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BOUNDARY TRIPLETS AND TITCHMARSH-WEYL FUNCTIONS OF DIFFERENTIAL OPERATORS WITH ARBITRARY DEFICIENCY INDICES

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Abstract. Let $l[y]$ be a formally selfadjoint differential expression of an even order on the interval $[0,b\rangle$, $b\le\infty$, with operator coefficients, acting in a separable Hilbert space H. We introduce the concept of deficiency indices $n_{b\pm}$ of the expression l at the point b and show that in the case $\dim H=\infty$ any values of $n_{b\pm}$ are possible. Moreover the decomposing selfadjoint boundary conditions exist if and only if $n_{b+}=n_{b-}$. Our considerations of differential operators with arbitrary (possibly unequal) deficiency indices are based on the concept of a decomposing D-boundary triplet. Such an approach enables to describe extensions of the minimal operator directly in terms of operator boundary conditions at the ends of the interval $[0,b\rangle$. In particular we describe in a compact form selfadjoint decomposing boundary conditions.

Associated to a D-triplet is an m-function, which can be regarded as a generalization of the classical characteristic (Titchmarsh-Weyl) function. Our definition enables to describe all m-functions (and, therefore, all spectral functions) directly in terms of boundary conditions at the right end b.

1. Introduction

Let $\Delta=[0,b\rangle$, $b\le\infty$, be an interval in $\mathbb{R}$, let H be a separable Hilbert space and let $[H]$ be the set of all bounded linear operators in H. The main objects of the paper are differential operators in the Hilbert space $\mathfrak{H}=L_2(\Delta;H)$, generated by a formally selfadjoint differential expression

$$\begin{aligned} l[y] &= l_H[y] \\ &= \sum_{k=1}^{n}(-1)^k((p_{n-k}y^{(k)})^{(k)}-\tfrac{i}{2}[(q_{n-k}^*y^{(k)})^{(k-1)}+(q_{n-k}y^{(k-1)})^{(k)}])+p_ny \end{aligned} \tag{1.1}$$

of an even order $2n$ with operator-valued coefficients $p_k(\cdot),q_k(\cdot):\Delta\to[H]$. Denote by L_0 and L minimal and maximal operators respectively, induced by the expression $l_H[y]$, and let $\mathcal{D}$ be the domain of the operator L. It is known that L_0 is a closed densely defined symmetric operator in $\mathfrak{H}$ and $L_0^*=L$. Moreover according to [12, 10] deficiency indices $n_\pm=n_\pm(L_0)$ of the operator L_0 are not necessarily equal.

Next recall that a closed operator $\widetilde{A}$ with the domain $\mathcal{D}(\widetilde{A})$ is called a proper extension of L_0 (and is referred to the class Ex_{L_0}) if $L_0\subset\widetilde{A}\subset L$. As is known a description of various classes of extensions $\widetilde{A}\in\mathrm{Ex}_{L_0}$ (selfadjoint, symmetric etc.) in terms of boundary conditions is an important problem in the spectral theory of differential operators. For a regular expression $l_H[y]$ with $\dim H\le\infty$ this problem was solved in a compact form by F. S. Rofe-Beketov in [21].

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The ideas of the paper [21] stimulated appearance of the method of boundary triplets and the corresponding Weyl functions, which has become a convenient tool in the extension theory of symmetric operators and its applications (for a densely defined operator see [8, 1, 2] and references therein). Note that the theory of boundary triplets and their Weyl functions was developed in [8, 1, 2] only for symmetric operators A with equal deficiency indices $n_\pm(A)$. In order to extend this theory to operators with unequal deficiency indices the concept of a D-boundary triplet and the corresponding Weyl function was introduced in our paper [18]. Let us briefly recall some definitions and results from this paper.

Assume that A is a densely defined symmetric operator and let $\mathcal{D}(A^*)$ be the domain of the adjoint A^*. Then a collection $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$, where $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$ and $\Gamma_j:\mathcal{D}(A^*)\to\mathcal{H}_j\ j\in\{0,1\}$ are linear mappings, is called a D-boundary triplet (or briefly D-triplet) for A^*, if the mapping $\Gamma=(\Gamma_0\quad\Gamma_1)^\top$ is surjective and the following abstract Green's identity holds

$$(1.2)\quad (A^*f,g)-(f,A^*g)=(\Gamma_1 f,\Gamma_0 g)-(\Gamma_0 f,\Gamma_1 g)+i(P_2\Gamma_0 f,P_2\Gamma_0 g),\quad f,g\in\mathcal{D}(A^*)$$

(here P_2 is the orthoprojector in $\mathcal{H}_0$ onto the subspace $\mathcal{H}_2:=\mathcal{H}_0\ominus\mathcal{H}_1$). Associated with the $\mathcal{D}$-triplet $\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ for A^* is the Weyl function $M_+(\cdot)$ defined by

$$(1.3)\qquad \Gamma_1 f_\lambda=M_+(\lambda)\Gamma_0 f_\lambda,\quad f_\lambda\in\mathfrak{N}_\lambda(A):=\operatorname{Ker}(A^*-\lambda),\quad \lambda\in\mathbb{C}_+.$$

The operator function $M_+(\cdot)$ is holomorphic on the upper half plane $\mathbb{C}_+$ and takes on values in the set $[\mathcal{H}_0,\mathcal{H}_1]$ of all bounded linear operators from $\mathcal{H}_0$ to $\mathcal{H}_1$ (see [18]).

It turns out that every D-triplet $\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ satisfies the relation $\dim\mathcal{H}_1=n_-(A)\le n_+(A)=\dim\mathcal{H}_0$. Furthermore in the case $\mathcal{H}_0=\mathcal{H}_1:=\mathcal{H}$ a D-triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is a boundary triplet (a boundary value space) for A^* [8], while the function $M(\lambda)=M_+(\lambda)$ coincides with the Weyl function introduced by V. A. Derkach and M. M. Malamud [1, 2]. Observe also that another approaches to generalization of the notion of a boundary triplet to the case of unequal deficiency indices were proposed in [14, 24] and the recent papers [3, 4].

In the present paper we apply the results of [18] to the differential operator L_0 with arbitrary (possibly unequal) deficiency indices $n_\pm$. The basic object here is a decomposing D-triplet for L, which is defined as follows. Let $\mathcal{H}_0'$ be a Hilbert space and let $\mathcal{H}_1'$ be a subspace in $\mathcal{H}_0'$. Then a D-triplet $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ for L is called decomposing, if $\mathcal{H}_j=H^n\oplus\mathcal{H}_j'$ and

$$(1.4)\quad \Gamma_0 y=\{y^{(2)}(0),\Gamma_0' y\}\,(\in H^n\oplus\mathcal{H}_0'),\quad \Gamma_1 y=\{-y^{(1)}(0),\Gamma_1' y\}\,(\in H^n\oplus\mathcal{H}_1'),\quad y\in\mathcal{D},$$

where $y^{(1)}(0),\ y^{(2)}(0)\in H^n$ are vectors of the quasi-derivatives at the point 0 (see (3.2)) and $\Gamma_j':\mathcal{D}\to\mathcal{H}_j',\ j\in\{0,1\}$, are linear maps. Clearly, in the case $\mathcal{H}_0'=\mathcal{H}_1':=\mathcal{H}'$ the decomposing D-triplet (1.4) turns into a decomposing boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ with $\mathcal{H}=H^n\oplus\mathcal{H}'$.

Let θ be a selfadjoint linear relation in H^n and let $\mathcal{D}_\theta$ be the set of all finite at the point b functions $y\in\mathcal{D}$ such that $\{y^{(1)}(0),y^{(2)}(0)\}\in\theta$. Denote by L_θ a symmetric extension of L_0 which is the closure of the operator $L\restriction\mathcal{D}_\theta$. Using a decomposing $\mathcal{D}$-triplet we show that the deficiency indices $n_\pm(L_\theta)$ of an operator L_θ do not depend on θ. This makes it possible to introduce the following definition.

Definition 1.1. The numbers $n_{b\pm}:=n_\pm(L_\theta)$ are called deficiency indices of the expression $l_H[y]$ at the point b.

Moreover the deficiency indices $n_\pm$ and $n_{b\pm}$ are connected via

$$(1.5)\qquad n_+=2n\,\dim H+n_{b+},\quad n_-=2n\,\dim H+n_{b-}.$$

It turns out that in the case $\dim H = \infty$ any deficiency indices at the point b are possible, while $n_+ = n_- = \infty$ (see Proposition 3.8). We also show that every decomposing D-triplet (1.4) satisfies the relation

$$\dim \mathcal{H}_1' = n_{b-} \le n_{b+} = \dim \mathcal{H}_0'. \tag{1.6}$$

Moreover for an expression $l_H[y]$ with arbitrary deficiency indices $n_{b-} \le n_{b+} \le \infty$ we construct a D-triplet (1.4) with operators Γ_0' and Γ_1', defined in the explicit form by means of boundary values at the point b.

Another object of our investigations is the Weyl function for the D-triplet (1.4), which can be written in the block-matrix form

$$M_+(\lambda) = \begin{pmatrix} m(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}_0' \to H^n \oplus \mathcal{H}_1', \quad \lambda \in \mathbb{C}_+. \tag{1.7}$$

The representation (1.7) induces the uniformly strict Nevanlinna function $m(\lambda)$, which we call an m-function. This function can be defined also by the following statement (i).

