A CHARACTERIZATION OF CLOSURE OF THE SET OF COMPACTLY SUPPORTED FUNCTIONS IN DIRICHLET GENERALIZED INTEGRAL METRIC AND ITS APPLICATIONS

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Dedicated to the blessed memory of A. Ya. Povzner

Abstract. We obtain conditions under which a function $u(x)$ with finite Dirichlet generalized integral over a domain $G$ ($u(x) \in H(G)$) belongs to the closure of the set $C_0^\infty(G)$ in the metrics of this Dirichlet integral (i.e., to the space $H_0(G)$). In the case where $G = \mathbb{R}^n$ ($n \geq 2$) using these conditions we construct examples of Dirichlet integrals such that $H(\mathbb{R}^n) \neq H_0(\mathbb{R}^n)$. For $n = 2$ these examples show that in the known Mazia theorem uniform positivity of the Dirichlet integral matrix cannot be replaced with its pointwise positivity. The characterization of the space $H_0(G)$ is also applied to the problem of relative equivalence of the spaces $H(G)$ and $H_0(G)$ concerning the part of the boundary $\Gamma (\Gamma \subseteq \partial G)$. This problem in fact coincides with the problem of possibility to set boundary conditions of corresponding boundary-value problems.

1. Introduction

One of A. Ya. Povzner’s remarkable achievements is a proof of essential self-adjointness of a Shrödinger semibounded operator with a continuous potential, which acts on the space $L_2(\mathbb{R}^3)$ with the domain of definition $C_0^\infty(\mathbb{R}^3)$ [18]. Later Yu. M. Berezansky [1] has generalized this result to elliptic operators of a general form, subordinated to the condition of global finiteness of propagation velocity (GFVP-condition). This condition is sufficient for coincidence of the closure of $C_0^\infty(\mathbb{R}^n)$ in the metric of the Dirichlet generalized integral with a set of functions for which this integral is finite. The aim of this paper is to study the closure of $C_0^\infty(G)$ in this metric when the domain $G$, generally speaking, does not coincide with $\mathbb{R}^n$. The quadratic functional

$$
D(u, u) = \int_G [(A(x)(\nabla u - i \overline{b} (x)u), (\nabla u - i \overline{b} (x)u) + q(x)|u|^2]dx
$$

is called the Dirichlet generalized integral. Here $G$ is an open set in $\mathbb{R}^n$, $A(x)$ is a positive Hermitian matrix-valued function, $\overline{b} (x)$ is a $n$-component vector-valued function with real components, $q(x)$ is a real function satisfying the condition $q(x) \geq \delta > 0$, $u(x)$ is a complex-valued function, and $(\cdot, \cdot), |\cdot|, \cdot, \cdot$ is the scalar product and the norm in the unitary space $E(\dim E < \infty)$.

Denote the closure in norm $\|u\| = (D(u, u))^{1/2}$ of the set $C_0^\infty(G)$ by $H_0(G)$ and by $H(G)$ denote the set of functions from $W_2^{1, loc}(G)$ for which the integral $D(u, u)$ is finite. It is easy to show that when elements of the matrices $A(x)$ and $A^{-1}(x)$, the components $\overline{b} (x)$, and $q(x)$ are functions measurable and locally bounded in $G$ then definitions of the spaces $H(G)$, $H_0(G)$ are reasonable and the inclusion $H_0(G) \subseteq H(G)$ is true. In
doing so, \( H(G) \) is Hilbert space with the scalar product \( \langle u, v \rangle_H = D(u, v) \) and \( H_0(G) \) is its subspace.

In the present work, the behavior of a function from \( H_0(G) \) at the boundary of a domain \( G \) is studied. There are new necessary, sufficient and in some cases necessary and sufficient conditions to claim that a function from \( H(G) \) belongs to the subset \( H_0(G) \).

These conditions may help to solve some of the problems discussed in publications repeatedly. Thus in some cases any function of \( H(G) \) coincides with a function from \( H_0(G) \) at the boundary or at its fixed part. Hence it is impossible to set boundary conditions repeatedly. Thus in some cases any function of \( H(G) \) may help to solve some of the problems discussed in publications repeatedly.

A similar problem is the problem of the equality \( H_0(G) = H(G) \). For some Dirichlet integrals in the case of \( G = \mathbb{R}^n \), this problem was studied in \([10–12], [19]\). In his work \([11]\) V. G. Mazia proved (for \( f(x) = 0 \), \( q(x) = \text{const} \)) that sufficient smoothness and uniform positivity of the matrix \( A(x) \) (\( A(x) \geq \varepsilon I, \varepsilon = \text{const} \)) provide the equality \( H_0(\mathbb{R}^n) = H(\mathbb{R}^n) \) if \( n = 1 \) or \( n = 2 \). As it is well known (see \([12]\), p. 133; \([10]\)) this statement is wrong if \( n \geq 3 \). It is easy to prove that for \( n = 1 \) in the theorem of V. G. Mazia the uniform positivity condition may be replaced by the ordinary pointwise positivity. Thus a natural question arises: is it true for \( n = 2 \)? Our characterization of the space \( H_0(G) \) gives a negative answer to this question (Example 2).

For a bounded domain \( G \), conditions of the equality \( H_0(G) = H(G) \) were considered in \([17]\). Conditions of essential self-adjointness of the minimal operator \( M \) corresponding to the functional (1) derived in \([4]\) give criteria of the equality \( H_0(G) = H(G) \) for an unbounded domain \( G \) too. The characterization of the space \( H_0(G) \) obtained in the present work is used for stating such conditions for an arbitrary domain \( G \), specifically, in a case when a suitable operator \( M \) may not be essentially self-adjoint (or when the operator \( M \) makes no sense at all ). In addition, this characterization permits to answer the question about coincidence of the space \( H_0(G) \) and \( H(G) \) for Dirichlet integrals for which the reduction to one-dimensional case used in \([17]\) is impossible.

If the minimal operator \( M \) is defined then the characterization of the space \( H_0(G) \) gives in fact a description of the definition domain \( H_0(G) \cap D_M \) of Friedrichs extension of this operator. Friedrichs extension for some types of the semibounded elliptical operators (under a less strict requirement on \( q(x) \) than here) were considered in \([7], [8]\).

The problem of density of the set \( C_0^\infty(G) \) in a Sobolev weight space with the order of the derivative derivative \( \geq 1 \) was considered in some works (e.g., see \([2]\)). Though the results given there don’t include the case with Dirichlet integral of type (1). The results of present work are partially announced in \([3]\).

2. Characterization of the spaces \( H_0(G) \)

Denote by \( D_\Omega(u, u) \) the integral (1) over the open subset \( \Omega \) of the domain \( G \) (\( \overline{\Omega} \subset G \)). Sometimes \( \Omega \) is supposed to be a bounded set and its boundary is supposed to be composed of a finite number of closed piecewise smooth hypersurfaces. Then \( \Omega \) is called a domain with piecewise-smooth boundary. In doing so, the boundary integrals are taken over external surface of the boundary \( \partial \Omega \). By \( \text{Lip}^{r, q}_{\text{loc}}(G) \) denote the set of \( r \)-component vector-valued functions \( \overline{f}(x) \) defined on \( G \) and satisfying the condition

\[
(*) \quad |\overline{f}(x_0 + y) - \overline{f}(x_0)| = O(|y|) \quad \text{as} \quad |y| \to 0
\]
for any point \(x_0 \in G\). In this case the constant in \(O(\cdot)\), generally speaking, depends on \(x_0\).
The set of vector-valued functions, for each of which (*) is satisfied with a constant in \(O(\cdot)\) independent of \(x_0\) is denoted by \(\text{Lip}_n(G)\). The set of functions from \(C^1(G)\) the gradients of which belong to \(\text{Lip}^{(n)}_\text{loc}(G)\) is denoted by \(C^{(1,1)}(G)\). Let us note that for components of the vector-valued function \(\mathbf{f} \in \text{Lip}^{(n)}_\text{loc}(G)\) the first order partial derivatives exist almost everywhere in \(G\) and for \(r = n\) the divergence \(\nabla \mathbf{f}\) also exists almost everywhere in \(G\) (see [21], p. 295). The following statement is a generalization of the theorem obtained in [5] (see also [6]).

