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ON THE GROUP OF FOLIATION ISOMETRIES

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Abstract. The purpose of our paper is to introduce some topology on the group $G^r_F(M)$ of all C^r-isometries of foliated manifold (M,F), which depends on a foliation F and coincides with compact-open topology when F is an n-dimensional foliation. If the codimension of F is equal to n, convergence in our topology coincides with pointwise convergence, where $n = \dim M$. It is proved that the group $G^r_F(M)$ is a topological group with compact-open topology, where $r \geq 0$. In addition it is showed some properties of F-compact-open topology.

Let M, N be n-dimensional smooth manifolds on which there are given k-dimensional smooth foliations F_1, F_2, respectively (where $0 < k < n$).

If for the some C^r-diffeomorphism $f : M \to N$ the image $f(L_\alpha)$ of any leaf L_α of the foliation F_1 is a leaf of the foliation F_2, we say that the pairs (M, F_1) and (N, F_2) are C^r-diffeomorphic. In this case the mapping f is called C^r-diffeomorphism, preserving foliation and is written as

$$f : (M, F_1) \to (N, F_2).$$

In the case where $M = N$, f is said to be a diffeomorphism of the foliated manifold (M, F).

Diffeomorphisms, preserving foliation, are investigated in [1], [2].

Definition. A diffeomorphism $\varphi : M \to M$ of the class $C^r (r \geq 0)$, preserving foliation, is called a foliation isometry F (an isometry of the foliated manifold (M, F)) if it is an isometry on each leaf of the foliation F, i.e. for each leaf L_α of the foliation, $\varphi : L_\alpha \to \varphi(L_\alpha)$ is an isometry.

Papers [3], [4] are devoted to isometric mappings of foliations. In these papers it is investigated the question under what conditions any isometry of the foliation is an isometry of the manifold and it is proved the existence of a diffeomorphism of a foliated manifold onto itself which is an isometry of the foliation, but it is not an isometry of the manifold. There is constructed an example of a diffeomorphism of a three-dimensional sphere which is an isometry of the Hopf fibration but is not an isometry of the three-dimensional sphere.

Let M be a n-dimensional smooth connected Riemannian manifold with a Riemannian metric g, F a smooth k-dimensional foliation on M. (In this paper manifolds and foliations have C^∞-smoothness). We denote by $L(p)$ a leaf of the foliation F passing through point the p, by T_pF the tangent space to the the leaf $L(p)$ at p, and by H_pF its orthogonal complement of T_pF in T_pM, $p \in M$. We get two subbundles (smooth distributions) $TF = \{T_pF : p \in M\}$, $\quad HF = \{H_pF : p \in M\}$ of the tangent bundle TM of the manifold M and, as a result, the tangent bundle TM of the manifold M decomposing into the sum of two orthogonal bundles, i.e., $TM = TF \oplus HF$. The restriction of the Riemannian metric g to T_pF for all p induces a Riemannian metric on the leaves. The induced Riemannian metric defines a distance function on every leaf. Further, everywhere

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in this paper, under the distance on a leaf, we understand this distance. This distance on a leaf is different from the distance induced by the distance on M.

Let us denote by $G^r_F(M)$ the set of all C^r isometries of a foliated manifold (M, F), where $r \geq 0$. The following remarks show that the notion of an isometry of a foliated manifold is correctly defined.

Remark 1. If $r \geq 1$, for each element $\varphi \in G^r_F(M)$ the differential $d\varphi$ preserves the length of each tangent vector $\nu \in T_pF$, i.e., $\mid d\varphi_p(\nu) \mid = \mid \nu \mid$ at any $p \in M$.

Remark 2. If $r = 0$, each element φ from $G^r_F(M)$ is a homeomorphism of the manifold M. A Riemannian metric on the manifold M induces a Riemannian metric on each leaf L_α which defines a distance on it. In this case, φ is an isometry between the metric spaces L_α and $\varphi(L_\alpha)$. Then, according to the known theorem, φ is a diffeomorphism of L_α onto $\varphi(L_\alpha)$ for each leaf L_α and its differential preserves the length of each tangent vector $\nu \in T_pF$, i. e., $\mid d\varphi_p(\nu) \mid = \mid \nu \mid$ at any $p \in M$ [5, page 74]. But as shown by a simple example, from differentiability of a mapping on each leaf does one can not infer its differentiability on the entire manifold M .