(i) Let $c(t,\lambda)$ and $s(t,\lambda)$ be operator solutions of the equation $l[Y] - \lambda Y = 0$ with the initial data

$$c^{(1)}(0,\lambda) = I_{H^n}, \quad c^{(2)}(0,\lambda) = 0, \quad s^{(1)}(0,\lambda) = 0, \quad s^{(2)}(0,\lambda) = I_{H^n}, \quad \lambda \in \mathbb{C}.$$

Then for every $\lambda \in \mathbb{C}_+$ there exists the unique operator $m(\lambda) \in [H^n]$ such that the operator function $v_0(t,\lambda) := -c(t,\lambda)m(\lambda) + s(t,\lambda)$ satisfy the relation $v_0(t,\lambda)\hat{h} \in \mathfrak{H}$ and the boundary condition $\Gamma_0'(v_0(t,\lambda)\hat{h}) = 0$ ($\hat{h} \in H^n$) at the point b.

If $\dim H = 1$ (the scalar case) and $n_+ = n_-$, then the function $m(\cdot)$ coincides with the classical characteristic (Titchmarsh - Weyl) function for decomposing boundary conditions [15, 20, 5]. Moreover in the case $\dim H \le \infty$ the function $m(\cdot)$ coincides with the characteristic function introduced for regular and quasi-regular expressions $l_H[y]$ in [7, 11]. Note in this connection that our definition of the m-function $m(\lambda)$ can be applied to differential operators with arbitrary (possibly unequal) deficiency indices. Moreover if $\dim H = \infty$ and $n_{b+} \ne n_{b-}$, then by (1.5) $n_+ = n_- = \infty$. At the same time in view of (1.6) there is not a decomposing boundary triplet (1.4) (with $\mathcal{H}_0' = \mathcal{H}_1'$). This shows that the concepts of a decomposing D-triplet and the corresponding m-function are useful even for operators L_0 with equal deficiency indices $n_+ = n_- = \infty$.

Observe also that our definition of $m(\lambda)$ contains in the explicit form the boundary condition at the right end b of the interval Δ. This enables to describe all m-functions (and, therefore, all spectral functions) directly in terms of such conditions [19].

We suppose that the above results on m-functions are new even in the scalar case for an operator L_0 with equal intermediate deficiency indices $n < n_+ = n_- < 2n$.

In the final part of the paper we characterize various classes of extensions $\widetilde{A} \in \mathrm{Ex}_{L_0}$ in terms of a decomposing D-triplet for L. Furthermore we describe spectrum of $\widetilde{A}$ by means of boundary conditions and the Weyl function $M_+(\lambda)$. In particular we show that selfadjoint decomposing boundary conditions exist if and only if $n_{b+} = n_{b-}$. Moreover if this criterion is satisfied and $\Pi = \{H^n \oplus \mathcal{H}', \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet (1.4), then the set of all selfadjoint decomposing conditions is described by the relations

$$\cos B_1\, y^{(1)}(0) + \sin B_1\, y^{(2)}(0) = 0, \quad \cos B_2\, \Gamma_0' y + \sin B_2\, \Gamma_1' y = 0, \tag{1.8}$$

where B_1 and B_2 are bounded selfadjoint operators in H^n and $\mathcal{H}'$ respectively. This implies that every selfadjoint boundary condition at the right end b is defined by the second equation in (1.8) (cf. [22]). For regular and quasi-regular expressions formula (1.8) was obtained in [21, 11]. Observe also that insufficiency of the condition $n_+ = n_-$ for existence of selfadjoint decomposing boundary conditions in the case $\dim H = \infty$ was noticed in [22] (see also [23]).

Finally, using the above statements we complement and generalize some results from [22] (see Corollary 4.7 and Remark 4.8).

In conclusion note that the results of the paper were partially announced in [19].

2. Preliminaries

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1,\mathcal{H}_2]$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H},\mathcal{H}]$; $A \restriction \mathcal{L}$ is the restriction of an operator A onto the linear manifold $\mathcal{L}$; $P_{\mathcal{L}}$ is the orthogonal projector in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plain of the complex plain.

Recall that a closed linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ is a closed subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (from $\mathcal{H}$ to $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ ($\widetilde{\mathcal{C}}(\mathcal{H})$). A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\mathrm{gr}T \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$.

For a relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathrm{Ker}T$ the domain, range and the kernel respectively. The inverse T^{-1} and adjoint T^* are relations defined by

$$T^{-1} = \{\{f',f\} : \{f,f'\} \in T\}, \quad T^{-1} \in \widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0),$$
$$T^* = \{\{g,g'\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (f',g) = (f,g'), \ \ \{f,f'\} \in T\}, \quad T^* \in \widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_0).$$

In the case $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ we write:

$0 \in \rho(T)$ if $\mathrm{Ker}\,T = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}_1$, or equivalently if $T^{-1} \in [\mathcal{H}_1,\mathcal{H}_0]$;

$0 \in \hat\rho(T)$ if $\mathrm{Ker}\,T = \{0\}$ and $\mathcal{R}(T)$ is a closed subspace in $\mathcal{H}_1$;

$0 \in \sigma_c(T)$ if $\mathrm{Ker}\,T = \{0\}$ and $\overline{\mathcal{R}(T)} = \mathcal{H}_1 \neq \mathcal{R}(T)$;

$0 \in \sigma_p(T)$ if $\mathrm{Ker}\,T \neq \{0\}$; $0 \in \sigma_r(T)$ if $\mathrm{Ker}\,T = \{0\}$ and $\overline{\mathcal{R}(T)} \neq \mathcal{H}_1$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T) = \{\lambda \in \mathbb{C} : \ 0 \in \rho(T-\lambda)\}$ and $\hat\rho(T) = \{\lambda \in \mathbb{C} : \ 0 \in \hat\rho(T-\lambda)\}$ the resolvent set and the set of regular type points of T respectively. Next, $\sigma(T) = \mathbb{C}\backslash\rho(T)$ stands for the spectrum of T. The spectrum $\sigma(T)$ admits the following classification:

$\sigma_c(T) = \{\lambda \in \mathbb{C} : 0 \in \sigma_c(T-\lambda)\}$ is the continuous spectrum;

$\sigma_p(T) = \{\lambda \in \mathbb{C} : 0 \in \sigma_p(T-\lambda)\}$ is the point spectrum;

$\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T)) = \{\lambda \in \mathbb{C} : 0 \in \sigma_r(T-\lambda)\}$ is the residual spectrum.

2.2. **Operator pairs.** Let $\mathcal{K},\mathcal{H}_0,\mathcal{H}_1$ be Hilbert spaces. A pair of operators $C_j \in [\mathcal{H}_j,\mathcal{K}]$, $j \in \{0,1\}$ will be called admissible if the range of the operator $C \in [\mathcal{H}_0 \oplus \mathcal{H}_1,\mathcal{K}]$ given by the block-matrix representation

$$C = (C_0 \quad C_1) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K},$$

coincides with $\mathcal{K}$. Two admissible pairs $C^{(j)} = (C_0^{(j)} \quad C_1^{(j)}) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K}_j$, $j \in \{1,2\}$ will be called equivalent if $C^{(2)} = XC^{(1)}$ with some isomorphism $X \in [\mathcal{K}_1,\mathcal{K}_2]$.

For a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ we write

$$\theta = \{(C_0,C_1);\mathcal{H}_0,\mathcal{H}_1;\mathcal{K}\} \tag{2.1}$$

if the operators $C_j \in [\mathcal{H}_j,\mathcal{K}]$, $j \in \{0,1\}$ form an admissible operator pair such that

$$\theta = \mathrm{Ker}\,C = \{\{h_0,h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0h_0 + C_1h_1 = 0\}. \tag{2.2}$$

Moreover in the case $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$ we write $\theta = \{(C_0,C_1);\mathcal{H};\mathcal{K}\}$. Clearly every $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ admits the unique representation (2.2) up to equivalence of operator pairs. This allows us to identify by means of the equality (2.1) a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and the corresponding class of equivalent admissible operator pairs $C_j \in [\mathcal{H}_j,\mathcal{K}]$, $j \in \{0,1\}$. Therefore in what follows we do not distinguish equivalent operator pairs.

Next recall some results and definitions from our paper [17].

Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$, let $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$ and let P_j be the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1,2\}$. With every linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we associate a $\times$-adjoint linear relation $\theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, which is defined as the set of all vectors $\hat{k} = \{k_0, k_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1$ such that

$$(k_1, h_0) - (k_0, h_1) + i(P_2 k_0, P_2 h_0) = 0, \quad \{h_0, h_1\} \in \theta. \tag{2.3}$$

According to (2.1) and (2.2) one can consider the relation $\theta^\times$ as the operator pair $\theta^\times = \{(C_{0\times}, C_{1\times}); \mathcal{H}_0, \mathcal{H}_1; \mathcal{K}_\times\}$. If $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$, then a linear relation (operator pair) $\theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H})$ coincides with θ^*. Moreover in the general case $\mathcal{H}_1 \subset \mathcal{H}_0$ the relation $\theta^\times$ has a number of properties similar to θ^* (see [17]).

