EXTENSION OF SOME LIONS-MAGENES THEOREMS

ALEKSANDR A. MURACH

Dedicated to the memory of A. Ya. Povzner

Abstract. A general form of the Lions-Magenes theorems on solvability of an elliptic boundary-value problem in the spaces of nonregular distributions is proved. We find a general condition on the space of right-hand sides of the elliptic equation under which the operator of the problem is bounded and has a finite index on the corresponding couple of Hilbert spaces. Extensive classes of the spaces satisfying this condition are constructed. They contain the spaces used by Lions and Magenes and many others spaces.

1. Introduction and statement of the problem

Let $\Omega$ be a bounded domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, with the boundary $\Gamma$ which is an infinitely smooth closed manifold of the dimension $n - 1$. The domain $\Omega$ is situated locally on the same side from $\Gamma$.

We consider the nonhomogeneous boundary-value problem in the domain $\Omega$:

(1) \[ Au = f \text{ in } \Omega, \quad B_j u = g_j \text{ on } \Gamma \text{ for } j = 1, \ldots, q. \]

In what follows $A$ is a linear differential expression on $\Omega$ of an arbitrary even order $2q \geq 2$, whereas $B_j$ with $j = 1, \ldots, q$ is a boundary linear differential expression on $\Gamma$ of order $m_j \leq 2q - 1$. All coefficients of $A$ and $B_j$ are complex-valued functions infinitely smooth on $\Omega := \Omega \cup \Gamma$ and on $\Gamma$ respectively.

Everywhere in the paper the boundary-value problem (1) is assumed to be regular elliptic. This means [1, Ch. 2, Sec. 5.1], [2, Ch. III, § 6, Sec. 5] that the expression $A$ is properly elliptic on $\Omega$ and the collection of boundary expressions $B := (B_1, \ldots, B_q)$ is normal and satisfies the complementing condition with respect to $A$ on $\Gamma$. It follows from the condition of normality that all orders $m_j$ with $j = 1, \ldots, q$ are mutually distinct.

Along with (1) we consider the boundary-value problem

(2) \[ A^+ v = \omega \text{ in } \Omega, \quad B^+_j v = h_j \text{ on } \Gamma, \quad j = 1, \ldots, q, \]

formally adjoint to the problem (1) with respect to the Green formula

\[
(Au, v)_{\Omega} + \sum_{j=1}^q (B_j u, C^+_j v)_{\Gamma} = (u, A^+ v)_{\Omega} + \sum_{j=1}^q (C_j u, B^+_j v)_{\Gamma}, \quad u, v \in C^\infty(\Omega). 
\]

Here, $A^+$ is the linear differential expression formally adjoint to $A$ and having the order $2q$ and coefficients from $C^\infty(\Omega)$. In addition, $\{B^+_j\}$, $\{C_j\}$, and $\{C^+_j\}$ are certain normal systems of linear differential boundary expression with coefficients from $C^\infty(\Gamma)$. They orders satisfy the condition

\[ \operatorname{ord} B_j + \operatorname{ord} C^+_j = \operatorname{ord} C_j + \operatorname{ord} B^+_j = 2q - 1. \]
In what follows we denote by $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_\Gamma$ the inner products in the spaces $L_2(\Omega)$ and $L_2(\Gamma)$ (formed by functions square-integrable over $\Omega$ or $\Gamma$ respectively) and natural extensions by continuity of these inner products.

We set

$$N := \{ u \in C^\infty(\overline{\Omega}) : Au = 0 \text{ in } \Omega, \ B_j u = 0 \text{ on } \Gamma \ \forall \ j = 1, \ldots, q \},$$

$$N^+ := \{ v \in C^\infty(\overline{\Omega}) : A^+ v = 0 \text{ in } \Omega, \ B_j^+ v = 0 \text{ on } \Gamma \ \forall \ j = 1, \ldots, q \}.$$  

Since both the problem (1) and (2) are regular elliptic, both the spaces $N$ and $N^+$ are finite-dimensional [1, Ch. 2, Theorem 5.3], [2, Ch. III, § 6, Sec. 4].

The fundamental property of every elliptic boundary-value problem consists in that the problem generates the bounded and Fredholm operator on appropriate couples of functional spaces. Note that a linear bounded operator $T : E_1 \to E_2$, with $E_1$ and $E_2$ being Banach spaces, is called the Fredholm operator if its kernel $\ker T$ and co-kernel $\ker T/\ker T$ are finite-dimensional. The Fredholm operator $T$ has the closed range in the space $E_2$ and the finite index $\text{ind} \ T := \ker T - \text{coker} T$. This operator naturally generates the homeomorphism $T : E_1/\ker T \to T(E_2)$.

Let us formulate the classical theorem on elliptic boundary-value problems (see, e.g., [2, Ch. 3, Sec. 6], [1, Ch. 2, Sec. 5.4]). In the paper, we restrict ourselves to the Hilbert spaces case, which is the most important for applications.

**Theorem 0.** The mapping

$$u \mapsto (A u, B u), \ u \in C^\infty(\overline{\Omega}),$$

can be extended by continuity to the bounded and Fredholm operator

$$\begin{align*}
(A, B) : H^{s+2q}(\Omega) &\to H^s(\Omega) \oplus \bigoplus_{j=1}^q H^{s+2q-m_j-1/2}(\Gamma) =: \mathcal{H}_s(\Omega, \Gamma),
\end{align*}$$

for every real $s \geq 0$. The kernel of this operator coincides with $N$, whereas the range consists of all vectors $(f, g_1, \ldots, g_q) \in \mathcal{H}_s(\Omega, \Gamma)$ satisfying the condition

$$\begin{align*}
(f, v)_\Omega + \sum_{j=1}^q (g_j, C_j^+ v)_\Gamma = 0 \quad &\text{for every } v \in N^+.
\end{align*}$$

The index of the operator (4) is equal to $\dim N - \dim N^+$ and independent of $s$.

In what follows $H^s(\Omega)$ and $H^s(\Gamma)$ are Hilbert spaces with the index $\sigma \in \mathbb{R}$ consisting of some distributions given in the domain $\Omega$ or on the manifold $\Gamma$ respectively (we will remind their definitions in Sec. 2). In addition, as usual $D'(\Omega)$ and $D'(\Gamma)$ stand for the linear topological spaces of all distributions given in $\Omega$ or on $\Gamma$. We always interpret distributions as antilinear functionals.

Theorem 0 has a generic nature because the spaces in which the operator (4) acts are common for all elliptic boundary-value problems of the same order. By this theorem, the operator $(A, B)$ establishes the homeomorphism of the factor space $H^{s+2q}(\Omega)/N$ onto the subspace

$$\{(f, g_1, \ldots, g_q) \in \mathcal{H}_s(\Omega, \Gamma) : (5) \text{ is true}\}$$

for each $s \geq 0$. Therefore the theorems on operators generated by elliptic boundary-value problems are called Homeomorphisms Theorems.

Generally, Theorem 0 is not true in the case $s < 0$ because the mapping $u \mapsto B_j u$ with $u \in C^\infty(\overline{\Omega})$ can not be extended to the bounded operator $B_j : H^{s+2q}(\Omega) \to D'(\Gamma)$ if $s + 2q \leq m_j + 1/2$. Therefore we have to use the space narrower than $H^{s+2q}(\Omega)$ as the domain of $(A, B)$, namely

$$D_{A, X}^{s+2q}(\Omega) := \{ u \in H^{s+2q}(\Omega) : Au \in X^s(\Omega) \},$$
where $X^*(\Omega)$ is a Hilbert space embedded continuously in $\mathcal{D}'(\Omega)$. In what follows the image $Au$ of $u \in \mathcal{D}'(\Omega)$ is understood in the theory of distributions. We endow the space (6) with the graphics inner product

$$
(u_1, u_2)_{D_{A,X}^{2q}\Omega} := (u_1, u_2)_{H^{s+2q}\Omega} + (Au_1, Au_2)_{X^*\Omega}
$$

and the corresponding norm.

The space $D_{A,X}^{s+2q}\Omega$ with the inner product (7) is complete. Indeed, if $(u_k)$ is a Cauchy sequence in $D_{A,X}^{s+2q}\Omega$, then by a completeness of $H^{s+2q}\Omega$ and $X^\ast\Omega$ there are two limits: $u := \lim u_k$ in $H^{s+2q}\Omega \hookrightarrow \mathcal{D}'\Omega$ and $f := \lim Au_k$ in $X^\ast\Omega \hookrightarrow \mathcal{D}'\Omega$ (embeddings are continuous). Since the differential operator $A$ is continuous in $\mathcal{D}'\Omega$, we deduce from the first limit that $Au = \lim Au_k$ in $\mathcal{D}'\Omega$. This implies by the second limit the equality $Au = f \in X^\ast\Omega$. Therefore $u \in D_{A,X}^{s+2q}\Omega$ and $Au_k = u$ in the space $D_{A,X}^{s+2q}\Omega$, that is this space is complete.