Example 1. Let $M = R^2(x, y)$ be the Euclidean plane with the Cartesian coordinates (x, y). Leaves L_α of foliation a F are given by the equations $y = \alpha = \text{const}$. Then the plane homeomorphism $\varphi : R^2 \to R^2$ determined by the formula

$$\varphi(x, y) = (x + y, y^{\frac{1}{3}})$$

is an isometry of the foliation F, but is not a diffeomorphism of the plane.

The set $\mathrm{Diff}^r(M)$ of all diffeomorphisms of a manifold M onto itself is a group with the operations of composition and taking the inverse. The set $G^r_F(M)$ is a subgroup of the group $\mathrm{Diff}^r(M)$.

It is known that if a manifold M is compact, the group $\mathrm{Diff}^r(M)$ is a topological group in the compact-open topology. This fact follows from Proposition 2 on page 269 and from Propositions 3, 4 on page 270 of [6]. It is also known that the group of isometries $I(M)$ is always a topological group with compact-open topology [5, page 192].

The purpose of this paper is to introduce some topology on the set $G^r_F(M)$, depending on the foliation F, such that it coincides with the compact-open topology if F is an n-dimensional foliation. If the codimension of the foliation F is equal to n, convergence in our topology coincides with the pointwise convergence.

Let $\{K_\lambda\}$ be a family of all compact sets where each K_λ is a subset of some leaf of the foliation F and let $\{U_\beta\}$ be a family of all open sets on M. We consider, for each pair K_λ and U_β, the set of all mappings $f \in G^r_F(M)$ such that $f(K_\lambda) \subset U_\beta$. This set of mappings is denoted by $[K_\lambda, U_\beta] = \{f : M \to M | f(K_\lambda) \subset U_\beta\}$.

It is not difficult to show that every possible finite intersections of sets of the form $[K_\lambda, U_\beta]$ forms a base for some topology. We will call this topology the foliated compact-open topology or, briefly, the F-compact-open topology.

The following proposition can be proved using a standard argument.

Proposition. *The set $G^r_F(M)$ with the F-compact open topology is a Hausdorff space.*

Theorem 1. *Let M be a smooth, connected and finite-dimensional manifold. Then the group of homeomorphisms* Homeo(M) *is a topological group with compact-open topology. In particular, the subgroups* $\mathrm{Diff}^r(M), G^r_F(M)$ *are topological groups in this topology.*

Proof. It is known that the mapping $(g, h) \to g \circ h$ is continuous for every Hausdorff and locally compact space [6, page 269].

By $d(x, y)$ we denote the distance between the points x and y, determined by some complete Riemannian metric. It is known that a smooth manifold M possesses a complete Riemannian metric [7, page 186].

Using a complete Riemannian metric on M we will prove that the mapping $\chi : f \to f^{-1}$ is continuous. For this purpose, we shall prove that the full preimage $\chi^{-1}(A)$ of the open set $A \subset \mathrm{Homeo}(M)$ is an open set. Indeed, it is enough to show this fact if A is an element of the prebase, i.e., $A = \{f \in \mathrm{Homeo}(M) : f(K) \subset V\}$, where K is compact and V is an open set. In this case, $\chi^{-1}(A) = \{f \in \mathrm{Homeo}(M) : f^{-1}(K) \subset V\}$. We will show that $\chi^{-1}(A)$ is an open set in the compact-open topology. Let $g \in \chi^{-1}(A)$, U be a neighborhood of $g^{-1}(K)$ with compact closure such that $\overline{U} \subset V$. We put $U(g) = \{g \in \mathrm{Diff}^r(M) : \quad d(g(x), h(x)) < \frac{\varepsilon}{2},\ \forall x \in \overline{U}\}$, where $\varepsilon = d(K, M\backslash g(U)) = \inf\{d(x, y) : x \in K,\ y \in (M\backslash g(U))\}$.

