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A CHARACTERIZATION OF CLOSURE OF THE SET OF COMPACTLY SUPPORTED FUNCTIONS IN DIRICHLET GENERALIZED INTEGRAL METRIC AND ITS APPLICATIONS

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Dedicated to the blessed memory of A. Ya. Povzner

Abstract. We obtain conditions under which a function $u(x)$ with finite Dirichlet generalized integral over a domain G ($u(x) \in H(G)$) belongs to the closure of the set $C_0^\infty(G)$ in the metrics of this Dirichlet integral (i.e., to the space $H_0(G)$). In the case where $G = R^n$ ($n \geq 2$) using these conditions we construct examples of Dirichlet integrals such that $H(R^n) \neq H_0(R^n)$. For $n = 2$ these examples show that in the known Mazia theorem uniform positivity of the Dirichlet integral matrix cannot be replaced with its pointwise positivity. The characterization of the space $H_0(G)$ is also applied to the problem of relative equivalence of the spaces $H(G)$ and $H_0(G)$ concerning the part of the boundary $\Gamma(\Gamma \subseteq \partial G)$. This problem in fact coincides with the problem of possibility to set boundary conditions of corresponding boundary-value problems.

1. Introduction

One of A. Ya. Povzner's remarkable achievements is a proof of essential self-adjointness of a Shrödinger semibounded operator with a continuous potential, which acts on the space $L_2(R^3)$ with the domain of definition $C_0^\infty(R^3)$ [18]. Later Yu. M. Berezansky [1] has generalized this result to elliptic operators of a general form, subordinated to the condition of global finiteness of propagation velocity (GFVP-condition). This condition is sufficient for coincidence of the closure of $C_0^\infty(R^n)$ in the metric of the Dirichlet generalized integral with a set of functions for which this integral is finite. The aim of this paper is to study the closure of $C_0^\infty(G)$ in this metric when the domain G, generally speaking, does not coincide with R^n. The quadratic functional

$$(1) \qquad D(u,u) = \int_G [(A(x)(\nabla u - i\overrightarrow{b}(x)u), (\nabla u - i\overrightarrow{b}(x)u) + q(x)|u|^2]dx$$

is called the Dirichlet generalized integral. Here G is an open set in R^n, $A(x)$ is a positive Hermitian matrix-valued function, $\overrightarrow{b}(x)$ is a n-component vector-valued function with real components, $q(x)$ is a real function satisfying the condition $q(x) \geq \delta > 0$, $u(x)$ is a complex-valued function, and $(\cdot,\cdot), |\cdot,\cdot|$ is the scalar product and the norm in the unitary space $E(\dim E < \infty)$.

Denote the closure in norm $\|u\| = (D(u,u))^{1/2}$ of the set $C_0^\infty(G)$ by $H_0(G)$ and by $H(G)$ denote the set of functions from $W^1_{2\mathrm{loc}}(G)$ for which the integral $D(u,u)$ is finite. It is easy to show that when elements of the matrices $A(x)$ and $A^{-1}(x)$, the components $\overrightarrow{b}(x)$, and $q(x)$ are functions measurable and locally bounded in G then definitions of the spaces $H(G)$, $H_0(G)$ are reasonable and the inclusion $H_0(G) \subseteq H(G)$ is true. In

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doing so, $H(G)$ is Hilbert space with the scalar product $\langle u, v\rangle_H = D(u, v)$ and $H_0(G)$ is its subspace.

In the present work, the behavior of a function from $H_0(G)$ at the boundary of a domain G is studied. There are new necessary, sufficient and in some cases necessary and sufficient conditions to claim that a function from $H(G)$ belongs to the subset $H_0(G)$. These conditions may help to solve some of the problems discussed in publications repeatedly. Thus in some cases any function of $H(G)$ coincides with a function from $H_0(G)$ at the boundary or at its fixed part. Hence it is impossible to set boundary conditions for the corresponding boundary-value problems. Impossibility to set boundary conditions on a part of dimension $\leq n-2$ of the boundary for some elliptic equations was considered in [20]. For some degenerate elliptic equations, criteria of impossibility to set boundary conditions were considered in [22, 23], [13–15] (see also [16], pp. 157–160). A characterization of the spaces $H_0(G)$ derived in the present work is used for a study of an equivalent problem of relative equivalence of the spaces $H(G)$ and $H_0(G)$ for Dirichlet integral of a general form for an arbitrary domain G.

A similar problem is the problem of the equality $H_0(G) = H(G)$. For some Dirichlet integrals in the case of $G = R^n$, this problem was studied in [10–12], [19]. In his work [11] V. G. Mazia proved (for $\overrightarrow{b}(x) = \overrightarrow{0}$, $q(x) = \text{const}$) that sufficient smoothness and uniform positivity of the matrix $A(x)$ ($A(x) \geq \varepsilon I$, $\varepsilon = \text{const}$) provide the equality $H_0(R^n) = H(R^n)$ if $n = 1$ or $n = 2$. As it is well known (see [12], p. 133; [10]) this statement is wrong if $n \geq 3$. It is easy to prove that for $n = 1$ in the theorem of V. G. Mazia the uniform positivity condition may be replaced by the ordinary pointwise positivity. Thus a natural question arises: is it true for $n = 2$? Our characterization of the space $H_0(G)$ gives a negative answer to this question (Example 2).

For a bounded domain G, conditions of the equality $H_0(G) = H(G)$ were considered in [17]. Conditions of essential self-adjointness of the minimal operator M corresponding to the functional (1) derived in [4] give criteria of the equality $H_0(G) = H(G)$ for an unbounded domain G too. The characterization of the space $H_0(G)$) obtained in the present work is used for stating such conditions for an arbitrary domain G, specifically, in a case when a suitable operator M may not be essentially self-adjoint (or when the operator M makes no sense at all). In addition, this characterization permits to answer the question about coincidence of the space $H_0(G)$ and $H(G)$ for Dirichlet integrals for which the reduction to one-dimensional case used in [17] is impossible.

If the minimal operator M is defined then the characterization of the space $H_0(G)$ gives in fact a description of the definition domain $H_0(G) \bigcap D_{M^*}$ of Friedrichs extension of this operator. Friedrichs extension for some types of the semibounded elliptical operators (under a less strict requirement on $q(x)$ than here) were considered in [7], [8].

The problem of density of the set $C_0^\infty(G)$ in a Sobolev weight space with the order of the derivative derivative ≥ 1 was considered in some works (e.g., see [2]). Though the results given there don't include the case with Dirichlet integral of type (1). The results of present work are partially announced in [3].

2. Characterization of the spaces $H_0(G)$

Denote by $D_\Omega(u, u)$ the integral (1) over the open subset Ω of the domain G ($\overline{\Omega} \subset G$). Sometimes Ω is supposed to be a bounded set and its boundary is supposed to be composed of a finite number of closed piecewise smooth hypersurfaces. Then Ω is called a domain with piecewise-smooth boundary. In doing so, the boundary integrals are taken over external surface of the boundary $\partial\Omega$. By $\mathrm{Lip}_{\mathrm{loc}}^{(r)}(G)$ denote the set of r-component vector-valued functions $\overrightarrow{f}(x)$ defined on G and satisfying the condition

$$(*) \qquad |\overrightarrow{f}(x_0 + y) - \overrightarrow{f}(x_0)| = O(|y|) \quad \text{as} \quad |y| \to 0$$

for any point $x_0 \in G$. In this case the constant in $O(\cdot)$, generally speaking, depends on x_0. The set of vector-valued functions, for each of which (*) is satisfied with a constant in $O(\cdot)$ independent of x_0 is denoted by $\mathrm{Lip}_r(G)$. The set of functions from $C^1(G)$ the gradients of which belong to $\mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$ is denoted by $C^{(1,1)}(G)$. Let us note that for components of the vector-valued function $\overrightarrow{f} \in \mathrm{Lip}^{(r)}_{\mathrm{loc}}(G)$ the first order partial derivatives exist almost everywhere in G and for $r = n$ the divergence $\nabla\overrightarrow{f}$ also exists almost everywhere in G (see [21], p. 295). The following statement is a generalization of the theorem obtained in [5] (see also [6]).

