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SOME PROPERTIES FOR BEURLING ALGEBRAS

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Abstract. Let G be a locally compact group and let ω be a weight function on G. In this paper, among other things, we show that the Beurling algebra $L^1(G,\omega)$ is super-amenable if and only if G is finite and it is biprojective if and only if G is compact.

1. Introduction

Super-amenable Banach algebras often go by the name of contractible Banach algebras in the literature [6]. The reason why we prefer to call them super-amenable is that the adjective contractible is also used in the K-theory of C^*-algebras.

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \longrightarrow E$ is called a *derivation* if

$$D(ab) = a \ . \ Db + \ (Da) \ . \ b \quad (a, b \in \mathcal{A}) \ .$$

A Banach algebra $\mathcal{A}$ is called *amenable* if for each Banach $\mathcal{A}$-bimodule E, every derivation $D : \mathcal{A} \longrightarrow E^*$ is inner. Also, a Banach algebra $\mathcal{A}$ is called *super-amenable* if for each Banach $\mathcal{A}$-bimodule E, each derivation $D : \mathcal{A} \longrightarrow E$ is inner. Every super-amenable Banach algebra is amenable and unital. Let G be a locally compact group. Then the *group algebra* $L^1(G)$ is super-amenable if and only if G is finite [10].

Biprojectivity is a notion that arises naturally in A. Ya. Helemskii Banach homology. The structure theory for biprojective Banach algebras is due to Yu. V. Selivanov [8]. For a locally compact group G, the group algebra $L^(G)$ is biprojective if and only if G is compact [5].

Like amenable, super-amenable and biprojective Banach algebras can be characterized through vanishing of certain cohomology groups [9].

Let ω be a weight function on a locally compact group G. Then the *Beurling algebra* $L^1(G,\omega)$ is the space of measurable functions $f : G \longrightarrow \mathbb{C}$ for which

$$||f||_\omega := \int_G |f(x)| \ \omega(x) \, d\lambda(x) < \infty \ ,$$

where λ is the left Haar measure on G. It is a Banach algebra with the convolution product

$$f \star g(x) := \int_G f(y) g(y^{-1}x) \, d\lambda(y) \ , \quad (f, g \in L^1(G,\omega)) \ .$$

The amenability of $L^1(G,\omega)$ has been studied by Grønbaek [4]. He proved that $L^1(G,\omega)$ is amenable if and only if G is amenable as a group and ω is diagonally bounded on G.

A Banach algebra $\mathcal{A}$ is called *weakly amenable* if every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is inner. The weak amenability of $L^1(G,\omega)$ is discussed in [3]. One result on this is: Let ω

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be a weight function on $\mathbb{Z}$. Then $\ell^1(\omega)$ is weakly amenable if and only if

$$\{\frac{\omega_n\omega_{-n}}{n} : n \in \mathbb{N}\} < \infty .$$

In this paper, we investigate the super-amenability and biprojectivity of $L^1(G,\omega)$. The result are surprisingly the same as in the non-weighted case.

2. Super-amenability

Let E be a Banach space. A *finite, biorthogonal system* for E is a set

$$\{(x_i,\varphi_j) : i,j = 1,\dots,n\},$$

where $x_1,\dots,x_n \in E$ and $\phi_1,\dots,\phi_n \in E^*$ satisfy

$$\langle x_i,\varphi_j\rangle = \delta_{i,j} \quad (i,j = 1,\dots,n).$$

Let $\mathcal{F}(E)$ be the set of all bounded finite rank operator on E. The map $\theta_n : M_n \longrightarrow \mathcal{F}(E)$ given by

$$\theta(A) := \sum_{i,j=1}^{n} a_{i,j}x_i \odot \varphi_j \quad (A = [a_{i,j}]_{i,j=1,\dots,n} \in M_n)$$

is a homomorphism where the map $x_i \odot \varphi_j$ is defined by

$$x_i \odot \varphi_j : E \longrightarrow \mathbb{C},\ x \longmapsto \langle x,\varphi_j\rangle x_i.$$