J.-L. Lions and E. Magenes [3, 4, 5, 1] found the certain important examples of $X^\ast\Omega$ such that the mapping (3) can be extended by continuity to the bounded and Fredholm operator

$$
(A, B) : D_{A,X}^{s+2q}\Omega \to X^\ast\Omega \oplus \bigoplus_{j=1}^q H^{s+2q-m_j-1/2}\Gamma =: \mathcal{X}_q(\Omega, \Gamma)
$$

if $s < 0$. In contrast to Theorem 0, the domain of the operator (8) depends on coefficients of the elliptic expression $A$. Therefore the theorems on properties of the operator (8) we naturally shall call the individual theorems. Let us formulate two individual theorems proved by Lions and Magenes.

**Theorem LM1.** [3, 4]. Let $s < 0$ and $X^\ast\Omega := L_2\Omega$. Then the mapping (3) can be extended by continuity to the bounded and Fredholm operator (8). The kernel of this operator coincides with $N$, whereas the domain consists of all vectors $(f, g_1, \ldots, g_q) \in \mathcal{X}_q(\Omega, \Gamma)$ satisfying (5). The index of the operator (8) is $\dim N - \dim N^\ast$ and independent of $s$.

Here, we especially note the case $s = -2q$, which is important in the spectral theory of elliptic operators with general boundary conditions [6–10]; see also the survey [11, Sec. 7.7, 9.6]. In this case the space

$$
D_{A,L^2}^0\Omega = \{ u \in L_2\Omega : Au \in L_2\Omega \}
$$

is the domain of the maximal operator $A_{\text{max}}$ corresponding to the differential expression $A$ [12, Sec. 1.2]. If $A$ is formally self-adjoint, then the minimal differential operator $A_{\text{min}} := A_{\text{max}}^\ast$ is semi-bounded symmetric operator in the Hilbert space $L_2\Omega$. Applying Theorem LM1 and some abstract results, one can describe the class of all self-adjoint extensions $\{\hat{A}\}$ of operator $A_{\text{min}}$ by means of boundary conditions. After that one can select among them self-adjoint realizations $\hat{A}$ with classical spectral properties (semi-boundedness, discrete spectrum, Weyl’s spectral asymptotics etc).

Even when all coefficients of $A$ are constant, the space (9) depends essentially on each of them. One can see it from the following result of L. Hörmander [12, Sec. 3.1, Theorem 3.1].

Let $A_1$ and $A_2$ be constant-coefficient linear differential expressions. If $D_{A_1,L^2}^0\Omega \subseteq D_{A_2,L^2}^0\Omega$, then either $A_2 = \alpha A_1 + \beta$ for some $\alpha, \beta \in \mathbb{C}$, or both $A_1$ and $A_2$ are certain polynomials in the derivation operator with respect to a vector $\epsilon$ and moreover $\text{ord } A_2 \leq \text{ord } A_1$. Note that the second possibility is excluded for elliptic operators.

To formulate the second Lions-Magenes theorem we need the following weighted space

$$
\mathcal{W}^\ast\Omega := \{ f = \varphi v : v \in H^\ast\Omega \},
$$
where \( s < 0 \), and a function \( \varrho \in C^\infty(\Omega) \) is positive. We endow this space with the inner product
\[
(f_1, f_2)_{\varrho H^s(\Omega)} := (\varrho^{-1} f_1, \varrho^{-1} f_2)_{H^s(\Omega)}
\]
and the corresponding norm. The space \( \varrho H^s(\Omega) \) is complete and embedded continuously in \( D'(\Omega) \). This follows from that the operator of multiplication by \( \varrho \) is continuous in \( D'(\Omega) \) and establishes the homeomorphism from the complete space \( H^s(\Omega) \) onto \( \varrho H^s(\Omega) \).

We consider a weight function \( \varrho := \varrho_1^{-s} \) such that
\[
\varrho_1 \in C^\infty(\overline{\Omega}), \quad \varrho_1 > 0 \text{ in } \Omega, \quad \varrho_1(x) = \text{dist}(x, \Gamma) \text{ in a neighborhood of } \Gamma.
\]

**Theorem LM2.** [1, Ch. 2, Sec. 7.3]. Let \( s < 0 \), \( s + 1/2 \notin \mathbb{Z} \), and \( X^s(\Omega) := \varrho_1^{-s} H^s(\Omega) \). Then the assertion of Theorem LM1 remains true.

We note that Lions and Magenes used a certain Hilbert space \( X^s(\Omega) \) as \( X^s(\Omega) \). This space coincides (up to equivalence of norms) with the weighted space \( \varrho_1^{-s} H^s(\Omega) \) for each non half-integer \( s < 0 \) [1, Ch. 2, Sec. 7.1]. Theorem LM2 also holds true for every half-integer \( s < 0 \) if we define the space \( X^s(\Omega) \) by means of the complex (holomorphic) interpolation, for instance,
\[
X^s(\Omega) := [X^{2s}(\Omega), L_2(\Omega)]_{1/2}.
\]
(See the definition and properties of this interpolation, e.g., in [1, Ch. 1, Sec. 14.1]).

In the paper, we find a general enough condition on the space \( X^s(\Omega) \) under which the operator (8) is well defined, bounded, and Fredholm if \( s < 0 \). The condition consists in the following.

**Condition I.** The set \( X^\infty(\Omega) := X^s(\Omega) \cap C^\infty(\overline{\Omega}) \) is dense in \( X^s(\Omega) \), and there exists a number \( c > 0 \) such that
\[
\|\mathcal{O} f\|_{H^s(\mathbb{R}^n)} \leq c \|f\|_{X^s(\Omega)} \text{ for each } f \in X^\infty(\Omega).
\]
Here, \( \mathcal{O} f(x) := f(x) \) for \( x \in \overline{\Omega} \), and \( \mathcal{O} f(x) := 0 \) for \( x \in \mathbb{R}^n \setminus \overline{\Omega} \).

In (12), we define by \( H^s(\mathbb{R}^n) \) the Hilbertian Sobolev space with index \( s \) and given over \( \mathbb{R}^n \). Note that if \( s \) is smaller, then Condition I is weaker for the same space \( X^s(\Omega) \).

Both of the spaces \( X^s(\Omega) := L_2(\Omega) \) and \( X^s(\Omega) := \varrho_1^{-s} H^s(\Omega) \) used by Lions and Magenes satisfy Condition I.

In the paper, we find all the Hilbertian Sobolev spaces \( X^s(\Omega) = H^s(\Omega) \) for which Condition I is fulfilled. In addition, we describe the class of all weights \( \varrho \in C^\infty(\Omega) \) such that the weighted space \( X^s(\Omega) := \varrho H^s(\Omega) \) satisfies Condition I. This class contains the weight \( \varrho := \varrho_1^{-s} \) as a special case. Thus, we get some generalizations of the Lions-Magenes theorems mentioned above to more extensive classes of the Hilbertian spaces \( X^s(\Omega) \) of right-hand sides of the elliptic equation.

Note that we generalize the Lions-Magenes theorems staying in classes of distributions given in the domain \( \Omega \). The different Homeomorphism Theorems for elliptic boundary-value problems were proved by Yu. M. Berezansky, S. G. Krein, Ya. A. Roitberg [13], M. Schechter [14], Roitberg [15–18], Yu. V. Kostarchuk and Roitberg [19] (see also the monograph [2, Ch. III, Sec. 6], survey [11, Sec. 7.9], and textbook [20, Ch. XVI, Sec. 1]).

In these theorems, the solution and/or the right-hand side of the elliptic equation are not distributions in \( \Omega \). Namely, the right-hand side is assumed to be in the negative space dual to \( H^{-s}(\Omega) \), \( s < 0 \), with respect to the inner product in \( L_2(\Omega) \). We denote this space by \( H^{s,0}(\Omega) \); it consists of some distributions in \( \mathbb{R}^n \) supported on \( \overline{\Omega} \). The solution is considered in \( H^{s+2k,0}(\Omega) \) [13, 14] or in the special space \( H^{s+2k,2}(\Omega) \) introduced by Roitberg [15–18] (see the definition in Sec. 3). Roitberg, Z. G. Sheftel’ and their disciples used the space \( H^{s+2k,2}(\Omega) \) systematically in the theory of elliptic problems (their results are summed up in Roitberg’s monographs [17, 18]).
These Homeomorphism Theorems have also different applications. Among them we mention the theorems on increase in smoothness of the solution up to the boundary, application to the investigation of Green function of an elliptic boundary-value problem, applications to elliptic problems with power singularities, the transmission problem, the Oshnoff problem, and many others (see the monographs [2, 17, 18], survey [11, Sec. 7.9], and references given therein).