**Theorem 1.** Let in Dirichlet integral (1) elements of the matrices \(A(x), A^{-1}(x)\), components \(\mathbf{b}(x)\), and \(q(x)\) be measurable and locally bounded in \(G\). Then the inequalities

\[
\int_\Omega \left( \nabla \mathbf{f} - (A^{-1} \mathbf{f}, \mathbf{f}) \right) |u|^2 \, dx \leq D_\Omega(u, u) + \int_{\partial \Omega} |u|^2 (\mathbf{f}, ds),
\]

\[
2 \left| \text{Re} \int_\Omega \left( \nabla u, u \mathbf{f} \right) \, dx \right| \leq D_\Omega(u, u) + \int_\Omega \left( A^{-1} \mathbf{f}, \mathbf{f} \right) |u|^2 \, dx
\]

hold true. Here, in formula (2), \(\mathbf{f}(x) \in \text{Lip}^{(n)}_\text{loc}(G)\) \((\mathbf{f}(x) : G \to \mathbb{R}^n)\), the function \(u(x) \in \text{Lip}^{(1)}_\text{loc}(G)\), the bounded domain \(\Omega (\Omega \subset G)\) has piecewise smooth boundary \(\partial \Omega\); in formula (3), the vector field \(\mathbf{f}(x)\) is measurable and locally bounded in \(G\), \(u(x)\) is an arbitrary function from \(W^{1,2}_\text{loc}(G)\), \(\Omega\) is an open set such that \(\Omega \subseteq G\).

**Proof.** In the inequality (3) it is possible to consider terms in the right-hand side as finite, otherwise this inequality is formally satisfied. We consider the integral

\[
J = \int_\Omega \left| A^\frac{1}{2} (\nabla u - i \mathbf{b} u) \right|^2 \, dx + \int_\Omega \left| A^{-\frac{1}{2}} u \mathbf{f} \right|^2 \, dx
\]

\[
= \int_\Omega \left( \left| A^\frac{1}{2} (\nabla u - i \mathbf{b} u) \right|^2 + \left| A^{-\frac{1}{2}} u \mathbf{f} \right|^2 \right) \, dx + 2 \text{Re} \left( (\nabla u, u \mathbf{f}) - i(\mathbf{b}, \mathbf{f}) |u|^2 \right) \, dx
\]

\[
= \int_\Omega A \left( \nabla u - i \mathbf{b} u, \nabla u - i \mathbf{b} u \right) \, dx + \int_\Omega \left( A^{-1} \mathbf{f}, \mathbf{f} \right) |u|^2 \, dx
\]

\[
+ \int_\Omega 2 \text{Re} \left( \nabla u, u \mathbf{f} \right) \, dx
\]

for any \(u(x) \in W^{1,2}_\text{loc}(G)\). Taking into account that \(J \geq 0, q(x) \geq 0\) we obtain validity of the inequality (3).

Further for \(u(x) \in \text{Lip}^{(n)}_\text{loc}(G), \mathbf{f}(x) \in \text{Lip}^{(n)}_\text{loc}(G)\) and for a bounded domain \(\Omega\) with piecewise smooth boundary \(\partial \Omega\),

\[
\int_\Omega 2 \text{Re} \left( \nabla u, u \mathbf{f} \right) \, dx = \int_\Omega \left( \nabla (|u|^2 \mathbf{f}) - |u|^2 \nabla \mathbf{f} \right) \, dx.
\]

From this equality and according to the Gauss-Ostrogradsky theorem it follows validity of the inequality (2) if we take into consideration that the integral \(J\) in which the plus sign is chosen is non-negative. Theorem 1 is proved.

The following necessary conditions for a function from \(H(G)\) to belong to the subspace \(H_0(G)\) result from Theorem 1.

**Theorem 2.** Let in Dirichlet integral (1) elements of the matrices \(A(x), A^{-1}(x)\), the components of \(\mathbf{b}(x)\), and \(q(x)\) be measurable and locally bounded in \(G\). Let \(\mathbf{g}(x) \in \text{Lip}^{(n)}_\text{loc}(G)\) \((\mathbf{g}(x) : G \to \mathbb{R}^n)\) be a vector field such that for some \(\varepsilon > 0\) almost everywhere in \(G\) the inequality

\[
\nabla \mathbf{g} \geq \varepsilon (A^{-1} \mathbf{g}, \mathbf{g}) - \text{const}
\]
holds. If \( u(x) \in H_0(G) \), then

\[
\int_G (\nabla \overline{g}) |u|^2 dx, \int_G (A^{-1} \overline{g}, \overline{g}) |u|^2 dx < +\infty. \tag{5}
\]

**Proof.** For \( u(x) \in H_0(G) \) there exists a sequence \( \{\varphi_k\}_{k=1}^\infty \), \( \varphi_k \in C_0^\infty(G) \) such that \( \varphi_k \to u \) in \( L_2(G) \) and \( D(\varphi_k, \varphi_k) \to D(u, u) \). We take \( f(x) = (\varepsilon/2) \overline{g}(x) \) in Theorem 1, where the constant \( \varepsilon > 0 \) is taken from (4). Under the condition (4) almost everywhere in \( G \) the following inequalities hold true:

\[
\nabla \overline{f} - (A^{-1} \overline{f}, \overline{f}) \geq (\varepsilon^2/4)(A^{-1} \overline{g}, \overline{g}) - \text{const},
\]

\[
\nabla \overline{f} - (A^{-1} \overline{f}, \overline{f}) \geq (\varepsilon/4)\nabla(\overline{g}) - \text{const}.
\]

Under the inequality (2) with \( \Omega = \text{supp} \varphi_k \), \( u = \varphi_k \) passing to the limit as \( k \to \infty \) we obtain (5). The theorem is proved. \( \square \)

**Corollary 1.** Let in Dirichlet integral (1) elements of the matrices \( A(x), A^{-1}(x) \), components of \( \overline{b}(x) \), and \( q(x) \) be measurable and locally bounded in \( G \). Let \( \overline{g} \in \operatorname{Lip}^{[n]}(G) (\overline{g}(x) : G \to \mathbb{R}^n) \) be a vector field such that for some \( \varepsilon > 0 \) almost everywhere in \( G \) the inequality (4) is valid. If the \( u(x) \in H_0(G) \cap \operatorname{Lip}^{(1)}(G) \) then the inequality

\[
\int_{\partial \Omega} |u|^2(\overline{g}, ds) \leq C_u
\]

is true. Here \( \Omega \) is an arbitrary bounded domain with piecewise smooth boundary \( \partial \Omega (\Omega \subset G) \), \( C_u \) is a constant independent of \( \Omega \).