Theorem 1. *Let in Dirichlet integral* (1) *elements of the matrices $A(x)$, $A^{-1}(x)$, components $\overrightarrow{b}(x)$, and $q(x)$ be measurable and locally bounded in G. Then the inequalities*

$$\int_\Omega \left(\nabla\overrightarrow{f} - (A^{-1}\overrightarrow{f},\overrightarrow{f})\right)|u|^2dx \le D_\Omega(u,u) + \int_{\partial\Omega}|u|^2(\overrightarrow{f},\overrightarrow{ds}), \tag{2}$$

$$2\left|\mathrm{Re}\int_\Omega\left(\nabla u, u\overrightarrow{f}\right)dx\right| \le D_\Omega(u,u) + \int_\Omega\left(A^{-1}\overrightarrow{f},\overrightarrow{f}\right)|u|^2dx \tag{3}$$

hold true. Here, in formula (2), $\overrightarrow{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$ ($\overrightarrow{f}(x) : G \to R^n$), the function $u(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$, the bounded domain Ω ($\overline{\Omega} \subset G$) has piecewise smooth boundary $\partial\Omega$; in formula (3), the vector field $\overrightarrow{f}(x)$ is measurable and locally bounded in G, $u(x)$ is an arbitrary function from $W^1_{2\mathrm{loc}}(G)$, Ω is an open set such that $\Omega \subseteq G$.

Proof. In the inequality (3) it is possible to consider terms in the right-hand side as finite, otherwise this inequality is formally satisfied. We consider the integral

$$\begin{aligned} J &= \int_\Omega \left|A^{\frac{1}{2}}(\nabla u - i\overrightarrow{b}u) \pm A^{-\frac{1}{2}}(u\overrightarrow{f})\right|^2 dx \\ &= \int_\Omega\left(\left|A^{\frac{1}{2}}(\nabla u - i\overrightarrow{b}u)\right|^2 + \left|A^{-\frac{1}{2}}u\overrightarrow{f}\right|^2 \pm 2\mathrm{Re}\left((\nabla u, u\overrightarrow{f}) - i(\overrightarrow{b},\overrightarrow{f})|u|^2\right)\right)dx \\ &= \int_\Omega\left(A\left(\nabla u - i\overrightarrow{b}u\right),\left(\nabla u - i\overrightarrow{b}u\right)\right)dx + \int_\Omega\left(A^{-1}\overrightarrow{f},\overrightarrow{f}\right)|u|^2dx \\ &\pm \int_\Omega 2\mathrm{Re}\left(\nabla u, u\overrightarrow{f}\right)dx \end{aligned}$$

for any $u(x) \in W^1_{2\mathrm{loc}}(G)$. Taking into account that $J \ge 0$, $q(x) \ge 0$ we obtain validity of the inequality (3).

Further for $u(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$, $\overrightarrow{f}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$ and for a bounded domain Ω with piecewise smooth boundary $\partial\Omega$,

$$\int_\Omega 2\mathrm{Re}\left(\nabla u, u\overrightarrow{f}\right)dx = \int_\Omega\left(\nabla(|u|^2\overrightarrow{f}) - \nabla\overrightarrow{f}|u|^2\right)dx.$$

From this equality and according to the Gauss-Ostrogradsky theorem it follows validity of the inequality (2) if we take into consideration that the integral J in which the plus sign is chosen is non-negative. Theorem 1 is proved. □

The following necessary conditions for a function from $H(G)$ to belong to the subspace $H_0(G)$ result from Theorem 1.

Theorem 2. *Let in Dirichlet integral* (1) *elements of the matrices $A(x)$, $A^{-1}(x)$, the components of $\overrightarrow{b}(x)$, and $q(x)$ be measurable and locally bounded in G. Let $\overrightarrow{g}(x) \in \mathrm{Lip}^{(n)}_{\mathrm{loc}}(G)$ ($\overrightarrow{g}(x) : G \to R^n$) be a vector field such that for some $\varepsilon > 0$ almost everywhere in G the inequality*

$$\nabla\overrightarrow{g} \ge \varepsilon(A^{-1}\overrightarrow{g},\overrightarrow{g}) - \mathrm{const} \tag{4}$$

holds. If $u(x) \in H_0(G)$, then

$$\int_G (\nabla \overrightarrow{g})|u|^2 dx, \ \int_G \left(A^{-1}\overrightarrow{g}, \overrightarrow{g}\right)|u|^2 dx < +\infty. \tag{5}$$

Proof. For $u(x) \in H_0(G)$ there exists a sequence $\{\varphi_k\}_{k=1}^{\infty}$, $\varphi_k \in C_0^{\infty}(G)$ such that $\varphi_k \to u$ in $L_2(G)$ and $D(\varphi_k, \varphi_k) \to D(u,u)$. We take $\overrightarrow{f}(x) = (\varepsilon/2)\overrightarrow{g}(x)$ in Theorem 1, where the constant $\varepsilon > 0$ is taken from (4). Under the condition (4) almost everywhere in G the following inequalities hold true:

$$\nabla \overrightarrow{f} - (A^{-1}\overrightarrow{f}, \overrightarrow{f}) \geq (\varepsilon^2/4)(A^{-1}\overrightarrow{g}, \overrightarrow{g}) - \text{const},$$
$$\nabla \overrightarrow{f} - (A^{-1}\overrightarrow{f}, \overrightarrow{f}) \geq (\varepsilon/4)\nabla(\overrightarrow{g}) - \text{const}.$$

Under the inequality (2) with $\overline{\Omega} = \operatorname{supp}\varphi_k$, $u = \varphi_k$ passing to the limit as $k \to \infty$ we obtain (5). The theorem is proved. □

Corollary 1. *Let in Dirichlet integral* (1) *elements of the matrices $A(x)$, $A^{-1}(x)$, components of $\overrightarrow{b}(x)$, and $q(x)$ be measurable and locally bounded in G. Let $\overrightarrow{g} \in \mathrm{Lip}_{\mathrm{loc}}^{(n)}(G)$ ($\overrightarrow{g}(x) : G \to R^n$) be a vector field such that for some $\varepsilon > 0$ almost everywhere in G the inequality* (4) *is valid. If the $u(x) \in H_0(G) \bigcap \mathrm{Lip}_{\mathrm{loc}}^{(1)}(G)$ then the inequality*

$$\int_{\partial\Omega} |u|^2 (\overrightarrow{g}, \overrightarrow{ds}) \leq C_u \tag{6}$$

is true. Here Ω is an arbitrary bounded domain with piecewise smooth boundary $\partial\Omega$ ($\overline{\Omega} \subset G$), C_u is a constant independent of Ω.