Next assume that θ is an operator pair (linear relation) (2.1), $C_0 = (C_{01} \quad C_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}$ is the block-matrix representation of C_0 and

$$\widetilde{S}_\theta := 2\mathrm{Im}(C_1 C_{01}^*) - C_{02} C_{02}^*, \quad \widetilde{S}_\theta \in [\mathcal{K}].$$

Definition 2.1. [17]. The operator pair (linear relation) (2.1) belongs to the class:

1) $\mathrm{Dis}(\mathcal{H}_0, \mathcal{H}_1)$ if $\widetilde{S}_\theta \geq 0$ and

$$0 \in \rho(C_{01} - \lambda C_1) \quad \text{for some (equivalently for all)} \quad \lambda \in \mathbb{C}_+; \tag{2.4}$$

2) $\mathrm{Ac}(\mathcal{H}_0, \mathcal{H}_1)$ if $\widetilde{S}_\theta \leq 0$ and

$$0 \in \rho(C_0 - \lambda C_1 P_1) \quad \text{for some (equivalently for all)} \quad \lambda \in \mathbb{C}_-; \tag{2.5}$$

3) $\mathrm{Sym}(\mathcal{H}_0, \mathcal{H}_1)$ ($\mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)$) if $\widetilde{S}_\theta = 0$ and at least one of the condition (respectively both the conditions) (2.4), (2.5) is satisfied.

Note that in the case $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$ classes Dis, Ac, Sym and Self coincide with sets of all maximal dissipative, maximal accumulative, maximal symmetric and selfadjoint linear relations in $\mathcal{H}$ respectively. Moreover every selfadjoint relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ admits the unique representation [21]

$$\theta = \{(\cos B, \sin B); \mathcal{H}; \mathcal{H}\}, \tag{2.6}$$

where $B = B^* \in [\mathcal{H}]$, $-\frac{\pi}{2} I \leq B \leq \frac{\pi}{2} I$ and $-\frac{\pi}{2} \notin \sigma_p(B)$.

2.3. **Boundary triplets and Weyl functions.** Let A be a closed densely defined symmetric operator in $\mathfrak{H}$. In what follows we will use the following notations:

$\mathfrak{N}_\lambda(A) := \mathrm{Ker}(A^* - \lambda)$ $(\lambda \in \hat{\rho}(A))$ is a defect subspace and $n_\pm(A) := \dim \mathfrak{N}_\lambda(A)$ $(\lambda \in \mathbb{C}_\pm)$ are deficiency indices of A;

Ex_A is the set of all proper extensions of A, i.e., the set of all closed operators $\widetilde{A}$ in $\mathfrak{H}$ such that $A \subset \widetilde{A} \subset A^*$.

Let $\mathcal{H}_0$ be a Hilbert space, let $\mathcal{H}_1$ be a subspace in $\mathcal{H}_0$ and let $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$. Denote by P_j the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1,2\}$.

Definition 2.2. [18]. A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where Γ_j are linear mappings from $\mathcal{D}(A^*)$ to $\mathcal{H}_j$ $(j \in \{0,1\})$, is called a D-boundary triplet (or briefly D-triplet) for A^*, if $\Gamma = (\Gamma_0 \quad \Gamma_1)^\top : \mathcal{D}(A^*) \to \mathcal{H}_0 \oplus \mathcal{H}_1$ is a surjective linear mapping onto $\mathcal{H}_0 \oplus \mathcal{H}_1$ and the following Green's identity holds

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) + i(P_2 \Gamma_0 f, P_2 \Gamma_0 g), \quad f, g \in \mathcal{D}(A^*). \tag{2.7}$$

In the following propositions some properties of D-triplets are specified (see [18]).

Proposition 2.3. *If $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a D-triplet for A^*, then*

$$\dim \mathcal{H}_1 = n_-(A) \leq n_+(A) = \dim \mathcal{H}_0.$$

Conversely for every symmetric densely defined operator A with $n_-(A) \leq n_+(A)$ there exists a D-triplet for A^*.

Proposition 2.4. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D-triplet for A^*. Then:*

1) $\mathrm{Ker}\Gamma_0 \cap \mathrm{Ker}\Gamma_1 = \mathcal{D}(A)$ *and the operators Γ_0, Γ_1 are bounded (in the graph norm);*

2) the equality

$$\mathcal{D}(\widetilde{A}_\theta) := \{f \in \mathcal{D}(A^*) : \{\Gamma_0 f, \Gamma_1 f\} \in \theta\} = \{f \in \mathcal{D}(A^*) : C_0\Gamma_0 f + C_1\Gamma_1 f = 0\} \tag{2.8}$$

establishes a bijective correspondence between all proper extensions $\widetilde{A} = \widetilde{A}_\theta \in \mathrm{Ex}_A$ with the domain $\mathcal{D}(\widetilde{A}_\theta)$ and all linear relations (admissible operator pairs) $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ given by (2.1), (2.2)*. Moreover $(\widetilde{A}_\theta)^* = \widetilde{A}_{\theta^\times}$ and an extension $\widetilde{A}_\theta \in \mathrm{Ex}_A$ is maximal dissipative, maximal accumulative, maximal symmetric or selfadjoint if and only if θ belongs to the class* Dis, Ac, Sym *or* $\mathrm{Self}(\mathcal{H}_0, \mathcal{H}_1)$ *respectively;*

3) The relations

$$\mathcal{D}(A_0) := \mathrm{Ker}\Gamma_0 = \{f \in \mathcal{D}(A^*) : \Gamma_0 f = 0\}, \quad A_0 = A^* \restriction \mathcal{D}(A_0) \tag{2.9}$$

define a maximal symmetric extension $A_0 \in \mathrm{Ex}_A$ such that $n_-(A_0) = 0$.

It turns out that for every $\lambda \in \mathbb{C}_+$ ($z \in \mathbb{C}_-$) the map $\Gamma_0 \restriction \mathfrak{N}_\lambda(A)$ ($P_1\Gamma_0 \restriction \mathfrak{N}_z(A)$) is an isomorphism. This makes it possible to introduce the operator functions (γ-fields) $\gamma_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathfrak{H}]$, $\gamma_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathfrak{H}]$ and the Weyl functions $M_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathcal{H}_1]$, $M_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathcal{H}_0]$ by

$$\gamma_+(\lambda) = (\Gamma_0 \restriction \mathfrak{N}_\lambda(A))^{-1}, \quad \lambda \in \mathbb{C}_+; \qquad \gamma_-(z) = (P_1\Gamma_0 \restriction \mathfrak{N}_z(A))^{-1}, \quad z \in \mathbb{C}_-, \tag{2.10}$$

$$\Gamma_1 \restriction \mathfrak{N}_\lambda(A) = M_+(\lambda)\Gamma_0 \restriction \mathfrak{N}_\lambda(A), \quad \lambda \in \mathbb{C}_+, \tag{2.11}$$

$$(\Gamma_1 + iP_2\Gamma_0) \restriction \mathfrak{N}_z(A) = M_-(z)P_1\Gamma_0 \restriction \mathfrak{N}_z(A), \quad z \in \mathbb{C}_-. \tag{2.12}$$

Let $\lambda \in \mathbb{C}_+$, $z \in \mathbb{C}_-$ and let

$$\gamma_+(\lambda) = (\gamma(\lambda) \quad \delta_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathfrak{H}, \tag{2.13}$$

$$M_+(\lambda) = (M(\lambda) \;\; N_+(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1, \tag{2.14}$$

$$M_-(z) = (M(z) \;\; N_-(z))^\top : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \tag{2.15}$$

be the block-matrix representations of $\gamma_+(\cdot)$ and $M_\pm(\cdot)$. Formulas (2.13) and (2.14), (2.15) induce the operator functions $\gamma(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_1, \mathfrak{H}]$ and $M(\cdot) : \mathbb{C}_+ \cup \mathbb{C}_- \to [\mathcal{H}_1]$.

Proposition 2.5. [18]. *All functions $\gamma_\pm(\cdot)$ and $M_\pm(\cdot)$ are holomorphic on their domains. Moreover $\gamma_+(\lambda)\mathcal{H}_0 = \mathfrak{N}_\lambda(A)$ ($\lambda \in \mathbb{C}_+$), $\gamma_-(z)\mathcal{H}_1 = \mathfrak{N}_z(A)$ ($z \in \mathbb{C}_-$) and*

$$M(\mu) - M^*(\lambda) = (\mu - \overline{\lambda})\, \gamma^*(\lambda)\gamma(\mu), \quad \mu, \lambda \in \mathbb{C}_+ \tag{2.16}$$

$$M_-(z) = M_+^*(\overline{z}), \quad M(z) = M^*(\overline{z})), \quad z \in \mathbb{C}_-. \tag{2.17}$$

Hence $M(\cdot)$ is a uniformly strict Nevanlinna function, that is $Im\lambda\, ImM(\lambda) \geq 0$ and $0 \in \rho(ImM(\lambda))$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

Remark 2.6. If a D-triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ satisfies the relation $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$ ($\Leftrightarrow A_0 = A_0^*$), then it is a boundary triplet. More precisely this means that the collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet (boundary value space) for A^* in the sense of [8]. In this case the function $M(\lambda) = M_\pm(\lambda)$ is defined for all $\lambda \in \rho(A_0)$ and coincides with the Weyl function introduced in [1].