A Banach space E has *property* **A** if there is a net of finite, bi-orthogonal systems

$$\{(x_i^{(\alpha)},\phi_j^{(\alpha)}) : i,j = 1,\dots,n_\alpha\}$$

for E with corresponding homomorphisms $\theta_\alpha : M_{n_\alpha} \longrightarrow \mathcal{F}(E)$ such that

(i) $\lim_\alpha \theta_\alpha(E_{n_\alpha}) = \mathrm{id}_E$, uniformly on compact subsets of E,

(ii) $\lim_\alpha \theta_\alpha(E_{n_\alpha})^* = \mathrm{id}_{E^*}$, uniformly on compact subsets of E^*, and

(iii) For each index α, there is a finite, irreducible $n_\alpha \times n_\alpha$ matrix group G_α such that $\sup_\alpha \sup_{g\in G_\alpha} \|\theta_\alpha(g)\| < \infty$.

An element $\mathrm{m} \in \mathcal{A}\hat{\otimes}\mathcal{A}$ is called a *diagonal* for $\mathcal{A}$ if

$$a\Delta\mathrm{m} = a, \quad a.\mathrm{m} = \mathrm{m}.a \quad (a \in \mathcal{A}),$$

where Δ is the *diagonal operator*

$$\Delta : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A},\ a\otimes b \longmapsto ab.$$

It is easy to see that $\mathcal{A}\hat{\otimes}\mathcal{A}$ becomes a Banach $\mathcal{A}$-bimodule through

$$a.(b\otimes c) := ab\otimes c \quad \text{and} \quad (b\otimes c).a := b\otimes ca \quad (a,b,c \in \mathcal{A}) .$$

It is clear that Δ is a bimodule homomorphism with respect to this module structure on $\mathcal{A}\hat{\otimes}\mathcal{A}$.

$\mathcal{A}$ is super-amenable if and only if it has a diagonal [7, Exercise 4.1.3]. Let G be a locally compact group with the left Haar measure λ and with identity e. A continuous map $\omega : G \longrightarrow \mathbb{R}^+$ is called a *weight function* on G if

$$\omega(xy) \le \omega(x)\omega(y), \quad \omega(e) = 1, \quad \omega(x) \ge 1 \quad (x,y \in G).$$

If ω is a weight function on G then the map

$$\omega\times\omega : G\times G \longrightarrow \mathbb{R}^+, (x,y) \longmapsto \omega(x)\omega(y)$$

is also a weight function on $G\times G$. The group algebra $L^1(G)$ has property **A** [7, Exercise 3.1.4] and it is super-amenable if and only if G is finite [7, Exercise 4.1.7]. We have the same results for $L^1(G,\omega)$:

Proposition 2.1. *If G is a locally compact group and ω is a weight function on G , then $L^1(G,\omega)$ has property* **A**.

Proof. First suppose that the Haar measure of G is finite. Consider the collection of all families τ consisting of finitely many, pairwise disjoint sets in $\mathbf{B}_G$, the Borel algebra on G, such that $\lambda(A) \neq 0$ for each $A \in \tau$. For two such families τ_1 and τ_2 define $\tau_1 < \tau_2$ if each member of τ_1 is the union of a subfamily of τ_2 . For each $\tau = \{A_1, \ldots, A_{n_\tau}\}$ we have a corresponding finite, bi-orthogonal systems,

$$\{(\frac{1}{\lambda(A_i)}\frac{\chi_{A_i}}{\omega}, \omega\chi_{A_j}) : i, j = 1, \ldots, n_\tau\}.$$