We have to note that Roitberg [21, Sec. 2.4] considered a condition on the space \( X^s(\Omega) \), which was somewhat stronger than our Condition I. He demanded additionally that \( C^\infty(\overline{\Omega}) \subset X^s(\Omega) \). Under this stronger condition, Roitberg [21, Sec. 2.4], [17, p. 190] proved the boundedness of the operator (8) for all \( s < 0 \). Homeomorphism Theorem for this operator was formulated in the survey [11, p. 85] provided that \( N = N^+ = \{0\} \). We also mention Homeomorphism Theorems [19, Theorem 4], [18, Sec. 1.3.8], in which this condition was used, but solutions of the elliptic boundary-value problem were considered in the space \( H^{s+2q, (2q)}(\Omega) \). Remark that Roitberg’s condition does not cover the important case where \( X^s(\Omega) = \{0\} \) as well as some weighted spaces \( X^s(\Omega) = \rho H^s(\Omega) \) which we consider.

We also note that Condition I is fulfilled for some classes of the Hilbertian Hörmander spaces [22, Sec. 2.2], [23, Sec. 10.1]. Their applications to elliptic operators and elliptic boundary-value problems were studied by V. A. Mikhailets and the author in [24–33].

The results of the paper are formulated in Section 2 as Theorems 1, 2, 3, and Corollaries 1, 2. The main result, Theorem 1, is proved in Section 4, all the rest in Section 5. In Section 3, we formulate the auxiliary propositions needed for our proofs. At the end of the paper, we give Appendix, in which a useful proposition on weight functions of the form \( \rho = \tilde{\rho}^s \) is established.

2. Results

We introduce some necessary function spaces. Let \( s \in \mathbb{R} \). Recall that

\[
H^s(\mathbb{R}^n) := \{ w \in \mathcal{S}'(\mathbb{R}^n) : \| w \|_{H^s(\mathbb{R}^n)} := \| (1 + | \xi |^2)^{s/2} \hat{w}(\xi) \|_{L^2(\mathbb{R}^n)} < \infty \},
\]

Here, \( \mathcal{S}'(\mathbb{R}^n) \) is the topological linear space of tempered distributions in \( \mathbb{R}^n \), whereas \( \hat{w} \) is the Fourier transform of \( w \). For a closed set \( Q \subset \mathbb{R}^n \), we put

\[
H^s_Q(\mathbb{R}^n) := \{ w \in H^s(\mathbb{R}^n) : \text{supp } w \subseteq Q \}.
\]

The space \( H^s_Q(\mathbb{R}^n) \) is Hilbert with respect to the inner product in \( H^s(\mathbb{R}^n) \). We are interested in the cases where \( Q \in \{ \mathbb{T}, \Omega, \Gamma \} \) with \( \Omega := \mathbb{R}^n \setminus \Omega \).

Following [1, Ch. 1, Sec. 12.1], we will define the Hilbert space \( H^s(\Omega) \). For arbitrary \( s \geq 0 \) we set

\[
H^s(\Omega) := H^s(\mathbb{R}^n) / H^s_\Omega(\mathbb{R}^n) = \overline{\{ w \mid \Omega : w \in H^s(\mathbb{R}^n) \}}.
\]

The space \( H^s(\Omega) \) is complete with respect to the Hilbertian norm

\[
\| u \|_{H^s(\Omega)} := \inf \{ \| u \|_{H^s(\mathbb{R}^n)} : w \in H^s(\mathbb{R}^n), \ w = u \text{ in } \Omega \}.
\]

The set \( C^\infty_0(\Omega) \) is dense in \( H^s(\Omega) \), each measurable function \( u \) over \( \Omega \) being identified with the antilinear functional \( (u, \cdot)_\Omega \). We denote by \( H^s_0(\Omega) \) the closure of the linear manifold

\[
C^\infty_0(\Omega) := \{ u \in C^\infty(\overline{\Omega}) : \text{supp } u \subset \Omega \}
\]

in the topology of \( H^s(\Omega) \). The space \( H^s_0(\Omega) \) is complete with respect to the inner product in \( H^s(\Omega) \).

For arbitrary \( s < 0 \), we denote by \( H^s(\Omega) \) the Hilbert space antidual to the space \( H^{-s}_0(\Omega) \) with respect to the inner product in \( L_2(\Omega) \). Since antilinear functionals from \( H^{-s}(\Omega) \) are defined uniquely by their values on functions from \( C^\infty_0(\Omega) \), we can correctly identify these functionals with distributions in \( \Omega \). It useful to keep in mind that \( H^s(\Omega) = \)
Theorem 1. Let \( s < 0 \), and \( X^s(\Omega) \) be an arbitrary Hilbert space embedded continuously in \( D'(\Omega) \) and satisfying Condition I. Then the elliptic boundary-value problem (1) possesses the following properties:

(i) The set
\[
D_{A,X}^\infty(\Omega) := \{ u \in C^\infty(\Omega) : Au \in X^s(\Omega) \}
\]
is dense in \( D_{A,X}^{s+2q}(\Omega) \).

(ii) The mapping \( u \mapsto (Au, Bu) \) with \( u \in D_{A,X}^\infty(\Omega) \) can be extended by continuity to the bounded linear operator (8).

(iii) The operator (8) is Fredholm. Its kernel is \( N \), and its range consists of all vectors \( \{ f, g_1, \ldots, g_q \} \in X_\sigma(\Omega, \Gamma) \) satisfying (5).

(iv) If the set \( \mathcal{O}(X^s(\Omega)) \) is dense in \( H^s(\mathbb{R}^n) \), then the index of (8) is \( \dim N - \dim N^+ \).

Let us consider some applications of Theorem 1 caused by a particular choice of the space \( X^s(\Omega) \). Evidently, the space \( X^s(\Omega) := \{ 0 \} \) satisfies Condition I. In this case, Theorem 1 describes properties of the semihomogeneous boundary-value problem (1) with \( f = 0 \) and holds true for every \( s \in \mathbb{R} \) (see also [25]).

All the Hilbertian Sobolev spaces satisfying Condition I are found in the next theorem.

Theorem 2. Let \( s < 0 \) and \( \sigma \in \mathbb{R} \). The space \( X^s(\Omega) := H^\sigma(\Omega) \) satisfies Condition I, if and only if
\[
\sigma \geq \max\{ s, -1/2 \}.
\]

The next result follows from Theorems 1 and 2.

Corollary 1. Let \( s < 0 \), and (13) be valid. Then the mapping \( u \mapsto (Au, Bu) \) with \( u \in C^\infty(\Omega) \) can be extended by continuity to the bounded and Fredholm operator
\[
(A, B) : \{ u \in H^{s+2q}(\Omega) : Au \in H^\sigma(\Omega) \} \to H^\sigma(\Omega) \oplus \bigoplus_{j=1}^q H^{s+2q-m_j-1/2}(\Gamma),
\]
its domain being the Hilbert space with respect to the norm
\[
\| u \|^2_{H^{s+2q}(\Omega)} + \| Au \|^2_{H^\sigma(\Omega)}^{1/2}.
\]
The index of (14) is \( \dim N - \dim N^+ \) and independent of \( s, \sigma \).

Here, we note the special case where \( \sigma = s \). If \( -1/2 < \sigma = s < 0 \), then the domain of (14) coincides with the space \( H^{s+2q}(\Omega) \) up to equivalence of norms. If \( \sigma = s = -1/2 \), then the domain is narrower than \( H^{2q-1/2}(\Omega) \) but does not depend on \( A \) as well (see Sec. 5.3 below).

We always have \( X^s(\Omega) \subseteq H^{-1/2}(\Omega) \) in Theorem 2. But we can get a space \( X^s(\Omega) \) containing an extensive class of some distributions \( f \notin H^{-1/2}(\Omega) \) and satisfying Condition I, if we use certain weighted spaces \( \varrho H^s(\Omega) \).
A function \( \varrho \) given in \( \Omega \) is called a \textit{multiplier} in the space \( H^s(\Omega) \) if the operator of multiplication by \( \varrho \) maps this space into itself and is bounded on it. A description of the class of all multipliers in \( H^s(\Omega) \) with \( s \geq 0 \) was given in [35, Sec. 6.3.3].

Let \( s < -1/2 \). We introduce the following condition on the weight function \( \varrho \).

\textbf{Condition II}. \textit{The function \( \varrho \) is a multiplier in the space \( H^{-s}(\Omega) \), and}

\begin{equation}
D_j^0 \varrho = 0 \quad \text{on} \quad \Gamma \quad \text{for every} \quad j \in \mathbb{Z}, \quad 0 \leq j < -s - 1/2.
\end{equation}

Here, \( D_j^0 \) is the derivation operator with respect to the unit vector \( \nu \) of inner normal to the boundary \( \Gamma \) of \( \Omega \). Note that if \( \varrho \) is a multiplier in \( H^{-s}(\Omega) \), then evidently \( \varrho \in H^{-s}(\Omega) \).

By the trace theorem [1, Ch. 1, Sec. 9.2], there is a trace \( (D_j^0 \varrho) | \Gamma \in H^{-s-j-1/2}(\Gamma) \) for every integer \( j \geq 0 \) such that \(-s - j - 1/2 > 0\). Hence, Condition II is formulated correctly.