Let us show that, if $h \in U(g)$, then $h^{-1}(K) \subset V$, i.e., $U(g) \subset \chi^{-1}(A)$. We will show that $h^{-1}(K) \subset U$. Assume it is not true. Let, for the some $h \in U(g)$, there exists a point $y \in K$ such that $h^{-1}(y) \in M\backslash U$, i.e., $y \in M\backslash h(U)$. Then, since $g^{-1}(y) \in U$, we have $d(y = g(g^{-1}(y)), h(g^{-1}(y))) < \frac{\varepsilon}{2}$. Let γ be the shortest geodesic (by virtue of completeness of M there exists a shortest geodesic between any two points) going from the point y to the point $h(g^{-1}(y))$, and $z \in \gamma \cap \partial(h(U))$. Then $h^{-1}(z) \in \overline{U}$ and besides $d(g(h^{-1}(z)), h(h^{-1}(z)))) < \frac{\varepsilon}{2}$. In addition $d(y, z) < \frac{\varepsilon}{2}$. Hence, $d(y, g(h^{-1}(z))) \leq d(y, z) + d(z, g(h^{-1}(z))) < \varepsilon$. But, on the other hand, since $z \notin h(U)$, we have $g(h^{-1}(z)) \in M\backslash g(U)$ and $d(y, g(h^{-1}(z))) \geq \varepsilon$. This contradiction shows that $h^{-1}(K) \subset U$. Hence, $U(g) \subset \chi^{-1}(A)$.

Now we will show that $U(g)$ is an open set in the compact-open topology. Let $f \in U(g)$, $\quad 0 < r < \frac{\varepsilon}{2} - a$, where $a = \max\limits_{x \in \overline{U}}\{d(f(x), g(x))\}$. We will show that f is an interior point of the set $U(g)$. For this purpose, we cover the compact set $f(\overline{U})$ with a finite number of balls B_i with the centers in the points $x_i \in \overline{U}$ and radius $0 < \delta < \frac{r}{2}$. If we put $K_i = f^{-1}(\overline{B_i})$, then $\bigcup\limits_{i=1}^{m} K_i \subset \overline{U}$. Let $h \in \bigcap\limits_{i=1}^{m} [K_i, \widetilde{B}_i]$, where $\widetilde{B}_i$ is a ball of radius $\frac{r}{2}$ with the center in the point x_i. If $x \in \overline{U}$, then it is not difficult show that $d(h(x), f(x)) < r + a < \frac{\varepsilon}{2}$, i.e., $h \in U(g)$. Thus f, together with the neighborhood $\bigcap\limits_{i=1}^{m} [K_i, \widetilde{B}_i]$, lies inside of $U(g)$. Hence, the set $U(g)$ is an open set in the compact-open topology. Theorem 1 is proved. □

The following theorem will be used later and deals with the theory of foliation.

Theorem 2. *Let M be a smooth complete Riemannian manifold of dimension n with a smooth foliation of dimension k, where $0 < k < n$.*

1) *Each leaf with the induced Riemannian metric is a complete Riemannian manifold.*

2) *Let $\gamma_m : (a, b) \to L_m$ be a sequence of geodesics (determined by the induced Riemannian metrics) on leaves L_m. If $\gamma_m(s_0) \to p$, $\quad \dot{\gamma}_m(s_0) \to v$ for $m \to \infty$ for the some $s_0 \in (a, b)$, then the sequence γ_m pointwise converges to the geodesic $\gamma : (a, b) \to L(p)$ of the leaf $L(p)$ that passes through the point p at $s = s_0$ in the direction of the vector v.*

Proof. 1) It is known that for a connected manifold M, the following conditions are equivalent [8, page 167]:

a) M is a complete Riemannian manifold;

b) M is a complete metric space with distance which is defined by a Riemannian metric. That is why we can prove the first part of Theorem 2 by using sequences.

Let L_α be some leaf of the foliation F, d_α the distance on L_α determined by the induced Riemannian metric. Let $\{p_m\}$ be a Cauchy sequence in L_α, i.e., for every $\varepsilon > 0$ there exists N such that $d_\alpha(p_m, p_i) < \varepsilon$ for $m, i \geq N$. Since $d(p_m, p_i) \leq d_\alpha(p_m, p_i)$, where d is the distance on M, the sequence $\{p_m\}$ is a Cauchy sequence in M . Since M is complete, this sequence converges in M.

By the definition of a foliation, for each point $p \in M$ there is a neighborhood U of the point p and a local system of coordinates $(x^1, x^2, \dots, x^k, y^{k+1}, y^{k+2}, \dots, y^n)$ on U such that the set $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^k}\}$ is basis for smooth sections $TF|_U$ (the restriction TF to U). Such a neighborhood is called a foliated neighborhood of the point p [1, page 122].