Let $\theta_\tau : M_{n_\tau} \longrightarrow F(L^1(G, \omega))$ be the corresponding homomorphism, then

$$\theta_\tau(E_{n_\tau})(\frac{\chi_L}{\omega}) = \frac{\chi_L}{\omega}$$

and

$$\theta_\tau(E_{n_\tau})^*(\frac{\chi_L}{\omega}) = \frac{\chi_L}{\omega},$$

for each $L \in \mathbf{B}_G, L < \tau$. Thus $\lim_\alpha \theta_\alpha(E_{n_\alpha}) = \mathrm{id}_E$ and $\lim_\alpha \theta_\alpha(E_{n_\alpha})^* = \mathrm{id}_{E^*}$ uniformly on compact subsets of E and E^*, respectively. Consider $\tau = \{A_1, \ldots, A_{n_\tau}\}$ and let $\mathbf{G}_\tau$ be the group of matrices of the form $\mathbf{D_t E}_\sigma$ where $\mathbf{D_t}$ is the diagonal matrix specified by $\mathbf{t} = (t_i\delta_{i,j})$, where $t_1, \ldots, t_{n_\tau} \in \{-1, 1\}$, and $\mathbf{E}_\sigma$ is the matrix corresponding to a permutation σ of $\{1, \ldots, n_\tau\}$. Certainly $\mathbf{G}_\tau$ is an irreducible $n_\tau \times n_\tau$ matrix group. For each $f \in L^1(G, \omega)$ and $g = \mathbf{D_t E}_\sigma \in \mathbf{G}_\tau$ we have

$$\begin{aligned}\|\theta_\tau(\mathbf{D_t E}_\sigma)f\|_\omega &= \Big\| \sum_{j=1}^{n_\tau} \Big(\int_{A_j} f(t)\omega(t)\, d\lambda(t)\Big) \frac{1}{\lambda(A_\sigma(j))} \frac{\chi_{A_\sigma(j)}}{\omega} \Big\|_\omega \\ &= \sum_{j=1}^{n_\tau} \Big| t_j \int_{A_j} f(t)\omega(t)\, d\lambda(t)\Big| \leq \sum_{j=1}^{n_\tau} \int_{A_j} |f(t)|\omega(t)\, d\lambda(t)| \leq \|f\|_\omega.\end{aligned}$$

Thus $\|\theta_\tau(g)\| \leq 1$.

Finally in general case, following [1, Corollary 5.6.64], we approximate the Haar measure λ with finite measures. □

Recall that a Banach space E has the *bounded approximation property* if there is a net $(T_\alpha)_\alpha$ in $\mathcal{F}(E)$ such that $\sup_\alpha ||T_\alpha|| \leq C$ for some $C \geq 1$, and $T_\alpha \longrightarrow \mathrm{id}_E$ uniformly on compact subsets of E.

Theorem 2.2. *The Beurling algebra $L^1(G, \omega)$ is super-amenable if and only if G is finite.*

Proof. Let G be a finite group of order n. Then

$$L^1(G, \omega) = \ell^1(G, \omega) = \{\Sigma_{g\in G}\ \alpha_g\ \delta_g\ :\ \Sigma_{g\in G}\ |\alpha_g|\ \omega(g)\ <\ \infty\ ,\ \alpha_g \in \mathbb{C}\ \},$$

where δ_g is the *characteristic function* of the singleton $\{g\}$.

Let $m := \frac{1}{n}\sum_{g\in G} \delta_g \otimes \delta_{g^{-1}}$ and let $h \in G$. Since $\delta_g \star \delta_{g^{-1}} = \delta_e$, $\delta_h \star \Delta m = \delta_h$. Also

$$\delta_h.m = \frac{1}{n}\sum_{g\in G} \delta_{hg} \otimes \delta_{g^{-1}} = \frac{1}{n}\sum_{g\in G} \delta_g \otimes \delta_{g^{-1}h} = m.\delta_h\ ,$$

so that m is a diagonal for $\ell^1(G, \omega)$, and therefore it is super-amenable.