**Proof.** Using the Gauss-Ostrogradsky theorem, supposing that \( \overline{f}(x) = \overline{g}(x) \) in the inequality (3) and taking into account Theorem 2 we obtain

\[
\int_{\partial \Omega} |u|^2(\overline{g}, ds) = \int_{\Omega} \nabla(|u|^2 \overline{g}) dx = 2\text{Re} \int_{\Omega} (\nabla u, u \overline{g}) dx + \int_{\Omega} \nabla \overline{g} |u|^2 dx \\
\leq D(\Omega(u, u) + \int_{\Omega} (A^{-1} \overline{g}, \overline{g}) |u|^2 dx + \int_{\Omega} \nabla \overline{g} |u|^2 dx \leq \text{const}.
\]

Corollary 1 is proved. \( \square \)

Now we pass to finding sufficient, and in certain cases necessary and sufficient conditions for a function \( u(x) \in H(G) \) to satisfy \( u(x) \in H_0(G) \). Below for a function \( \eta(x) \) defined in \( G \), we sometimes require that the condition

\[
0 \leq \eta(x) \to \infty, \quad \text{as} \quad x \to \partial G \tag{7}
\]

be satisfied. This means that for any \( N > 0 \) there is a compact set \( \mathcal{R}_N \subset G \) that for \( x \in G \setminus \mathcal{R}_N \) the inequality \( \eta(x) > N \) is true.

**Theorem 3.** Let in Dirichlet integral (1) elements of the matrices \( A(x), A^{-1}(x) \), components of \( \overline{b}(x) \) and \( q(x) \) be measurable and locally bounded functions in \( G \), and the matrix-valued function \( A(x) \) be symmetric (real).

\( 1^0. \) In order that an element \( u \in H(G) \) belongs to \( H_0(G) \), it is sufficient that there exists a function \( \eta(x) \) satisfying (7), and at least one of the conditions:

\( i) \quad \eta(x) \in \operatorname{Lip}^{(1)}(G) \), \( \tau^{-2} \int_{\Omega_\tau} (A \nabla \eta, \nabla \eta) |u|^2 dx \leq C; \)

\( ii) \quad \eta(x) \in C^{(1,1)}(G) \), \( a_{ij}(x) \in \operatorname{Lip}^{(1)}(G) \), \( \tau^{-2} \int_{\Omega_\tau} (\tau - \eta)(\nabla A \nabla \eta) \cdot |u|^2 dx \leq C. \)

Here \( \Omega_\tau = \{ x : x \in G, \eta(x) < \tau \} \) and \( C \) is a constant independent of \( \tau \geq \tau_0 > 0. \)
20. If \( a_{ij}(x) \in \text{Lip}^{(1)}_{\text{loc}}(G) \) and the function \( \eta(x) \in C^{(1,1)}(G) \) satisfies almost everywhere in \( G \) the inequality

\[
\nabla(A\nabla \eta) + K \geq \varepsilon(A\nabla \eta, \nabla \eta)
\]

with constants \( K \geq 0, \varepsilon \geq 0 \) then for each function \( u(x) \in H_0(G) \)

\[
\text{i)} \int_G (A\nabla \eta, \nabla \eta) |u|^2 \, dx < +\infty,
\]

\[
\text{iv)} \int_G (\nabla(A\nabla \eta)) |u|^2 \, dx < +\infty.
\]

If the function \( \eta(x) \) in addition satisfies the condition (7) then convergence of either of the integrals \( \text{i)} \), \( \text{iv)} \) is necessary and sufficient for an element \( u(x) \in H(G) \) to belong to the subspace \( H_0(G) \).

**Proof.** 1°. Consider the Dirichlet integral as a closed symmetric semibounded quadratic form in the space \( L_2(G) \) with the domain of definition \( H_0(G) \). If for some element \( u \in L_2(G) \) there exists a sequence \( \{u_k\} \) of elements from \( H_0(G) \) such that \( u_k \to u \) in \( L_2(G) \) and the number sequence \( D(u_k, u_k) \) is bounded then the element \( u \in L_2(G) \) belongs to the domain of this quadratic form (see [9], p. 395, Theorem 1.16). Let \( \psi \in C_0(G) \cap \text{Lip}^{(1)}_{\text{loc}}(G) \). If \( u \in H(G) \) then \( \psi \cdot u \in H_0(G) \). The latter results from that averaging \( \varphi_t(x) = ((\psi \cdot u + \omega_t)(x) \) with sufficiently small radius \( t \) enters in \( H_0(G) \). This averaging is uniformly bounded for \( t > 0 \) in the metric of the space \( W_2^1(\Omega_1) \) for a fixed bounded domain \( \Omega_1 \supset \text{supp} \psi = \overline{\Omega} \). Since, under our conditions,

\[
D(\varphi_t, \varphi_t) \leq C_{\Omega_1} \|\varphi_t\|^2_{W_2^1(\Omega_1)}
\]

and also \( \varphi_t \to \psi \cdot u \) as \( t \to 0 \) in \( L_2(G) \), we have \( \psi \cdot u \in H_0(G) \). For a real function \( \psi(x) \in \text{Lip}_1(G) \) the equality

\[
D(\psi u, \psi u) = D^\psi(u, u) + \int_G (A\nabla \psi, \nabla \psi) |u|^2 \, dx + 2\text{Re} \int_G (\nabla u, u \psi A\nabla \psi) \, dx
\]

is true. Here

\[
D^\psi(u, u) = \int_G \psi^2((A\nabla \psi - i \overline{b} u), (\nabla \psi - i \overline{b} u)) + q|u|^2 \, dx,
\]

\( u \in H(\Omega) \), and \( \Omega \) is a subdomain of \( G \) such that \( \psi(x) = 0 \) for \( x \in G \setminus \Omega \). Letting, in the inequality (3) of Theorem 1, \( \overline{f} = \psi A\nabla \psi \) we obtain from (9) that for such a subdomain \( \Omega \), the inequality

\[
D(\psi u, \psi u) \leq D^\psi(u, u) + D_\Omega(u, u) + \int_\Omega (1 + \psi^2)(A\nabla \psi, \nabla \psi) |u|^2 \, dx
\]

is true. Let \( \psi = \psi(x, \tau) = (1 - \eta(x)/\tau)_+ \). Here \( \eta(x) \in \text{Lip}^{(1)}_{\text{loc}}(G) \) satisfies (7) and the parameter \( \tau > \tau_0 > 0 \). It is obvious that \( \psi(x) \in C_0(G) \cap \text{Lip}^{(1)}_{\text{loc}}(G) \) and \( \psi \cdot u \in H_0(G) \). As \( (A\nabla \psi, \nabla \psi) = \tau^{-2}(A\nabla \eta, \nabla \eta) \) for \( x \in \Omega_\tau = \{x : x \in G, \eta(x) < \tau\} \), from the last inequality and condition \( i) \) taking into account that \( u \in H(G) \) we obtain validity of the inequality \( D(\psi u, \psi u) \leq \text{const} \) for all \( \tau > \tau_0 > 0 \). Since \( \psi(x, \tau) u(x) \to u(x) \) as \( \tau \to \infty \) in \( L_2(G) \), we have \( u(x) \in H_0(G) \). Sufficiency of condition \( i) \) is proved. For \( \psi \in C_0(G) \cap C^{(1,1)}(G) \) taking into account that

\[
2\text{Re}(\nabla u, u \psi A\nabla \psi) = (\nabla |u|^2, \psi A\nabla \psi)
\]

and integrating by parts the equality (9) we obtain

\[
D(\psi u, \psi u) = D^\psi(u, u) - \int_\Omega \psi(\nabla(A\nabla \psi)) |u|^2 \, dx.
\]

If we select \( \psi \) just as above then \( \text{supp} \psi = \Omega_\tau = \{x : x \in G, \eta(x) < \tau\} \). Therefore,

\[
-\int_{\Omega_\tau} \psi(\nabla(A\nabla \psi)) |u|^2 \, dx = \tau^{-2} \int_{\Omega_\tau} (\tau - \eta)(\nabla(A\nabla \eta)) \cdot |u|^2 \, dx.
\]
Boundedness of $D(\psi u, \psi u)$ results from the equality (11) and condition ii) and so does the inclusion $u(x) \in H_0(G)$. Item i) of the theorem is proved.