Proof. Using the Gauss-Ostrogradsky theorem, supposing that $\overrightarrow{f}(x) = \overrightarrow{g}(x)$ in the inequality (3) and taking into account Theorem 2 we obtain

$$\int_{\partial\Omega} |u|^2(\overrightarrow{g}, \overrightarrow{ds}) = \int_{\Omega} \nabla(|u|^2\overrightarrow{g})dx = 2\mathrm{Re}\int_{\Omega}(\nabla u, u\overrightarrow{g})dx + \int_{\Omega}\nabla\overrightarrow{g}|u|^2 dx$$
$$\leq D_{\Omega}(u,u) + \int_{\Omega}(A^{-1}\overrightarrow{g}, \overrightarrow{g})|u|^2 dx + \int_{\Omega}\nabla\overrightarrow{g}|u|^2 dx \leq \text{const}.$$

Corollary 1 is proved. □

Now we pass to finding sufficient, and in certain cases necessary and sufficient conditions for a function $u(x) \in H(G)$ to satisfy $u(x) \in H_0(G)$. Below for a function $\eta(x)$ defined in G, we sometimes require that the condition

$$0 \leq \eta(x) \to \infty, \quad \text{as} \quad x \to \partial G \tag{7}$$

be satisfied. This means that for any $N > 0$ there is a compact set $\Re_N \subset G$ that for $x \in G \setminus \Re_N$ the inequality $\eta(x) > N$ is true.

Theorem 3. *Let in Dirichlet integral* (1) *elements of the matrices $A(x)$, $A^{-1}(x)$, components of $\overrightarrow{b}(x)$ and $q(x)$ be measurable and locally bounded functions in G, and the matrix-valued function $A(x)$ be symmetric (real).*

1^0. *In order that an element $u \in H(G)$ belongs to $H_0(G)$, it is sufficient that there exists a function $\eta(x)$ satisfying* (7)*, and at least one of the conditions:*

i) $\eta(x) \in \mathrm{Lip}_{\mathrm{loc}}^{(1)}(G), \quad \tau^{-2}\int_{\Omega_\tau}(A\nabla\eta, \nabla\eta)|u|^2 dx \leq C;$

ii) $\eta(x) \in C^{(1,1)}(G), \quad a_{ij}(x) \in \mathrm{Lip}_{\mathrm{loc}}^{(1)}(G), \quad \tau^{-2}\int_{\Omega_\tau}(\tau - \eta)(\nabla(A\nabla\eta)) \cdot |u|^2 dx \leq C.$

Here $\Omega_\tau = \{x : \ x \in G, \ \eta(x) < \tau\}$ and C is a constant independent of $\tau \geq \tau_0 > 0$.

2^0. *If* $a_{ij}(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ *and the function* $\eta(x) \in C^{(1,1)}(G)$ *satisfies almost everywhere in* G *the inequality*

$$\nabla(A\nabla\eta) + K \geq \varepsilon(A\nabla\eta, \nabla\eta) \tag{8}$$

with constants $K \geq 0$, $\varepsilon \geq 0$ *then for each function* $u(x) \in H_0(G)$

$$iii)\ \int_G (A\nabla\eta, \nabla\eta)|u|^2 dx < +\infty,$$
$$iv)\ \int_G (\nabla(A\nabla\eta))|u|^2 dx < +\infty.$$

If the function $\eta(x)$ *in addition satisfies the condition* (7) *then convergence of either of the integrals iii), iv) is necessary and sufficient for an element* $u(x) \in H(G)$ *to belong to the subspace* $H_0(G)$.

Proof. 1^0. Consider the Dirichlet integral as a closed symmetric semibounded quadratic form in the space $L_2(G)$ with the domain of definition $H_0(G)$. If for some element $u \in L_2(G)$ there exists a sequence $\{u_k\}$ of elements from $H_0(G)$ such that $u_k \to u$ in $L_2(G)$ and the number sequence $D(u_k, u_k)$ is bounded then the element $u \in L_2(G)$ belongs to the domain of this quadratic form (see [9], p. 395, Theorem 1.16). Let $\psi \in C_0(G) \bigcap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$. If $u \in H(G)$ then $\psi \cdot u \in H_0(G)$. The latter results from that averaging $\varphi_t(x) = ((\psi \cdot u) * \omega_t)(x)$ with sufficiently small radius t enters in $H_0(G)$. This averaging is uniformly bounded for $t > 0$ in the metric of the space $W^1_2(\Omega_1)$ for a fixed bounded domain $\Omega_1 \supset \mathrm{supp}\,\psi = \overline{\Omega}$. Since, under our conditions,

$$D(\varphi_t, \varphi_t) \leq C_{\Omega_1} \|\varphi_t\|^2_{W^1_2(\Omega_1)}$$

and also $\varphi_t \to \psi \cdot u$ as $t \to 0$ in $L_2(G)$, we have $\psi \cdot u \in H_0(G)$. For a real function $\psi(x) \in \mathrm{Lip}_1(G)$ the equality

$$D(\psi u, \psi u) = D^\psi(u, u) + \int_G (A\nabla\psi, \nabla\psi)|u|^2 dx + 2\mathrm{Re}\int_G (\nabla u, u\psi A\nabla\psi)dx \tag{9}$$

is true. Here

$$D^\psi(u, u) = \int_g \psi^2[(A(\nabla\psi - i\overrightarrow{b}u), (\nabla\psi - i\overrightarrow{b}u)) + q|u|^2]dx,$$

$u \in H(\Omega)$, and Ω is a subdomain of G such that $\psi(x) = 0$ for $x \in G \setminus \Omega$. Letting, in the inequality (3) of Theorem 1, $\overrightarrow{f} = \psi A\nabla\psi$ we obtain from (9) that for such a subdomain Ω, the inequality

$$D(\psi u, \psi u) \leq D^\psi(u, u) + D_\Omega(u, u) + \int_\Omega (1 + \psi^2)(A\nabla\psi, \nabla\psi)|u|^2 dx \tag{10}$$

is true. Let $\psi = \psi(x, \tau) = (1 - \eta(x)/\tau)_+$. Here $\eta(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ satisfies (7) and the parameter $\tau > \tau_0 > 0$. It is obvious that $\psi(x) \in C_0(G) \bigcap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ and $\psi \cdot u \in H_0(G)$. As $(A\nabla\psi, \nabla\psi) = \tau^{-2}(A\nabla\eta, \nabla\eta)$ for $x \in \Omega_\tau = \{x : x \in G,\ \eta(x) < \tau\}$, from the last inequality and condition $i)$ taking into account that $u \in H(G)$ we obtain validity of the inequality $D(\psi u, \psi u) \leq \mathrm{const}$ for all $\tau > \tau_0 > 0$. Since $\psi(x, \tau)u(x) \to u(x)$ as $\tau \to \infty$ in $L_2(G)$, we have $u(x) \in H_0(G)$. Sufficiency of condition $i)$ is proved. For $\psi \in C_0(G) \bigcap C^{(1,1)}(G)$ taking into account that

$$2\mathrm{Re}(\nabla u, u\psi A\nabla\psi) = (\nabla|u|^2, \psi A\nabla\psi)$$

and integrating by parts the equality (9) we obtain

$$D(\psi u, \psi u) = D^\psi(u, u) - \int_\Omega \psi(\nabla(A\nabla\psi))|u|^2 dx. \tag{11}$$

If we select ψ just as above then $\mathrm{supp}\,\psi = \Omega_\tau = \{x : x \in G,\ \eta(x) < \tau\}$. Therefore,

$$-\int_{\Omega_\tau} \psi(\nabla(A\nabla\psi))|u|^2 dx = \tau^{-2}\int_{\Omega_\tau} (\tau - \eta)(\nabla(A\nabla\eta)) \cdot |u|^2 dx.$$

Boundedness of $D(\psi u, \psi u)$ results from the equality (11) and condition $ii)$ and so does the inclusion $u(x) \in H_0(G)$. Item 1^0 of the theorem is proved.

