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ABOUT $*$-REPRESENTATIONS OF POLYNOMIAL SEMILINEAR RELATIONS

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Dedicated to the memory of A. Ya. Povzner

ABSTRACT. In the present paper we study $*$-representations of semilinear relations with polynomial characteristic functions. For any finite simple non-oriented graph Γ we construct a polynomial characteristic function such that Γ is its graph. Full description of graphs which satisfy polynomial (degree one and two) semilinear relations is obtained. We introduce the G-orthoscalarity condition and prove that any semilinear relation with quadratic characteristic function and condition of G-orthoscalarity is $*$-tame. This class of relations contains, in particular, $*$-representations of $U_q(so(3))$.

1. Introduction

Pairs of self-adjoint linear operators in a Hilbert space which satisfy semilinear relations arise in different problems of mathematics and physics (see, for example [3, 5]) and were studied in [2, 10, 11, 12, 13, 14] and others.

Following [2, 10, 14] we will say that a pair of bounded selfadjoint operators (A, B) in a Hilbert space H satisfies a semilinear relation if

$$\sum_{i=1}^{m} f_i(A)\, B\, g_i(A) = h(A), \tag{1.1}$$

where $f_i(t)$, $g_i(t)$, $h(t)$ are polynomials on $\mathbb{R}$ for all $i = \overline{1, m}$. A pair of operators (A, B), which satisfies (1.1) is called a representation of the semilinear relation.

With any semilinear relation the following two objects are associated:

- a characteristic function, which is a polynomial of two variables on $\mathbb{R}^2$:
$$P_n(t, s) := \sum_{i=1}^{m} f_i(t) g_i(s) = \sum_{0 \le i+j \le n} a_{ij}\, t^i s^j, \quad (t, s) \in \mathbb{R}^2, \quad a_{ij} \in \mathbb{R}.$$
In the present paper we assume that for the characteristic function, $P_n(t, s) = 0$ if and only if $P_n(s, t) = 0$;
- a simple non-oriented graph Γ, for which the set of vertices in $\mathbb{R}$, and a vertex $t \in \mathbb{R}$ is connected with a vertex $s \in \mathbb{R}$ if and only if $P_n(t, s) = 0$.

A standard way for describing all $*$-representation of a semilinear relation is to describe all irreducible $*$-representation up to a unitary equivalence and decompose any representation into a direct sum or an integral of irreducible ones. Thus in the present paper we study only irreducible representations.

The structure of irreducible representations crucially depends on the structure of connected components of the graph Γ. Applying the well-known facts about $*$-representations of graphs, we see that the problem of unitary description of all $*$-representations for pairs

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of operators witch satisfy semilinear relations is not ∗-wild (see [10]) if and only if all connected components of the corresponding graph are of the forms: .

In Section 2 we show that arbitrarily complicated graphs can arise even for polynomial semilinear relations: for any finite simple non-oriented graph Γ we construct a polynomial characteristic function such that Γ is its graph (Theorem 2.1).

In Section 3 we list graphs which arise as connected components of characteristic functions of degree one and two (Theorem 3.1). To do it, we construct a dynamical system on the set of zeroes of the polynomial such that the connected components are described in terms of orbits of the dynamical system.

As one could expect, only few of them are ∗-tame, so it is natural to study pairs (A, B) satisfying semilinear relations with additional conditions. In Section 4 we introduce the condition of G-orthoscalarity and show that the description of G-orthoscalar pairs satisfying a semilinear relation with a quadratic characteristic function is ∗-tame (Theorem 4.1). The G-orthoscalarity condition is closely related to the orthoscalarity condition in representations of graphs (see [6, 7]). On the other hand, the Fairlie algebra [3, 5, 13] generated by self-adjoint elements $a = a^*$, $b = b^*$ and two semilinear relations $[a, [a, b]_q]_{q^{-1}} = -b$, $[b, [b, a]_q]_{q^{-1}} = -a$, $(q \in \mathbb{T} \cup \mathbb{R} \backslash \{0\})$ gives an important example of semilinear relation with a quadratic characteristic function and the G-orthoscalarity condition.

2. Graphs of polynomials

To begin with, we discuss some general properties of polynomial characteristic functions of semilinear relations.

