

## MORSE FUNCTIONS AND FLOWS ON NONORIENTABLE SURFACES

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**ABSTRACT.** The present paper deals with the correspondence between Morse functions and flows on nonorientable surfaces. It is proved that for every Morse flow with an indexing of saddle points on a nonorientable surface there is a unique Morse function, up to a fiber equivalence, such that its gradient flow is trajectory equivalent to the initial flow, and the values of the function in the saddle points are ordered according to the indexing. The algorithm for constructing the Morse function from a Morse flow with an indexing is given. Reeb graphs and 3-graphs, which assign Morse functions and the corresponding Morse flows with the number of the saddle points less than 3 are presented.

### INTRODUCTION

In the work smooth functions and smooth vector fields on closed 2-manifolds are considered.

Sharko in [1] and [2] and Kulinich in [3] have obtained a topological classification of Morse functions on surfaces. They have used the Reeb graphs. Fomenko has introduced the notions of atom and molecule and used them for a classification of Morse-Smale flows and Morse functions on surfaces (see [4]). Oshemkov in [5] has developed this method and introduced the notion of an  $f$ -graph for assigning the atoms.

Peixoto in [6] has introduced a distinguishing graph which is a complete topological invariant for Morse-Smale flows without closed orbits (Morse flows), and classified them up to trajectory equivalence. In [7], Sharko and Oshemkov introduced a three colored graph, which is an invariant for Morse flows on surfaces. In this work, we use it for assigning the Morse flows.

In [8], Smale proved that Morse flows are gradient flows without separatrices from a saddle to a saddle. Hence, the class of Morse functions corresponds to a class of Morse flows. But it is possible that fiber equivalent functions correspond to trajectory nonequivalent Morse flows, and vice versa. Thus the correspondence between the functions and the flows depends on the metric.

In [9] it was proved that every Morse flow with an indexing of the saddle points on an orientable surface corresponds uniquely, up to fiber equivalence, to a Morse function. An algorithm for constructing a Morse function to the flow with indexing was formulated. The Reeb graph was used for assigning Morse functions. But in the nonorientable case, the Reeb graph does not assign a Morse function, so the following problems arise: what additional information is essential for the assignment of a Morse function and how is it possible to construct a Morse function from a Morse flow with an indexing? In this work we use the Reeb graph with signs in saddle points for an assignment of a Morse function and generalize the results of [9] to the nonorientable case.

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## 1. PRELIMINARIES

Let  $M$  be a smooth closed two-dimensional manifold,  $f : M \rightarrow \mathbf{R}$  a smooth function,  $v : M \rightarrow TM$  a smooth vector field.

**Definition 1.** A point  $x \in M$  is called a *critical point* of the function  $f : M \rightarrow \mathbf{R}$  if the differential of the function  $f$  at this point is equal to 0,  $df(x) = 0$ , that is,  $\frac{\partial f(x)}{\partial x_1} = \frac{\partial f(x)}{\partial x_2} = 0$ . A critical point  $x \in M$  is called *non-degenerate*, if the matrix  $H = \left( \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)_{i,j=1,2}$  in some local coordinates  $x_1, x_2$  is non-degenerate.

On a two-dimensional manifold there are 3 types of non-degenerate critical points, — a minimum (local), a saddle, and a maximum (local).

**Definition 2.** A smooth function  $f : M \rightarrow \mathbf{R}$  is called a *Morse function*, if all its critical points are non-degenerate. A Morse function is called *simple* if all its critical points lie on different levels,  $f(p) \neq f(q)$ , if the critical points are distinct,  $p \neq q$ .

**Definition 3.** A component of the level line  $f^{-1}(y)$  of the Morse function is called a *fiber*. Two Morse functions are called *fiber equivalent* if there is a homeomorphism of the surface onto itself which maps fibers of one function to fibers of another one, and the local minima to the local minima. A neighborhood of the critical fiber which is foliated into level lines of the function and considered to within the fiber equivalence is called an *atom*.

We consider only simple Morse functions.