2. We use Theorem 2 with $\mathcal{G}(x) = A \nabla \eta$. Condition (8) implies validity of (4). If $u \in H_0(G)$ then validity of conditions iii), iv) results from Theorem 2. If the condition (7) is valid then sufficiency of each of these conditions results from the proved item i). Theorem 3 is proved.

Corollary 2. Let in Dirichlet integral (1) elements of the real matrices $A(x), a_{ij}(x) \in \text{Lip}_{loc}^{(1)}(G)$, all the rest of its coefficients be locally bounded in $G$, $\eta(x) \in C^{(1,1)}(G)$ and satisfies the condition (8). Then for each function $u(x) \in H_0(G) \cap \text{Lip}_{loc}^{(1)}(G)$ the inequality

$$
\int_{\partial \Omega} |u|^2 (A \nabla \eta, \overline{n}) \, ds \leq C_u
$$

is true. Here $\Omega$ is an arbitrary bounded domain with piecewise smooth boundary $\partial \Omega$ ($\overline{\Omega} \subset G$), and $C_u$ is a constant independent of $\Omega$. If the function $\eta(x)$ in addition satisfies the condition (7) then validity of the inequality (12) for a sequence of bounded domains $\{\Omega_n\}_{n=1}^{\infty}$ with piecewise smooth boundaries exhausting the domain $G$ is a necessary and sufficient condition for the function $u(x) \in H_0(G) \cap \text{Lip}_{loc}^{(1)}(G)$ to belong to the subspace $H_0(G)$.

Proof. We use Corollary 1 with $\mathcal{G} = A \nabla \eta$. Condition (8) implies validity of (4) and finiteness of the integral over $\partial \Omega$ under consideration. Let now the condition (7) be also satisfied. We use the inequality (2) of Theorem 1 for the domain $\Omega_1$ with $\mathcal{T} = (\varepsilon/2)A \nabla \eta$ where the constant $\varepsilon > 0$ is taken from (8). From the inequality (2), and also from the condition (8) it follows that both conditions iii), iv) of Theorem 3 are satisfied and, therefore, $u(x) \in H_0(G)$. Sufficiency is established. Necessity, as shown earlier, directly results from Corollary 1.

Remark 1. It is possible to construct a function $\eta(x)$ possessing only one of properties (7) or (8) for any domain $G$ and the matrix-valued function $A(x) > 0$. However, existence of such a function satisfying both these conditions simultaneously is possible only when special restrictions on the matrix-valued function $A(x)$ and the domain $G$ are imposed. While applying Theorem 3 the following obvious remark is often useful.

Remark 2. For two Dirichlet integrals that are different only in the matrices $A_1(x)$ and $A_2(x)$, the statement

$$
C_1 A_2(x) \leq A_1(x) \leq C_2 A_2(x) \Rightarrow H_1(G) = H_2(G), H_01(G) = H_02(G)
$$

is valid. Here the matrix inequalities are assumed to hold true almost everywhere in $G$ with constants $C_1, C_2 > 0$, and the subspaces corresponding to Dirichlet integrals are understood as sets of functions in $L_2(G)$.

Example 1. Let us consider the Dirichlet integral in the case when the domain $G$ is a bounded open set in $R^n$ and the coefficients are locally bounded in $G$. The following statement results from Theorem 3.

Let $\partial G$ be a closed hypersurface of the class $C^2$ and $I_n$ be the identity matrix. If the matrix of the Dirichlet integral satisfies the inequalities $C_1 I_n \leq A(x) \leq C_2 I_n (x \in G)$ for some $C_1, C_2 > 0$, then a function $u(x) \in H(G)$ belongs to the subspace $H_0(G)$ if and only if

$$
N = \int_{G} (d(x))^{-2} |u(x)|^2 \, dx < +\infty.
$$

Here $d(x)$ is the distance from the point $x$ to the set $R^n \setminus G$.

Indeed, consider the Dirichlet integral with the matrix $A(x) = I_n$. If $G$ is an arbitrary bounded open set then it is possible to assume that $\eta(x) = -\ln(\delta(x)/R)$ in
Theorem 3, where \( \delta(x) \) is a regularized distance from the point \( x \in G \) to the set \( R^n \setminus G \) (see [21], p. 203), \( R \) is a sufficiently large constant guaranteeing nonnegativity of the functions \( \eta(x) \), satisfying the condition (7). It is obvious that \( (\nabla \eta, \nabla \eta) = \| \nabla \eta \|^2 = |\nabla \delta|^2/\delta^2 \), \( \nabla (\nabla \eta) = \Delta \eta = (|\nabla \delta|^2 - \Delta \delta)/\delta^2 \). The condition (8) is equivalent to the inequality \( |\nabla \delta|^2 \geq (1 + \varepsilon)\delta \Delta \delta - C\varepsilon \delta^2 \) where \( \varepsilon, C > 0 \). As it is shown in [6, Lemmas 6.1, 6.2] in the case when \( \partial G \) is a closed hypersurface of the class \( C^2 \), the usual distance \( d(x) \) from the point \( x \) to set \( R^n \setminus G \) in some neighborhood \( \partial G \) belongs to \( C^2 \) and \( |\nabla d(x)| = 1 \), \( |\Delta d(x)| \leq \text{const} \). Thus the regularized distance can be chosen to coincide with \( d(x) \) in some neighborhood \( \partial G \). In this case for \( \eta(x) \) the condition (8) is satisfied. Applying item 2 of Theorem 3, we obtain that convergence of the integral \( N \) completely characterizes the subspace \( H_0(G) \) for the case when \( A(x) = I_n \), and \( \partial G \) is a closed hypersurface of the class \( C^2 \). From here and from Remark 2 validity of our statement results.

3. The example of the necessary conditions application

It is convenient to use Corollary 1 for establishing noncoincidence of the spaces \( H_0(R^n) \) and \( H(R^n) \). The following example shows that the uniform positivity condition can not be replaced by pointwise positivity in V. G. Maz’ya result [11].

**Example 2.** Consider the Dirichlet integral for the domain \( G = R^n \) \((n \geq 2)\) with \( A(x) = \text{diag}(a_1(x), a_2(x), \ldots, a_n(x)) \), \( b(x) = \tilde{0} \), \( q(x) = 1 \). Introduce two unbounded domains,

\[
D_{1,l} = \{ x \in R^n : x_1 > 1; r < x_1^{-1} \}, \quad D_{2,l} = \{ x \in R^n : x_1 > 1; x_1^{-l} < r < 2x_1^{-1} \}.
\]

Here \( r = \sqrt{x_2^2 + x_3^2 + \ldots + x_n^2} \), \( l \) is a number such that \( l > 1/(n-1) \). We will show that the following.

**Statement.** If \( a_i(x) \) and \( a_i^{-1}(x) \) are functions positive, locally bounded in \( R^n \) and, for \( x \in D_{1,l} \), the inequalities

\[
(13) \quad a_i(x) \geq \delta x_1^n(1 - r x_1^l); \quad a_i(x) \geq \delta x_1^{-n-2l+2}(1 - r x_1^l), \quad i = \frac{2}{l}, n
\]

are satisfied with constants \( \alpha > (n - 1)l + 1, \delta > 0 \) and for \( x \in D_{2,l} \) the inequalities

\[
(14) \quad a_i(x) \leq C x_1^eta; \quad a_i(x) \leq C x_1^{-\gamma}, \quad i = \frac{2}{l}, n,
\]

hold with constants \( \beta < (n - 1)l + 1, \gamma > (3 - n)l + 1 \) and \( C > 0 \), then \( H_0(R^n) \neq H(R^n) \).