2^0. We use Theorem 2 with $\overrightarrow{g}(x) = A\nabla\eta$. Condition (8) implies validity of (4). If $u \in H_0(G)$ then validity of conditions $iii), iv)$ results from Theorem 2. If the condition (7) is valid then sufficiency of each of these conditions results from the proved item 1^0. Theorem 3 is proved. □

Corollary 2. *Let in Dirichlet integral* (1) *elements of the real matrices $A(x)$, $a_{ij}(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$, all the rest of its coefficients be locally bounded in G, $\eta(x) \in C^{(1,1)}(G)$ and satisfies the condition* (8). *Then for each function $u(x) \in H_0(G) \bigcap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ the inequality*

$$\int_{\partial\Omega} |u|^2 (A\nabla\eta, \overrightarrow{ds}) \le C_u \tag{12}$$

is true. Here Ω is an arbitrary bounded domain with piecewise smooth boundary $\partial\Omega$ $(\overline{\Omega} \subset G)$, and C_u is a constant independent of Ω. If the function $\eta(x)$ in addition satisfies the condition (7) *then validity of the inequality* (12) *for a sequence of bounded domains $\{\Omega_k\}_{k=1}^{\infty}$ with piecewise smooth boundaries exhausting the domain G is a necessary and sufficient condition for the function $u(x) \in H(G) \bigcap \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ to belong to the subspace $H_0(G)$.*

Proof. We use Corollary 1 with $\overrightarrow{g} = A\nabla\eta$. Condition (8) implies validity of (4) and finiteness of the integral over $\partial\Omega$ under consideration. Let now the condition (7) be also satisfied. We use the inequality (2) of Theorem 1 for the domain Ω_k with $\overrightarrow{f} = (\varepsilon/2)A\nabla\eta$ where the constant $\varepsilon > 0$ is taken from (8). From the inequality (2), and also from the condition (8) it follows that both conditions $iii)$, $iv)$ of Theorem 3 are satisfied and, therefore, $u(x) \in H_0(G)$. Sufficiency is established. Necessity, as shown earlier, directly results from Corollary 1. □

Remark 1. It is possible to construct a function $\eta(x)$ possessing only one of properties (7) or (8) for any domain G and the matrix-valued function $A(x) > 0$. However, existence of such a function satisfying both these conditions simultaneously is possible only when special restrictions on the matrix-valued function $A(x)$ and the domain G are imposed.

While applying Theorem 3 the following obvious remark is often useful.

Remark 2. For two Dirichlet integrals that are different only in the matrices $A_1(x)$ and $A_2(x)$, the statement

$$C_1 A_2(x) \le A_1(x) \le C_2 A_2(x) \Rightarrow H_1(G) = H_2(G), H_{01}(G) = H_{02}(G)$$

is valid. Here the matrix inequalities are assumed to hold true almost everywhere in G with constants $C_1, C_2 > 0$, and the subspaces corresponding to Dirichlet integrals are understood as sets of functions in $L_2(G)$.

Example 1. Let us consider the Dirichlet integral in the case when the domain G is a bounded open set in R^n and the coefficients are locally bounded in G. The following statement results from Theorem 3.

Let ∂G be a closed hypersurface of the class C^2 and I_n be the identity matrix. If the matrix of the Dirichlet integral satisfies the inequalities $C_1 I_n \le A(x) \le C_2 I_n$ $(x \in G)$ for some $C_1, C_2 > 0$, then a function $u(x) \in H(G)$ belongs to the subspace $H_0(G)$ if and only if

$$N = \int_G (d(x))^{-2} |u(x)|^2 dx < +\infty.$$

Here $d(x)$ is the distance from the point x to the set $R^n \setminus G$.

Indeed, consider the Dirichlet integral with the matrix $A(x) = I_n$. If G is an arbitrary bounded open set then it is possible to assume that $\eta(x) = -\ln(\delta(x)/R)$ in

Theorem 3, where $\delta(x)$ is a regularized distance from the point $x \in G$ to the set $R^n \setminus G$ (see [21], p. 203), R is a sufficiently large constant guaranteeing nonnegativity of the functions $\eta(x)$, satisfying the condition (7). It is obvious that $(A\nabla\eta, \nabla\eta) = |\nabla\eta|^2 = |\nabla\delta|^2/\delta^2$, $\nabla(A\nabla\eta) = \Delta\eta = (|\nabla\delta|^2 - \delta\Delta\delta)/\delta^2$. The condition (8) is equivalent to the inequality $|\nabla\delta|^2 \geq (1+\varepsilon)\delta\Delta\delta - C\delta^2$ where ε, $C > 0$. As it is shown in [6, Lemmas 6.1, 6.2] in the case when ∂G is a closed hypersurface of the class C^2, the usual distance $d(x)$ from the point x to set $R^n \setminus G$ in some neighborhood ∂G belongs to C^2 and $|\nabla d(x)| = 1$, $|\Delta d(x)| \leq \text{const}$. Thus the regularized distance can be chosen to coincide with $d(x)$ in some neighborhood ∂G. In this case for $\eta(x)$ the condition (8) is satisfied. Applying item 2^0 of Theorem 3, we obtain that convergence of the integral N completely characterizes the subspace $H_0(G)$ for the case when $A(x) = I_n$, and ∂G is a closed hypersurface of the class C^2. From here and from Remark 2 validity of our statement results.

3. The example of the necessary conditions application

It is convenient to use Corollary 1 for establishing noncoincidence of the spaces $H_0(R^n)$ and $H(R^n)$. The following example shows that the uniform positivity condition can not be replaced by pointwise positivity in V. G. Mazia' result [11].

Example 2. Consider the Dirichlet integral for the domain $G = R^n$ $(n \geq 2)$ with $A(x) = \text{diag}\{a_1(x), a_2(x), \dots, a_n(x)\}$, $\overrightarrow{b}(x) = \overrightarrow{0}$, $q(x) = 1$. Introduce two unbounded domains,

$$D_{1,l} = \{x \in R^n : x_1 > 1; r < x_1^{-l}\}, \quad D_{2,l} = \{x \in R^n : x_1 > 1; x_1^{-l} < r < 2x_1^{-l}\}.$$

Here $r = \sqrt{x_2^2 + x_3^2 + \ldots + x_n^2}$, l is a number such that $l > 1/(n-1)$. We will show that the following.

Statement. *If $a_i(x)$ and $a_i^{-1}(x)$ are functions positive, locally bounded in R^n and, for $x \in D_{1,l}$, the inequalities*

$$a_1(x) \geq \delta x_1^{\alpha}(1 - rx_1^l); \quad a_i(x) \geq \delta x_1^{\alpha-2l-2}(1 - rx_1^l), \quad i = \overline{2,n} \tag{13}$$

are satisfied with constants $\alpha > (n-1)l + 1$, $\delta > 0$ and for $x \in D_{2,l}$ the inequalities

$$a_1(x) \leq Cx_1^{\beta}; \quad a_i(x) \leq Cx_1^{-\gamma}, \quad i = \overline{2,n}, \tag{14}$$

hold with constants $\beta < (n-1)l+1$, $\gamma > (3-n)l+1$ and $C > 0$, then $H_0(R^n) \neq H(R^n)$.

Note that we do not impose additional constraints on the behavior of the functions $a_i(x)$ in the set $D_{1,l} \bigcup D_{2,l}$. As it follows from (14), the condition of uniform positivity can not be satisfied for $n = 2$ and $n = 3$ for the matrix $A(x)$.