**Definition 4.** The quotient space  $M/\sim$  with orientation of edges according to the direction of the increase of the function is called a *Reeb graph*, where  $f : M \rightarrow \mathbf{R}$  is a Morse function,  $x_1 \sim x_2$  if  $x_1$  and  $x_2$  belongs to one fiber. Reeb graphs are considered to within the isomorphism of oriented graphs.

The atom can be one of three types, — trousers, inverted trousers, and a nonorientable atom. The first atom corresponds to a vertex of the Reeb graph with two edges one of which is directed toward and the other outwards the vertex. The second atom corresponds to a vertex of the Reeb graph with one edge directed towards and two edges directed outwards the vertex. The nonorientable atom corresponds to the vertex of valency 2.

**Proposition 1.** ([4], Theorem 2.4, p. 71). *Two Morse functions on the orientable surface are fiber equivalent if and only if their Reeb graphs are isomorphic.*

**Definition 5.** A singular point of the vector field  $v = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}$  is called *non-degenerate* if the matrix  $\left( \frac{\partial v_i}{\partial x_j} \right)_{i,j=1,2}$  has no eigenvalues the real part of which equals 0.

A vector field on a surface can have three types of non-degenerate singular points, — a source, a saddle, and a sink.

**Definition 6.** A smooth vector field  $v$  on a two-dimensional manifold  $M$  is called a *Morse vector field*, if

- (1)  $v$  has a finite number of singular points and all of them are non-degenerate;
- (2) each trajectory starts and ends in a singular point;
- (3) there are no trajectories, which connect saddles.

Two Morse fields are called *trajectory equivalent* if there is a homeomorphism of the surface onto itself which maps trajectories of one field into trajectories of another one preserving their orientation.

*Remark 1.* There is exactly one Morse vector field without saddle points on a closed two-dimensional manifold. It is the gradient field of the height function on the sphere  $S^2$  standardly embedded in  $\mathbf{R}^3$  with the metric induced from  $\mathbf{R}^3$ . We consider Morse fields with at least one saddle.

**Definition 7.** A graph is called a *three-colored graph* if the degrees of its vertices are equal to 3, and edges are colored in three colors (s, u, t) in such a way that each vertex is the end of edges of three different colors. Two three-colored graphs are called *isomorphic* if they are isomorphic without coloring, and the isomorphism preserves the coloring.

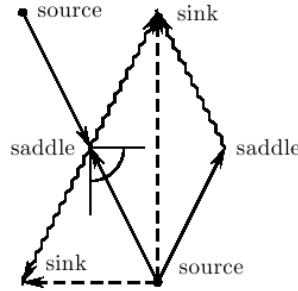


FIGURE 1. Triangulation of a surface

We assign a three-colored graph to the Morse vector field (details see in [7]). For this purpose we construct separatrices. They divide the surface into *canonical quadrilaterals* (see fig. 1). We construct one trajectory from the source to the sink for each canonical quadrilateral. Thus, we obtain a triangulation of the surface (we call these triangles *canonical*). Each triangle corresponds to a vertex of the three-colored graph, two vertices are joined by an *s*-edge (*u*-edge, *t*-edge) if the corresponding triangles have a common side (separatrix) from a source to a saddle (the separatrix from a saddle to a sink, the trajectory from a source to a sink). We call such separatrices *s-separatrices* (*u-separatrices*, *t-trajectories*). A cycle on the graph in which *s*-edges and *u*-edges alternate is called an *su-cycle*.

**Proposition 2.** *Two Morse fields are trajectory equivalent if and only if their three-colored graphs are isomorphic.*

**Proposition 3.** *Three-colored graph corresponds to some Morse field if and only if all its *su*-cycles have length 4.*

For proofs of these statements, see [7].

We call such graphs *3-graphs*. That is a 3-graph is a collection of  $n$  *su*-squares (where  $n$  is the number of saddles) the vertices of which are connected with *t*-edge. By a *3-graph with indexing*, we call a 3-graph with indexing of *su*-squares.

We consider Morse functions to within fiber equivalence, and Morse fields to within trajectory equivalence.

