28) holds for  $L_m^2(\mathcal{I})$ , if one changes it as follows: a)  $\sigma(\mu)$  is a spectral function corresponding to  $PM(\lambda)P$  ( $\Im PM(\lambda)P \geq 0$  as  $\Im \lambda > 0$  [20]); b) remove  $\mathring{P}$  from (28); c)  $f(t) \in C^s(\bar{\mathcal{I}})$  and  $\varphi(\mu, f) = \int_{\mathcal{I}} X_\mu^*(t)W_\mu(t)F_\mu(t)dt$ , if the interval  $\mathcal{I}$  is finite or  $f(t) \in C_0^s(\bar{\mathcal{I}})$ , if  $L_m^2(\mathcal{I}) = L_m^2(\bar{\mathcal{I}})$ .

The following theorem establishes a relationship between the generalized resolvents of the relations  $\mathcal{L}_0$  that are given by Theorem 1, and the boundary value problems for the equation (1) with boundary conditions depending on the spectral parameter. Already in the simplest case, where  $l$  and  $m$  that generate (1) are self-adjoint differential operators we see that the pair  $\{y, f\}$  satisfies the boundary conditions that contain both  $y$  derivatives and  $f$  derivatives of corresponding orders at the ends of interval.

**Theorem 2.** *Let the interval  $\mathcal{I} = (a, b)$  be finite.*

*Let the operator-functions  $\mathcal{M}_\lambda, \mathcal{N}_\lambda \in B(\mathcal{H}^p)$ , depend analytically on the non-real  $\lambda$ ,*

$$(41) \quad \mathcal{M}_\lambda^* [\Re Q(a)] \mathcal{M}_\lambda = \mathcal{N}_\lambda^* [\Re Q(b)] \mathcal{N}_\lambda \quad (\Im \lambda \neq 0),$$

*where  $Q(t)$  is the coefficient of the equation (2) corresponding by Lemma 3 to the equation (1) (see (10), (11)),*

$$(42) \quad \|\mathcal{M}_\lambda h\| + \|\mathcal{N}_\lambda h\| > 0 \quad (0 \neq h \in \mathcal{H}^p, \Im \lambda \neq 0),$$

*the lineal  $\{\mathcal{M}_\lambda h \oplus \mathcal{N}_\lambda h \mid h \in \mathcal{H}^p\} \subset \mathcal{H}^{2p}$  is a maximal  $\mathcal{Q}$ -nonnegative subspace since  $\Im \lambda \neq 0$ , where  $\mathcal{Q} = (\Im \lambda) \text{diag}(\Re Q(a), -\Re Q(b))$  (and therefore*

$$(43) \quad \Im \lambda (\mathcal{N}_\lambda^* [\Re Q(b)] \mathcal{N}_\lambda - \mathcal{M}_\lambda^* [\Re Q(a)] \mathcal{M}_\lambda) \leq 0 \quad (\Im \lambda \neq 0).$$

*Then for any  $f(t) \in C^s(\bar{\mathcal{I}})$  the boundary problem that is obtained by adding the boundary conditions*

$$(44) \quad \exists h = h(\lambda, f) \in \mathcal{H}^p : x(a, \lambda, f) = \mathcal{M}_\lambda h, \quad x(b, \lambda, f) = \mathcal{N}_\lambda h,$$

*to the equation (1), where  $x(t, \lambda, f)$  is defined by (23), has the unique solution  $R_\lambda f$  as  $\Im \lambda \neq 0$ . It is generated by the generalized resolvent  $R_\lambda$  of the relation  $\mathcal{L}_0$  that is constructed, as in item 1° of Theorem 1, using the c.o.*

$$M(\lambda) = -\frac{1}{2} (X_\lambda^{-1}(a) \mathcal{M}_\lambda + X_\lambda^{-1}(b) \mathcal{N}_\lambda) (X_\lambda^{-1}(a) \mathcal{M}_\lambda - X_\lambda^{-1}(b) \mathcal{N}_\lambda)^{-1} (iG)^{-1},$$

*where*

$$(X_\lambda^{-1}(a) \mathcal{M}_\lambda - X_\lambda^{-1}(b) \mathcal{N}_\lambda)^{-1} \in B(\mathcal{H}^p) \quad (\Im \lambda \neq 0),$$

*$X_\lambda(t)$  is an operator solution of the homogeneous equation (2) with coefficients (10), (11) and such that  $X_\lambda(0) = I_p$ .*

*Proof.* Proof follows from Lemma 3, Theorem 1 and from from [20, Remark 1.1].  $\square$

For  $s = 0$ , Theorem 2 is known (see [28, 5] as  $\dim \mathcal{H} < \infty$ , [20] as  $r = 1$ ,  $\dim \mathcal{H} = \infty$ ).