In order to prove our statement we employ Corollary 1 with a vector field $\overrightarrow{g}(x)$ defined by $\overrightarrow{g}(x) = \theta(x_1)\overrightarrow{f}(x)$ on the domain $G_{1,l} = \{x \in R^n : x_1 > 1/2;\ r < x_1^{-l}\}$, where

$$\overrightarrow{f}(x) = \{x_1^{\alpha-1}(1 - rx_1^l);\ -lx_1^{\alpha-2}x_2(1 - rx_1^l);\ \dots\ ; -lx_1^{\alpha-2}x_i(1 - rx_1^l)\ \dots; \\ -lx_1^{\alpha-2}x_n(1 - rx_1^l)\}$$

and $0 \leq \theta(x_1) \leq 1$ is a smooth function such that $\theta(x_1) = 1$ for $x_1 > 1$, $\theta(x_1) = 0$ for $x_1 \leq 1/2$. Suppose that $\overrightarrow{g}(x) = 0$ for $x \in R^n \setminus G_{1,l}$. The vector field $\overrightarrow{g}(x) \in \text{Lip}_{\text{loc}}^{(n)}(R^n)$ is defined similarly. We show that for some $\varepsilon > 0$, for the matrix-valued function $A(x)$ described above almost everywhere in R^n the inequality $\nabla\overrightarrow{g} \geq \varepsilon(A^{-1}\overrightarrow{g}, \overrightarrow{g}) - \text{const}$ holds true. It is sufficient to prove this inequality for the domain $D_{1,l}$.

For $x \in D_{1,l}$, on the one hand,

$$\begin{aligned}
\nabla\overrightarrow{g} &= \nabla\overrightarrow{f} \\
&= (\alpha-1)x_1^{\alpha-2}(1 - rx_1^l) - lrx_1^{\alpha+l-2} + \Sigma_{i=2}^n\left[-lx_1^{\alpha-2}(1 - rx_1^l) + lx_1^{\alpha-2}x_i\frac{x_i}{r}x_1^l\right] \\
&= (\alpha - (n-1)l - 1)x_1^{\alpha-2}(1 - rx_1^l).
\end{aligned}$$

On the other hand owing to conditions (13), for $x \in D_{1,l}$,

$$(A^{-1}\overrightarrow{g}, \overrightarrow{g}) = a_1^{-1}(x)x_1^{2\alpha-2}(1-rx_1^l)^2 + \Sigma_{i=2}^n a_i^{-1}(x)l^2x_1^{2\alpha-4}x_i^2(1-rx_1^l)^2$$
$$\le \frac{1+l^2}{\delta}x_1^{\alpha-2}(1-rx_1^l).$$

Thus, our inequality is valid for $\varepsilon = \delta(\alpha-(n-1)l-1)/(l^2+1)$.

We construct a function $u(x) \in H(R^n)$ that does not belong to $H_0(R^n)$. Together with the domain $G_{1,l}$ and the function $\theta(x_1)$ defined above, we consider the domain

$$G_{2,l} = \{x \in R^n : x_1 > 1/2; x_1^l < r < 2x_1^l\}$$

and let

$$u(x) = \begin{cases} \theta(x_1), & x \in \overline{G_{1,l}}, \\ \theta(x_1)(2-rx_1^l), & x \in G_{2,l}, \\ 0, & x \in R^n \setminus (\overline{G_{1,l}} \bigcup G_{2,l}). \end{cases}$$

It is obvious that $u(x) \in \mathrm{Lip}_{\mathrm{loc}}^{(1)}(R^n)$. Let us show that $u(x) \in H(R^n)$.

$$D(u,u) \le \int_{D_{1,l}\bigcup D_{2,l}} \left(\sum_{i=1}^n a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^2 + |u|^2\right)dx + \text{const}.$$

Recalling that $\frac{\partial u}{\partial x_i} = 0$ for $x \in D_{1,l}$ and $\frac{\partial u}{\partial x_1} = -lrx_1^{l-1}$, $\frac{\partial u}{\partial x_i} = -x_1^l\frac{x_l}{r}$, $(i>1)$ for $x \in D_{2,l}$ and using conditions (14) we then obtain that

$$I = \int_{D_{1,l}\bigcup D_{2,l}} \left(\sum_{i=1}^n a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^2 + |u|^2\right)dx$$
$$= \int_{D_{2,l}} \left(a_1(x)l^2r^2x_1^{2l-2} + \sum_{i=2}^n a_i(x)x_1^{2l}\frac{x_i^2}{r^2}\right)dx + \int_{D_{1,l}\bigcup D_{2,l}} |u|^2dx$$
$$\le C\int_1^\infty dx_1 \int_{x_1^{-l}\le r\le 2x_1^{-l}} \left(l^2r^2x_1^{2l+\beta-2} + x_1^{2l-\gamma}\right)dv_{n-1} + \int_1^\infty dx_1 \int_{r\le 2x_1^{-l}} dv_{n-1}.$$

Here dv_{n-1} is a volume element in R^{n-1}.

Denote the area of the hypersphere of radius r in R^{n-1} by S_{n-2}^r. $S_{n-2}^r = r^{n-2}S_{n-2}^1 = r^{n-2}\frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}$. We do not eliminate the case $n=2$ taking $S_0^r = 2$ for all r. We obtain the following estimate:

$$I \le CS_{n-2}^1\int_1^\infty dx_1\int_{x_1^{-l}}^{2x_1^{-l}} (l^2r^2x_1^{2l+\beta-2} + x_1^{2l-\gamma})r^{n-2}dr$$
$$+ S_{n-2}^1\int_1^\infty dx_1\int_0^{2x_1^{-l}} r^{n-2}dr = \frac{CS_{n-2}^1 l^2(2^{n+1}-1)}{n+1}\int_1^\infty x_1^{2l+\beta-2-l(n+1)}dx_1$$
$$+ \frac{CS_{n-2}^1(2^{n-1}-1)}{n-1}\int_1^\infty x_1^{2l-\gamma-l(n-1)}dx_1 + \frac{S_{n-2}^1 2^{n-1}}{n-1}\int_1^\infty x_1^{-l(n-1)}dx_1.$$

Three integrals in the right-hand side in the last equality converge, since the conditions imposed on the constants l, β, and γ guarantee that $2l+\beta-2-l(n+1), 2l-\gamma-l(n-1), -l(n-1) < -1$. Thus $u(x) \in H(R^n)$.

Let $\Omega_t = \{x \in R^n : -t < x_i < t, i = \overline{1,n}\}$ be a hypercube with the edge length $2t$. Consider the integral

$$\begin{aligned} J(t) &= \int_{\partial\Omega_t} |u|^2(\overrightarrow{g}, \overrightarrow{ds}) = \int_{r \le t^{-l}} t^{\alpha-1}(1 - rt^l) dv_{n-1} \\ &= S^1_{n-2}\left(t^{\alpha-1}\int_0^{t^{-l}} r^{n-2}dr - t^{\alpha+l-1}\int_0^{t^{-l}} r^{n-1}dr\right) \\ &= \frac{S^1_{n-2}}{n(n-1)} t^{\alpha-(n-1)l-1} \end{aligned}$$

for $t > 2^l$. Since $\alpha > (n-1)l + 1$, we have $J(t) \to \infty$ as $t \to \infty$. Owing to Corollary 1, $H_0(R^n) \ne H(R^n)$ and the statement of Example 2 is proved.

Remark 3. The matrix $A(x)$ in Example 2 can be constructed as to satisfy the inequality $|A(x)| \le Cx_1^{2+\varepsilon} + C_1$ with arbitrary constants $C, C_1, \varepsilon > 0$. For $\varepsilon = 0$, this inequality means that the GFVP-condition is satisfied, it guarantees that $H_0(R^n) = H(R^n)$ if the coefficients of the Dirichlet integral are sufficiently smooth.