Proposition 1. *Let* $P_n(t, s) = \sum_{0 \le i+j \le n} a_{ij}\, t^i s^j$ *satisfies the following properties:*

- $P_n(t, s)$ *is an irreducible polynomial on* $\mathbb{R}^2$, *that is* $P_n(t, s)$ *cannot be decomposed into a product of two real polynomials,*
- $P_n(t, s) = 0$ *if and only if* $P_n(s, t) = 0$,
- *the set* $\mathcal{V} = \{(t, s) \in \mathbb{R} |\ \ P_n(t, s) = 0\}$ *contains at least one regular point of* $P_n(t, s)$ *(a point* $(t_0, s_0) \in \mathcal{V}$ *in which* $\operatorname{grad} P_n|_{(t_0,s_0)} \neq 0$*),*

then either $P_n(t, s) = P_n(s, t)$, *or* $P_n(t, s) = c(t - s)$.

Proof. The proof is based in the following lemma [8].

Lemma 1. *Let* $\mathcal{V}$ *be a real algebraic set defined by an irreducible polynomial equation* $f(x) = 0$, $x \in \mathbb{R}^n$, *and the set* $\mathcal{V}$ *contain at least one regular point of the polynomial* f. *Then any polynomial which vanishes in* $\mathcal{V}$ *is a multiple of the polynomial* f.

It follows from the lemma above that under the conditions of the theorem, $P_n(t, s) = cP_n(s, t)$. If $c \neq 1$, then $P_n(t, t) = 0$ for all t and therefore, by the irreducibility, $P_n(t, s) = c(t - s)$. □

By using singular polynomials (the set $\mathcal{V}$ contains only singular points) we can build polynomial with any preassigned finite graph.

Theorem 2.1. *For any finite simple non-oriented graph* Γ *there exists a polynomial* $P_n(t, s) = P_n(s, t)$ *such that is* Γ *is a graph of* $P_n(t, s)$.

Proof. Let $\Gamma = (V, E)$ be a given graph with sets of vertices V and edges E,

$$V = \{v_1, v_2, \dots, v_r\}, \quad v_i \in \mathbb{R}, \quad r \in \mathbb{N}, \quad i = \overline{1, r},$$
$$E = \{e_1, e_2, \dots, e_w\} \subseteq (V \times V) \subseteq (\mathbb{R} \times \mathbb{R}), \quad e_j = (v_{j_1}, v_{j_2}), \quad w \in \mathbb{N}, \quad j = \overline{1, w}.$$

Construct the polynomial $P_n(t,s)$ as follows:

$$P_n(t,s) = \prod_{i=1}^{w} [(t - v_{i_1})^2 + (s - v_{i_2})^2][(t - v_{i_2})^2 + (s - v_{i_1})^2] \tag{2.1}$$

We see that only elements of the set E are roots of this polynomial. $\square$

Proposition 2. *Let $P_n(t,s)$ be a polynomial characteristic function of a semilinear relation of degree n, Γ be the corresponding graph, then the valency of any vertex of Γ is less or equal to n, if for any $x_0 \in \mathbb{R}$ $P_n(x_0, s) \not\equiv 0$.*

Proof. If we fix $t = t_0 \in \mathbb{R}$, then $P_n(t_0, s)$ is a polynomial of degree n on $\mathbb{R}$. Then $P_n(t_0, s)$ has no more than n real roots, that is, any vertex t_0 of the graph is connected with not more than n vertex. $\square$

Remark 1. If there exist $x_0 \in \mathbb{R}$ such that $P_n(x_0, s) \equiv 0$, then the valency of its vertex is equal to $\aleph_1$.

3. Graphs of polynomials of degrees one and two

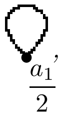
3.1. Polynomials of degree one.

Proposition 3. *Symmetric and antisymmetric polynomials of degree one has the following connected components of its graph:*

(1) *For $P_1^-(t,s) := t - s$,*

, $t \in \mathbb{R}$;

(2) *For $P_1^+(t,s) := t + s - a_1$, $a_1 \in \mathbb{R}$,*



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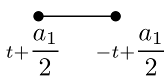


, $t \in \mathbb{R}\setminus\{0\}$.