**Example 2.** Let, in the equation (1),  $r = 4$ ,  $s = 2$ .

a) Let  $\mathcal{M}_\lambda = \mathcal{N}_\lambda = I_p$ . Then the boundary conditions (44) can be represented in the form

$$(45) \quad \begin{aligned} y(a) &= y(b), & y'(a) &= y'(b), \\ y^{[2]}(a|l) &= y^{[2]}(b|l), \\ y^{[3]}(a|l - \lambda m) - f^{[1]}(a|m) &= y^{[3]}(b|l - \lambda m) - f^{[1]}(b|m). \end{aligned}$$

In particular for the equation

$$(46) \quad y^{(IV)} - \lambda(-y'' + y) = -f'' + f$$

conditions (45) have the form

$$(47) \quad \begin{aligned} y(a) = y(b), \quad y'(a) = y'(b), \quad y''(a) = y''(b), \\ y'''(a) + f'(a) = y'''(b) + f'(b). \end{aligned}$$

b) If  $\dim \mathcal{H} = 1$ ,

$$\mathcal{M}_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{N}_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then the boundary conditions (44) can be written in the form

$$(48) \quad y(a) = y(b) = 0, \quad y'(a) = y^{[2]}(b|l), \quad y^{[2]}(a|l) = -y'(b).$$

In particular for the equation (46) conditions (48) have the form

$$(49) \quad y(a) = y(b) = 0, \quad y'(a) = y''(b), \quad y''(a) = -y'(b).$$

On functions satisfying either the boundary condition (47) with  $f(t) \equiv 0$  or the boundary conditions (49), expressions  $l[y] = y^{(IV)}$  and  $m[y] = -y'' + y$  define a self-adjoint and symmetric operators correspondingly.

When the boundary conditions are such that  $l[y]$  defines a self-adjoint operator and  $m[y]$  defines only a symmetric operator on functions satisfying these conditions, then the eigenfunction expansion of the special scalar equation (6) in the regular case is constructed in [7, 14]. We note that, in the general case, the boundary conditions (44) are not reduced to the boundary condition of [7, 14] type, since it is possible that conditions (44) in the case [7, 14] do not imply  $s$  boundary conditions containing only the derivatives of order up to  $s - 1$ .

In the next theorem,  $\mathcal{I} = R$  and condition (14) hold both on the negative semi-axis  $R_-$  (i.e. as  $\mathcal{I} = R_+$ ) and on the positive semi-axis  $R_+$  (i.e. as  $\mathcal{I} = R_-$ ).

**Theorem 3.** *Let  $\mathcal{I} = R$ , the coefficient of the equation (6) be periodic on each of the semi-axes  $R_+$  and  $R_-$  with periods  $T_+ > 0$  and  $T_- > 0$  correspondingly. Then the spectrums of the monodromy operators  $X_\lambda(\pm T_\pm)$  ( $X_\lambda(t)$  is from Theorem 2) do not intersect the unit circle as  $\Im \lambda \neq 0$ , the c.o.  $M(\lambda)$  of the equation (1) is unique and equal to*

$$(50) \quad M(\lambda) = \left( \mathcal{P}(\lambda) - \frac{1}{2} I_p \right) (iG)^{-1} \quad (\Im \lambda \neq 0)$$

where the projection  $\mathcal{P}(\lambda) = P_+(\lambda)(P_+(\lambda) + P_-(\lambda))^{-1}$ ,  $P_\pm(\lambda)$  are Riesz projections of the monodromy operators  $X_\lambda(\pm T_\pm)$  that correspond to their spectrums lying inside the unit circle,  $(P_+(\lambda) + P_-(\lambda))^{-1} \in B(\mathcal{H}^p)$  as  $\Im \lambda \neq 0$ .