## 2. THE ASSIGNMENT OF MORSE FUNCTIONS AND FLOWS

According to Proposition 1, to a Morse function on an orientable surface there can be assigned a Reeb graph. We shall consider now a nonorientable case. From the Morse lemma it follows that the transformation of fibers in saddle levels lies in attaching a rectangle. In an orientable case, attaching happens so that the surface remains oriented, therefore it is unique. In general, it is possible to attach the usual rectangle and overwound one to one boundary circle, the last leads to occurrence of a nonorientable saddle

atom (the vertex of valency 2 of the Reeb graph). Also, it is possible to paste a rectangle on two boundary circles in two, generally speaking, not homeomorphic ways. The latter leads to that in a nonorientable case, Reeb graphs, generally speaking, does not assign Morse function, an additional information is necessary. For example, Reeb graphs represented in Fig. 2, corresponds to two nonequivalent Morse functions on a torus and on a Klein bottle. For the correspondence to be unique, we equip the Reeb graphs with signs, as shown in Fig. 2. For this purpose, we set an arbitrary orientation of the tubes that connect atoms. Further we set an orientation of saddle atoms that correspond to vertices of valency 3 of the Reeb graph, according to the orientation of the tube which is attached to the upper boundary, if the type of the atom is the trousers, or to the bottom boundary, if the type of the atom is the turned trousers. And near with the ends of the edges we put a plus if the corresponding tube is attached to the rest of the boundaries of the atoms with the coordination of orientations, and a minus otherwise. That is, the sign is set to the pair of incident edges, one of which corresponds to the tube which is attached to the top boundary of the saddle atom, and another corresponds to the tube which is attached to the bottom boundary of the same atom. The arrangement of the signs is ambiguous, that is, Morse functions which correspond to two different arrangements of the signs on Reeb graph can be equivalent. For local extrema and nonorientable saddle atoms the arrangement of signs is not required.

**Definition 8.** We call such graphs with the additional information *equipped Reeb graphs*. Two equipped Reeb graph are called *isomorphic* if they are isomorphic as usual graphs, and it is possible to obtain identical signs at the junctions of the corresponding edges by several realization of the following operation: the replacement of the signs (or a sign, if it is only one) from one end of an edge with a simultaneous replacement of the signs (or a sign, if it is only one) on its other end.

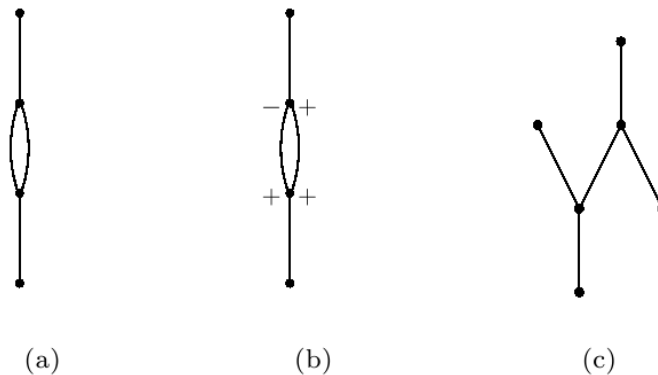


FIGURE 2. Reeb graphs

**Proposition 4.** *Two Morse functions on a surface are fiber equivalent if and only if their equipped Reeb graphs are isomorphic.*

*Proof.* The only reason of the ambiguity of the signs on Reeb graph is the choice of arbitrary orientation of the tubes which correspond to the edges. The operation of the replacement of signs from Definition 8 eliminates this ambiguity. Therefore, necessity follows from Statement 1. Sufficiency is proved similarly to the orientable case, because the equipped Reeb graph assigns from what pieces it is necessary to paste together a surface, what components of their boundaries must be pasted together and how (coordinating orientations or not).  $\square$

We assign Morse functions by the equipped Reeb graphs. According to the statement 2, Morse flows can be assigned by 3-graphs.