Note that we do not impose additional constraints on the behavior of the functions \( a_i(x) \) in the set \( D_{1,l} \cup D_{2,l} \). As it follows from (14), the condition of uniform positivity can not be satisfied for \( n = 2 \) and \( n = 3 \) for the matrix \( A(x) \).

In order to prove our statement we employ Corollary 1 with a vector field \( \overline{g}(x) \) defined by \( \overline{g}(x) = \theta(x_1) \overline{f}(x) \) on the domain \( G_{1,l} = \{ x \in R^n : x_1 > 1/2; r < x_1^{-1} \} \), where

\[
\overline{f}(x) = \{ x_1^{n-1}(1 - r x_1^l); -lx_1^{n-2}x_2(1 - r x_1^l); \ldots; -lx_1^{n-2}x_{n-1}(1 - r x_1^l) \ldots; -lx_1^{n-2}x_n(1 - r x_1^l) \}
\]

and \( 0 \leq \theta(x_1) \leq 1 \) is a smooth function such that \( \theta(x_1) = 1 \) for \( x_1 > 1, \theta(x_1) = 0 \) for \( x_1 \leq 1/2 \). Suppose that \( \overline{g}(x) = 0 \) for \( x \in R^n \setminus G_{1,l} \). The vector field \( \overline{g}(x) \in \text{Lip}_{\text{loc}}(R^n) \) is defined similarly. We show that for some \( \varepsilon > 0 \), for the matrix-valued function \( A(x) \) described above almost everywhere in \( R^n \) the inequality \( \nabla \overline{g} \geq \varepsilon (A^{-1} \overline{g}, \overline{g}) - \text{const} \) holds true. It is sufficient to prove this inequality for the domain \( D_{1,l} \).

For \( x \in D_{1,l} \), on the one hand,

\[
\nabla \overline{g} = \nabla \overline{f}
\]

\[
= (\alpha - 1)x_1^{n-2}(1 - r x_1^l) - lr x_1^{n-l-2} + \Sigma_{i=2}^{n} \left[ -lx_1^{n-2}(1 - r x_1^l) + lx_1^{n-2}x_i x_i x_1^l \right]
\]

\[
= (\alpha - (n - 1)l)x_1^{n-2}(1 - r x_1^l).
\]
On the other hand owing to conditions (13), for \( x \in D_{1,l} \),
\[
(A^{-1} \vec{g}, \vec{y}) = a_1^{-1}(x)x^{2\alpha - 2}(1 - rx_1^l)^2 + \sum_{i=2}^{n} a_i^{-1}(x)x_i^{2\alpha - 4}x_i^2(1 - rx_1^l)^2 \\
\leq \frac{1 + l^2}{\delta} x_1^{\alpha - 2}(1 - rx_1^l).
\]
Thus, our inequality is valid for \( \varepsilon = \delta(\alpha - (n - 1)l - 1)/(l^2 + 1) \).

We construct a function \( u(x) \in H(R^n) \) that does not belong to \( H_0(R^n) \). Together with the domain \( G_{1,l} \) and the function \( \theta(x_1) \) defined above, we consider the domain
\[
G_{2,l} = \{ x \in R^n : x_1 > 1/2; x_1 < r < 2x_1^l \}
\]
and let
\[
u(x) = \begin{cases} 
\theta(x_1), & x \in \overline{G_{1,l}}, \\
\theta(x_1)(2 - rx_1^l), & x \in G_{2,l}, \\
0, & x \in R^n \setminus (G_{1,l} \cup G_{2,l}).
\end{cases}
\]
It is obvious that \( u(x) \in \text{Lip}_{\text{loc}}^{1}(R^n) \). Let us show that \( u(x) \in H(R^n) \).

\[
D(u, u) \leq \int_{D_{1,l} \cup D_{2,l}} \left( \sum_{i=1}^{n} a_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 + |u|^2 \right) dx + \text{const.}
\]
Recalling that \( \frac{\partial u}{\partial x_i} = 0 \) for \( x \in D_{1,l} \) and \( \frac{\partial u}{\partial x_i} = -lrx_1^{l-1}, \frac{\partial u}{\partial x_1} = -x_1^l/r, \) (\( i > 1 \)) for \( x \in D_{2,l} \) and using conditions (14) we then obtain that
\[
I = \int_{D_{1,l} \cup D_{2,l}} \left( \sum_{i=1}^{n} a_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 + |u|^2 \right) dx
\]
\[
= \int_{D_{2,l}} \left( a_1(x)l^2r^2x_1^{2l-2} + \sum_{i=2}^{n} a_i(x)x_i^{2l-1}r^2 \right) dx + \int_{D_{1,l} \cup D_{2,l}} |u|^2 dx
\]
\[
\leq C \int_{1}^{\infty} dx_1 \int_{x_1^{-l}}^{2x_1^{-l}} \left( l^2r^2x_1^{2l+\beta-2} + x_1^{2l-\gamma} \right) dv_{n-1} + \int_{1}^{\infty} dx_1 \int_{r \leq 2x_1^{-l}} dv_{n-1}.
\]
Here \( dv_{n-1} \) is a volume element in \( R^{n-1} \).

Denote the area of the hypersphere of radius \( r \) in \( R^{n-1} \) by \( \mathcal{S}^{n-2}_r \). \( \mathcal{S}^{n-2}_r = r^{n-2} S^{1}_{n-2} = \frac{r^{n-2} 2\pi^{(n-1)/2}}{\Gamma(n-1/2)} \). We do not eliminate the case \( n = 2 \) taking \( S^{1}_{0} = 2 \) for all \( r \). We obtain the following estimate:
\[
I \leq C S^{1}_{n-2} \int_{1}^{\infty} dx_1 \int_{x_1^{-l}}^{2x_1^{-l}} \left( l^2r^2x_1^{2l+\beta-2} + x_1^{2l-\gamma} \right) r^{n-2} dr
\]
\[
+ S^{1}_{n-2} \int_{1}^{\infty} dx_1 \int_{0}^{2x_1^{-l}} r^{n-2} dr = \frac{C S^{1}_{n-2} l^2(2^{n+1} - 1)}{n + 1} \int_{1}^{\infty} x_1^{2l+\beta-2-l(n+1)} dx_1
\]
\[
+ \frac{C S^{1}_{n-2}(2^{n-1} - 1)}{n - 1} \int_{1}^{\infty} x_1^{2l-\gamma-l(n-1)} dx_1 + \frac{C S^{1}_{n-2} 2^{n-1}}{n - 1} \int_{1}^{\infty} x_1^{-l(n-1)} dx_1.
\]
Three integrals in the right-hand side in the last equality converge, since the conditions imposed on the constants \( l, \beta, \) and \( \gamma \) guarantee that \( 2l + \beta - 2 - l(n + 1), 2l - \gamma - l(n - 1), -l(n - 1) < -1 \). Thus \( u(x) \in H(R^n) \).
Let $\Omega_t = \{x \in \mathbb{R}^n : -t < x_i < t, i = 1, n\}$ be a hypercube with the edge length $2t$.
Consider the integral
\[ J(t) = \int_{\partial \Omega_t} |u|^2 (\nabla, ds) = \int_{r \leq t^{-1}} t^{n-1} (1 - rt^l) dv_{n-1} \]
\[ = \frac{S_{n-2}^l}{n(n-1)} t^{n-(n-1)l-1} \]
for $t > 2^l$. Since $\alpha > (n-1)l + 1$, we have $J(t) \to \infty$ as $t \to \infty$. Owing to Corollary 1, $H_0(R^n) \neq H(R^n)$ and the statement of Example 2 is proved.