Conversely suppose that $L^1(G, \omega)$ is super-amenable. By Proposition 2.1 it has property **A**. Similar to [7, Example C.1.2(c)], we can show that it has bounded approximation property. By [7, Theorem 4.1.5] there are $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$L^1(G, \omega) \backsimeq M_{n_1} \oplus \cdots \oplus M_{n_k}.$$

Thus $L^1(G, \omega)$ has finite dimension and since it is unital [7, Exercise 4.1.1] so G is finite. □

3. Biprojectivity

A Banach algebra $\mathcal{A}$ is *biprojective* if the diagonal operator Δ has a bounded right inverse which is an $\mathcal{A}$-bimodule homomorphism. Similar to [1, Proposition 3.3.20] we have the following proposition.

Proposition 3.1. *Let G be a locally compact group and ω is a weight function on G. Then there is an isometric isomorphism $T : L^1(G,\omega)\hat{\otimes}L^1(G,\omega) \longrightarrow L^1(G\times G,\omega\times\omega)$ such that*

$$T(f\otimes g)(x,y) := f(x)g(y) \quad (f,g\in L^1(G,\omega),(x,y)\in G\times G). \tag{1}$$

Proof. By [1, A.3.69], there is a unique continuous linear map T such that (1) holds. It is easy to check that T is a homomorphism and $||T||\leq 1$.

Take $\mathcal{S}$ to be the linear subspace of $L^1(G,\omega)\hat{\otimes}L^1(G,\omega)$ spanned by elements of the form $f\otimes g$, where f and g are simple functions in $L^1(G,\omega)$. Then $\mathcal{S}$ is dense in $L^1(G,\omega)\hat{\otimes}L^1(G,\omega)$. Each element a of $\mathcal{S}$ can be written as a finite sum $a=\sum_{i,j}a_{i,j}\dfrac{\chi_{E_i}}{\omega}\otimes\dfrac{\chi_{F_j}}{\omega}$, where $E_i,F_j\in\mathbf{B}_G$, for each i,j and the rectangles $E_i\times F_j$ are pairwise disjoint in $G\times G$. It follows that $||Ta||_{\omega\times\omega}\geq||a||_\pi$ and so T is an isometry.

The range of T contains $\dfrac{\chi_{E\times F}}{\omega\times\omega}$ for each rectangle $E\times F$ in $\mathbf{B}_{G\times G}$, and we claim that the linear span of such functions is dense in $L^1(G\times G,\omega\times\omega)$. To see this, it suffices to show that $\dfrac{\chi_U}{\omega\times\omega}$ can be approximated for each open set U in $G\times G$ of finite measure. For each such U, its measure is the supremum of the measures of the compact sets contained in U, and each compact subset of U is contained is the union of finitely many open rectangles each contained in U. Thus $\dfrac{\chi_U}{\omega\times\omega}$ can indeed be approximated, giving the claim. Thus T is surjective. $\square$

Lemma 3.2. *If ω is a weight function on G, then G is compact if and only if $\omega\in L^1(G)$.*

Proof. Since G is compact if and only if $\lambda(G)<\infty$, the proof is trivial. $\square$

The map

$$\varphi_0 : L^1(G,\omega)\longrightarrow\mathbb{C},\ f\longmapsto\int_G f(x)\omega(x)\,d\lambda(x)$$

is called *the augmentation* character on $L^1(G,\omega)$ and its kernel $L^1_0(G,\omega)$ is called the augmentation ideal of $L^1(G,\omega)$. It is a closed ideal of $L^1(G,\omega)$ with codimension one. Also $L^1_0(G,\omega)$ is *essential* as a left Banach $L^1(G,\omega)$-module, that is the linear hull of $\{g\star f : g\in L^1(G,\omega), f\in L^1_0(G,\omega)\}$ is dense in $L^1_0(G,\omega)$.