3. Boundary triplets for differential operators

3.1. **Differential operators.** Let $\Delta = [0, b\rangle$ ($b \leq \infty$) be an interval on the real axis (in the case $b < \infty$ the point b may or may not belong to Δ), let H be a separable Hilbert space with $\dim H \leq \infty$ and let

$$l[y] = l_H[y] = \sum_{k=1}^{n} (-1)^k ((p_{n-k}y^{(k)})^{(k)} - \tfrac{i}{2}[(q_{n-k}^* y^{(k)})^{(k-1)} + (q_{n-k}y^{(k-1)})^{(k)}]) + p_n y, \tag{3.1}$$

be a differential expression of an even order $2n$ with operator-valued coefficients $p_k(\cdot), q_k(\cdot) : \Delta \to [H]$ satisfying the conditions

$$p_k(\cdot), q_k(\cdot) \in C^{n-k}(\Delta), \quad p_k(t) = p_k^*(t), \quad 0 \in \rho(p_0(t)), \quad t \in \Delta, \ \ k = 0 \div n$$

(here we put $q_n \equiv 0$). Denote by $y^{[k]}(\cdot), \ k = 0 \div 2n$ the quasi-derivatives of a vector-function $y(\cdot) : \Delta \to H$, corresponding to the expression (3.1). Moreover for every operator function $Y(\cdot) : \Delta \to [\mathcal{K}, H]$ ($\mathcal{K}$ is a Hilbert space) introduce quasi-derivatives $Y^{[k]}(\cdot)$ by the same formulas as $y^{[k]}$ (see [20, 21, 13]).

Let $\mathcal{D}(l)$ be the set of all functions $y(\cdot)$ such that $y^{[k]}(\cdot), \ k = 0 \div (2n-2)$ has a continuous derivative on Δ and $y^{[2n-1]}$ is absolutely continuous on every finite segment $[0, \beta] \subset \Delta$. Furthermore for a given Hilbert space $\mathcal{K}$ denote by $\mathcal{D}_{\mathcal{K}}(l)$ the set of all operator-functions $Y(\cdot)$ with values in $[\mathcal{K}, H]$ such that $Y^{[k]}(\cdot), \ k = 0 \div (2n-1)$ has a continuous derivative on Δ. Clearly for every $y \in \mathcal{D}(l)$ and $Y \in \mathcal{D}_{\mathcal{K}}(l)$ the functions $y^{[k]}(\cdot) : \Delta \to H, \ k = 0 \div (2n-1)$ and $Y^{[k]}(\cdot) : \Delta \to [\mathcal{K}, H], \ k = 0 \div 2n$ are continuous on Δ, the function $y^{[2n]}(t)(\in H)$ is defined almost everywhere on Δ and

$$l[y] = y^{[2n]}(t), \quad y \in \mathcal{D}(l); \qquad l[Y] = Y^{[2n]}(t), \quad Y \in \mathcal{D}_{\mathcal{K}}(l).$$

This makes it possible to introduce the vector functions $y^{(j)}(\cdot) : \Delta \to H^n, \ j \in \{1, 2\}$ and $\widetilde{y}(\cdot) : \Delta \to H^n \oplus H^n$,

$$y^{(1)}(t) := \{y^{[k-1]}(t)\}_{k=1}^n (\in H^n), \quad y^{(2)}(t) := \{y^{[2n-k]}(t)\}_{k=1}^n (\in H^n), \tag{3.2}$$

$$\widetilde{y}(t) = \{y^{(1)}(t), y^{(2)}(t)\} (\in H^n \oplus H^n), \quad t \in \Delta, \tag{3.3}$$

which correspond to every $y \in \mathcal{D}(l)$. Similarly with each $Y \in \mathcal{D}_{\mathcal{K}}(l)$ we associate the operator-functions $Y^{(j)}(\cdot) : \Delta \to [\mathcal{K}, H^n]$ and $\widetilde{Y}(\cdot) : \Delta \to [\mathcal{K}, H^n \oplus H^n]$ given by

$$Y^{(1)}(t) = (Y(t) \ \ Y^{[1]}(t) \ \ \ldots \ \ Y^{[n-1]}(t))^\top, \quad Y^{(2)}(t) = (Y^{[2n-1]}(t) \ \ Y^{[2n-2]}(t) \ \ \ldots \ \ Y^{[n]}(t))^\top,$$

$$\widetilde{Y}(t) = (Y^{(1)}(t) \ \ \ Y^{(2)}(t))^\top : \mathcal{K} \to H^n \oplus H^n, \quad t \in \Delta.$$

It is clear that for every $Y \in \mathcal{D}_{\mathcal{K}}(l)$ and $h \in \mathcal{K}$ the function $y(t) := Y(t)h$ belongs to $\mathcal{D}(l)$ and $y^{(j)}(t) = Y^{(j)}(t)h, \ j \in \{1, 2\}$.

Next for a given $\lambda \in \mathbb{C}$ consider the equation

$$l[y] - \lambda y = 0. \tag{3.4}$$

It is known that for every pair of vectors $y_j \in H^n$ (operators $Y_j \in [\mathcal{K}, H^n]$) there exists the unique vector-function $y \in \mathcal{D}(l)$ (operator-function $Y \in \mathcal{D}_{\mathcal{K}}(l)$) such that $y^{[2n]}(t) - \lambda y(t) = 0$ ($Y^{[2n]}(t) - \lambda Y(t) = 0$) and $y^{(j)}(0) = y_j$ (respectively, $Y^{(j)}(0) = Y_j$), $j \in \{1, 2\}$. These functions are called solutions of (3.4) with the initial data y_j (or Y_j). Moreover we distinguish the two "canonical" operator solutions $c(\cdot, \lambda)$ and $s(\cdot, \lambda) : \Delta \to [H^n, H], \ \lambda \in \mathbb{C}$ of the equation (3.4) with the initial data

$$c^{(1)}(0, \lambda) = I_{H^n}, \quad c^{(2)}(0, \lambda) = 0, \quad s^{(1)}(0, \lambda) = 0, \quad s^{(2)}(0, \lambda) = I_{H^n}, \quad \lambda \in \mathbb{C}. \tag{3.5}$$

In what follows we will denote by $\mathfrak{H}(= L_2(\Delta; H))$ the Hilbert space of all measurable functions $f(\cdot) : \Delta \to H$ such that $\int_0^b ||f(t)||^2 \, dt < \infty$. It is known [20, 21] that the expression (3.1) generate the maximal operator L in $\mathfrak{H}$, defined on the domain

$$\mathcal{D} = \mathcal{D}(L) := \{y \in \mathcal{D}(l) \cap \mathfrak{H} : l[y] \in \mathfrak{H}\} \tag{3.6}$$

by the equality $Ly = l[y], \ y \in \mathcal{D}$. Moreover the Lagrange's identity

$$(Ly, z)_{\mathfrak{H}} - (y, Lz)_{\mathfrak{H}} = [y, z](b) - [y, z](0), \quad y, z \in \mathcal{D} \tag{3.7}$$

holds with

$$[y,z](t) = (y^{(1)}(t), z^{(2)}(t))_{H^n} - (y^{(2)}(t), z^{(1)}(t))_{H^n}, \quad [y,z](b) = \lim_{t\uparrow b}[y,z](t). \tag{3.8}$$

The minimal operator L_0 is defined as a restriction of L onto the domain $D_0 = \mathcal{D}(L_0)$ of all functions $y \in \mathcal{D}$ such that $\widetilde{y}(0) = 0$ and $[y,z](b) = 0$ for all $z \in \mathcal{D}$. As is known [20, 21] L_0 is a closed densely defined symmetric operator in $\mathfrak{H}$ and $L_0^* = L$. In the sequel we denote by $n_\pm := n_\pm(L_0)$ ($n_\pm \in \mathbb{Z}_+ \cup \{\infty\}$) the deficiency indices of the operator L_0. According to [20, 13] the following relation holds

$$n \cdot \dim H \le n_\pm \le 2n \cdot \dim H. \tag{3.9}$$

Next denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ linear manifolds in $\mathfrak{H}$ given by

$$\mathcal{D}_1 := \{y \in \mathcal{D} : \widetilde{y}(0) = 0\}, \quad \mathcal{D}_2 := \{y \in \mathcal{D} : [y,z](b) = 0,\ z \in \mathcal{D}\}. \tag{3.10}$$

It is easily to check that

$$\mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{D}_0, \quad \mathcal{D}_1 + \mathcal{D}_2 = \mathcal{D}. \tag{3.11}$$

Definition 3.1. An extension $\widetilde{A} \in \mathrm{Ex}_{L_0}$ is referred to the class Dex_{L_0} if

$$\mathcal{D}(\widetilde{A}) = (\mathcal{D}(\widetilde{A}) \cap \mathcal{D}_1) + (\mathcal{D}(\widetilde{A}) \cap \mathcal{D}_2). \tag{3.12}$$

It is clear that $\widetilde{A} \in \mathrm{Dex}_{L_0}$ if and only if $\widetilde{A} \in \mathrm{Ex}_{L_0}$ and for every $y \in \mathcal{D}(\widetilde{A})$ there exists $z \in \mathcal{D}(\widetilde{A})$ such that $\widetilde{z}(0) = 0$ and $z(t) = y(t)$ on some interval $(\eta, b) \in \Delta$. Therefore Dex_{L_0} is the set of all extensions defined by decomposing boundary conditions (see [6, 23]).

Definition 3.2. A symmetric extension $\widetilde{A}_b \in \mathrm{Ex}_{L_0}$ is referred to the class $\mathrm{Sym}_{L_0,b}$ if there exists a selfadjoint extension $\widetilde{A} \in \mathrm{Dex}_{L_0}$ such that $\mathcal{D}(\widetilde{A}_b) = \mathcal{D}(\widetilde{A}) \cap \mathcal{D}_1$.

In Section 4 we will define the class $\mathrm{Sym}_{L_0,b}$ in terms of internal properties of extensions $\widetilde{A}_b \in \mathrm{Ex}_{L_0}$ (see Definition 4.3).