4. Relative equivalence of the spaces $H(G)$ and $H_0(G)$

We call a subdomain $F \subset G$ adjacent to a part of the boundary $\Gamma \subseteq \partial G$ if for each point $x \in \Gamma$ (including the infinite point) there exists a neighborhood U such that $U \bigcap G \subset F$.

Definition 1. We call the space $H(G)$ and the space $H_0(G)$ relatively equivalent with respect to the given part of the boundary $\Gamma \subseteq \partial G$ and we write

$$H(G) = H_0(G) \ (\mathrm{mod}\, \Gamma) \tag{15}$$

if there exists a subdomain F adjacent to Γ such that any function from $H(G)$ coincides almost everywhere in F with some function from $H_0(G)$.

In the case where $\Gamma = \partial G$, it is possible to consider that $F = G$. Therefore, the equality $H(G) = H_0(G)$ (mod ∂G) is equivalent to the simple equality $H(G) = H_0(G)$.

Let $F_1,\ F\ (F_1 \subseteq F)$ be a subdomain of G. Suppose there exists a function $\mu(x)$ such that $\mu(x) \in \mathrm{Lip}_1(G),\ 0 \le \mu(x) \le 1,\ \mu(x) \equiv 1$ for $x \in F_1$ and $\mu(x) \equiv 0$ for $x \in G \backslash F$, and for $\mu(x)$-almost everywhere in G,

$$(A\nabla\mu, \nabla\mu) \le \mathrm{Const.}$$

Definition 2. A subdomain F adjacent to Γ is called a subdomain separating Γ if there exist one more subdomain $F_1 \subseteq F$ adjacent to Γ and the function $\mu(x)$ defined above.

It is obvious that if the subdomain F separates Γ it also separates any its part $\Gamma_1 \subset \Gamma$. Note also that if the matrix-valued function $A(x)$ of the Dirichlet integral is locally bounded, then the whole boundary ∂G is automatically separated by any adjacent subdomain, in particular by the whole domain G.

Corollary 3. *Assume that, in the Dirichlet integral* (1), *elements of the matrices* $A(x)$, $A^{-1}(x)$, *components of* $\overrightarrow{b}(x)$, *and* $q(x)$ *are measurable and locally bounded in* G *and the matrix-valued function* $A(x)$ *is symmetric (real).*

1^0. *Suppose there exists a function* $\eta(x)$ *such that* $\eta(x)$ *satisfies the condition* (7) *and for* $\eta(x)$ *at least one of conditions*

$$\eta(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}, \quad q(x) + K \ge \varepsilon\tau^{-2}(A\nabla\eta, \nabla\eta); \tag{16}$$

$$\eta(x) \in C^{(1,1)}(G), \quad a_{ij}(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}, \quad q(x) + K \ge \varepsilon\tau^{-2}(\tau - \eta)(\nabla(A\nabla\eta)) \tag{17}$$

is satisfied almost everywhere in $\Omega_\tau \bigcap F = \{x : x \in G, \eta(x) < \tau\} \bigcap F$ *with constants* $K \ge 0$, $\varepsilon > 0$ *independent of* $\tau \ge \tau_0 > 0$. *Here* F *is a subdomain of* G *adjacent to* Γ *and separating* Γ. *Then the equality* (15) *holds.*

2^0. *If the equality* (15) *with the subdomain F separating Γ is valid and the condition*

$$\int_F [(A(x)\overrightarrow{b}(x), \overrightarrow{b}(x)) + q(x)]dx < +\infty \tag{18}$$

is satisfied then, for any function $\eta(x)$ satisfying the condition (8),

$$\int_{F_1} (\nabla(A\nabla\eta))dx < +\infty. \tag{19}$$

Here F_1 is the subdomain from Definition 2.

Proof. 1^0. Let $\mu(x) \in \mathrm{Lip}_1(G)$ be a function from Definition 2 and let $u(x)$ be an arbitrary function from $H(G)$. For $\psi(x) = \mu(x)$, the inequality (10) implies $\mu u \in H(G)$. Conditions (16), (17) ensure that the conditions *i*) or *ii*) of Theorem 3 are satisfied for the function μu, i.e., $\mu u \in H_0(G)$. But $\mu u = u$ for $x \in F_1$ therefore the equality (15) is true.

2^0. Assume the converse, that is, for some function $\eta(x)$ satisfying the condition (8) the condition (19) is not valid. From the inequality (10) with $u(x) \equiv 1$, $\psi(x) = \mu(x)$, and from the condition (18) it follows that $\mu \in H(G)$. Since the equality (15) holds true, there is a function $\varphi(x) \in H_0(G)$ such that $\varphi(x) = \mu(x)$ for $x \in F_1 \subset F$. Let $\varphi_k(x) \in C_0^\infty(G)$ be a sequence of functions such that $\varphi_k(x) \to \varphi(x)$ in the metric of the space $H_0(G)$. Then $\mu\varphi_k \in H_0(G)$, $\mu\varphi_k \to \mu\varphi$ in the metric of $L_2(G)$. Taking into account again the inequality (10) with $u(x) = \varphi(x)$, $\psi(x) = \mu(x)$, and Theorem 1.16 of [9, p. 395], we conclude that $\mu\varphi \in H_0(G)$. According to Theorem 3 (item 2^0), $\int_G (\nabla(A\nabla\eta))|\mu\varphi|^2 dx < +\infty$, which is impossible since

$$\int_G (\nabla(A\nabla\eta))|\mu\varphi|^2 dx \geq \int_{F_1} (\nabla(A\nabla\eta))dx - \mathrm{const}.$$

The Corollary 3 is proved. □

Example 3. Denote a linear manifold of the dimension k $(0 \leq k < n)$ by $Ł^k$. Consider the case $\partial G = Ł^k$. Let the coordinate system be such that the its origin and k of the first bases vectors belong to $Ł^k$. Consider the boundary part, $\Gamma = \{x : \ |x_i| < r_i, \ i = 1, 2, \ldots, k\} \bigcap Ł^k$, and the adjacent domain $U_\varepsilon = \{x : \ |x_i| < r_i + \varepsilon, \ i = 1, 2, \ldots, k; d(x) < \varepsilon\} \bigcap G$. Here $d(x) = \sqrt{x_{k+1}^2 + x_{k+2}^2 + \ \ldots \ + x_n^2}$ is the distance from the point x to the manifold $Ł^k$. We assume that the matrix $A(x)$ in the Dirichlet integral for $x \in U_\varepsilon$ is a block-diagonal matrix $A(x) = A_1(x) \bigoplus A_2(x)$. Here $A_1(x)$, $A_2(x)$ are positive matrix-valued functions of orders k and $n - k$ accordingly. Note that the case of $k = 0$ is not excluded. In this case, $A(x) = A_2(x)$. Let the matrix $A_1(x)$ be globally bounded and let the matrix $A_1^{-1}(x)$ be locally bounded in U_ε. The rest of the coefficients of the Dirichlet integral are assumed to be locally bounded. For some constants C_1, $C_2 > 0$ inequalities

$$C_1 a(d(x)) I_{n-k} \leq A_2(x) \leq C_2 a(d(x)) I_{n-k}$$

are assumed to hold. Here the function $a(t)$ is defined for $t \in (0, \varepsilon]$ as positive and continuous. It is obvious that the subdomain U_ε separates Γ ($F = U_\varepsilon$, $F_1 = U_{\varepsilon 1} = \{x : |x_i| < r_i, i = 1, 2, \ \ldots, \ k; \ d(x) < \varepsilon/2\} \bigcap G$).