Theorem 3.3. *If ω is a weight function on G , then $L^1(G,\omega)$ is biprojective if and only if G is compact.*

Proof. Let G be a compact group and define the map

$$\rho : L^1(G,\omega)\longrightarrow L^1(G\times G,\omega\times\omega), \rho(f)(x,y) := f(xy) \quad (f\in L^1(G,\omega), x,y\in G).$$

We have $\Delta(f_1\otimes f_2)=\int_G f_1\otimes f_2(xy^{-1},y)d\lambda(y)$ for each f_1 and f_2 in $L^1(G,\omega)$ and $x\in G$. So $\Delta(F)(x)=\int_G F(xy^{-1},y)d\lambda(y)$ for each $F\in L^1(G\times G,\omega\times\omega)$ and $x\in G$. If $f\in L^1(G,\omega)$ and $x\in G$, then

$$(\Delta\rho)(f)(x)=\int_G\rho(f)(xy^{-1},y)\,d\lambda(y)=\int_G f(x)\,d\lambda(y)=f(x).$$

Thus $\Delta\rho = \mathrm{id}_{L^1(G,\omega)}$ and ρ is a $L^1(G,\omega)$-bimodule homomorphism and therefore $L^1(G,\omega)$ is biprojective. Let's $\mathcal{A} := L^1(G,\omega)$ and $\mathbf{L} := L^1_0(G,\omega)$. By [7, Lemma 4.3.10], the module map

$$\Theta : \mathcal{A}\hat{\otimes}\frac{\mathcal{A}}{\mathbf{L}} \longrightarrow \frac{\mathcal{A}}{\mathbf{L}},\ f \otimes g + \mathbf{L} \longmapsto f \star g + \mathbf{L}$$

has a bounded right inverse ρ_1 which is also a left-$\mathcal{A}$-module homomorphism. By [7, Exercise 5.1.2 and Proposition 5.1.6], $\frac{\mathcal{A}}{\mathbf{L}}$ is *projective* and there is a left $\mathcal{A}$-module homomorphism $\tilde{\rho} : \frac{\mathcal{A}}{\mathbf{L}} \longrightarrow \mathcal{A}$ such that $\pi\tilde{\rho} = \mathrm{id}_{\frac{\mathcal{A}}{\mathbf{L}}}$, where $\pi : \mathcal{A} \longrightarrow \frac{\mathcal{A}}{\mathbf{L}}$ is canonical epimorphism.

The map

$$\tilde{\phi_0} : \frac{\mathcal{A}}{\mathbf{L}} \longrightarrow \mathbb{C},\ f + \mathbf{L} \longmapsto \int_G f(x)\omega(x)\, d\lambda(x)$$

is an isomorphism. Now set $\rho := \tilde{\rho}\tilde{\phi_0}^{-1}$ and $f_0 := \rho(1) \in \mathcal{A}$. We have

$$\phi_0(f_0) = \phi_0(\rho(1)) = \phi_0(\tilde{\rho}\tilde{\phi_0}^{-1})(1) = 1.$$

$\mathbb{C}$ is a left Banach $\mathcal{A}$-module with the module action

$$f.\alpha := \phi_0(f)\alpha \quad (\alpha \in \mathbb{C}, f \in \mathcal{A}).$$

Since ρ is a left $\mathcal{A}$-module homomorphism, for each $f \in \mathcal{A}$ we have

$$f \star f_0 = f \star \rho(1) = \rho(f.1) = \rho(\phi_0(f)1) = \phi_0(f)\rho(1) = \phi_0(f)f_0,$$

and for each $x \in G$ and $f \in \mathcal{A} - \mathbf{L}$

$$\phi_0(f)f_0 = \phi_0(\delta_x \star f)f_0 = (\delta_x \star f) \star f_0 = \delta_x \star (f \star f_0) = \delta_x \star (\phi_0(f)f_0) = \phi_0(f)(\delta_x \star f_0).$$

Thus f_0 is non-zero, constant function in $\mathcal{A}$, and by Lemma 3.2, G is compact. $\square$

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