Corollary 1. *The problem of describing all irreducible pairs of self-adjoint operators (A, B) up to unitary equivalence satisfying the semilinear relations*

$$AB - BA = h_1(A), \quad AB + BA - a_1 B = h_2(A),$$

$$a_1 \in \mathbb{R}, \quad h_1(t), h_2(t) \text{ are Borel function on } \mathbb{R}.$$

is $*$-tame (for a complete description, see [10]*).*

Consider the algebraic curve

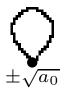
$$P_2(t,s) := a_0 + 2a_1(t+s) + a_2(t^2 + s^2) + 2a_{11}ts = 0, \quad a_0, a_1, a_2, a_{11} \in \mathbb{R}. \tag{3.1}$$

Consider two different cases.

3.2. **Polynomials without squares.** First, let $a_2 = 0$. Making a shift of variables, $t \to t - a_1/a_{11}$, $s \to s - a_1/a_{11}$, and normalizing by setting $a_{11} \neq 0$, we can assume that $a_1 = 0$ and $a_{11} = 1$, and have the equation: $P_2(t,s) := ts = a_0$.

Two different cases arise:

(1) if $a_0 \neq 0$, then any connected component of the graph of $P_2(t,s)$ is one of the following:



(if $a_0 > 0$),

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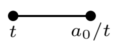


, $t \in \mathbb{R}\setminus\{\pm\sqrt{a_0}, 0\}$.

Corollary 2. *The problem of describing all irreducible pairs of self-adjoint operators (A, B) up to unitary equivalence satisfying the semilinear relation*

$$a_0 B + ABA = h(A),$$

$h(t)$ is Borel function on $\mathbb{R}$, $a_0 \in \mathbb{R}$, $a_0 \neq 0$, is $*$-tame (for a full description see [10]*).*

(2) if $a_0 = 0$, then the whole graph $P_2(t, s)$ is connected, any point is connected to zero.

Corollary 3. *The problem of describing all irreducible pairs of self-adjoint operators (A, B) up to unitary equivalence satisfying the semilinear relation*

$$ABA = h(A), \qquad h(t) \text{ is a Borel function on } \mathbb{R},$$

is $*$-wild.

3.3. **Polynomials with square members.** Now let $a_2 \neq 0$. Assume that $a_2 = 1$, then we have the curve

$$P_2(t, s) := a_0 + 2a_1(t + s) + (t^2 + s^2) + 2a_{11}ts = 0. \tag{3.2}$$

To describe the graph corresponding to (3.2), we introduce a dynamical system on the set $\mathcal{V} = \{(t, s) \in \mathbb{R}^2 | \quad P_2(t, s) = 0\}$. For any $(t, s) \in \mathcal{V}$ construct two maps:

$$f_1(t, s) := (s, t), \quad f_2(t, s) := (t, \varphi(t, s)), \quad \text{where} \quad \varphi(t, s) : \mathcal{V} \to \mathbb{R}. \tag{3.3}$$

Since $P_2(t, s) = 0$ iff $P_2(s, t) = 0$, it follows that $\varphi(t, \varphi(t, s)) = s$ and $f_1 f_1(t, s) = (t, s)$, $f_2 f_2(t, s) = (t, s)$ for any $(t, s) \in \mathcal{V}$.

Consider the following map

$$F(t, s) := f_1 f_2(t, s) = (\varphi(t, s), t). \tag{3.4}$$

Proposition 4. *The dynamical system generated by the maps f_1, f_2 is completely defined by the dynamical system generated by the map F.*

Proof. For any $(t, s) \in \mathbb{R}^2$ and $n \in \mathbb{N}_0$, this map has the following properties:

$$(f_1 f_2)^n(t, s) = F^n(t, s), \qquad f_1(f_2 f_1)^n(t, s) = F^n(s, t),$$
$$(f_2 f_1)^n(t, s) = f_1 F^n(s, t), \qquad f_2(f_1 f_2)^n(t, s) = f_1 F^{n+1}(t, s).$$

Then the trajectory of any point $(t_0, s_0) \in \mathbb{R}^2$ with respect to the action of the maps f_1, f_2 can be uniquely recovered from the trajectory of points $(t_0, s_0), (s_0, t_0) \in \mathbb{R}^2$, with respect to the action of the map F as follows:

$$\mathrm{Tr}_{f_1, f_2}(t_0, s_0) = \{F^n(t_0, s_0), f_1 F^n(t_0, s_0), F^n(s_0, t_0), f_1 F^n(s_0, t_0),\ n = 0, 1, \ldots\}. \qquad \square$$

The proposition above implies the following fact.