Consider the integral

$$J = \int_0^\varepsilon t^{k+1-n} a^{-1}(t) dt.$$

The following statements hold true.

1^0. *If the integral J diverges, then $H(G) = H_0(G) \, (\mathrm{mod}\, \Gamma)$.*

2^0. *If the integral J converges and the condition* (18) *with $F = U_\varepsilon$ is satisfied, then the space $H(G)$ is not equivalent to $H_0(G)$ with respect to Γ.*

We prove statements 1^0 and 2^0 for the case $A_2(x) = a(d(x)) I_{n-k}$. In the general case, 1^0, 2^0 directly result from Remark 2.

Suppose that the integral J diverges. Consider the function

$$s(t) = \int_t^{\varepsilon} \tau^{k+1-n} a^{-1}(\tau) d\tau$$

and also the function $\eta_1(x) = \ln \frac{s(d(x))}{s(\varepsilon/2)}$ for $d(x) \le \varepsilon/2$ and $\eta_1(x) = 0$ for $d(x) > \varepsilon/2$. Assume $\eta(x) = \eta_1(x) + \rho(x)$ in Corollary 3. Here the sufficiently smooth nonnegative function $\rho(x) \to \infty$ as $|x| \to \infty$ and it is equal to 0 in a neighborhood of U_ε. By an immediate calculation subject to $|\nabla d| = 1$, $\Delta d = (n-k-1)/d(x)$, we obtain

$$\nabla(A\nabla\eta) = -(d(x))^{2(k+1-n)} a^{-1}(d(x)) s^{-2}(d(x)) \quad \text{for} \quad d(x) \le \varepsilon/2$$

and $\nabla(A\nabla\eta) = 0$ in the other part of the set $F = U_\varepsilon$. The condition (17) of Corollary 3 is valid. Whence we conclude that $H(G) = H_0(G) \pmod{\Gamma}$.

Assume that the integral J converges and the numbers r_i, $\varepsilon > 0$ are arbitrarily small. Consider the function $\sigma(t) = \int_0^t \tau^{k+1-n} a^{-1}(\tau) d\tau$ and also the function $\eta(x) = -\ln \frac{\sigma(d(x))}{\sigma(\varepsilon)}$ for $x \in U_\varepsilon$ that is extended to the domain $R^n \setminus U_\varepsilon$ to be sufficiently smooth nonnegative and satisfying the condition (8). For $x \in U_\varepsilon$ the condition (8) is also satisfied, because

$$\nabla(A\nabla\eta) = (A\nabla\eta, \nabla\eta) = (d(x))^{2(k+1-n)} a^{-1}(d(x)) \sigma^{-2}(d(x)).$$

We show that for the chosen function $\eta(x)$ the condition (19) is not satisfied. We consider the set $\Omega_\varepsilon(\delta) = U_{\varepsilon 1} \bigcap \{x : \ d(x) \ge \delta\}$ and calculate the following integral:

$$\begin{aligned}
I(\delta) &= \int_{\Omega_\varepsilon(\delta)} \nabla(A\nabla)\eta) dx \\
&= \left(\Pi_{i=1}^k r_i\right) \int_{\delta \le d(x) \le \varepsilon/2} (d(x))^{2(k+1-n)} a^{-1}(d(x)) \sigma^{-2}(d(x)) dv_{n-k} \\
&= \left(\Pi_{i=1}^k r_i\right) \int_{\delta}^{\varepsilon/2} t^{k+1-n} a^{-1}(t) \sigma^{-2}(t) S_{n-k-1}^1 dt \\
&= S_{n-k-1}^1 \left(\Pi_{i=1}^k r_i\right) \int_{\delta}^{\varepsilon/2} \sigma^{-2}(t) \sigma'(t) dt = S_{n-k-1}^1 \left(\Pi_{i=1}^k r_i\right) (\sigma^{-1}(\delta) - \sigma^{-1}(\varepsilon/2)).
\end{aligned}$$

Here dv_{n-k} is a volume element and S_{n-k-1}^1 is the area of the hypersphere in the space R^{n-k}. Since $I(\delta) \to \infty$ as $\delta \to 0$, the condition (19) is not satisfied and statement 2^0 is proved.

Note that in the work [14] (see also [16], pp. 157–160) for $k = n-1$ the convergence (divergence) of the integral J is considered as a criterion of possibility (impossibility) to set boundary conditions on a boundary part.

5. Use of the operator M for studying conditions for the equality $H_0(G) = H(G)$

If the coefficients of the functional (1) are sufficiently smooth then the symmetric differential expression

$$(\nabla - i\overrightarrow{b}(x))^*(A(x)(\nabla - i\overrightarrow{b}(x))u) + q(x)u$$

can be constructed for the functional (1). This expression defines an operator M defined on $C_0^\infty(G)$ in the space $L_2(G)$. If such an operator exists then it is possible to weaken the sufficient conditions of the equality $H_0(G) = H(G)$ that are contained in Corollary 3. The following proposition differs from the known ones (see [12]; p. 133) only by the form which is convenient for us.

Proposition 1. *If, in the Dirichlet integral* (1),

$$a_{ij}(x), b_j(x) \in \mathrm{Lip}_{\mathrm{loc}}^{(1)}(G), \quad q(x) \in L_{\infty\,\mathrm{loc}}, \tag{20}$$

then the space $H(G)$ *can be represented in the form of the orthogonal sum* $H(G) = H_0(G) \bigoplus D$, *where* $D = \{ u : u \in H(G) \bigcap D_{M^*};\ M^*u = 0\}$. *Therefore the statements*

$$1)\ D \setminus \{0\} \neq \emptyset \Leftrightarrow H_0(G) \neq H(G),$$

$$2)\ H(G) \bigcap D_{M^*} \subseteq H_0(G) \Leftrightarrow H_0(G) = H(G)$$

are valid. From essential self-adjointness of the corresponding operator M *the equality* $H_0(G) = H(G)$ *results.*

Proof. Assume that $H_0(G) \neq H(G)$. Since $H(G)$ is a Hilbert space with the scalar product $\langle u, v\rangle_H = D(u, v)$, it follows that $H(G)$ contains a nonzero element $u \in W^1_{2\mathrm{loc}}(G)$ such that, for all $\varphi \in C_0^\infty(G)$, the equality $\langle \varphi,\ u\rangle_H = 0$ is true. From this it follows that

$$\langle M\varphi,\ u\rangle_{L_2(G)} = \langle \varphi,\ u\rangle_H = 0 = \langle \varphi,\ 0\rangle_{L_2(G)},$$

i.e., $u \in D_{M^*},\ M^*u = 0$ and $u \in D$. The converse is obvious, $u \in D \Rightarrow \langle \varphi, u\rangle_{H(G)} = \langle M\varphi, u\rangle_{L_2(G)} = \langle \varphi, M^*u\rangle_{L_2(G)} = 0$ ($\varphi \in C_0^\infty(G)$). Therefore the equality $H(G) = H_0(G) \oplus D$ and statements 1), 2) hold. Under the condition $q(x) \geq \delta > 0$ the point $\lambda = 0$ is a point of regular type for the operator $\overline{M}$. Therefore the equality $D = \{0\}$ follows from self-adjointness of this operator. This means that $H_0(G) = H(G)$. Proposition 1 is proved. □

Thus, applying Theorem 3 for proving the equality $H_0(G) = H(G)$ with sufficiently smooth coefficients of Dirichlet integral it is possible to be limited with functions from $H(G) \bigcap D_{M^*}$. Here it is possible to apply Theorem 3.1 of [6], which gives an priori estimates for functions from D_{M^*}. Let $\rho(x)$ and $\sigma(x)$ be functions from $\mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ such that

$$0 \leq \rho(x) \to \infty \quad \text{as} \quad x \to \partial G, \quad 0 \leq \sigma(x) \leq \text{const}. \tag{21}$$

Let the condition

$$\sigma^2(A\nabla\rho, \nabla\rho) + (A\nabla\sigma, \nabla\sigma) \leq C\rho^m \cdot e^{2\alpha\rho} \tag{22}$$

be satisfied. Here the constants $C, m > 0, \alpha \geq 0$.