Also let  $\dim \mathcal{H} < \infty$ ,  $\mathbf{A} = \{\mu \in R : \det p(t, \mu) \neq 0 \forall t \in (-T_-, T_+)\}$ , a finite interval  $\Delta \subset \mathbf{A}$ . Then in item 2° of Theorem 1  $d\sigma(\mu) = d\sigma_{ac}(\mu) + d\sigma_d(\mu)$ ,  $\mu \in \Delta$ . Here  $\sigma_{ac}(\mu) \in AC(\Delta)$  and, for  $\mu \in \Delta$ ,

$$(51) \quad \sigma'_{ac}(\mu) = \frac{1}{2\pi} G^{-1} (Q_-^*(\mu) G Q_-(\mu) - Q_+^*(\mu) G Q_+(\mu)) G^{-1}$$

where the projections  $Q_\pm(\mu) = q_\pm(\mu)(P_+(\mu) + P_-(\mu))^{-1}$ ,  $q_\pm(\mu)$  are Riesz projections of the monodromy matrixes  $X_\mu(\pm T_\pm)$  corresponding to the multipliers equal to 1 such that they are shifted inside the unit circle as  $\mu$  is shifted to the upper half plane,  $P_\pm(\mu) = P_\pm(\mu + i0)$ ;  $\sigma_d(\mu)$  is a jump function.

Let us notice that the sets on which  $q_\pm(\mu)$ ,  $P_\pm(\mu)$ ,  $(P_+(\mu) + P_-(\mu))^{-1}$  are not infinitely differentiable do not have finite limit points  $\in \mathbf{A}$  as well as the set of points of increase of  $\sigma_d(\mu)$ .

*Proof.* Let the operator  $G$  be indefinite (otherwise the proof is modified in an obvious way). The unitary dichotomy of the operators  $X_\lambda(\pm T_\pm)$  and the fact that  $M(\lambda)$  (50) is a c.o. of the equation (1) on  $(-T_-, T_+)$  follow from [20, p. 161, 162]. Since  $X_\lambda(t \pm T_\pm) = X_\lambda(t)X_\lambda(\pm T_\pm)$ ,  $t \in R_\pm$ , and  $\Im \lambda U[X_\lambda(t)]$  does not decrease as  $\Im \lambda \neq 0$ , we have that  $M(\lambda)$  (50) is a c.o. of the equation (1) on any finite  $\mathcal{I}$  and therefore it is a c.o. on the axis.

Let for some non-real  $\lambda_0$  the homogeneous equation (2) with coefficients (10), (11) have a solution  $x(t) \in L^2_{W_{\lambda_0}}(R^1)$ .

Since, for  $k \in Z_+$ ,

$$\begin{aligned} \|x(t)\|_{L^2_{W_{\lambda_0}}(R^1)}^2 &= \sum_{j=-\infty}^0 (\Delta_{\lambda_0}(-kT_-, 0)x(-jkT_-), x(-jkT_-)) \\ &\quad + \sum_{j=0}^{\infty} (\Delta_{\lambda_0}(0, kT_+)x(jkT_+), x(jkT_+)), \end{aligned}$$

and using condition (14) on  $\mathcal{I} = R_\pm$  and estimates of the type [8, p. 290], we see that  $x(0) \in H_- \cap H_+$ , where  $H_\pm$  are the invariant subspaces of the operators  $X_\lambda(\pm T_\pm)$  that correspond to their spectrums lying inside the unit circle. But  $H_- \cap H_+ = \{0\}$  [20, p. 162]. Therefore in view of Lemma 1.5 from [20] the c.o.  $M(\lambda)$  (50) is unique.

Formula (51) follows from [19, Theorem 13]. Decomposition  $d\sigma(\mu) = d\sigma_{ac}(\mu) + d\sigma_d$ ,  $\mu \in \Delta$  as well as the remark after the formulating of Theorem 3 are proved in the same way as similar statements in [18]. In the proof for  $r \leq s$  one should take into account that Krein-Lyubarsky theory [22] for homogeneous periodic system (2) is still valid for  $\lambda \in \mathcal{A} \cap R$  and when  $H_\lambda(t)$  contains  $\lambda$  is a Nevanlinna manner; it can be seen analysing the statement of this theory in [29, p. 147–150, 181–183] and the proof of Theorem 1.2 from [12, p. 305]. Theorem is proved.  $\square$

**Example 3.** Let  $\dim \mathcal{H} = 1$ ,  $l[y] = (i)^n y^{(n)}$ ,  $m[y] = (i)^{2n} y^{(2n)} + y$ ,  $\mathcal{I} = R$  (and therefore  $L_m^2(\mathcal{I}) = L_m^2(\mathcal{I})$ ). In this case,  $E_0 = E_{+0}$ , the spectral matrix  $\sigma(\mu) \in AC_{loc}$ , and, in view of Theorem 3 for  $n = 1, 3, \dots$ ,

$$(52) \quad \begin{aligned} \sigma'(\mu) &= \frac{1}{2\pi n(k^{2n} + 1)} \left\{ 2k^n A^n + (1 - k^{2n})I_{2n} + \sum_{j=1}^{n-1} (k^{2n-j} \right. \\ &\quad \left. + (-1)^{j+1} k^j) (A^j + (-1)^{j+1} A^{-j}) \right\} (iJ)^{-1}, \end{aligned}$$