Proposition 5. *Let $F^n(t, s) := (t_n, s_n)$, $n \in \mathbb{N}_0$. If $\Gamma = (V, E)$ is the graph of the polynomial $P_2(t, s)$, then the set of its vertices is $V = V_1 \cup V_2$ and the set of its edges is $E_1 \cup E_2$, where $V_1 = \{s_n, n \in \mathbb{N}_0\}$, $E_1 = \{(s_n, s_{n+1}), n \in \mathbb{N}_0\}$, with starting points $(s_0, s_1) = (\lambda, \mu)$ and $V_2 = \{s_n, n \in \mathbb{N}_0\}$, $E_2 = \{(s_n, s_{n+1}), n \in \mathbb{N}_0\}$, with starting points $(s_0, s_1) = (\mu, \lambda)$, $(\lambda, \mu) \in \mathbb{R}^2$, such that $P_2(\lambda, \mu) = 0$.*

Now we apply these facts to describe connected components of the graph of a quadratic characteristic function. Consider the dynamical system $(\mathcal{V}, \mathbb{N}_0, F)$, where

$$\mathcal{V} = \{(t, s) \in \mathbb{R}^2 \mid a_0 + 2a_1(t + s) + (t^2 + s^2) + 2a_{11}ts = 0,\ a_0, a_1, a_{11} \in \mathbb{R}\} \tag{3.5}$$
$$F(t, s) = (-2a_{11}t - s - 2a_1, t), \quad \text{for any } (t, s) \in \mathcal{V}.$$

Let $F^n(t, s) = (t_n, s_n)$, then it follows from (3.5) that

$$\begin{cases} t_{n+1} = -2a_{11}t_n - s_n - 2a_1, \\ s_{n+1} = t_n \end{cases} \Rightarrow \begin{cases} t_{n+1} = -2a_{11}t_n - s_n - 2a_1, \\ t_{n+1} = s_{n+2} \end{cases}$$

and we have the difference equation

$$s_{n+2} + 2a_{11}s_{n+1} + s_n = -2a_1, \quad n \in \mathbb{N}_0. \tag{3.6}$$

Solutions of this equation are the following.

- The hyperbolic case ($a_{11}^2 > 1$),

$$s_n = \frac{s_1 - s_0\lambda_1}{2\sqrt{a_{11}^2 - 1}}\,\lambda_2^n - \frac{s_1 - s_0\lambda_2}{2\sqrt{a_{11}^2 - 1}}\,\lambda_1^n - \frac{a_1}{a_{11}+1}, \quad \lambda_{1,2} = -a_{11} \mp \sqrt{a_{11}^2 - 1}.$$

- The parabolic case ($a_{11}^2 = 1$),

$$\begin{aligned} &\text{if } a_{11} = -1, \text{ then } s_n = s_0 + (s_1 - s_0)n - a_1 n^2,\\ &\text{if } a_{11} = 1, \text{ then } s_n = (-1)^n(s_0 - (s_1 + s_0)n) - \frac{a_1}{2}. \end{aligned}$$

- The elliptic case ($a_{11}^2 < 1$),

$$s_n = s_0\cos(n\,\psi) + \frac{s_1 + s_0a_{11}}{\sqrt{1 - a_{11}^2}}\sin(n\,\psi) - \frac{a_1}{a_{11}+1}, \quad \psi = \arccos(-a_{11}). \tag{3.7}$$

It is easy to prove that a dynamical system of hyperbolic type has no cycles. A dynamical system of parabolic type has cycles if and only if $a_{11} = -1$ and $a_0 = a_1 = 0$ or $a_{11} = 1$ and $a_0 = a_1^2$.