\textbf{Theorem 3}. \textit{Let} \( s < -1/2 \), \textit{and a function} \( \varrho \in C^\infty(\Omega) \) \textit{be positive}. \textit{The space} \( X^s(\Omega) := gH^s(\Omega) \) \textit{satisfies Condition I, if and only if the function} \( \varrho \) \textit{satisfies Condition II}.

The next result follows from Theorems 1 and 3.

\textbf{Corollary 2}. \textit{Let} \( s < -1/2 \), \textit{and a positive function} \( \varrho \in C^\infty(\Omega) \) \textit{satisfy Condition II}. \textit{Then the mapping} \( u \rightarrow (Au, Bu) \) \textit{with} \( u \in C^\infty(\Omega) \), \( Au \in gH^s(\Omega) \) \textit{can be extended by continuity to the bounded and Fredholm operator}

\begin{equation}
(A, B) : \{ u \in H^{s+2q}(\Omega) : Au \in gH^s(\Omega) \} \rightarrow gH^s(\Omega) \oplus \bigoplus_{j=1}^q H^{s+2q-m_j-1/2}(\Gamma),
\end{equation}

\textit{its domain being the Hilbert space with respect to the norm}

\[ \left\| (u^H, u^\Gamma) \right\|_{H^s(\Omega)} = \left( \|u\|_{H^{s+2q}(\Omega)}^2 + \|\varrho^{-1} Au\|_{H^s(\Omega)}^2 \right)^{1/2}. \]

\textit{The index of} (16) \textit{is} \( \dim N - \dim N^+ \) \textit{and independent of} \( s, \varrho \).

We give an important example of a function \( \varrho \) satisfying Condition II for fixed \( s < -1/2 \) if we set \( \varrho := \varrho_1^s \), where \( \varrho_1 \) meets (10), and the number \( \delta \) is such that \( \delta \geq -s - 1/2 \in \mathbb{Z} \) or \( \delta > -s - 1/2 \notin \mathbb{Z} \) (we will prove it in Appendix).

Let us compare Theorem 1 and its Corollaries 1, 2 with the Lions-Magenes theorems [1, 3, 4, 5] on elliptic boundary-value problems in the spaces of distributions.

We restrict ourselves to the case of Hilbertian spaces, whereas the non-Hilbertian Sobolev spaces were considered in [3, 4, 5] as well.

A proposition similar to Theorem 1 was proved in [5, Sec. 6.10] for non-half-integers \( s \leq -2q \) and the Dirichlet problem, the space \( X^s(\Omega) \) obeying some different conditions depending on the problem. Our Condition I does not depend on it.

Theorem LM1 is a special case of Corollary 1 where \( \sigma = 0 \), i.e. \( X^s(\Omega) = L_2(\Omega) \). Note that some spaces \( X^s(\Omega) \) containing \( L_2(\Omega) \) are permissible in Theorem 2 and Corollary 1. The space \( X^s(\Omega) = H^{-1/2}(\Omega) \) is the most extensive among them provided that \( s \leq -1/2 \).

If \(-1/2 < \sigma < s < 0\), then Corollary 1 coincides with Theorem 7.5 of Lions and Magenes [1, Ch. 2] proved under the additional assumption that \( N = N^+ = \{0\} \).

Theorem LM2 for \( s < -1/2 \) is a special case of Corollary 2 because the function \( \varrho := \varrho_1^s \) satisfies Condition II. As we have mentioned, in the case where \(-1/2 < s < 0\) one can use the more extensive space \( H^s(\Omega) \) instead of the weighted space \( gH^s(\Omega) \) in (16).

3. \textbf{Auxiliary propositions}

First we note [1, Ch. 1, Theorem 11.5] that for every \( s > 1/2 \)

\begin{equation}
H^s_0(\Omega) := \{ u \in H^s(\Omega) : D_j^0 u = 0 \quad \text{on} \quad \Gamma \quad \forall \quad j \in \mathbb{Z}, \quad 0 \leq j < s - 1/2 \}.
\end{equation}
In addition [1, Ch. 1, Theorem 11.1]
\[(18)\] \( H_0^s(\Omega) = H^s(\Omega) \quad \text{for} \quad 0 \leq s \leq 1/2. \]

Further we will use some results of Roitberg [15, 17] on properties of the problem (1) in the Hilbert scale
\[(19)\] \( H^{s,(2q)}(\Omega) := \tilde{H}^{s,2(2q)}(\Omega), \quad s \in \mathbb{R}, \)
introduced by him. We also need some properties of this scale.

Let us give the definition of the scale (19). Let \( s \in \mathbb{R} \). In the case were \( s \geq 0 \) we denote by \( H^{s,(0)}(\Omega) \) the space \( H^s(\Omega) \). In the case were \( s < 0 \) we denote by \( H^{s,(0)}(\Omega) \) the space \( H^s_0(\mathbb{R}^n) \) antidual to \( H^s(\Omega) \) with respect to the inner product in \( L_2(\Omega) \). The space \( H^{s,(0)}(\Omega) \) is Hilbert for every \( s \in \mathbb{R} \), with the set \( C^\infty(\overline{\Omega}) \) being dense in it. Here as usual, the function \( f \in C^\infty(\overline{\Omega}) \) is identified with the functional \( (f, \cdot)_{\Omega} \).

In view of (18)
\[(20)\] \( H^{s,(0)}(\Omega) = H^s(\Omega) \quad \text{with equality of norms for} \quad s \geq -1/2. \)

If \( s \in \mathbb{R} \) and \( s \neq -1/2 \), then the spaces \( H^{s,(0)}(\Omega) \) and \( H^s(\Omega) \) are different.

Let \( s \in \mathbb{R} \) and \( s \neq -1/2 \) for all \( j = 1, \ldots, 2q \). We denote by \( H^{s,(2q)}(\Omega) \) the completion of the linear system \( C^\infty(\overline{\Omega}) \) with respect to the norm
\[
\| u \|_{H^{s,(2q)}(\Omega)} := \left( \| u \|^2_{H^{s,(0)}(\Omega)} + \sum_{j=1}^{2q} \| D_j^{s-j} u \|_{\tilde{H}^{s-j,1/2}(\Gamma)}^2 \right)^{1/2}.
\]
The space \( H^{s,(2q)}(\Omega) \) is separable Hilbert.

In the case were \( s \in \{ j - 1/2 : j = 1, \ldots, 2q \} \) we define the separable Hilbert space \( H^{s,(2q)}(\Omega) \) by means of the complex interpolation
\[
H^{s,(2q)}(\Omega) := \left[ H^{s-1/2,(2q)}(\Omega), H^{s+1/2,(2q)}(\Omega) \right]_{1/2}.
\]
We note that by the trace theorem [1, Ch. 1, Sec. 9.2]
\[(21)\] \( H^{s,(2q)}(\Omega) = H^s(\Omega) \quad \text{with equivalence of norms for} \quad s > 2q - 1/2. \)
The spaces \( H^{s,(2q)}(\Omega) \) and \( H^s(\Omega) \) are different if \( s \leq 2q - 1/2 \).

The embeddings \( H^{s_2,(0)}(\Omega) \hookrightarrow H^{s_1,(0)}(\Omega) \) and \( H^{s_2,(2q)}(\Omega) \hookrightarrow H^{s_1,(2q)}(\Omega) \) are compact and dense for arbitrary \( s_1, s_2 \in \mathbb{R}, \ s_1 < s_2 \). This follows from the compactness of the embeddings \( H^{s_1}(\Omega) \hookrightarrow H^{s_2}(\Omega) \) and \( H^{s_2}(\Gamma) \hookrightarrow H^{s_1}(\Gamma) \).

**Proposition 1.** [[17], Theorems 4.1.1, 5.3.1]. Let \( s \in \mathbb{R} \). The mapping \( u \mapsto (Au, Bu) \) with \( u \in C^\infty(\overline{\Omega}) \) can be extended by continuity to the linear bounded operator
\[(22)\] \( (A, B) : H^{s+2q,(2q)}(\Omega) \to H^{s,(0)}(\Omega) \oplus \bigoplus_{j=1}^{q} H^{s+2q-m_j-1/2}(\Gamma) =: H_{s,(0)}(\Omega, \Gamma). \)

This operator is Fredholm. Its kernel coincides with \( N \), whereas its range consists of all vectors \( (f, g_1, \ldots, g_q) \in H_{s,(0)}(\Omega, \Gamma) \) satisfying condition (5). The index of (22) is \( \dim N - \dim N^\perp \).

Proposition 1 is another example of the generic theorem on elliptic boundary-value problems. If \( s \geq 0 \), then Proposition 1 coincides with Theorem 0 by (21). But if \( s < -1/2 \), then both the spaces \( H^{s+2q,(2q)}(\Omega) \) and \( H^{s,(0)}(\Omega) = H^s_0(\mathbb{R}^n) \) consist of the elements which are not distributions in the domain \( \Omega \).