3.2. **Decomposing boundary triplets.** Let $\mathcal{H}_1'$ be a subspace in a Hilbert space $\mathcal{H}_0'$, let $\mathcal{H}_2' := \mathcal{H}_0' \ominus \mathcal{H}_1'$ and let P_j' be the orthoprojector in $\mathcal{H}_0'$ onto $\mathcal{H}_j'$, $j \in \{1,2\}$.

Definition 3.3. A D-boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for L will be called decomposing, if $\mathcal{H}_0 = H^n \oplus \mathcal{H}_0'$, $\mathcal{H}_1 = H^n \oplus \mathcal{H}_1'$ and

$$\Gamma_0 y = \{y^{(2)}(0), \Gamma_0' y\}\, (\in H^n \oplus \mathcal{H}_0'), \quad \Gamma_1 y = \{-y^{(1)}(0), \Gamma_1' y\}\, (\in H^n \oplus \mathcal{H}_1'), \quad y \in \mathcal{D} \tag{3.13}$$

where $\Gamma_j' : \mathcal{D} \to \mathcal{H}_j'$, $j \in \{0,1\}$ are linear maps.

A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for L will be called decomposing, if $\mathcal{H} = H^n \oplus \mathcal{H}'$ and the maps $\Gamma_j : \mathcal{D} \to H^n \oplus \mathcal{H}'$, $j \in \{0,1\}$ are given by (3.13) with $\mathcal{H}_0' = \mathcal{H}_1' := \mathcal{H}'$.

Clearly every decomposing boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for L is a decomposing D-triplet with $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}(\iff \mathcal{H}_0' = \mathcal{H}_1' := \mathcal{H}')$.

Lemma 3.4. *Let $\mathcal{H}_1'$ be a subspace in a Hilbert space $\mathcal{H}_0'$, let $\Gamma_j' : \mathcal{D} \to \mathcal{H}_j'$ be a linear map, let $\mathcal{H}_j = H^n \oplus \mathcal{H}_j'$ and let $\Gamma_j : \mathcal{D} \to \mathcal{H}_j$, $j \in \{0,1\}$ be operators given by* (3.13). *Then the following statements are equivalent:*

(i) the map $\Gamma' = (\Gamma_0' \ \ \Gamma_1')^\top : \mathcal{D} \to \mathcal{H}_0' \oplus \mathcal{H}_1'$ is surjective and

$$[y,z](b) = (\Gamma_1' y, \Gamma_0' z) - (\Gamma_0' y, \Gamma_1' z) + i(P_2'\Gamma_0' y, P_2'\Gamma_0' z), \quad y,z \in \mathcal{D}, \tag{3.14}$$

(ii) the collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a (decomposing) D-triplet for L.

Proof. The equivalence of (3.14) and the abstract Green's identity (2.7) for operators (3.13) directly follows from the Lagrange's identity (3.7).

Next assume that the statement (i) is valid and let $\mathcal{D}_1, \mathcal{D}_2$ be linear manifolds (3.10). Then for every $y \in \mathcal{D}_2$ and $z \in \mathcal{D}$ the right hand part of (3.14) is equal to 0, which implies that $\Gamma' \restriction \mathcal{D}_2 = 0$. Hence

$$\Gamma'\mathcal{D}_1 = \Gamma'(\mathcal{D}_1 + \mathcal{D}_2) = \Gamma'\mathcal{D} = \mathcal{H}_0' \oplus \mathcal{H}_1' \tag{3.15}$$

and by (3.13)

$$\Gamma\mathcal{D}_1 = (\{0\} \oplus \mathcal{H}_0') \oplus (\{0\} \oplus \mathcal{H}_1'), \quad \Gamma\mathcal{D}_2 = (H^n \oplus \{0\}) \oplus (H^n \oplus \{0\}), \tag{3.16}$$

where $\Gamma = (\Gamma_0 \;\; \Gamma_1)^\top$. Therefore $\Gamma\mathcal{D} = (H^n \oplus \mathcal{H}_0') \oplus (H^n \oplus \mathcal{H}_1') = \mathcal{H}_0 \oplus \mathcal{H}_1$.

Conversely, if $\Gamma\mathcal{D} = \mathcal{H}_0 \oplus \mathcal{H}_1$, then the equality $\Gamma'\mathcal{D} = \mathcal{H}_0' \oplus \mathcal{H}_1'$ is immediately implied by (3.13). Thus the equivalence (i) $\Longleftrightarrow$ (ii) is valid. $\square$

For every $\theta = \theta^* \in \widetilde{\mathcal{C}}(H^n)$ we put $L_\theta := L \restriction \mathcal{D}(L_\theta)$ and $L_\theta' := L \restriction \mathcal{D}(L_\theta')$, where

$$\mathcal{D}(L_\theta') := \{y \in \mathcal{D} : \widetilde{y}(0) \in \theta\}, \quad \mathcal{D}(L_\theta) := \{y \in \mathcal{D}(L_\theta') : [y,z](b) = 0,\, z \in \mathcal{D}\}. \tag{3.17}$$

It is easily seen that L_θ is a closed symmetric operator in $\mathfrak{H}$ and $L_\theta^* = L_\theta'$.

Lemma 3.5. *Let* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a decomposing D-triplet* (3.13) *for L, let* $\theta = \theta^* \in \widetilde{\mathcal{C}}(H^n)$ *and let* $\Gamma_{j,\theta} := \Gamma_j' \restriction \mathcal{D}(L_\theta'),\ j \in \{0,1\}$*.Then a collection* $\Pi_\theta := \{\mathcal{H}_0' \oplus \mathcal{H}_1', \Gamma_{0,\theta}, \Gamma_{1,\theta}\}$ *is a D-triplet for* $L_\theta'(= L_\theta^*)$*.*

Conversely, let $\Pi' = \{\mathcal{H}_0' \oplus \mathcal{H}_1', G_0, G_1\}$ *be a D-triplet for* L_θ'*. Then there exists a decomposing D-triplet* (3.13) *for L such that* $G_j = \Gamma_j' \restriction \mathcal{D}(L_\theta'),\ j \in \{0,1\}$*.*

Proof. 1) Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a D-triplet (3.13) for L. Since $\theta = \theta^*$, it follows from the first equality in (3.17) that $[y,z](0) = 0,\ y, z \in \mathcal{D}(L_\theta')$. Moreover according to Lemma 3.4 the identity (3.14) holds. Therefore by (3.7) one has

$$(L_\theta' y, z) - (y, L_\theta' z) = [y,z](b) = (\Gamma_1' y, \Gamma_0' z) - (\Gamma_0' y, \Gamma_1' z) + i(P_2'\Gamma_0' y, P_2'\Gamma_0' z), \;\; y, z \in \mathcal{D}(L_\theta').$$

Moreover (3.15) and the inclusion $\mathcal{D}_1 \subset \mathcal{D}(L_\theta')$ imply that $\Gamma'\mathcal{D}(L_\theta') = \mathcal{H}_0' \oplus \mathcal{H}_1'$. Hence Π_θ is is a D-triplet for L_θ'.

2) Conversely let $\Pi' = \{\mathcal{H}_0' \oplus \mathcal{H}_1', G_0, G_1\}$ be a D-triplet for L_θ'. Since $\mathcal{D}_1 \subset \mathcal{D}(L_\theta')$, the second relation in (3.11) yields

$$\mathcal{D}(L_\theta') = \mathcal{D}_1 + \mathcal{D}(L_\theta). \tag{3.18}$$

Introduce the operators $G_j' := G_j \restriction \mathcal{D}_1,\ j \in \{0,1\}$ and $G' := (G_0' \;\; G_1')^\top : \mathcal{D}_1 \to \mathcal{H}_0' \oplus \mathcal{H}_1'$. It follows from (3.7) and the identity (2.7) (for Π') that

$$[y,z](b) = (G_1' y, G_0' z) - (G_0' y, G_1' z) + i(P_2' G_0' y, P_2' G_0' z), \quad y, z \in \mathcal{D}_1. \tag{3.19}$$

Moreover in view of (3.18) $G'\mathcal{D}_1 = G\mathcal{D}(L_\theta') = \mathcal{H}_0' \oplus \mathcal{H}_1'$, so that the map G' is surjective.

Next assume that $y \in \mathcal{D}$. Then by (3.11) there exist $y_j \in \mathcal{D}_j,\ j \in \{1,2\}$ such that

$$y = y_1 + y_2. \tag{3.20}$$

Let $y = u_1 + u_2$ be the representation (3.20) by means of another pair $u_j \in \mathcal{D}_j$. Then by the first relation in (3.11) $y_1 - u_1 \in \mathcal{D}_0 \subset \mathcal{D}(L_\theta)$ and consequently $G_j(y_1 - u_1) = 0,\ j \in \{0,1\}$. Hence $G_j' y_1 = G_j' u_1$ which allows us to introduce the operators $\Gamma_j' : \mathcal{D} \to \mathcal{H}_j'$ by $\Gamma_j' y := G_j' y_1,\ y \in \mathcal{D},\ j \in \{0,1\}$ (here y_1 is taken from (3.20)).