**Remark 3.** The matrix $A(x)$ in Example 2 can be constructed as to satisfy the inequality $|A(x)| \leq Cx_1^{2+\varepsilon} + C_1$ with arbitrary constants $C, C_1, \varepsilon > 0$. For $\varepsilon = 0$, this inequality means that the GFVP-condition is satisfied, it guarantees that $H_0(R^n) = H(R^n)$ if the coefficients of the Dirichlet integral are sufficiently smooth.

### 4. Relative equivalence of the spaces $H(G)$ and $H_0(G)$

We call a subdomain $F \subset G$ adjacent to a part of the boundary $\Gamma \subseteq \partial G$ if for each point $x \in \Gamma$ (including the infinite point) there exists a neighborhood $U$ such that $U \cap G \subset F$.

**Definition 1.** We call the space $H(G)$ and the space $H_0(G)$ relatively equivalent with respect to the given part of the boundary $\Gamma \subseteq \partial G$ and we write
\[ H(G) = H_0(G) \mod \Gamma \]
if there exists a subdomain $F$ adjacent to $\Gamma$ such that any function from $H(G)$ coincides almost everywhere in $F$ with some function from $H_0(G)$.

In the case where $\Gamma = \partial G$, it is possible to consider that $F = G$. Therefore, the equality $H(G) = H_0(G) \mod \partial G$ is equivalent to the simple equality $H(G) = H_0(G)$.

Let $F_1, F (F_1 \subseteq F)$ be a subdomain of $G$. Suppose there exists a function $\mu(x)$ such that $\mu(x) \in \text{Lip}_1(G)$, $0 \leq \mu(x) \leq 1$, $\mu(x) \equiv 1$ for $x \in F_1$ and $\mu(x) \equiv 0$ for $x \in G \setminus F$, and for $\mu(x)$-almost everywhere in $G$, 
\[ (A\nabla \mu, \nabla \mu) \leq \text{Const}. \]

**Definition 2.** A subdomain $F$ adjacent to $\Gamma$ is called a subdomain separating $\Gamma$ if there exist one more subdomain $F_1 \subseteq F$ adjacent to $\Gamma$ and the function $\mu(x)$ defined above.

It is obvious that if the subdomain $F$ separates $\Gamma$ it also separates any its part $\Gamma_1 \subset \Gamma$. Note also that if the matrix-valued function $A(x)$ of the Dirichlet integral is locally bounded, then the whole boundary $\partial G$ is automatically separated by any adjacent subdomain, in particular by the whole domain $G$.

**Corollary 3.** Assume that, in the Dirichlet integral (1), elements of the matrices $A(x)$, $A^{-1}(x)$, components of $\overline{b(x)}$, and $q(x)$ are measurable and locally bounded in $G$ and the matrix-valued function $A(x)$ is symmetric (real).

1°. Suppose there exists a function $\eta(x)$ such that $\eta(x)$ satisfies the condition (7) and for $\eta(x)$ at least one of conditions
\[ (\eta(x) \in \text{Lip}_1^{(1)}G, q(x) + K \geq \alpha \tau^{-2} (A\nabla \eta, \nabla \eta); \]
\[ (\eta(x) \in C^{(1,1)}(G), a_{ij}(x) \in \text{Lip}_1^{(1)}G, q(x) + K \geq \alpha \tau^{-2} (\tau - \eta)(\nabla(A\nabla \eta)) \]
is satisfied almost everywhere in $\Omega_t \cap F = \{x : x \in G, \eta(x) < \tau\} \cap F$ with constants $K \geq 0$, $\varepsilon > 0$ independent of $\tau \geq \tau_0 > 0$. Here $F$ is a subdomain of $G$ adjacent to $\Gamma$ and separating $\Gamma$. Then the equality (15) holds.
2). If the equality (15) with the subdomain $F$ separating $\Gamma$ is valid and the condition
\begin{equation}
(18) \quad \int_F [(A(x) \overline{b}(x), \overline{b}(x))] + q(x)]dx < +\infty
\end{equation}
is satisfied then, for any function $\eta(x)$ satisfying the condition (8),
\begin{equation}
(19) \quad \int_{F_1} (\nabla(A\nabla\eta))dx < +\infty.
\end{equation}
Here $F_1$ is the subdomain from Definition 2.

Proof. 1). Let $\mu(x) \in \operatorname{Lip}_1(G)$ be a function from Definition 2 and let $u(x)$ be an arbitrary function from $H(G)$. For $\psi(x) = \mu(x)$, the inequality (10) implies $\mu u \in H(G)$. Conditions (16), (17) ensure that the conditions $i)$ or $ii)$ of Theorem 3 are satisfied for the function $\mu u$, i.e., $\mu u \in H_0(G)$. But $\mu u = u$ for $x \in F_1$ therefore the equality (15) is true.

2). Assume the converse, that is, for some function $\eta(x)$ satisfying the condition (8) the condition (19) is not valid. From the inequality (10) with $u(x) \equiv 1$, $\psi(x) = \mu(x)$, and from the condition (18) it follows that $\mu \in H(G)$. Since the equality (15) holds true, there is a function $\varphi(x) \in H_0(G)$ such that $\varphi(x) = \mu(x)$ for $x \in F_1 \subset F$. Let $\varphi_k(x) \in C_0(G)$ be a sequence of functions such that $\varphi_k(x) \to \varphi(x)$ in the metric of the space $H_0(G)$. Then $\mu \varphi_k \in H_0(G)$, $\mu \varphi_k \to \mu \varphi$ in the metric of $L_2(G)$. Taking into account again the inequality (10) with $u(x) = \varphi(x)$, $\psi(x) = \mu(x)$, and Theorem 1.16 of [9, p. 395], we conclude that $\mu \varphi \in H_0(G)$. According to Theorem 3 (item 2), $\int_G (\nabla(A\nabla\eta))|\mu \varphi|^2dx < +\infty$, which is impossible since
\begin{equation}
\int_G (\nabla(A\nabla\eta))|\mu \varphi|^2dx \geq \int_{F_1} (\nabla(A\nabla\eta))dx - \text{const}.
\end{equation}
The Corollary 3 is proved. □

Example 3. Denote a linear manifold of the dimension $k$ ($0 \leq k < n$) by $L^k$. Consider the case $\partial G = L^k$. Let the coordinate system be such that the its origin and $k$ of the first bases vectors belong to $L^k$. Consider the boundary part, $\Gamma = \{x : |x_i| < r_i, i = 1, 2, \ldots, k\} \cap L^k$, and the adjacent domain $U_\varepsilon = \{x : |x_i| < r_i + \varepsilon, i = 1, 2, \ldots, k; d(x) < \varepsilon\} \cap G$. Here $d(x) = \sqrt{x_{k+1}^2 + x_{k+2}^2 + \ldots + x_n^2}$ is the distance from the point $x$ to the manifold $L^k$. We assume that the matrix $A(x)$ in the Dirichlet integral for $x \in U_\varepsilon$ is a block-diagonal matrix $A(x) = A_1(x) \bigoplus A_2(x)$. Here $A_1(x)$, $A_2(x)$ are positive matrix-valued functions of orders $k$ and $n - k$ accordingly. Note that the case of $k = 0$ is not excluded. In this case, $A(x) = A_2(x)$. Let the matrix $A_1(x)$ be globally bounded and let the matrix $A_1^{-1}(x)$ be locally bounded in $U_\varepsilon$. The rest of the coefficients of the Dirichlet integral are assumed to be locally bounded. For some constants $C_1$, $C_2 > 0$ inequalities
\begin{equation}
C_1 a(d(x))I_{n-k} \leq A_2(x) \leq C_2 a(d(x))I_{n-k}
\end{equation}
are assumed to hold. Here the function $a(t)$ is defined for $t \in (0, \varepsilon]$ as positive and continuous. It is obvious that the subdomain $U_\varepsilon$ separates $\Gamma$ ($F = U_\varepsilon$, $F_1 = U_{\varepsilon_1} = \{x : |x_i| < r_i, i = 1, 2, \ldots, k; d(x) < \varepsilon/2\} \cap G$).