Proposition 6. *The map* (3.7) *has cycles if and only if $\psi = \pi l/k$, where $l \in \mathbb{Z}$, $k \in \mathbb{N}$. If l/k is an irreducible fraction then the length of the cycle is equal to k if l is even and $2k$ if l is odd. If the map* (3.7) *does not have cycles then the set $\{(s_{n+1}, s_n),\, n \in \mathbb{N}_0\}$ is dense in the set $\mathcal{V}$ for any $(s_0, s_1) \in \mathcal{V}$.*

Proof. Find the angle ψ such that there exist $m_1, m_2 \in \mathbb{N}_0$ for which

$$s_0\cos(m_1\,\psi) + \frac{s_1 + s_0a_{11}}{\sqrt{1 - a_{11}^2}}\sin(m_1\,\psi) = s_0\cos(m_2\,\psi) + \frac{s_1 + s_0a_{11}}{\sqrt{1 - a_{11}^2}}\sin(m_2\,\psi). \tag{3.8}$$

There exist such an angle α that the equality (3.8) can be rewritten as follows:

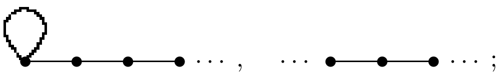
$$\cos(\alpha)\cos(m_1\,\psi) + \sin(\alpha)\sin(m_1\,\psi) = \cos(\alpha)\cos(m_2\,\psi) + \sin(\alpha)\sin(m_2\,\psi),$$

that is, $\psi = \frac{2\pi l}{m_2 - m_1}$ with some $l \in \mathbb{Z}$. Suppose that $m_1 = 0$, $m_2 = 2k$; we obtain that if l/k is an irreducible fraction then the length of the cycle is equal to k if l is even and $2k$ if l is odd. If the map (3.7) does not have cycles then $\psi = p\pi$, $p \in (-1/\pi, 1/\pi)\setminus(\mathbb{Q} \cup \pi\mathbb{Q})$ and the map (3.7) is adjoint to the irrational rotation of the unit circle. Thus we have that the set $\{(s_{n+1}, s_n),\, n \in \mathbb{N}_0\}$ is dense in the set $\mathcal{V}$ for any $(s_0, s_1) \in \mathcal{V}$. □

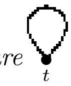
Summing up the above, we obtain a theorem about graphs of polynomials of degree two.

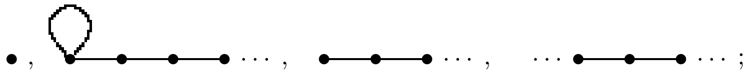
Theorem 3.1. *Any connected component of $P_2(t, s) = a_0 + 2a_1(t+s) + (t^2+s^2) + 2a_{11}ts$, $a_0, a_1, a_{11} \in \mathbb{R}$ is one of the following.*

(1) *The hyperbolic case ($a_{11}^2 > 1$).*
For $a_{11}a_0 > 0$ the connected components are

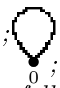


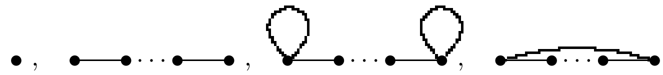
for $a_{11}a_0 < 0$ the connected components are

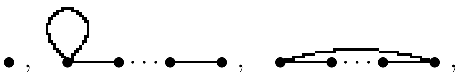
(2) *The parabolic case* ($a_{11}^2 = 1$).
For $a_{11} = -1$ *and* $a_1 = a_0 = 0$ *the connected components are*  , $t \in \mathbb{R}$*;*
for $a_{11} = -1$ *and* $a_1 a_0 \neq 0$ *the connected components are*



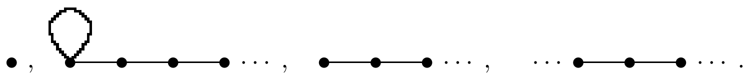
for $a_{11} = 1$ *and* $a_1^2 \neq a_0$ *the connected components are*

(3) *The elliptic case* ($a_{11}^2 < 1$).
For $a_0 > 0$ *the connected components are* $\bullet$*;*
for $a_0 = 0$ *the connected components are* $\bullet$*,*  *;*
for $a_0 < 0$ *the connected components are the following: if* $\arccos(-a_{11}) = \frac{l\pi}{k}$, $\frac{l}{k} \in \mathbb{Q}$ *(rational ellipse),*

 , *l is odd,*

 , *l is even;*

if $\arccos(-a_{11}) = p\pi$*,* $p \in (-\frac{1}{\pi}, \frac{1}{\pi}) \setminus (\mathbb{Q} \cup \pi\mathbb{Q})$ *(irrational ellipse),*



Remark 2. Irreducibility of pairs of the operators (A, B) that satisfy semilinear relation (1.1) implies that the spectrum of the operator A is discrete in any case with an exception of the case of an irrational ellipse. Below we will show that if the characteristic function of a semilinear relation is an irrational ellipse then there exists an irreducible representations with continuous spectrum of the operator A.