### 3. THE CONSTRUCTION OF A MORSE FUNCTION FROM A MORSE FLOW WITH INDEXING

**Theorem 1.** *Let  $M$  be a smooth closed two-dimensional manifold,  $\Phi$  be a Morse flow with an indexing of saddle points on  $M$ . Then there exists a Morse function  $f : M \rightarrow \mathbf{R}$ , unique up to fiber equivalence, such that its gradient flow  $\text{grad } f$  is trajectory equivalent to the flow  $\Phi$  in some Riemannian metric and values of the function  $f$  in saddle points are ordered according to the the indexing of the saddle points of the flow  $\Phi$ .*

*Proof.* The existence of the function in a nonorientable case is proved similarly to orientable case (see [9]).

We prove uniqueness. We show uniqueness of the construction of the equipped Reeb graph from a 3-graph.

Let a Morse function be defined on the manifold. All regular fibers of the Morse function are circles. From the definition of a Reeb graph it follows that two critical points are connected in Reeb graph by an edge if and only if there is a smooth path on the manifold connecting the points along which the function increases and which does not intersect critical fibers, but its ends. Two saddles are joined on Reeb graph with two edges if and only if there are two increasing smooth paths on the manifold which connect these saddles and do not intersect critical fibers, but its ends, and these paths cannot be connected with a constant path on manifold. That is the interior points of the paths belong to different fibers.

Since the function increases along the trajectories of the gradient flow, the increasing paths for local extrema are separatrices. Hence, on the Reeb graph the minimum is joined with a saddle which has the least index among those saddles which are connected by separatrices to a source which corresponds to the minimum. The similar construction is for maxima. The increasing path (or paths) for saddle points passes on canonical quadrilaterals, intersecting  $s$ - and  $u$ -separatrices. It is an increasing path if and only if the function increases along it on each passed quadrilateral. And it fulfills if the value of the function in the entry point in a quadrilateral is less than the value in the exit point. The latter is determined only by the numbers of saddles (vertex) of the passed quadrilaterals, because there are two choices for the entry and the exit (through an  $s$ - or a  $u$ -separatrix), and the number of the passed quadrilaterals is finite. To check the path for intersection of critical fibers, it is enough to examine the existence of a constant path (along which the function is constant) from each saddle with the number between the numbers of the ends of the increasing path. It can be checked similarly to the examination of the paths for increasing. The examination for the multiplicity of the edges on the Reeb graph is also reduced to the searching a constant path.

For an arrangement of the signs it is enough to set the orientation on the canonical triangles, coordinated in the neighborhoods of the saddle points, and to compare the orientations of the first and the last triangles for the increasing path, corresponding to the edge on the Reeb graph. If the orientations can be coordinated (that is, they are opposite), we put a plus, otherwise we put a minus.  $\square$

*Remark 2.* Though only one function corresponds to a flow with an indexing, but functions which corresponds to different indexing of saddle points of the flow can be fiber equivalent. Moreover, one function can correspond to different (trajectory nonequivalent) flows.

### 4. AN ALGORITHM FOR FINDING MORSE FUNCTIONS

In this section we use 3-graphs and equipped Reeb graphs for the assignment of Morse flows and functions.

We describe an algorithm for constructing a Morse function to within trajectory equivalence by Morse flow with the indexing of the saddle points. For the construction of the Reeb graph we find the paths from the proof of Theorem 1. For a local extremum, this path is the separatrix to the saddle with the nearest index. For a minimum it is the least, for a maximum it is the greatest index. For finding the paths between saddles we start from saddles with smaller indices and construct increasing paths. The algorithm of the construction of all admissible increasing paths from a saddle  $a$  is the following. We have the interval  $(a, b)$ , in the beginning  $b = n + 1$ , where  $n$  is the number of saddles. We start passing from the saddle  $a$  to one of the four canonical quadrilaterals (it is necessary to examine all). We check the opposite saddle with the index  $i$ . If  $i < a$  then we can exit from the quadrilateral through a  $u$ -separatrix. If  $i > b$  then we can exit through an  $s$ -separatrix. If  $i \in (a, b)$  we assign  $b := i$ , remember  $i$  as a candidate for the end of the path (we throw the previous candidate), and we exit through an  $s$ -separatrix. This procedure stops when the index of the current saddle is  $i = a$  or  $i = b$ . If we have not found a candidate at the end of the path, then it is impossible to construct an admissible increasing path to a saddle through this quadrilateral. If the candidate  $c$  has been found, the path from  $a$  to  $c$  may be admissible. It remains to check whether the found paths do not intersect the critical fibers and correspond to different edges on the Reeb graph.