Consider the integral
\begin{equation}
J = \int_0^\varepsilon t^{k+1-n}a^{-1}(t)dt.
\end{equation}
The following statements hold true.

1). If the integral $J$ diverges, then $H(G) = H_0(G) \pmod{\Gamma}$.

2). If the integral $J$ converges and the condition (18) with $F = U_\varepsilon$ is satisfied, then the space $H(G)$ is not equivalent to $H_0(G)$ with respect to $\Gamma$.

We prove statements 1) and 2) for the case $A_2(x) = a(d(x))I_{n-k}$. In the general case, 1), 2) directly result from Remark 2.
Suppose that the integral $J$ diverges. Consider the function
\[ s(t) = \int_t^\infty x^{k+1-n} a^{-1}(\tau) d\tau \]
and also the function $\eta_1(x) = \ln \frac{s(d(x))}{\sigma(d(x))}$ for $d(x) \leq \varepsilon/2$ and $\eta_1(x) = 0$ for $d(x) > \varepsilon/2$.
Assume $\eta(x) = \eta_1(x) + \rho(x)$ in Corollary 3. Here the sufficiently smooth nonnegative function $\rho(x) \to \infty$ as $|x| \to \infty$ and it is equal to 0 in a neighborhood of $U_\varepsilon$. By an immediate calculation subject to $|\nabla d| = 1$, $\Delta d = (n-k-1)/d(x)$, we obtain
\[ \nabla(A \nabla \eta) = -(d(x))^{2(k+1-n)} a^{-1}(d(x)) s^{-2}(d(x)) \text{ for } d(x) \leq \varepsilon/2 \]
and $\nabla(A \nabla \eta) = 0$ in the other part of the set $F = U_\varepsilon$. The condition (17) of Corollary 3 is valid. Whence we conclude that $H(G) = H_0(G)$ (mod $\Gamma$).
Assume that the integral $J$ converges and the numbers $r_i$, $\varepsilon > 0$ are arbitrarily small. Consider the function $\sigma(t) = \int_0^t x^{k+1-n} a^{-1}(\tau) d\tau$ and also the function $\eta(x) = -\ln \frac{\sigma(d(x))}{\sigma(\varepsilon)}$ for $x \in U_\varepsilon$ that is extended to the domain $R^n \setminus U_\varepsilon$ to be sufficiently smooth nonnegative and satisfying the condition (8). For $x \in U_\varepsilon$ the condition (8) is also satisfied, because
\[ \nabla(A \nabla \eta) = (A \nabla \eta, \nabla \eta) = (d(x))^{2(k+1-n)} a^{-1}(d(x)) \sigma^{-2}(d(x)). \]
We show that for the chosen function $\eta(x)$ the condition (19) is not satisfied. We consider the set $\Omega_2(\delta) = U_\varepsilon \cap \{x : d(x) \geq \delta\}$ and calculate the following integral:
\[ I(\delta) = \int_{\Omega_2(\delta)} \nabla(A \nabla \eta) dx \]
\[ = (\Pi_{i=1}^k r_i) \int_\delta^{\varepsilon/2} \frac{d(x)}{x^{k+1-n}} a^{-1}(d(x)) \sigma^{-2}(d(x)) d\sigma_{n-k} \]
\[ = (\Pi_{i=1}^k r_i) \int_\delta^{\varepsilon/2} \frac{x^{k+1-n} a^{-1}(t) \sigma^{-2}(t) S^1_{n-k-1} dt}{\sigma^{-1}(t) - \sigma^{-1}(\varepsilon/2)}. \]
Here $d\sigma_{n-k}$ is a volume element and $S^1_{n-k-1}$ is the area of the hypersphere in the space $R^{n-k}$. Since $I(\delta) \to \infty$ as $\delta \to 0$, the condition (19) is not satisfied and statement $2^0$ is proved.
Note that in the work [14] (see also [16], pp. 157–160) for $k = n-1$ the convergence (divergence) of the integral $J$ is considered as a criterion of possibility (impossibility) to set boundary conditions on a boundary part.

5. USE OF THE OPERATOR $M$ FOR STUDYING CONDITIONS FOR THE EQUALITY $H_0(G) = H(G)$

If the coefficients of the functional (1) are sufficiently smooth then the symmetric differential expression
\[ (\nabla - i \vec{b}(x))^*(A(x)(\nabla - i \vec{b}(x))u) + q(x)u \]
can be constructed for the functional (1). This expression defines an operator $M$ defined on $C_0^\infty(G)$ in the space $L_2(G)$. If such an operator exists then it is possible to weaken the sufficient conditions of the equality $H_0(G) = H(G)$ that are contained in Corollary 3. The following proposition differs from the known ones (see [12]; p. 133) only by the form which is convenient for us.

**Proposition 1.** If, in the Dirichlet integral (1),
\[ a_{ij}(x), b_{ij}(x) \in \text{Lip}_{\text{loc}}(G), \quad q(x) \in L_{\text{loc}}. \]
then the space $H(G)$ can be represented in the form of the orthogonal sum $H(G) = H_0(G) \oplus D$, where $D = \{ u : u \in H(G) \cap D_{M^*} : M^* u = 0 \}$. Therefore the statements
\begin{enumerate}
    \item $D \setminus \{0\} \neq \emptyset \Leftrightarrow H_0(G) \neq H(G)$,
    \item $H(G) \cap D_{M^*} \subseteq H_0(G) \Leftrightarrow H_0(G) = H(G)$
\end{enumerate}
are valid. From essential self-adjointness of the corresponding operator $M$ the equality $H_0(G) = H(G)$ results.

Proof. Assume that $H_0(G) \neq H(G)$. Since $H(G)$ is a Hilbert space with the scalar product $(u, v)_H = D(u, v)$, it follows that $H(G)$ contains a nonzero element $u \in W_{2loc}^1(G)$ such that, for all $\varphi \in C_0^\infty(G)$, the equality $(\varphi, u)_H = 0$ is true. From this it follows that
$$
    \langle M \varphi, u \rangle_{L_2(G)} = \langle \varphi, u \rangle_H = 0 = \langle \varphi, 0 \rangle_{L_2(G)},
$$
i.e., $u \in D_{M^*}$, $M^* u = 0$ and $u \in D$. The converse is obvious, $u \in D \Rightarrow (\varphi, u)_H = \langle M \varphi, u \rangle_{L_2(G)} = \langle \varphi, M^* u \rangle_{L_2(G)} = 0$ ($\varphi \in C_0^\infty(G)$). Therefore the equality $H(G) = H_0(G) \oplus D$ and statements 1), 2) hold. Under the condition $q(x) \geq \delta > 0$ the point $\lambda = 0$ is a point of regular type for the operator $M$. Therefore the equality $D = \{0\}$ follows from self-adjointness of this operator. This means that $H_0(G) = H(G)$. Proposition 1 is proved. \hfill $\Box$

Thus, applying Theorem 3 for proving the equality $H_0(G) = H(G)$ with sufficiently smooth coefficients of Dirichlet integral it is possible to be limited with functions from $H(G) \cap D_{M^*}$. Here it is possible to apply Theorem 3.1 of [6], which gives an a priori estimates for functions from $D_{M^*}$. Let $\rho(x)$ and $\sigma(x)$ be functions from Lip\textsuperscript{(1)}\textsubscript{loc}(G) such that
\begin{equation}
    0 \leq \rho(x) \rightarrow \infty \text{ as } x \rightarrow \partial G, \quad 0 \leq \sigma(x) \leq \text{const}.
\end{equation}
Let the condition
\begin{equation}
    \sigma^2(A \nabla \rho, \nabla \rho) + (A \nabla \sigma, \nabla \sigma) \leq C \rho^m \cdot e^{2\alpha \rho}
\end{equation}
be satisfied. Here the constants $C, m > 0$, $\alpha \geq 0$.