Remark 3. Irreducibility of the pair (A, B) implies that in the hyperbolic and the parabolic (except when $a_{11} = -1$ and $a_0 = a_1 = 0$ or $a_{11} = 1$ and $a_0 = a_1^2$) cases the operator A is unbounded, and in the elliptic case the operator A is bounded.

Following [14] we define unbounded representations of the polynomial semilinear relation (1.1). We will say that a pair of symmetric operators A, B satisfies the semilinear relation (1.1) if there exist a dense set $K \subset H$ such that:

- K is invariant with respect to $A, B, E_A(\Delta)$, $\quad \Delta \in \mathfrak{B}(\mathbb{R})$,
- K consists of bounded vectors for A, $\quad K \subset H_B(A) \subset D(A)$,
- relation (1.1) holds on K.

Then the following holds.

Corollary 4. *The problem of describing all (bounded and unbounded) irreducible pairs of self-adjoint operators (A, B) up to a unitary equivalence satisfying the semilinear relation*

$$a_0 B + 2a_1(AB + BA) + A^2 B + BA^2 + 2a_{11} ABA = h(A),$$

for any Borel on $\mathbb{R}$ function $h(t)$ is ∗-wild.

4. Representations of semilinear relations with condition of G-orthoscalarity

It follows from Corollary 4 that it is natural to study pairs of operators (A, B) satisfying semilinear relations with additional conditions. Here we give a condition of G-orthoscalarity such that under this condition the unitary classification of representations of semilinear relations with any quadratic characteristic function can be done.

Let H be a Hilbert space, (A, B) be a pair of linear self-adjoint operators in H. Let $G_0(t), G_1(t), G_2(t)$ be a Borel function on $\mathbb{R}$. We will say that a pair of operators (A, B) satisfy a semilinear relation with the condition of G-orthoscalarity, if they satisfy the relations

$$\begin{cases} \sum\limits_{i=1}^{m} f_i(A)\, B\, g_i(A) = 0, \\ \\ G_1(A)B^2 + 2BG_2(A)B + B^2G_1(A) = G_0(A), \end{cases} \tag{4.1}$$

where $f_i(t)$, $g_i(t)$, $i = \overline{1, m}$.

Following Remark 3 we can consider representations (A, B) of semilinear relations with the condition of G-orthoscalarity (4.1) by unbounded operators A, B, assuming that the relations hold on the domain K described in Remark 3.

Theorem 4.1. *The problem of describing all irreducible pairs of self-adjoint (bounded or unbounded) operators (A, B) in a Hilbert space H up to a unitary equivalence satisfying*

$$\begin{cases} A^2B + BA^2 + 2a_{11}ABA + 2a_1(AB + BA) + a_0B = 0, \\ G_1(A)B^2 + 2BG_2(A)B + B^2G_1(A) = G_0(A) \end{cases} \tag{4.2}$$

is $*$-tame for any Borel functions $G_0(t)$, $G_1(t)$, $G_2(t)$, the numbers a_0, a_1, $a_{11} \in \mathbb{R}$; if $a_{11} = -\cos(p\pi)$, $p \in (-1/\pi, 1/\pi) \setminus (\mathbb{Q} \cup \pi\mathbb{Q})$ then we additionally assume that $\sigma(A)$ is discrete.

Proof. The proof of this theorem is based on following lemma [1] (here we give an equivalent formulation of this lemma).

Lemma 2. *Let (A, B) be a pair of linear self-adjoint operators in a Hilbert space H which satisfies the semilinear relation (1.1) with the graph A_k $(\bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet)$. Let the spectrum $\sigma(A)$ of the operator A be discrete. If, with respect to the decomposition $H = H_{\lambda_1} \oplus H_{\lambda_2} \oplus \ldots \oplus H_{\lambda_k}$, where $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, H_{λ_i} is the eigenspace corresponding to the eigenvalue λ_i, $i = \overline{1, k}$, $k \in \mathbb{N} \cup \{\infty\}$, the matrix blocks of the operator B satisfy the following relations*

$$\begin{gathered} p_2B_{21}B_{12} + q_2B_{23}B_{32} = r_2I_{H_2}, \\ p_3B_{32}B_{23} + q_3B_{34}B_{43} = r_3I_{H_3}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{k-1}B_{k-1\,k-2}B_{k-2\,k-1} + q_{k-1}B_{k-1\,k}B_{k\,k-1} = r_{k-1}I_{H_{k-1}}, \\ \\ p_i, q_i, r_i \in \mathbb{R}, \quad q_i \neq 0, \quad B_{ij} : H_j \to H_i, \quad i, j = 2, \ldots, (k-1), \end{gathered} \tag{4.3}$$

then, in irreducible representation of its semilinear relation, $\dim H_i = 1$, $i = \overline{1, k}$.