To discover the type of the saddle atom, we start a constant path from the saddle  $a$  to one of the canonical quadrilaterals. We check the opposite saddle with the index  $i$ . If  $i < a$  then it is possible to exit through a  $u$ -separatrix. If  $i > a$  then it is possible to exit through an  $s$ -separatrix. The procedure stops, when we return to the saddle  $a$ . We check through what quadrilateral we return to  $a$ . If it has a common  $s$ -separatrix of the saddle  $a$  with the initial quadrilateral, then the type of the atom is the trousers, if it has a common  $u$ -separatrix of the saddle  $a$ , then the type of the atom is the turned trousers, if there are no common separatrices, then it is a nonorientable saddle atom. This statement follows from that it is possible to connect the increasing paths which pass on the quadrilaterals with a common  $s$ -separatrix with a constant path (it intersects this separatrix) (see Fig. 1). Similarly, it is possible to connect the increasing ways which pass on the quadrilaterals with common a  $u$ -separatrix with a constant path, slightly raising the initial constant path (as we consider only simple Morse functions, no saddle can prevent).

Since two (or four) paths correspond to one edge of the Reeb graph, the number of paths is even. If there is no increasing path, there is no increasing edge from this saddle to another saddle. If there are two increasing paths, then there is one edge to their end. Indeed, if there is the edge to the maximum and there is a path on the quadrilateral incident to the corresponding sink, then there is a saddle which is connected with a separatrix to this sink and the index of which is greater than  $a$ . And this contradicts to the rule of the junction of maxima. If there are 4 paths and the type of the atom is equal 1 then there are two edges (one to minimal and one to maximal end of the paths). Otherwise there is one edge to the minimal end of the paths.

It is possible to consider all aforesaid paths as paths on a 3-graph. Intersecting an  $s$ -separatrix, we pass to the corresponding  $s$ -edge, passing on a canonical quadrilateral from a saddle to a saddle, we pass on the  $t$ -edge. The canonical quadrilaterals (and triangles) which are incident to a saddle correspond to the vertices of the  $su$ -square. Two increasing paths which correspond to one edge of the Reeb graph start in the vertices of the  $su$ -square, which are connected with an  $s$ -edge, a  $u$ -edge or are opposite, if the type of the saddle atom is the trousers, the turned trousers or the nonorientable saddle atom, respectively.

For an arrangement of the signs, it is necessary to set a sign for each vertex of the 3-graph in such a way that the signs alternate at the bypass of everyone  $s - u$  cycle.

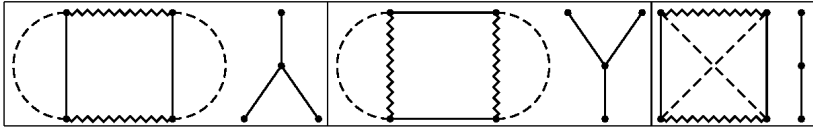


TABLE 1. 3-graphs and Reeb graphs for Morse functions and flows with one saddle

Saddles with nonorientable atom of the equipped Reeb graph is not equipped with signs. For a saddle with the index 1 of the equipped Reeb graph we assign pluses. Further we look over the edges which connect the saddles of Reeb graph, in ascending order of the numbers of their beginnings. If the ends of the corresponding increasing path on 3-graph have different signs on the ends, then we put a plus to the end of the edge, and a minus, otherwise. Only for nonorientable saddle atom the sign can depend on the choice of the path (from two or four variants), but in this case it is unnecessary to put signs. For other two types of the saddle atoms the sign does not depend on the choice of the path. Since one vertex of the 3-graph corresponds to canonical triangle and the number of vertices in the increasing path on the 3-graph is even, the inequality of the signs on the ends implies the possibility to set the coordinated orientation to the canonical triangles and to the tube with saddle atom which are glued together from these triangles.