Next we show that the map $\Gamma' := (\Gamma_0' \;\; \Gamma_1')^\top$ satisfies the statement (i) of Lemma 3.4. It follows from definition of Γ_j' that $\Gamma' y = G' y,\ y \in \mathcal{D}_1$ and $\Gamma' \restriction \mathcal{D}_2 = 0$. Hence

$$\Gamma'\mathcal{D} = \Gamma'(\mathcal{D}_1 + \mathcal{D}_2) = G'\mathcal{D}_1 = \mathcal{H}_0' \oplus \mathcal{H}_1'.$$

Moreover (3.19) and the second equality in (3.11) give the identity (3.14). Thus by Lemma 3.4 the operators (3.13) form a decomposing D-triplet for L. Finally the equalities

$G_j = \Gamma'_j \upharpoonright \mathcal{D}(L'_\theta)$ are implied by (3.18) and the relations $\Gamma'_j \upharpoonright \mathcal{D}_1 = G_j \upharpoonright \mathcal{D}_1$, $\Gamma'_j \upharpoonright \mathcal{D}(L_\theta) = G_j \upharpoonright \mathcal{D}(L_\theta) = 0$, $j \in \{0, 1\}$. □

Proposition 3.6. *Deficiency indices $n_\pm(L_\theta)$ of an operator L_θ do not depend on θ.*

Proof. Without loss of generality one can suppose that $n_-(L_{\theta_0}) \le n_+(L_{\theta_0})$ for some $\theta_0 = \theta_0^*$. In this case let $\Pi' = \{\mathcal{H}'_0 \oplus \mathcal{H}'_1, G_0, G_1\}$ be a D-triplet for L'_{θ_0} (such a D-triplet exists according to Proposition 2.3). Then by Lemma 3.5 there exists a decomposing D-triplet $\Pi = \{(H^n \oplus \mathcal{H}'_0) \oplus (H^n \oplus \mathcal{H}'_1), \Gamma_0, \Gamma_1\}$ for L. Now assume that θ is an arbitrary selfadjoint linear relation in H^n. Then by Lemma 3.5 there exists a D-triplet $\Pi_\theta = \{\mathcal{H}'_0 \oplus \mathcal{H}'_1, \Gamma_{0,\theta}, \Gamma_{1,\theta}\}$ for L'_θ. This and Proposition 2.3 imply that $n_-(L_\theta) = n_-(L_{\theta_0}) = \dim \mathcal{H}'_1$, $n_+(L_\theta) = n_+(L_{\theta_0}) = \dim \mathcal{H}'_0$. □

Proposition 3.6 allows us to introduce the following definition.

Definition 3.7. The numbers $n_{b+}, n_{b-} \in \mathbb{Z}_+ \cup \{\infty\}$ defined by

$$n_{b+} = n_{b+}(l) := n_+(L_\theta), \quad n_{b-} = n_{b-}(l) := n_-(L_\theta), \quad \theta = \theta^* \in \widetilde{\mathcal{C}}(H^n) \tag{3.21}$$

will be called deficiency indices of the differential expression $l = l[y]$ at the right end b of the interval $\Delta = [0, b\rangle$.

It follows from (3.17) that $\mathfrak{N}_\lambda(L_\theta)$ ($\lambda \in \mathbb{C}_\pm$) is the set of all functions $y \in \mathfrak{H}$, which are solutions of (3.4) with $\widetilde{y}(0) \in \theta$. Hence $\dim \mathfrak{N}_\lambda(L_\theta) \le \dim \theta = n \cdot \dim H$ and therefore

$$0 \le n_{b\pm} \le n \cdot \dim H. \tag{3.22}$$

It turns out that in the case $\dim H = \infty$ all values $n_{b\pm}$ satisfying (3.22) are possible. More precisely the following proposition holds.

Proposition 3.8. *For every $n \in \mathbb{N}$ and for every pair $N_+, N_- \in \mathbb{Z}_+ \cup \{\infty\}$ there exists a differential expression $l = l_H[y]$ of the order $2n$ on the half-line $[0, \infty)$ (see (3.1)) such that $\dim H = \infty$ and $n_{\infty+}(l) = N_+$, $n_{\infty-}(l) = N_-$.*

Proof. Let $l_j = l_{H_j}[y]$ be differential expressions (3.1) with coefficients $p_{k,j}$, $q_{k,j}$, $j \in \{1, 2\}$ and let $l := l_1 \oplus l_2$ be the expression $l[y] = l_{H_1 \oplus H_2}[y]$ with coefficients

$$p_k(t) = p_{k,1}(t) \oplus p_{k,2}(t), \quad q_k(t) = q_{k,1}(t) \oplus q_{k,2}(t), \quad t \in \Delta, \quad k = 0 \div n.$$

It is easily seen that in this case

$$n_{b+}(l) = n_{b+}(l_1) + n_{b+}(l_2), \quad n_{b-}(l) = n_{b-}(l_1) + n_{b-}(l_2). \tag{3.23}$$

Next assume that $l = l_H[y]$ is the expression (3.1) with $\dim H < \infty$ and let H' be an infinite-dimensional Hilbert space. Consider the expression $l' = l_{H'}[y]$ of the order $2n$ such that $n_{b\pm}(l') = 0$ (for example, one can put $l_{H'}[y] = y^{(2n)}$). It follows from (3.23) that the expressions $l \oplus l'$ and l have the same deficiency indices at the point b. Hence to prove the proposition it is enough to show that for given $n \in \mathbb{N}$ and $N_\pm \in \mathbb{Z}_+ \cup \{\infty\}$ there exist a Hilbert space H and an expression $l = l_H[y]$ such that $n_{\infty\pm}(l) = N_\pm$.

Now without loss of generality suppose that $0 \le N_- \le N_+ \le \infty$, $N_+ \ne 0$ and let $r := N_+ - N_-$(in the case $N_+ = N_- = \infty$ we put $r = 0$). Assume also that $H_0 := \mathbb{C}^2$, if $n = 1$, and $H_0 = \mathbb{C}$, if $n \ge 2$. According to [10, 12] for every $n \in \mathbb{N}$ there exist expressions $l' = l'_{H_0}[y]$ and $l'' = l''_{H_0}[y]$ of the order $2n$ on the half-line $[0, \infty)$ such that

$$n_{\infty+}(l') = n_{\infty-}(l') = 1, \quad n_{\infty+}(l'') = 1, \quad n_{\infty-}(l'') = 0. \tag{3.24}$$

Let p'_k, q'_k and p''_k, q''_k be coefficients of the expressions l' and l'' respectively. Introduce Hilbert spaces $\widetilde{H} := \bigoplus_{j=1}^{N_-} H_0$, $\hat{H} := \bigoplus_{j=1}^{r} H_0$ and operator-functions $\widetilde{p}_k(\cdot), \widetilde{q}_k(\cdot) : [0, \infty) \to$

$[\widetilde{H}]$ and $\hat{p}_k(\cdot), \hat{q}_k(\cdot) : [0,\infty) \to [\hat{H}]$ by

$$\widetilde{p}_k(t)\widetilde{h} = \{p'_k(t)h_j\}_{j=1}^{N_-}, \quad \widetilde{q}_k(t)\widetilde{h} = \{q'_k(t)h_j\}_{j=1}^{N_-}, \quad \widetilde{h} = \{h_j\}_1^{N_-} \in \widetilde{H};$$
$$\hat{p}_k(t)\hat{h} = \{p''_k(t)h_j\}_{j=1}^{r}, \quad \hat{q}_k(t)\hat{h} = \{q''_k(t)h_j\}_{j=1}^{r}, \quad \hat{h} = \{h_j\}_1^{r} \in \hat{H}, \quad t \in [0,\infty).$$

Denote by $\widetilde{l} = l_{\widetilde{H}}[y]$ and $\hat{l} = l_{\hat{H}}[y]$ differential expressions (3.1) with coefficients $\widetilde{p}_k$, $\widetilde{q}_k$ and $\hat{p}_k$, $\hat{q}_k$ respectively. Taking (3.24) into account one can easily check that

$$n_{\infty+}(\widetilde{l}) = n_{\infty-}(\widetilde{l}) = N_-, \quad n_{\infty+}(\hat{l}) = r, \quad n_{\infty-}(\hat{l}) = 0. \tag{3.25}$$

Finally in view of (3.23) and (3.25) the expression $l := \widetilde{l} \oplus \hat{l}$ satisfies the required equality $n_{\infty\pm}(l) = N_\pm$. □

The following proposition is an analog of Proposition 2.3.

Proposition 3.9. *If* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ $\big(\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}\big)$ *is a decomposing D-boundary triplet (boundary triplet) for L, then*

$$\dim \mathcal{H}'_1 = n_{b-} \le n_{b+} = \dim \mathcal{H}'_0 \quad (n_{b-} = n_{b+} = \dim \mathcal{H}'). \tag{3.26}$$

Conversely for every differential expression (3.1) *with* $n_{b-} \le n_{b+}$ $(n_{b-} = n_{b+})$ *there exists a decomposing D-boundary triplet (boundary triplet)* Π *for L.*

Proof. The statements of the proposition are immediately implied by Lemma 3.5 and Proposition 2.3. □

Recall that the expression (3.1) is called regular, if its coefficients are defined on the finite segment $\Delta = [0, b]$. It is clear that for a regular expression $n_{b-} = n_{b+} = 2n \dim H$ and the collection $\Pi = \{H^n \oplus H^n, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 y = \{y^{(2)}(0), y^{(2)}(b)\}, \quad \Gamma_1 y = \{-y^{(1)}(0), y^{(1)}(b)\}, \quad y \in \mathcal{D}, \tag{3.27}$$

is a decomposing boundary triplet for L.

In the next proposition we construct in the explicit form a decomposing D-triplet for an arbitrary expression (3.1) with $n_{b-} \le n_{b+} \le \infty$.