**Proposition 2.** [[17], Theorem 7.1.1]. Let \( s \in \mathbb{R}, \, \delta > 0, \) and \( u \in H^{s+2q,(2q)}(\Omega) \). If \( (A, B)u \in H_{s+\delta,(0)}(\Omega, \Gamma) \), then \( u \in H^{s+2q+\delta,(2q)}(\Omega) \).

**Proposition 3.** [[17], Theorem 6.1.1]. Let \( s \in \mathbb{R} \). The following assertions are true:
The norm in the space $H^{s+2q,(2q)}(\Omega)$ is equivalent to the norm
\[
\left( \|u\|_{H^{s+2q,(2q)}(\Omega)}^2 + \|Au\|_{H^{-s,(0)}(\Omega)}^2 \right)^{1/2}
\]
on the set of all functions $u \in C^\infty(\Omega)$. Therefore the space $H^{s+2q,(2q)}(\Omega)$ coincides with the completion of the linear system $C^\infty(\Omega)$ with respect to the norm (23).

(ii) The mapping $I_A : u \mapsto (u, Au)$ with $u \in C^\infty(\Omega)$ can be extended by continuity to the homeomorphism
\[
I_A : H^{s+2q,(2q)}(\Omega) \leftrightarrow K_{s+2q,A}(\Omega).
\]

Here,
\[
K_{s+2q,A}(\Omega) := \{ (u_0, f) : u_0 \in H^{s+2q,(0)}(\Omega), f \in H^{s,(0)}(\Omega) \}
\]

is a closed subspace in $H^{s+2q,(0)}(\Omega) \oplus H^{s,(0)}(\Omega)$.

**Proposition 4.** ([17], Theorem 6.2.1). Let $s < -2q - 1/2$. For each couple of distributions $u_0 \in H^{s+2q,(0)}(\Omega)$ and $f \in H^{s,(0)}(\Omega)$ satisfying the condition
\[
(u_0, A^+v)_{\Omega} = (f, v)_{\Omega} \quad \forall \ v \in H_0^{2q}(\Omega) \cap H^{-s,(0)}(\Omega)
\]
there exists a unique couple $(u_0^*, f) \in K_{s+2q,A}(\Omega)$ such that
\[
(u_0, v)_{\Omega} = (u_0^*, v)_{\Omega} \quad \forall \ v \in C^\infty(\Omega).
\]

Furthermore,
\[
\|u_0\|_{H^{s+2q,(0)}(\Omega)} \leq c \left( \|u_0\|_{H^{s+2q,(0)}(\Omega)}^2 + \|f\|_{H^{-s,(0)}(\Omega)}^2 \right)^{1/2},
\]
with number $c > 0$ being independent of $u_0$, $f$, and $u_0^*$.

**Remark 1.** ([17], Sec. 6.2). The conditions (24) and (25) are equivalent for $s \geq -2q - 1/2$, but they are not equivalent if $s < -2q - 1/2$.

4. PROOF OF THE MAIN RESULT

Now we will prove the main result of the paper, Theorem 1. We assume that its condition be fulfilled; i.e., $s < 0$ and the Hilbert space $X^s(\Omega)$ is embedded continuously in $D'(\Omega)$ and satisfies Condition 1. It follows that the mapping $f \mapsto \mathcal{O}f$ with $f \in X^\infty(\Omega)$ can be extended by continuity to the bounded linear operator
\[
\mathcal{O} : X^s(\Omega) \to H_{H^s}^{\infty}(\mathbb{R}^n) = H^{s,(0)}(\Omega).
\]
This operator is injective. Indeed, let $\mathcal{O}f = 0$ for a distribution $f \in X^s(\Omega)$. Chose a sequence $(f_k) \subset X^\infty(\Omega)$ such that $f_k \to f$ in $X^s(\Omega) \hookrightarrow D'(\Omega)$. Then $\mathcal{O}f_k \to 0$ in $H_{H^s}^{\infty}(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$ that implies
\[
(f, v)_{\Omega} = \lim(f_k, v)_{\Omega} = \lim(\mathcal{O}f_k, v)_{\Omega} = 0 \quad \text{for every } v \in C^\infty(\Omega).
\]
Thus, $f = 0$ as a distribution belonging to $X^s(\Omega) \hookrightarrow D'(\Omega)$; i.e., the operator (28) is injective. This operator defines the continuous embedding $X^s(\Omega) \hookrightarrow H^{s,(0)}(\Omega)$.

According to Proposition 1, the element $Au \in H^{s,(0)}(\Omega)$ is correctly defined for an arbitrary $u \in H^{s+2q,(2q)}(\Omega)$ by means of passing to the limit. We set
\[
D_{A,X}^{s+2q,(2q)}(\Omega) := \{ u \in H^{s+2q,(2q)}(\Omega) : Au \in X^s(\Omega) \}.
\]
We also endow the space $D_{A,X}^{s+2q,(2q)}(\Omega)$ with the graphs inner product
\[
\langle u_1, u_2 \rangle_{D_{A,X}^{s+2q,(2q)}(\Omega)} = \langle u_1, u_2 \rangle_{H^{s+2q,(2q)}(\Omega)} + \langle Au_1, Au_2 \rangle_{X^s(\Omega)}.
\]
The space $D_{A,X}^{s+2q,(2q)}(\Omega)$ is complete with respect to it. Indeed, let $(u_k)$ be a Cauchy sequence in $D_{A,X}^{s+2q,(2q)}(\Omega)$. Since both the spaces $H^{s+2q,(2q)}(\Omega)$ and $X^s(\Omega)$ are complete,
the limits \( u := \lim u_k \) in \( H^{s+2q,(2q)}(\Omega) \) and \( f := \lim Au_k \) in \( H^{s}(\Omega) \) exist. The first of them implies that \( Au = \lim Au_k \) in \( H^{s,(0)}(\Omega) \). We have from this in view of the second limit and the continuity of (28) that \( Au = f \in X^s(\Omega) \). Hence, \( u \in D^{s+2q,(2q)}_{A,X}(\Omega) \) and \( \lim u_k = u \in D^{s+2q,(2q)}_{A,X}(\Omega) \); i.e., the space \( D^{s+2q,(2q)}_{A,X}(\Omega) \) is complete.

By Proposition 1, the restriction of the operator (22) to \( D^{s+2q,(2q)}_{A,X}(\Omega) \) gives the bounded operator

\[
(A,B) : D^{s+2q,(2q)}_{A,X}(\Omega) \to \mathcal{H}_s(\Omega,\Gamma).
\]

The kernel of (29) is \( N \), and the range consists of all vectors \( (f,g_1,\ldots,g_q) \in \mathcal{H}_s(\Omega,\Gamma) \) satisfying condition (5). Hence, the operator (29) is Fredholm, with its co-kernel being of a dimension \( \beta \leq \dim N^+ \).

Moreover, if \( \mathcal{O}(X^\infty(\Omega)) \) is dense in \( H^{2\alpha}(\mathbb{R}^n) \), then \( \beta = \dim N^+ \). Indeed, denoting the operator (22) by \( \Lambda \) and the narrower operator (29) by \( \Lambda_0 \), we consider the operators \( \Lambda^* \) and \( \Lambda_0^* \) adjoint to them. Since the embedding \( \mathcal{H}_s(\Omega,\Gamma) \hookrightarrow H^{s,(0)}(\Omega,\Gamma) \) is continuous and dense, we have \( \ker \Lambda_0^* \supseteq \ker \Lambda^* \). Hence

\[
\beta = \dim \text{coker} \Lambda_0 = \dim \ker \Lambda_0^* \geq \dim \ker \Lambda^* = \dim \text{coker} \Lambda = \dim N^+.
\]

Therefore \( \beta = \dim N^+ \) and the index of (29) is equal to \( \dim N - \dim N^+ \) if \( \mathcal{O}(X^\infty(\Omega)) \) is dense in \( H^{2\alpha}(\mathbb{R}^n) \).

Let us show that the set \( D^{\infty}_{A,X}(\Omega) \) is dense in \( D^{s+2q,(2q)}_{A,X}(\Omega) \). Since \( X^\infty(\Omega) \times (C^\infty(\Gamma))^q \) is dense in \( \mathcal{H}_s(\Omega,\Gamma) \), we can write by the Gohberg-Krein lemma [36, Lemma 2.1]

\[
\mathcal{H}_s(\Omega,\Gamma) = (A,B)(D^{s+2q,(2q)}_{A,X}(\Omega)) + \mathcal{Q}(\Omega,\Gamma),
\]

where \( \mathcal{Q}(\Omega,\Gamma) \) is a finite-dimensional subspace satisfying the condition

\[
\mathcal{Q}(\Omega,\Gamma) \subset X^\infty(\Omega) \times (C^\infty(\Gamma))^q.
\]

Denote by \( \Pi \) the projector of the space \( \mathcal{H}_s(\Omega,\Gamma) \) onto the first term in (30) parallel to the second term.