Let U be a foliated neighborhood of the point p with local coordinates $(x^1, x^2, \dots, x^k, y^{k+1}, y^{k+2}, \dots, y^n)$, where $p = \lim_{m\to\infty} p_m$. Then connected components of the intersection $U \bigcap L_\beta$ for any leaf L_β are given by the equations $y^{k+1} = \text{const}$, $y^{k+2} = \text{const}$, $\dots$, $y^n = \text{const}$.

Let $\varepsilon > 0$ be a small number such that the open ball $B_\varepsilon(p) = \{q \in M : d(p, q) < \varepsilon\}$ is contained in U and m_0 be an integer such that $d(p, p_l) < \frac{\varepsilon}{4}$ for $l \geq m_0$ and $d_\alpha(p_{l'}, p_{l''}) < \frac{\varepsilon}{2}$ at $l', l'' \geq m_0$. If two points p_l, p_m belong to different connected components of $L_\alpha \cap U$ then any curve in L_α which goes from p_l to p_m leaves the ball $B_\varepsilon(p)$ and returns into this ball before coming to the point p_m. From here it follows that $d_\alpha(p_l, p_m) \geq \varepsilon$, where $d_\alpha(p_l, p_m)$ is the distance between the points p_l and p_m on the leaf L_α. This contradiction shows that all points p_m for $m \geq m_0$ belong to the same connected component L_0 of the intersection $L_\alpha \cap U$. The component L_0 is defined by the equations $y^{k+1} = \text{const}$, $y^{k+2} = \text{const}$, $\dots$, $y^n = \text{const}$ and, therefore, the limit point p also belongs to L_0. From here it follows that $p \in L_\alpha$ and p_m converges to p in L_α.

2) Let $\pi : TM \to TF$ be an orthogonal projection, $V(M), V(F), V(H)$ a set of smooth sections of the bundles TM, TF, HF, respectively. We set $\widetilde{\nabla}_X Y = \pi(\nabla_X Y)$ for vector fields $X \in V(M)$, $Y \in V(F)$, where ∇ is a Levi-Civita connection determined by the Riemannian metric g on M. It is known that $\widetilde{\nabla}_X Y$ is a connection on TF, and its restriction to each leaf L_α coincides with a connection on L_α, determined by the induced Riemannian metric on L_α from M ([9, page 20], [10, page 59]). Therefore, the smooth parametric curve $\mu : (a, b) \to M$ lying on a leaf L_α of the foliation F is geodesic on L_α (determined by the induced Riemannian metric) if and only if

$$\widetilde{\nabla}_{\dot\mu}\dot\mu = 0. \tag{1}$$

If μ lies in the foliated neighborhood U, its equations have the form

$$\begin{cases} x^i = x^i(s) \\ y^\alpha = \text{const} \end{cases},$$

where $1 \leq i \leq k,\ k+1 \leq \alpha \leq n$. So, for ∇, we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^l_{i,j} \frac{\partial}{\partial x^l} + \Gamma^\alpha_{i,j} \frac{\partial}{\partial y^\alpha}, \tag{2}$$

hence,

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^i}} = \Gamma^l_{i,j} \frac{\partial}{\partial x^l}, \tag{3}$$

where $1 \leq i, j, l \leq k$, $k+1 \leq \alpha \leq n$, $\Gamma^\beta_{i,j}$ are Christoffel symbols. From here using properties of the operator $\widetilde{\nabla}$ it follows that equation (1) is equivalent to the following system of differential equations of the 2nd order:

$$\frac{d^2 x^i}{ds} + \Gamma^i_{l,j} \frac{dx^l}{ds} \frac{dx^j}{ds} = 0. \tag{4}$$

If we put $u^i = \frac{dx^i}{ds}$, then it is possible to write this system as

$$\begin{cases} \frac{dx^i}{ds} = u^i \\ \frac{du^i}{ds} = -\Gamma^i_{l,j} u^l u^j \end{cases}. \tag{5}$$

Let us consider a geodesic $\gamma : (a, b) \to L(p)$, leaving a point p at $s = s_0$ in the direction of a vector v. This curve satisfies equation (1). Let $K_0 \subset (a, b)$ be a compact set containing s_0 such that $\gamma(K_0) \subset V$, $\{x^i(s)\}$, $\{u^i(s)\}$ are the first k coordinates of the point $\gamma(s)$ and the velocity vector $\dot{\gamma}(s)$, respectively, where $s \in K_0$, $1 \leq i \leq k$. Then these functions satisfy a system of differential equations (5) with the initial conditions: $x^i(s_0) = p^i$, $u^i(s_0) = v^i$, where $i = 1, 2, \dots, k$, $p = (p^1, p^2, \dots, p^n)$, $v = (v^1, v^2, \dots, v^n)$. Since $\gamma_m(s_0) \to p$, $\dot{\gamma}_m(s_0) \to v$ for $m \to \infty$ under the theorem of continuous dependence of a solution of the differential equation on the initial data, the sequence γ_m converges to γ uniformly on the compact $K_0 \subset (a, b)$. Further for every compact set $K \subset (a, b)$ containing K_0, and covering $\gamma(K)$ with foliated neighborhoods, we will obtain that γ_m converges to γ uniformly on the compact set K. Theorem 2 is proved. □