It follows from Theorem 3.1 and the first relation above that under the condition of the theorem the spectrum of the operator A in any irreducible representation is discrete. Any connected component of the graph of the semilinear relation contains the graph A_k, thus the operator B can be written as the corresponding block matrix. The second relation (G-orthoscalarity) gives conditions on the blocks of the operator B which contain the conditions 4.3. A routine analysis of the additional relations on the blocks of the operator B gives us the result. □

An example of a semilinear relation with G-orthoscalarity condition is given by the Fairlie algebra [3, 13]. Consider the following $*$-algebra:

$$A_{q,\mu} := \mathbb{C}\langle a = a^*, b = b^* | \quad [a, [a, b]_q]_{q^{-1}} = \operatorname{sign}(\mu) b, \quad [b, [a, b]_q]_{q^{-1}} = \operatorname{sign}(\mu) a\rangle,$$

where $[x, y]_q := xy - qyx$, $q \in \mathbb{R} \cup \mathbb{T}, \mu \in \mathbb{R}$.

Irreducible $*$-representations of $A_{q\,\mu}$ are pairs of self-adjoint operators (A, B) which satisfy the relations

$$\begin{cases} A^2B + BA^2 + 2a_{11}ABA + a_0B = 0, \\ B^2A + BA^2 + 2a_{11}BAB + a_0A = 0, \quad a_0 \in \mathbb{R}, \quad a_{11} = -\dfrac{q + q^{-1}}{2}. \end{cases} \tag{4.4}$$

Note that, if $a_0 = 1$, then this algebra is the Fairlie algebra [3], and if $a_0 = a_{11} = 1$ then it is the universal enveloping algebra of the Lie algebra $so(3)$, if $a_0 = a_{11} = -1$ then it is a graded analogue of the Lie algebra $so(3)$ [4, 10].

Let A and B be operators of a $*$-representation of the $*$-algebra $A_{q,\,\mu}$. Suppose that the spectrum $\sigma(A)$ of the operator A is discrete. A description of all irreducible $*$-representations up to unitary equivalence of the $*$-algebra $A_{q,\,\mu}$ can be obtained as $*$-representations of the first relation of (4.4) with the condition of G-orthoscalarity (for $G = 0$, $G_0(t) = -a_0t$, $G_1(t) = t$, $G_2(t) = a_{11}t$. For a description of $*$-representations of A_q, μ, see [13].

Consider the semilinear relation (4.4) with

$$\arccos(-a_{11}) = p\pi, \qquad p \in \left(-\frac{1}{\pi}, \frac{1}{\pi}\right) \setminus (\mathbb{Q} \cup \pi\mathbb{Q}).$$

From Theorem 3.1 it follows that any orbit of the corresponding dynamical system is dense in the set of zeroes of the characteristic function of the semilinear relation. There is no measurable section which contains one point from each orbit of the dynamical system. We show that there exists an irreducible $*$-representation of (4.4) with continuous spectrum of the operator A, $\sigma(A) = \left[-\dfrac{1}{i\,\mathrm{sh}(-p\pi)}, \dfrac{1}{i\,\mathrm{sh}(-p\pi)}\right]$.

Example 1. *Let* $H = L_2\left(\left[-\dfrac{1}{i\,\mathrm{sh}(-p\pi)}, \dfrac{1}{i\,\mathrm{sh}(-p\pi)}\right], dx\right)$. *Consider operators in* H :

$$Af(x) = xf(x), \quad Bf(x) = b_1(x)f(F_1^{-1}(x)) + b_2(x)f(F_2^{-1}(x)),$$

where $F_{1,2}(x) = -a_{11}x \pm \sqrt{x^2(a_{11}^2 - 1) + 1}, \quad b_1(x), b_2(x) \in H.$

The pair (A,B) satisfies (4.4) and is irreducible.

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