Let \( u \in D^{s+2q,(2q)}_{A,X}(\Omega) \). Approximate \( F := (A,B)u \) by a sequence \( (F_k) \subset X^\infty(\Omega) \times (C^\infty(\Gamma))^q \) in the the topology of \( \mathcal{H}_s(\Omega,\Gamma) \). We have

\[
\lim \Pi F_k = \Pi F = F = (A,B)u \quad \text{in} \quad \mathcal{H}_s(\Omega,\Gamma),
\]

and by (31)

\[
(\Pi F_k) \subset X^\infty(\Omega) \times (C^\infty(\Gamma))^q.
\]

The Fredholm operator (29) naturally generates the topological isomorphism

\[
\Lambda_0 := (A,B) : D^{s+2q,(2q)}_{A,X}(\Omega)/N \cong \Pi(\mathcal{H}_s(\Omega,\Gamma)).
\]

In view of (32),

\[
\lim \Lambda_0^{-1} \Pi F_k = \{ u + w : w \in N \} \quad \text{in} \quad D^{s+2q,(2q)}_{A,X}(\Omega)/N.
\]

Hence, there is a sequence of representatives \( u_k \in D^{s+2q,(2q)}_{A,X}(\Omega) \) of cosets \( \Lambda_0^{-1} \Pi F_k \) such that

\[
\lim u_k = u \quad \text{in} \quad D^{s+2q,(2q)}_{A,X}(\Omega).
\]

In addition, by (33) we have

\[
(A,B)u_k = \Pi F_k \in C^\infty(\Omega \setminus \Gamma) \times (C^\infty(\Gamma))^q.
\]

It follows in view of Proposition 2, equality (20) and the Sobolev embedding theorem that

\[
u_k \in \bigcap_{\delta>0} H^{s+2q+\delta,(2q)}(\Omega) = \bigcap_{\delta>0} H^{s+2q+\delta}(\Omega) = C^\infty(\Omega \setminus \Gamma).
\]

Thus, in (34) we have \( (u_k) \subset D^{\infty}_{A,X}(\Omega) \); therefore \( D^{\infty}_{A,X}(\Omega) \) is dense in \( D^{s+2q,(2q)}_{A,X}(\Omega) \).
Next, we will consider the cases $-2q - 1/2 \leq s < 0$ and $s < -2q - 1/2$ separately.

The first case: $-2q - 1/2 \leq s < 0$. Then $H^{s+2q,0}(\Omega) = H^{s+2q}(\Omega)$ in view of (20).

We use Proposition 3 and consider the mapping $I_0 : u \mapsto u_0$ in which $u \in D^{s+2q}_{A,X}(\Omega)$ and $(u_0, f) := I_A u$. This mapping establishes the homeomorphism

$$I_0 : D^{s+2q}_{A,X}(\Omega) \leftrightarrow D^{s+2q}_{A,X}(\Omega).$$

Indeed, note that (24) $\Leftrightarrow$ (25) for arbitrary $u_0 \in H^{s+2q,0}(\Omega) = H^{s+2q}(\Omega)$ and $f \in X'(\Omega) \hookrightarrow H^{s,0}(\Omega)$ (see remark 1). Condition (25) means that $A u_0 = f$ as distributions in $\Omega$. It follows by Proposition 3 that $I_0(D^{s+2q}_{A,X}(\Omega)) = D^{s+2q}_{A,X}(\Omega)$. Moreover, we have the equivalence of the norms:

$$\|u\|_{D^{s+2q}_{A,X}(\Omega)} = \|u\|_{H^{s+2q,0}(\Omega)} + \|f\|_{X'(\Omega)}$$

Hence, the mapping $I_0$ establishes the homeomorphism (35).

It follows from properties of the operator (29) denoted by $\Lambda_0$ and the operator (35) that

$$\Lambda_0^{-1} : D^{s+2q}_{A,X}(\Omega) \to \mathcal{A}_s(\Omega, \Gamma)$$

is a bounded and Fredholm operator, with the range and index being the same as for (29). Since $I_0$ bijectively maps the set $D^\infty_{A,X}(\Omega)$ onto itself, this set is dense in $D^{s+2q}_{A,X}(\Omega)$, and (36) is an extension by continuity of the mapping $u \mapsto (A u, B u)$ with $u \in D^\infty_{A,X}(\Omega)$. Theorem 1 is proved in the first case.

The second case: $s < -2q - 1/2$. Then $H^{s+2q,0}(\Omega) = H^{s+2q}_{TT}(\mathbb{R}^n)$. In addition

$$H^{s+2q}(\Omega) = \{w \mid \Omega : w \in H^{s+2q}_{TT}(\mathbb{R}^n)\}$$

This follows immediately from the equality $H^{s+2q}(\Omega) = H^{s+2q}_{TT}(\mathbb{R}^n)/H^{s+2q}_{TT}(\mathbb{R}^n)$ mentioned in Section 2.

Let us denote $R w := w | \Omega$ for $w \in \mathcal{S}'(\mathbb{R}^n)$. We will prove that the mapping $I_0 : u \mapsto R u_0$ with $u \in D^{s+2q}_{A,X}(\Omega)$ and $(u_0, f) := I_A u$ establishes the topological isomorphism (35) in the case under consideration. (In the first case, $R u_0 = u_0$.) We use Proposition 3 and note that (24) $\Rightarrow$ (25). For an arbitrary $u \in D^{s+2q}_{A,X}(\Omega)$, we have: $R u_0 \in H^{s+2q}(\Omega)$ (see (37)), $f = A u \in X'(\Omega)$, and

$$(R u_0, A^+ v)(\Omega) = (u_0, A^+ v)(\Omega) = (f, v)(\Omega) \quad \text{for every} \quad v \in C_0^\infty(\Omega);$$

i. e., $A R u_0 = f$ as distributions in $\Omega$. Therefore $I_0 u = R u_0 \in D^{s+2q}_{A,X}(\Omega)$. Moreover, in view of (38) and the definition of $H^{s+2q}_{TT}(\mathbb{R}^n)$ we have:

$$\|I_0 u\|_{D^{s+2q}_{A,X}(\Omega)} = \|R u_0\|_{H^{s+2q}(\Omega)} + \|f\|_{X'(\Omega)}$$

Hence, the operator $I_0 : D^{s+2q}_{A,X}(\Omega) \to D^{s+2q}_{A,X}(\Omega)$ is bounded.

Now we will show that this operator is bijective. Let $\omega \in D^{s+2q,0}_{A,X}(\Omega)$ and $f := A \omega \in X'(\Omega)$. Due to (37), there is a distribution $u_0 \in H^{s+2q}_{TT}(\mathbb{R}^n)$ such that $\omega = R u_0$. Distributions $u_0$ and $f$ satisfy condition (25) because

$$(u_0, A^+ v)(\Omega) = (\omega, A^+ v)(\Omega) = (f, v)(\Omega) \quad \text{for every} \quad v \in C_0^\infty(\Omega).$$
According to Proposition 4, for \( u_0 \in H^{s+2q_0}(\Omega) \) and \( f \in X^s(\Omega) \hookrightarrow H^{s_0}(\Omega) \) there exists a unique couple \((u_0^*, f)\) in \( K_{s+2q_0}(\Omega) \) such that condition (26) is fulfilled. This implies by Proposition 3 that
\[
\left.u^* := I_A^{-1}(u_0^*, f)\right) \in D^{s+2q_0}_{A,X}(\Omega), \quad \text{and} \quad I_0u^* = Ru_0^* = \omega.
\]
The element \( u^* \) is a unique preimage of \( \omega \) under the mapping \( I_0 \). Indeed, if \( I_0u' = \omega \) for some \( u' \in D^{s+2q_0}_{A,X}(\Omega) \), then the couple \((u_0', f') := I_Au' \in K_{s+2q_0}(\Omega) \) satisfies the following conditions:
\[
f' = ARu_0 = A\omega = f \quad \text{and} \quad (u_0', v)_\Omega = (\omega, v)_\Omega = (u_0, v)_\Omega \quad \forall \ v \in C_0^\infty(\Omega).
\]
Therefore by Proposition 4, the couples \((u_0^*, f') = (u_0', f)\) and \((u_0^*, f)\) are equal that implies the equality of their preimages \( u' \) and \( u^* \) under the mapping \( I_A \).

Thus, the linear bounded operator (35) is bijective in the case examined and therefore is a topological isomorphism by the Banach theorem on inverse operator. Now using the Fredholm property of (29) and reasoning as in the first case, we complete our proof in the second case.

Theorem 1 is proved.

5. PROOFS OF THEOREMS 2, 3, AND COROLLARIES

5.1. Proof of Theorem 2. By the condition, \( s < 0, \sigma \in \mathbb{R} \), and \( X^s(\Omega) := H^\sigma(\Omega) \). Then the set \( X^\infty(\Omega) = C^\infty(\overline{\Omega}) \) is dense in the space \( H^\sigma(\Omega) \).