Remark. With a view of simplification of designations, in terms of a kind (2), (3) which have a summation with a repeating index, the symbol of summation is omitted.

The following theorem shows some property of the group $G^r_F(M)$ with the F-compact-open topology.

Theorem 3. *Let M be a complete smooth n-dimensional Riemannian manifold with a smooth k-dimensional foliation F, $f_m \in G^r_F(M)$, $r \geq 1$. Suppose that for each leaf L_α there exists a point $o_\alpha \in L_\alpha$ such that $f_m(o_\alpha) \to f(o_\alpha)$, $d(f_m(o_\alpha)) \to d(f(o_\alpha))$, where $f \in G^r_F(M)$. Then the sequence f_m converges to f in F-compact-open topology.*

Proof. Let $p \in L_\alpha$ for some leaf L_α. Under the conditions of the theorem there exists a point $o_\alpha \in L_\alpha$ such that $f_m(o_\alpha) \to f(o_\alpha)$. Let $\gamma : [0; l] \to L_\alpha$ be a geodesics determined by the induced Riemannian metric on L_α such that $\gamma(0) = o_\alpha$, $\gamma(l) = p$. By virtue of completeness of the leaf L_α, without loss of generality, we can assume that γ realizes the distance $d_0 = d_\alpha(o_\alpha, p)$ on L_α, and let γ be parametrized by the length of the arc. We put $\gamma^{'}(s) = f(\gamma(s))$ for all $s \in [0; l]$. Since f is an isometry, $\gamma^{'}$ is a geodesic on $f(L_\alpha)$, moreover its length is equal to the distance between the points $f(o_\alpha)$ and $f(p)$. If we consider $\gamma_m = f_m(\gamma)$, than they are geodesics on $f_m(L_\alpha)$. From the conditions of the theorem, we have $\gamma_m(0) \to \gamma^{'}(0)$, $\dot{\gamma}_m(0) \to \dot{\gamma}^{'}(0)$ for at $m \to \infty$, where $\dot{\gamma}_m(0), \dot{\gamma}^{'}(0)$ are tangential vectors. Then from Theorem 2 we get that the sequence $\gamma_m(s)$ converges to $\gamma^{'}(s)$ for each $s \in [0; l]$. Therefore it follows that $\lim\limits_{m\to\infty} f_m(p) = f(p)$.

Now we show that $f_m \to f$ uniformly on each compact set lying in a leaf of the foliation F. We denote by $d(x, y)$ the distance between points x and y, determined by the Riemannian metric. Let K be a compact set in a leaf L and $\varepsilon > 0$. Since K is compact there exists a finite number of points $p_1, p_2, \dots, p_m$ in L such that each point $p \in K$ has distance less than ε from some p_i .

For each point p_i there is a number N_i such that $d(f_m(p_i), f(p_i)) < \frac{\varepsilon}{3}$ for any $m \geq N_i$. Besides, for each point $p \in K$ there exists p_i such that $d_L(p, p_i) < \frac{\varepsilon}{3}$ where $d_L(p, p_i)$ is the distance between the points p and p_i determined by the induced Riemannian metric on L. Therefore it follows that

$$\begin{aligned} d(f_m(p), f(p)) &\leq d(f_m(p), f_m(p_i)) + d(f_m(p_i), f(p_i)) + d(f(p_i), f(p)) \\ &\leq d_m(f_m(p), f_m(p_i)) + d(f_m(p_i), f(p_i)) + d_0(f(p_i), f(p)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for $m > N = \max\limits_{1\leq i\leq m} \{N_i\}$, where d_m- is the distance on the leaf $f_m(L)$, d_0 is the distance on the leaf $f(L)$. From here it follows that $f_m \to f$ in the F-compact-open topology. Theorem 3 is proved. □

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