## 5. EXAMPLES

There are three 3-graphs with one  $su$ -square. Hence, there are three Morse flows with one saddle. Two of them are assigned on sphere and the third flow is assigned on  $\mathbf{RP}^2$ . The 3-graphs and the corresponding Reeb graphs are shown in Table 1.

There are 11 3-graphs with two  $su$ -squares. So, there are 11 Morse flows with two saddles and 22 flows with indexing. But some flows are symmetric and they have only one unique numeration of saddles. Hence, there are 15 different Morse flows with numeration of 2 saddles. 3-graphs and equipped Reeb graphs of corresponding Morse functions are shown in Table 2. The signs are drawn only for one graph, because in other cases all signs are +.

Some flows correspond one function, so there are 11 different functions with 2 saddles. One can compute the number of Morse functions with 2 saddles as number of different equipped Reeb graphs and get the same result. Indeed, there are two saddle atoms and there are 3 types for each atoms. So there are 9 possibilities, but one of them corresponds 3 different graphs (see Figure 2).

The number of Morse flows and functions with 1, 2 and 3 saddles are resulted in the Table 3. It follows from these results that some nonequivalent Morse flows with the numeration correspond the equivalent functions but not the reverse.

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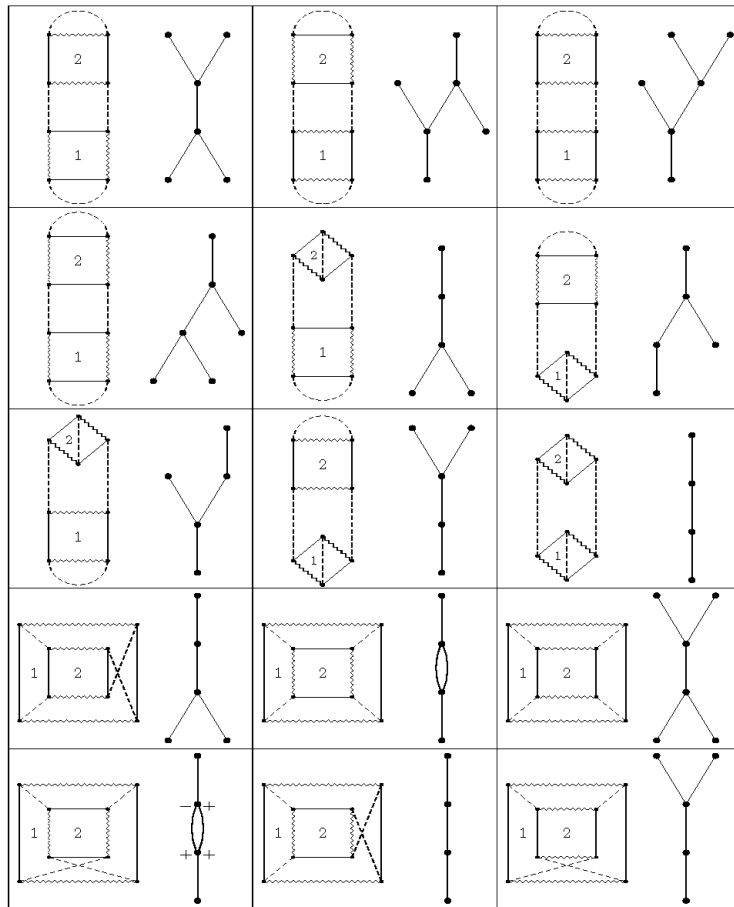


TABLE 2. 3-graphs and the Reeb graphs of Morse functions and flows with two saddles

Number of saddles	1	2	3
Number of Morse flows	3	11	63
Number of Morse flows with numeration	3	15	216
Number of Morse functions	3	11	62

TABLE 3. The number of Morse flows with  $n$  numbered saddles

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