Sufficiency. Let (13) be fulfilled, i.e. \( \sigma \geq \max\{s, -1/2\} \). Then by (20) we have
\[
H^\sigma(\Omega) = H^{s_0}(\Omega) \hookrightarrow H^{s_0}(\Omega),
\]
with the embedding being continuous. We remark that each function \( f \in C^\infty(\overline{\Omega}) \) is identified with the functional \((f, \cdot)_\Omega\), the last being identified with the function \( \mathcal{O}f \) from the space \( H^s_0(\mathbb{R}^n) = H^{s_0}(\Omega) \). Hence
\[
\|\mathcal{O}f\|_{H^s(\mathbb{R}^n)} \leq c \|f\|_{H^{s_0}(\Omega)} \quad \text{for every} \quad f \in C^\infty(\overline{\Omega}),
\]
with number \( c > 0 \) being independent of \( f \). Sufficiency is proved.

Necessity. Let \( X^s(\Omega) := H^\sigma(\Omega) \) satisfy Condition 1. We assume that \( \sigma < 0 \). (If \( \sigma \geq 0 \), then (13) holds true). The operator (28) establishes the continuous dense embedding \( H^\sigma(\Omega) \hookrightarrow H^{s_0}(\Omega) \). This implies that
\[
H^{-s}(\Omega) = (H^{s_0}(\Omega))^\prime \subseteq (H^\sigma(\Omega))^\prime = H^{-\sigma}_0(\Omega),
\]
we denoting by \( H^\prime \) the space antidual to \( H \) with respect to the inner product in \( L_2(\Omega) \).
Hence, \(-s \geq -\sigma \). Moreover \(-\sigma \leq 1/2 \), because if \(-\sigma > 1/2 \), then the function \( f \equiv 1 \in H^{-s}(\Omega) \) would not belong to \( H^{-\sigma}_0(\Omega) \) by virtue of (17). Thus, \( \sigma \) satisfies (13). Necessity is proved.

5.2. Proof of Corollary 1. Let numbers \( s < 0 \) and \( \sigma \) satisfy inequality (13). The boundedness and the Fredholm property of (14) follow immediately from Theorems 1 and 2 in which \( X^s(\Omega) := H^\sigma(\Omega) \) and \( D^\infty_{A,X}(\Omega) = C^\infty(\overline{\Omega}) \). Moreover, since the set \( \mathcal{O}(C^\infty(\overline{\Omega})) \) identified with \( C^\infty(\overline{\Omega}) \) is dense in \( H^s_0(\mathbb{R}^n) = H^{s_0}(\Omega) \), the index of (14) is equal to \( \dim N - \dim N^+ \) by Theorem 1(iv) and therefore independent of \( s, f \).

5.3. Remark to Corollary 1. Here we consider the special case of Corollary 1 where \(-1/2 \leq \sigma = s < 0 \). In this case the domain of the operator (14) does not depend on \( A \). Indeed, if \(-1/2 < \sigma = s < 0 \), then we have the bounded operator \( A : H^{s+2q_0}(\Omega) \rightarrow H^{s}(\Omega) \) because \( s \) is not half-integer [1, Ch. 1, Proposition 12.1]. It follows that the domain of (14) coincides with \( H^{s+2q_0}(\Omega) \) and therefore does not depend on \( A \).

If \( s = \sigma = -1/2 \), we cannot reason as above because the space \( H^{-1/2}(\Omega) \) is narrower than \( A(H^{2q-1/2}(\Omega)) \). However, in view of Theorem 1(i), equality (20), and Proposition 3
we draw a conclusion that the domain of (14) is the completion of the set of all \( u \in C^\infty(\Omega) \) with respect to the norm
\[
\|u\|^2_{H^{2q-1/2}(\Omega)} + \|Au\|^2_{H^{-1/2}(\Omega)} = \|u\|^2_{H^{2q-1/2}(\Omega)} + \|Au\|^2_{H^{-1/2}(\Omega)} \leq \|u\|^2_{H^{2q-1/2}(\Omega)}.
\]
Hence, the domain coincides with the space \( H^{2q-1/2}(\Omega) \) independent of \( A \).

5.4. **Proof of Theorem 3.** By the condition, \( s < -1/2 \) whereas \( \varrho \in C^\infty(\Omega) \) is positive. Let us denote by \( M_\varrho \) and \( M_{\varrho^{-1}} \) the operators of multiplication by \( \varrho \) and \( \varrho^{-1} \) respectively. We have the isometric isomorphism \( M_\varrho : H^s(\Omega) \leftrightarrow \varrho H^s(\Omega) \). It follows from this and from the density of \( C^\infty_0(\Omega) \) in \( H^s(\Omega) \) that the set \( C^\infty_0(\Omega) \) is dense in \( H^s(\Omega) \). Hence, the more extensive set \( X^s(\Omega) \) is dense in \( H^s(\Omega) \).

We need the following lemma.

**Lemma 1.** Let \( s < -1/2 \). The multiplication by \( \varrho \in C^\infty(\Omega) \) is a bounded operator
\[
M_\varrho : H^{-s}(\Omega) \to H_{0}^{-s}(\Omega)
\]
if and only if \( \varrho \) satisfies Condition \( \text{II}_s \).

**Proof.**

**Necessity.** If the multiplication by \( \varrho \) defines the bounded operator (39), then \( \varrho \) is a multiplier in \( H^{-s}(\Omega) \) and belongs to \( H_{0}^{-s}(\Omega) \). Therefore \( \varrho \) satisfies Condition \( \text{II}_s \) in view of (17).

**Sufficiency.** Let \( \varrho \) satisfy Condition \( \text{II}_s \). We only need to prove that \( \varrho u \in H_{0}^{-s}(\Omega) \) for every \( u \in H^{-s}(\Omega) \). Condition \( \text{II}_s \) implies that \( \varrho \in H_{0}^{-s}(\Omega) \) in view of (17). We chose sequences \( (u_k) \subset C^\infty(\Omega) \) and \( (\varrho_j) \subset C^\infty_0(\Omega) \) such that \( u_k \to u \) and \( \varrho_j \to \varrho \) in \( H^{-s}(\Omega) \).

Since both the functions \( \varrho \) and \( u_k \) are multipliers in the space \( H^{-s}(\Omega) \), we have therein
\[
\lim_{k \to \infty} (\varrho u_k) = \varrho u \quad \text{and} \quad \lim_{j \to \infty} (\varrho_j u_k) = \varrho u_k \quad \text{for every} \quad k.
\]
This in view of \( \varrho_j u_k \in C^\infty_0(\Omega) \) implies that \( \varrho u \in H_{0}^{-s}(\Omega) \). Sufficiency is proved. \( \square \)

Now let us define the following space with inner product:
\[
\varrho^{-1} H_{0}^{-s}(\Omega) := \{ f = \varrho^{-1} v : v \in H_{0}^{-s}(\Omega) \},
\]
\[
(f_1, f_2) = (\varrho f_1, \varrho f_2)_{H^{-s}(\Omega)}.
\]
We have the isometric isomorphism
\[
M_{\varrho^{-1}} : H_{0}^{-s}(\Omega) \leftrightarrow \varrho^{-1} H_{0}^{-s}(\Omega).
\]
Hence, the space \( \varrho^{-1} H_{0}^{-s}(\Omega) \) is complete, with \( C^\infty_0(\Omega) \) being dense in it.

Note that
\[
(\varrho^{-1} H_{0}^{-s}(\Omega))' = \varrho H^s(\Omega) \quad \text{with equality of norms}.
\]
Indeed, passing in (40) to adjoint operator, we get the isometric isomorphism
\[
M_{\varrho^{-1}} : (\varrho^{-1} H_{0}^{-s}(\Omega))' \leftrightarrow (H_{0}^{s}(\Omega))' = H^s(\Omega).
\]
This by the definition of \( \varrho H^s(\Omega) \) implies the isometric isomorphism
\[
I = M_{\varrho}M_{\varrho^{-1}} : (\varrho^{-1} H_{0}^{-s}(\Omega))' \leftrightarrow \varrho H^s(\Omega),
\]
where \( I \) is the identity operator. Thus, (41) is proved.

Now we can complete the proof of Theorem 3 in the following way. According to Lemma 1, Condition \( \text{II}_s \) is equivalent to the boundedness of the operator (39) that by (40) is equivalent to the continuous embedding \( H^{-s}(\Omega) \hookrightarrow \varrho^{-1} H_{0}^{-s}(\Omega) \). This embedding is dense and by (40) is equivalent to the continuous dense embedding
\[
\varrho H^s(\Omega) = (\varrho^{-1} H_{0}^{-s}(\Omega))' \hookrightarrow (H^{-s}(\Omega))' = H^s(\Omega).
\]
Finally, the embedding \( \varrho H^s(\Omega) \hookrightarrow H^s(\mathbb{R}^n) \) is equivalent to Condition \( \text{I}_s \). Note that the last embedding is dense because \( C^\infty_0(\Omega) \) is dense in \( H^s_{0}(\mathbb{R}^n) \) [34, Sec. 4.3.2, Theorem 1(b)]. Thus, Conditions \( \text{II}_s \) and \( \text{I}_s \) are equivalent for \( X^s(\Omega) \) is dense in \( H^s(\Omega) \).