Proposition 3.10. *Let* $l[y]$ *be the expression* (3.1) *with* $n_{b-} \le n_{b+} \le \infty$. *In the case* $n_{b-} = n_{b+} = \infty$ *denote by* m *an arbitrary (finite or infinite) nonnegative integer and let* $m = n_{b+} - n_{b-}$ *in the case* $n_{b-} < \infty$. *Moreover let* $\mathcal{H}'_1 = \mathbb{C}^{n_{b-}}$, $\mathcal{H}'_2 = \mathbb{C}^m$ *and* $\mathcal{H}'_0 = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ *(here we put* $\mathbb{C}^\infty = l_2$*). Then there exist sequences of functions* $\{f_j\}_1^{n_{b-}}$, $\{g_j\}_1^{n_{b-}}$ *and* $\{h_j\}_1^m$ $(f_j, g_j, h_j \in \mathcal{D})$ *such that the Hilbert spaces* $\mathcal{H}_j = H^n \oplus \mathcal{H}'_j$ *and the linear maps* $\Gamma_j : \mathcal{D} \to \mathcal{H}_j$, $j \in \{0, 1\}$,

$$\Gamma_0 y := \{y^{(2)}(0), \{[y, f_j](b)\}_{j=1}^{n_{b-}}, \{[y, h_j](b)\}_{j=1}^{m}\} \, (\in H^n \oplus (\mathcal{H}'_1 \oplus \mathcal{H}'_2)), \tag{3.28}$$
$$\Gamma_1 y := \{-y^{(1)}(0), \{[y, g_j](b)\}_{j=1}^{n_{b-}}\} (\in H^n \oplus \mathcal{H}'_1), \quad y \in \mathcal{D} \tag{3.29}$$

form a decomposing D-triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *for L.*

Proof. Let $\widetilde{\mathfrak{N}}_{\pm i} = \mathrm{Ker}(L^*_\theta \pm i)$ be defect subspaces of the operator L_θ (see (3.17)) and let $\widetilde{\mathfrak{N}}'_i$, $\widetilde{\mathfrak{N}}''_i$ be subspaces in $\widetilde{\mathfrak{N}}_i$ such that $\dim \widetilde{\mathfrak{N}}'_i = n_{b-}$, $\dim \widetilde{\mathfrak{N}}''_i = m$ and $\widetilde{\mathfrak{N}}_i = \widetilde{\mathfrak{N}}'_i \oplus \widetilde{\mathfrak{N}}''_i$. Then by von Neumann's formula the following decomposition holds

$$\mathcal{D}(L^*_\theta) = \mathcal{D}(L_\theta) \dotplus \widetilde{\mathfrak{N}}'_i \dotplus \widetilde{\mathfrak{N}}''_i \dotplus \widetilde{\mathfrak{N}}_{-i}. \tag{3.30}$$

Next assume that $\{v'_j\}_1^{n_{b-}}$, $\{v''_j\}_1^{m}$ and $\{u_j\}_1^{n_{b-}}$ are orthonormal bases in $\widetilde{\mathfrak{N}}'_i$, $\widetilde{\mathfrak{N}}''_i$ and $\widetilde{\mathfrak{N}}_{-i}$ respectively. Our aim is to show that the statement of the proposition is valid with the sequences $\{f_j\}_1^{n_{b-}}$, $\{g_j\}_1^{n_{b-}}$ and $\{h_j\}_1^m$ given by

$$f_j = \tfrac{1}{2i}(u_j - v'_j), \quad g_j = \tfrac{1}{2}(u_j + v'_j), \quad h_j = \tfrac{1}{\sqrt{2}} v''_j. \tag{3.31}$$

Denote by P', P'' and Q the projectors in $\mathcal{D}(L_\theta^*)$ onto the subspaces $\widetilde{\mathfrak{N}}_i'$, $\widetilde{\mathfrak{N}}_i''$ and $\widetilde{\mathfrak{N}}_{-i}$ respectively, corresponding to the decomposition (3.30). It follows from the Lagrange's identity (3.7) that

$$[y,z](b) = 2i\big((P'y,P'z)+(P''y,P''z)-(Qy,Qz)\big), \quad y,z\in\mathcal{D}(L_\theta^*). \tag{3.32}$$

Therefore

$$[y,v_j'](b)=2i(P'y,v_j'), \quad [y,v_j''](b)=2i(P''y,v_j''), \quad [y,u_j](b)=-2i(Qy,u_j), \quad y\in\mathcal{D}(L_\theta^*)$$

and the equalities (3.31) yield

$$\{[y,f_j](b)\}_1^{n_{b-}} = F_y'+G_y, \quad \{[y,g_j](b)\}_1^{n_{b-}} = i(F_y'-G_y), \quad \{[y,h_j](b)\}_1^{m} = i\sqrt{2}F_y'', \tag{3.33}$$

where the sequences F_y', $G_y\in\mathcal{H}_1'$ and $F_y''\in\mathcal{H}_2'$ are defined by

$$F_y'=\{(P'y,v_j')\}_{j=1}^{n_{b-}}, \quad F_y''=\{(P''y,v_j'')\}_{j=1}^{m}, \quad G_y=\{(Qy,u_j)\}_{j=1}^{n_{b-}}, \quad y\in\mathcal{D}(L_\theta^*). \tag{3.34}$$

Next we show that the operators $\Gamma_0':\mathcal{D}\to\mathcal{H}_1'\oplus\mathcal{H}_2'$ and $\Gamma_1':\mathcal{D}\to\mathcal{H}_1'$ given by

$$\Gamma_0'y=\big\{\{[y,f_j](b)\}_{j=1}^{n_{b-}},\ \{[y,h_j](b)\}_{j=1}^{m}\big\}, \quad \Gamma_1'y=\{[y,g_j](b)\}_{j=1}^{n_{b-}}, \quad y\in\mathcal{D} \tag{3.35}$$

satisfy the identity (3.14).

It follows from (3.33) that

$$\Gamma_0'y=\{F_y'+G_y,\ i\sqrt{2}F_y''\}(\in\mathcal{H}_1'\oplus\mathcal{H}_2'), \quad \Gamma_1'y=i(F_y'-G_y)(\in\mathcal{H}_1'), \quad y\in\mathcal{D}(L_\theta^*). \tag{3.36}$$

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be linear manifolds (3.10). Then $\Gamma_j'\upharpoonright\mathcal{D}_2=0,\ j\in\{0,1\}$ and in view of (3.11) it is sufficient to prove (3.14) only for all $y,z\in\mathcal{D}_1$. Since $\mathcal{D}_1\subset\mathcal{D}(L_\theta^*)$, the equalities (3.36) hold for all $y\in\mathcal{D}_1$. Moreover by (3.34) for every $y,z\in\mathcal{D}(L_\theta^*)$ one has

$$(F_y',F_z')=(P'y,P'z), \quad (F_y'',F_z'')=(P''y,P''z), \quad (G_y,G_z)=(Qy,Qz). \tag{3.37}$$

Now the direct calculation with taking (3.32) and (3.36) into account leads to the required identity (3.14) for all $y,z\in\mathcal{D}_1$.

Let further $h=\{h_j\}_1^{n_{b-}}$, $k=\{k_j\}_1^{n_{b-}}\in\mathcal{H}_1'$ and $s=\{s_j\}_1^m\in\mathcal{H}_2'$. Then the function

$$y:=\Sigma_{j=1}^{n_{b-}}h_jv_j'+\Sigma_{j=1}^{m}s_jv_j''+\Sigma_{j=1}^{n_{b-}}k_ju_j, \quad y\in\mathcal{D}(L_\theta^*)$$

satisfies the equalities $F_y'=h$, $F_y''=s$ and $G_y=k$, which in view of (3.36) shows that the map $\Gamma'=(\Gamma_0'\ \ \Gamma_1')^\top$ is surjective. Finally combining (3.35) with (3.28), (3.29) and taking Lemma 3.4 into account we arrive at the required statement. $\square$

Remark 3.11. 1) Let L_θ be a symmetric extension (3.17). Since $n_\pm=\dim(\mathcal{D}(L_\theta)/\mathcal{D}_0)+n_\pm(L_\theta)$ and by (3.17) $\dim(\mathcal{D}(L_\theta)/\mathcal{D}_0)=\dim\theta=n\cdot\dim H$, it follows that

$$n_\pm=n\cdot\dim H+n_\pm(L_\theta)=n\cdot\dim H+n_{b\pm}. \tag{3.38}$$

Note that in the case $\dim H<\infty$ the statement of Proposition 3.6 is a well-known consequence of the first equality in (3.38).

2) The notion of a boundary triplet for a regular differential expression was first introduced in [21] by means of the equalities (3.27). Afterwards various constructions of boundary triplets for the expression (3.1) on the half-line $[0,\infty)$ with $n_+=n_-$ were suggested in [11, 9, 16]. Note that in these papers some additional restrictions are imposed on the expression $l[y]$. Observe also that for the case $\dim H<\infty$ and $n_+=n_-$ formulas (3.28), (3.29) were obtained in [16].

3.3. m**-functions.** For a pair of Hilbert spaces $\mathcal{K}$, H denote by $L_2'[\mathcal{K},H]$ the set of all operator-functions $Y(\cdot):\Delta\to[\mathcal{K},H]$ such that $Y(t)h\in\mathfrak{H}$ for all $h\in\mathcal{K}$.