since  $|\mu| < \frac{1}{2}$ , where  $\sum_{j=1}^0 = 0$ ,  $rg\sigma'(\mu) = 2$ , and  $\sigma'(\mu) = 0$  as  $|\mu| > \frac{1}{2}$ . For  $n = 2, 4, 6, \dots$ , one has

$$(53) \quad \sigma'(\mu) = \frac{1}{\pi n(k^{2n} - 1)} \sum_{j=1}^{n/2} (k^{2n-2j+1} - k^{2j-1}) (A^{2j-1} + A^{1-2j}) (iJ)^{-1}$$

since  $0 < \mu < \frac{1}{2}$ , where  $rg\sigma'(\mu) = 4$ , and  $\sigma'(\mu) = 0$  as  $\mu \notin [0, 1/2]$ . In (52), (53)  $k = -i \sqrt{\frac{1+(-1)^n \sqrt{1-4\mu^2}}{2\mu}}$ ,  $A = (iJ)^{-1} H_\mu(t)$ , where the matrices  $J$  and  $H_\mu(t)$  are independent of  $t$  and defined by (10).

In particular, for  $n = 1$ ,  $|\mu| < \frac{1}{2}$ ,

$$\sigma'(\mu) = \frac{1}{2\pi} \begin{pmatrix} \frac{2}{\sqrt{1-4\mu^2}} & 0 \\ 0 & \frac{1}{2}\sqrt{1-4\mu^2} \end{pmatrix}.$$

And for  $n = 2$ ,  $0 < \mu < \frac{1}{2}$ ,

$$\sigma'(\mu) = \frac{1}{\pi} \sqrt{\frac{1 + \sqrt{1 - 4\mu^2}}{2\mu(1 - 2\mu)}} \cdot \frac{1}{\sqrt{1 - 2\mu} + \sqrt{1 + 2\mu}} \begin{pmatrix} 1 & 0 & 0 & \mu \\ 0 & 1 & \mu & 0 \\ 0 & \mu & \mu(1 - \mu) & 0 \\ \mu & 0 & 0 & \mu(1 - \mu) \end{pmatrix}.$$

**Remark 3.** In the case  $r \leq s$  in contrast to the case  $r > s$ , the point spectrum of the relation  $\mathcal{L}'_0$  can be non-empty including the case when  $\mathcal{L}'_0$  corresponds to the scalar equation (1) with periodic coefficients on the axis.

Indeed let  $m[y] = -y'' + y$ ,  $l[y] = p(t)y$ , where  $p(t + 4) = p(t) \in C(R)$ ,  $p(t) = \begin{cases} 1, & 4k \leq t \leq 4k + 1 \\ 0, & 4k + 2 \leq t \leq 4k + 3 \end{cases}$ . Then for any function  $y(t) \in C^2_0(R)$  such that  $\text{supp}y(t) \subset \bigcup_k [4k + 2, 4k + 3]$ , the pair  $\{y(t), 0\} \in \mathcal{L}'_0$ , i.e. the point spectrum of  $\mathcal{L}'_0$  contains  $\lambda = 0$ . Similarly an example for  $r = 1$ ,  $r = 2$ ,  $s = 2$  is constructed.

The following remark is proved similarly to Theorem 3.

**Remark 4.** Let  $\mathcal{I} = R_+$ , the coefficients of the equation (6) be periodic with the period  $T > 0$ . Then

1) Any c.o. of the equation (1) is found by combining the formula (4.4) from [20] and the formula (50).

2) If  $\infty > \dim \mathcal{H}^p = 2k$ ,  $L$  is a  $k$  dimensional  $G$ -neutral subspace (see [8]), then the c.o. of the equation (1), which corresponds to a self-adjoint boundary condition in zero,  $x(0, \lambda, f) \in L$ , is given by the formula (36) from [18]. The corresponding spectral matrix-function  $\sigma(\mu)(d\sigma_{ac}(\mu), \mu \in \Delta_+)$  is given for  $r > s$  ( $r \leq s$ ) by Theorem 6 from [18], starting from the monodromy matrix  $X_\lambda(T)$  of the equation (6), where  $\Delta_+$  is an analog of  $\Delta$  from Theorem 3.

*Acknowledgments.* The author is grateful to Professor F. S. Rofe-Beketov for his great attention to this work.

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Received 19/03/2009