**Theorem 4.** Assume the coefficients of the Dirichlet integral with a real matrix $A(x)$ satisfy conditions (20) and functions $\rho(x)$, $\sigma(x) \in \text{Lip}^{(1)}_{\text{loc}}(G)$ satisfy requirements (21), (22). Let also for $\varphi \in C_0^\infty(G)$ the inequality
\begin{equation}
    D(\sigma \varphi, \sigma \varphi) + C_1 \| \sigma \varphi \|^2_{L_2(G)} + C_2 \| \varphi \|^2_{L_2(G)} \geq \| Q_{\alpha, \varepsilon} \varphi \|^2_{L_2(G)}
\end{equation}
be valid. Here the constants $C_1, C_2 \geq 0$, $\varepsilon > 0$ and
$$
    Q_{\alpha, \varepsilon}(x) = (\alpha + \varepsilon) e \sigma (A \nabla \rho, \nabla \rho)^{1/2} + (A \nabla \sigma, \nabla \sigma)^{1/2},
$$
the constant $\alpha \geq 0$ coincides with the corresponding constant in (22) ($e$ is the base of natural logarithm). If there exists a function $\eta(x) \in C^{(1,1)}(G)$ satisfying the condition (7) such that at least one of the conditions
\begin{equation}
    \sigma^2(A \nabla \rho, \nabla \rho) + q(x) + K \geq \varepsilon \tau^{-2}(A \nabla \eta, \nabla \eta),
\end{equation}
\begin{equation}
    \sigma^2(A \nabla \rho, \nabla \rho) + q(x) + K \geq \varepsilon \tau^{-2}(\tau - \eta) \nabla (A \nabla \eta)
\end{equation}
is satisfied almost everywhere in $\Omega_\tau = \{ x : x \in G, \eta(x) < \tau \}$ with constants $\varepsilon > 0$, $K \geq 0$ independent of $\tau$ for $\tau \geq \tau_0 > 0$ then $H_0(G) = H(G)$.

Proof. According to Proposition 1, it suffices to prove that $H(G) \cap D_{M^*} \subseteq H_0(G)$, where $M$ is the elliptic operator corresponding to the Dirichlet integral. Under Theorem 3.1 from [6],
$$
    \int_G \sigma^2(A \nabla \rho, \nabla \rho) |u|^2 dx < +\infty
$$
for each \( u(x) \in D_{M^*} \). Therefore, owing to one of the conditions (23) or (24) for \( u(x) \in D_{M^*} \), the conditions i) or ii) of Theorem 3 are satisfied. From here it follows that \( H_0(G) = H(G) \). The theorem is proved.

Note that in the case where \( \sigma(x) \equiv 0 \) and \( \Gamma = \partial G \) if the coefficients of the Dirichlet integral are sufficiently smooth, then Theorem 4 becomes a special case of item 1 of Corollary 3.

We give one more criterion of the equality \( H_0(G) = H(G) \) which also results from Theorem 3 with the use of existence of the operator \( M \).

**Lemma.** Let the coefficients of the Dirichlet integral (1) with a real matrix \( A(x) \) satisfy conditions (20). If the function \( u(x) \in H(G) \cap D_{M^*} \) then for any function \( \eta(x) \in C^{(1,1)}(G) \) satisfying condition (7) the inequality

\[
\frac{1}{\tau^2} \left| \int_{\Omega} [(A\nabla\eta, \nabla\eta) - (\tau - \eta) \cdot \nabla (A\nabla\eta)]|u|^2 \, dx \right| \leq C_u
\]

holds true. Here \( \Omega_\tau = \{ x : x \in G, \, \eta(x) < \tau \} \) and \( C_u \) is a constant independent of \( \tau \geq \tau_0 > 0 \).

**Proof.** Using integration by parts it is easy to show validity of the equality

\[
\int_{\Omega} \psi^2 (A(\nabla u - i \, b_u), (\nabla u - i \, b_u)) + q|u|^2 \, dx
\]

\[
= \Re \int_{\Omega} \psi^2 \overline{u} M^* u \, dx + \frac{1}{2} \int_{\Omega} \nabla (A \nabla \psi^2)|u|^2 \, dx,
\]

\[\psi(x) \in C_0(G) \bigcap C^{(1,1)}(\Omega).\]

Here \( \overline{\Omega} = \supp \psi, \, u(x) \in H(G) \cap D_{M^*} \). Assume \( \eta(x) \in C^{(1,1)}(G) \) and satisfies (7). From (26) it follows that, for \( \psi = \psi(x, \tau) = (1-\eta(x)/\tau)_+ \), the inequality \( \left| \int_{\Omega_\tau} \nabla (A \nabla \psi^2)|u|^2 \, dx \right| \leq \mathrm{Const} \) is true. Therefore taking into account that for \( x \in \Omega_\tau \)

\[
\nabla (A \nabla \psi^2) = 2(A \nabla \psi, \nabla \psi) + 2 \psi (\nabla (A \nabla \psi)) = (2/\tau^2)((A \nabla \eta, \nabla \eta) - (\tau - \eta)(\nabla (A \nabla \eta)))
\]

we obtain validity of inequality (25). The lemma is proved.

**Theorem 5.** Let the coefficients of the Dirichlet integral (1) with a real matrix \( A(x) \) satisfy conditions (20). Suppose there exists a function \( \eta(x) \in C^{(1,1)}(G) \) such that \( \eta(x) \) satisfies the condition (7) and the inequality

\[
K \tau^2 + k \tau^2 q(x) + (A \nabla \eta, \nabla \eta) \geq (1 + \varepsilon)(\tau - \eta)(\nabla (A \nabla \eta))
\]

is true for each \( \tau \geq \tau_0 > 0 \) with constants \( K, k \geq 0, \varepsilon > 0 \), which are independent of \( \tau \), almost everywhere in \( \Omega_\tau = \{ x : x \in G, \eta(x) < \tau \} \). Then the equality \( H_0(G) = H(G) \) is true.

**Proof.** Let \( u(x) \) be an arbitrary function from \( H(G) \cap D_{M^*} \). Owing to condition (27) inequality

\[
\frac{1}{\tau^2} \left| \int_{\Omega_\tau} [(A\nabla\eta, \nabla\eta) - (\tau - \eta) \cdot \nabla (A\nabla\eta)]|u|^2 \, dx \right|
\]

\[
\geq \frac{1}{\tau^2} \int_{\Omega_\tau} \varepsilon (\tau - \eta) \cdot \nabla (A\nabla\eta)|u|^2 \, dx - \int_{\Omega_\tau} (K + k q(x))|u|^2 \, dx
\]

is true. According to the lemma, this implies for \( u(x) \) that the condition ii) of Theorem 3 is satisfied and, therefore, \( u(x) \in H_0(G) \), i.e., \( H(G) \cap D_{M^*} \subseteq H_0(G) \). From item 2) of Proposition 1 we obtain validity of the equality \( H_0(G) = H(G) \). The theorem is proved.
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