Theorem 4. *Assume the coefficients of the Dirichlet integral with a real matrix* $A(x)$ *satisfy conditions* (20) *and functions* $\rho(x)$, $\sigma(x) \in \mathrm{Lip}^{(1)}_{\mathrm{loc}}(G)$ *satisfy requirements* (21), (22). *Let also for* $\varphi \in C_0^\infty(G)$ *the inequality*

$$D(\sigma\varphi, \sigma\varphi) + C_1 \parallel \sigma\varphi \parallel^2_{L_2(G)} + C_2 \parallel \varphi \parallel^2_{L_2(G)} \geq \parallel Q_{\alpha,\varepsilon}\varphi \parallel^2_{L_2(G)}$$

be valid. Here the constants $C_1, C_2 \geq 0, \varepsilon > 0$ *and*

$$Q_{\alpha,\varepsilon}(x) = (\alpha + \varepsilon)e\sigma(A\nabla\rho, \nabla\rho)^{1/2} + (A\nabla\sigma, \nabla\sigma)^{1/2},$$

the constant $\alpha \geq 0$ *coincides with the corresponding constant in* (22) *(e is the base of natural logarithm). If there exists a function* $\eta(x) \in C^{(1,1)}(G)$ *satisfying the condition* (7) *such that at least one of the conditions*

$$\sigma^2(A\nabla\rho, \nabla\rho) + q(x) + K \geq \varepsilon\tau^{-2}(A\nabla\eta, \nabla\eta), \tag{23}$$

$$\sigma^2(A\nabla\rho, \nabla\rho) + q(x) + K \geq \varepsilon\tau^{-2}(\tau - \eta)\nabla(A\nabla\eta) \tag{24}$$

is satisfied almost everywhere in $\Omega_\tau = \{x : x \in G,\ \eta(x) < \tau\}$ *with constants* $\varepsilon > 0, K \geq 0$ *independent of* τ *for* $\tau \geq \tau_0 > 0$ *then* $H_0(G) = H(G)$.

Proof. According to Proposition 1, it suffices to prove that $H(G) \bigcap D_{M^*} \subseteq H_0(G)$, where M is the elliptic operator corresponding to the Dirichlet integral. Under Theorem 3.1 from [6],

$$\int_G \sigma^2(A\nabla\rho, \nabla\rho)|u|^2 dx < +\infty$$

for each $u(x) \in D_{M^*}$. Therefore, owing to one of the conditions (23) or (24) for $u(x) \in D_{M^*}$, the conditions $i)$ or $ii)$ of Theorem 3 are satisfied. From here it follows that $H_0(G) = H(G)$. The theorem is proved. □

Note that in the case where $\sigma(x) \equiv 0$ and $\Gamma = \partial G$ if the coefficients of the Dirichlet integral are sufficiently smooth, then Theorem 4 becomes a special case of item 1^0 of Corollary 3.

We give one more criterion of the equality $H_0(G) = H(G)$ which also results from Theorem 3 with the use of existence of the operator M. For a proof of this, the following lemma is needed.

Lemma. *Let the coefficients of the Dirichlet integral* (1) *with a real matrix* $A(x)$ *satisfy conditions* (20). *If the function* $u(x) \in H(G) \bigcap D_{M^*}$ *then for any function* $\eta(x) \in C^{(1,1)}(G)$ *satisfying condition* (7) *the inequality*

$$\frac{1}{\tau^2}\left|\int_{\Omega_\tau} [(A\nabla\eta, \nabla\eta) - (\tau - \eta)\cdot\nabla(A\nabla\eta)]|u|^2 dx\right| \le C_u \tag{25}$$

holds true. Here $\Omega_\tau = \{x : x \in G,\ \eta(x) < \tau\}$ *and* C_u *is a constant independent of* $\tau \ge \tau_0 > 0$.

Proof. Using integration by parts it is easy to show validity of the equality

$$\begin{gathered}\int_\Omega \psi^2[(A(\nabla u - i\overrightarrow{b}u), (\nabla u - i\overrightarrow{b}u)) + q|u|^2]dx \\ = \mathrm{Re}\int_\Omega \psi^2\overline{u}M^*u dx + \frac{1}{2}\int_\Omega \nabla(A\nabla\psi^2)|u|^2dx, \\ \psi(x) \in C_0(G)\bigcap C^{(1,1)}(\Omega).\end{gathered} \tag{26}$$

Here $\overline{\Omega} = \text{supp}\ \psi,\ u(x) \in H(G)\bigcap D_{M^*}$. Assume $\eta(x) \in C^{(1,1)}(G)$ and satisfies (7). From (26) it follows that, for $\psi = \psi(x,\tau) = (1-\eta(x)/\tau)_+$, the inequality $\left|\int_{\Omega_\tau} \nabla(A\nabla\psi^2)|u|^2dx\right| \le$ Const is true. Therefore taking into account that for $x \in \Omega_\tau$

$$\nabla(A\nabla\psi^2) = 2(A\nabla\psi, \nabla\psi) + 2\psi(\nabla(A\nabla\psi)) = (2/\tau^2)((A\nabla\eta, \nabla\eta) - (\tau - \eta)(\nabla(A\nabla\eta)))$$

we obtain validity of inequality (25). The lemma is proved. □

Theorem 5. *Let the coefficients of the Dirichlet integral* (1) *with a real matrix* $A(x)$ *satisfy conditions* (20). *Suppose there exists a function* $\eta(x) \in C^{(1,1)}(G)$ *such that* $\eta(x)$ *satisfies the condition* (7) *and the inequality*

$$K\tau^2 + k\tau^2 q(x) + (A\nabla\eta, \nabla\eta) \ge (1+\varepsilon)(\tau - \eta)(\nabla(A\nabla\eta)) \tag{27}$$

is true for each $\tau \ge \tau_0 > 0$ *with constants* $K, k \ge 0$, $\varepsilon > 0$, *which are independent of* τ, *almost everywhere in* $\Omega_\tau = \{x : x \in G, \eta(x) < \tau\}$. *Then the equality* $H_0(G) = H(G)$ *is true.*

Proof. Let $u(x)$ be an arbitrary function from $H(G)\bigcap D_{M^*}$. Owing to condition (27) inequality

$$\begin{aligned}&\frac{1}{\tau^2}\int_{\Omega_\tau}[(A\nabla\eta, \nabla\eta) - (\tau - \eta)\cdot\nabla(A\nabla\eta)]|u|^2]dx \\ &\quad \ge \frac{1}{\tau^2}\int_{\Omega_\tau} \varepsilon(\tau - \eta)\cdot\nabla(A\nabla\eta)|u|^2dx - \int_{\Omega_\tau}(K + kq(x))|u|^2dx\end{aligned}$$

is true. According to the lemma, this implies for $u(x)$ that the condition $ii)$ of Theorem 3 is satisfied and, therefore, $u(x) \in H_0(G)$, i.e., $H(G)\bigcap D_{M^*} \subseteq H_0(G)$. From item 2) of Proposition 1 we obtain validity of the equality $H_0(G) = H(G)$. The theorem is proved. □

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