Theorem 3 is proved.
5.5. Proof of Corollary 2. Let \( s < -1/2 \), and a positive function \( \varphi \in C^\infty(\Omega) \) satisfy Condition II. The boundedness and Fredholm property of the operator (16) follow from Theorems 1 and 3 with \( X^s(\Omega) := \varphi H^s(\Omega) \). In addition, since the set \( \mathcal{O}(X^s(\Omega)) \) is dense in \( H^0_{\text{loc}}(\mathbb{R}^n) \), Theorem 1(iv) implies that the index of (16) is \( \dim N - \dim N^+ \) and independent of \( s, \varphi \).

Acknowledgments. The author would like to thank Yu. M. Berezansky and V. A. Mikhailets for their valuable remarks and interest in this work.

Appendix

A.1. In Appendix we will prove the following proposition, which gives an important example of the function satisfying Condition II.

Proposition A. Let a number \( s < -1/2 \) and a function \( \varphi_1 \) satisfying condition (10) be given. Assume that \( \delta \geq -s - 1/2 \in \mathbb{Z} \) or \( \delta > -s - 1/2 \notin \mathbb{Z} \). Then the function \( \varphi := \varphi_1^\delta \) satisfies Condition II.

A.2. Proof of Proposition A. Condition (15) is fulfilled for the function \( \varphi = \varphi_1^\delta \) because \( \varphi_1 = 0 \) on \( \Gamma \), and \( \delta \geq -s - 1/2 \). Therefore we only need to prove that \( \varphi_1^\delta \) is a multiplier in the space \( H^{-s}(\Omega) \). If the positive number \( \delta \) is integer, then the function \( \varphi_1^\delta \) belongs to \( C^\infty(\overline{\Omega}) \) and therefore is a multiplier in \( H^{-s}(\Omega) \). Further we assume that \( \delta \notin \mathbb{Z} \). Then by the condition, \( \delta > -s - 1/2 \).

It is not difficult to verify that the function \( \eta_b(t) := t^\delta, \ 0 < t < 1 \), belongs to \( H^{-s}((0, 1)) \) (we will do it in the next paragraph). Hence, this function has an extension from the interval \( (0, 1) \) to \( \mathbb{R} \) pertaining to \( H^{-s}(\mathbb{R}) \). Let us retain the notation \( \eta_b \) for the extension. By the Strichartz theorem [37], [35, Sec. 2.2.9], every function from the space \( H^{-s}(\mathbb{R}) \) is a multiplier therein if \( -s > 1/2 \). Hence, \( \eta_b \) is a multiplier in \( H^{-s}(\mathbb{R}) \). Then [35, Sec. 2.4, Proposition 5] the function \( \eta_{b,n}(t', t_n) := \eta_b(t_n) \) of arguments \( t' \in \mathbb{R}^{n-1}, \ t_n \in \mathbb{R} \) is a multiplier in \( H^{-s}(\mathbb{R}^n) \). This function coincides with \( \varphi_1^\delta \) in the special local coordinates \( (x', t_n) \) near the boundary \( \Gamma \). Here, \( x' \) is a coordinate of a point on \( \Gamma \), and \( t_n \) is the distance from \( \Gamma \). It follows by [35, Sec. 6.4.1, Lemma 3] that \( \varphi_1^\delta \) is a multiplier in each space \( H^{-s}(\Omega \setminus V_j) \), where \( \{V_j : j = 1, \ldots, r\} \) is a finite collection of balls in \( \mathbb{R}^n \) with a sufficiently small radius \( \varepsilon \), and the collection covers the boundary \( \Gamma \). By supplementing this collection with the set \( V_0 := \{x \in \Omega : \text{dist}(x, \Gamma) > \varepsilon/2\} \), we get the finite open covering of the closed domain \( \overline{\Omega} \). Let certain functions \( \chi_j \in C_0^\infty(V_j), \ j = 0, 1, \ldots, r \), form the partition of unity on \( \overline{\Omega} \) corresponding to this covering. Since the multiplication by a function from \( C_0^\infty(V_j) \) is a bounded operator in the space \( H^{-s}(\Omega \setminus V_j) \), the function \( \varphi_1^\delta \chi_j \) has to be a multiplier in this space and therefore in \( H^{-s}(\Omega) \). Hence, \( \varphi_1^\delta = \sum_{j=0}^r \chi_j \varphi_1^\delta \) is a multiplier in \( H^{-s}(\Omega) \).

It remains to proof that \( \eta_b \in H^{-s}((0, 1)) \). We use the inner description of the space \( H^{-s}((0, 1)) \). If \( -s \in \mathbb{Z} \), the inclusion \( \eta_b \in H^{-s}((0, 1)) \) is equivalent to that \( \eta_b \in L_2((0, 1)) \) and \( \eta_b^{(-s)} \in L_2((0, 1)) \). The last two inclusions are fulfilled because \( \delta > -s - 1/2 \). Hence, \( \eta_b \in H^{-s}((0, 1)) \) in the case examined. If \( -s \notin \mathbb{Z} \), then by [38, p. 214, Sec. 7.48] the inclusion \( \eta_b \in H^{-s}((0, 1)) \) is equivalent to that \( \eta_b \in H^{[-s]}((0, 1)) \) and

\[
\int_0^1 \int_0^1 \frac{|D^{|[-s]|} \varphi^\delta - D^{|[-s]|} \varphi^\delta|}{|t - \tau|^{1+2(|[-s]|)}} dt d\tau < \infty.
\]

Here as usual, \( |[-s]| \) and \( \{s\} \) are the integral and fractional parts of \( -s \) respectively.

Since \( \delta > [-s] - 1/2 \), we have \( \eta_b \in H^{[-s]}((0, 1)) \), that was proved above. In addition, inequality (42) holds true by virtue of the following elementary lemma, which we will prove in Subsection A.3.
Lemma A. Let $\alpha, \beta, \gamma \in \mathbb{R}$, and in addition, $\alpha \neq 0$, $\gamma > 0$. Then
\begin{equation}
I(\alpha, \beta, \gamma) := \int_0^1 \int_0^t \frac{t^\alpha - \tau^\alpha \gamma}{|t - \tau|^\beta} \, dt \, d\tau < \infty
\end{equation}
if and only if the following inequalities are fulfilled:
\begin{equation}
\alpha \gamma - \beta > -2, \quad \gamma - \beta > -1, \quad \alpha \gamma > -1.
\end{equation}
Indeed, the double integral in (42) is equal to $c \cdot I(\alpha, \beta, \gamma)$, where $c$ is a positive number, whereas $\alpha = \delta - \lfloor -s \rfloor$, $\beta = 1 + 2\lfloor -s \rfloor$, and $\gamma = 2$. Equalities (44) are fulfilled for these numbers $\alpha$, $\beta$, and $\gamma$, because
\[ \alpha \gamma - \beta = 2(\delta + s) - 1 > -2, \quad \gamma - \beta = 1 - 2\lfloor -s \rfloor > -1, \quad \alpha \gamma = 2(\delta - \lfloor -s \rfloor) > -1. \]
We have used the condition $\delta > -s - 1/2$ in the first and the third inequalities. Thus, the inclusion $\eta \in H^{\gamma}(\mathbb{R})$ is also valid in the case of non-integer $s < -1/2$.

Proposition A is proved.

A.3. Proof of Lemma A. Changing the variable $\lambda := \tau/t$ in the inner integral, we can write the following in view of evident transformations:
\[ I(\alpha, \beta, \gamma) = 2 \int_0^1 dt \int_0^t \frac{t^\alpha - \tau^\alpha \gamma}{|t - \tau|^\beta} \, d\tau = 2 \int_0^1 t^{\alpha \gamma - \beta + 1} \, dt \int_0^1 \frac{1 - \lambda^\alpha \gamma}{|1 - \lambda|^\beta} \, d\lambda. \]
Here, the integral in variable $t$ is finite if and only if $\alpha \gamma - \beta > -2$, whereas the integral in $\tau$ is finite if and only if both the inequalities $\alpha \gamma > -1$ and $\gamma - \beta > -1$ hold. Hence (43) $\Leftrightarrow$ (44), which is what had to prove.

References


INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 3 TERESHCHENKIVSKA, KYIV, 01601, UKRAINE; CHERNING STATE TECHNOLOGICAL UNIVERSITY, 95 SHEVCHENKA, CHERNING, 14027, UKRAINE

E-mail address: murach@imath.kiev.ua

Received 29/10/2008; Revised 16/03/2009