Theorem 3.12. *Let $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a decomposing D-triplet* (3.13) *and let*

$$\gamma_+(\lambda)=(\gamma_1(\lambda)\ \ \gamma_{2+}(\lambda)):H^n\oplus\mathcal{H}_0'\to\mathfrak{H},\quad \lambda\in\mathbb{C}_+,\tag{3.39}$$

$$\gamma_-(z)=(\gamma_1(z)\ \ \gamma_{2-}(z)):H^n\oplus\mathcal{H}_1'\to\mathfrak{H},\quad z\in\mathbb{C}_-,\tag{3.40}$$

$$M_+(\lambda)=\begin{pmatrix} m(\lambda) & M_{2+}(\lambda)\\ M_{3+}(\lambda) & M_{4+}(\lambda)\end{pmatrix}:H^n\oplus\mathcal{H}_0'\to H^n\oplus\mathcal{H}_1',\quad\lambda\in\mathbb{C}_+,\tag{3.41}$$

$$M_-(z)=\begin{pmatrix} m(z) & M_{2-}(z)\\ M_{3-}(z) & M_{4-}(z)\end{pmatrix}:H^n\oplus\mathcal{H}_1'\to H^n\oplus\mathcal{H}_0',\quad z\in\mathbb{C}_-,\tag{3.42}$$

be the block-matrix representations of the corresponding γ-fields and Weyl functions (see (2.10)–(2.12)*). Then:*

1) the extension $A_0\in\mathrm{Ex}_{L_0}$ (see Proposition 2.4, 3)) has the domain

$$\mathcal{D}(A_0)=\{y\in\mathcal{D}:y^{(2)}(0)=0,\ \Gamma_0'y=0\};\tag{3.43}$$

2) for every $\lambda\in\mathbb{C}_+\cup\mathbb{C}_-$ there exists the unique operator function $v_0(\cdot,\lambda)\in L_2'[H^n,H]$, satisfying the equation (3.4)*and the boundary conditions*

$$v_0^{(2)}(0,\lambda)=I_{H^n},\quad\lambda\in\mathbb{C}_+\cup\mathbb{C}_-,\tag{3.44}$$

$$\Gamma_0'(v_0(t,\lambda)\hat h)=0,\quad\lambda\in\mathbb{C}_+;\quad P_1'\Gamma_0'(v_0(t,z)\hat h)=0,\quad z\in\mathbb{C}_-,\quad\hat h\in H^n.\tag{3.45}$$

Moreover for every $\lambda\in\mathbb{C}_+$ $(z\in\mathbb{C}_-)$ there exists the unique operator function $u_+(\cdot,\lambda)\in L_2'[\mathcal{H}_0',H]$ $(u_-(\cdot,z)\in L_2'[\mathcal{H}_1',H])$, satisfying (3.4) *and the boundary conditions*

$$u_+^{(2)}(0,\lambda)=0,\quad\Gamma_0'(u_+(t,\lambda)h_0')=h_0',\quad\lambda\in\mathbb{C}_+,\quad h_0'\in\mathcal{H}_0';\tag{3.46}$$

$$u_-^{(2)}(0,z)=0,\quad P_1'\Gamma_0'(u_-(t,z)h_1')=h_1',\quad z\in\mathbb{C}_-,\quad h_1'\in\mathcal{H}_1'.\tag{3.47}$$

For every fixed $t\in\Delta$ the operator-functions $v_0(t,\cdot)$, $u_+(t,\cdot)$ and $u_-(t,\cdot)$ are holomorphic on their domains.

3) formulas (3.39) *and* (3.40) *define the holomorphic operator-functions $\gamma_1(\cdot):\mathbb{C}_+\cup\mathbb{C}_-\to[H^n,\mathfrak{H}]$, $\gamma_{2+}(\cdot):\mathbb{C}_+\to[\mathcal{H}_0',\mathfrak{H}]$ and $\gamma_{2-}(\cdot):\mathbb{C}_-\to[\mathcal{H}_1',\mathfrak{H}]$ satisfying the relations*

$$(\gamma_1(\lambda)\hat h)(t)=v_0(t,\lambda)\hat h,\quad\lambda\in\mathbb{C}_+\cup\mathbb{C}_-,\quad\hat h\in H^n;\tag{3.48}$$

$$(\gamma_{2+}(\lambda)h_0')(t)=u_+(t,\lambda)h_0',\quad\lambda\in\mathbb{C}_+,\quad h_0'\in\mathcal{H}_0';\tag{3.49}$$

$$(\gamma_{2-}(z)h_1')(t)=u_-(t,z)h_1',\quad z\in\mathbb{C}_-,\quad h_1'\in\mathcal{H}_1'.\tag{3.50}$$

4) the block-matrix representations (3.41) *and* (3.42) *generate the holomorphic operator-function $m(\cdot):\mathbb{C}_+\cup\mathbb{C}_-\to[H^n]$, which can be also defined in terms of the "canonical" solutions* (3.5) *by the following statement:*

(i) there exists the unique operator-function $m(\cdot):\mathbb{C}_+\cup\mathbb{C}_-\to[H^n]$ such that for every $\lambda\in\mathbb{C}_+\cup\mathbb{C}_-$ the operator -function

$$v_0(t,\lambda):=-c(t,\lambda)m(\lambda)+s(t,\lambda)\tag{3.51}$$

(of the variable t) belongs to $L_2'[H^n,H]$ and satisfies the boundary condition (3.45)*.*

Moreover formula (3.41) *gives the holomorphic operator-functions $M_{2+}(\lambda)(\in[\mathcal{H}_0',H^n])$ and $M_{3+}(\lambda)(\in[H^n,\mathcal{H}_1'])$ $(\lambda\in\mathbb{C}_+)$, which can be also defined by the following statement:*

(ii) there exists the unique operator-function $M_{2+}(\cdot):\mathbb{C}_+\to[\mathcal{H}_0',H^n]$ $(M_{3+}(\cdot):\mathbb{C}_+\to[H^n,\mathcal{H}_1'])$ such that the operator-function

$$u_+(t,\lambda):=-c(t,\lambda)M_{2+}(\lambda),\quad\lambda\in\mathbb{C}_+\quad(u_-(t,z):=-c(t,z)M_{3+}^*(\overline{z}),\ \ z\in\mathbb{C}_-)\tag{3.52}$$

belongs to $L_2'[\,\cdot,H]$ and satisfies the second equality in (3.46) *(respectively in* (3.47)*).*

Finally the operator function $M_{4+}(\cdot):\mathbb{C}_+\to[\mathcal{H}_0',\mathcal{H}_1']$ *in* (3.41) *can be defined via*

$$M_{4+}(\lambda)h_0'=\Gamma_1'(u_+(t,\lambda)h_0'),\quad h_0'\in\mathcal{H}_0',\quad \lambda\in\mathbb{C}_+. \tag{3.53}$$

5) $m(\cdot)$ *is a uniformly strict Nevanlinna function satisfying the relation*

$$m(\mu)-m^*(\lambda)=(\mu-\overline{\lambda})\int_0^b v_0^*(t,\lambda)v_0(t,\mu)\,dt,\quad \mu,\lambda\in\mathbb{C}_+. \tag{3.54}$$

The integral in (3.54) *converges strongly, that is*

$$\int_0^b v_0^*(t,\lambda)v_0(t,\mu)\,dt=s-\lim_{\eta\uparrow b}\int_0^\eta v_0^*(t,\lambda)v_0(t,\mu)\,dt. \tag{3.55}$$

Proof. The statement 1) is immediately implied by (3.13).

2) Let

$$S_+(t,\lambda)=(v_0(t,\lambda)\ \ u_+(t,\lambda)):H^n\oplus\mathcal{H}_0'\to H,\quad \lambda\in\mathbb{C}_+, \tag{3.56}$$

$$S_-(t,z)=(v_0(t,z)\ \ u_-(t,z)):H^n\oplus\mathcal{H}_1'\to H,\quad z\in\mathbb{C}_- \tag{3.57}$$

be the operator solutions of (3.4) with the initial data

$$\widetilde{S}_+(0,\lambda)=\begin{pmatrix}v_0^{(1)}(0,\lambda) & u_+^{(1)}(0,\lambda)\\ v_0^{(2)}(0,\lambda) & u_+^{(2)}(0,\lambda)\end{pmatrix}:=\begin{pmatrix}-m(\lambda) & -M_{2+}(\lambda)\\ I_{H^n} & 0\end{pmatrix}, \tag{3.58}$$

$$\widetilde{S}_-(0,z)=\begin{pmatrix}v_0^{(1)}(0,z) & u_-^{(1)}(0,z)\\ v_0^{(2)}(0,z) & u_-^{(2)}(0,z)\end{pmatrix}:=\begin{pmatrix}-m(z) & -M_{2-}(z)\\ I_{H^n} & 0\end{pmatrix}. \tag{3.59}$$

It follows from (3.5) and (3.58), (3.59) that

$$v_0(t,\lambda)=-c(t,\lambda)m(\lambda)+s(t,\lambda),\quad \lambda\in\mathbb{C}_+\cup\mathbb{C}_-, \tag{3.60}$$

$$u_+(t,\lambda)=-c(t,\lambda)M_{2+}(\lambda),\quad \lambda\in\mathbb{C}_+;\qquad u_-(t,z)=-c(t,z)M_{2-}(z),\quad z\in\mathbb{C}_-. \tag{3.61}$$

Next we show that the operator-functions v_0 and $u_\pm$, defined by (3.56) and (3.57) possess the required properties.

For fixed $h_0\in\mathcal{H}_0,\ \lambda\in\mathbb{C}_+$ and $h_1\in\mathcal{H}_1,\ z\in\mathbb{C}_-$ consider the vector-functions

$$y_+=y_+(t,h_0,\lambda):=(\gamma_+(\lambda)h_0)(t),\quad y_-=y_-(t,h_1,z):=(\gamma_-(z)h_1)(t). \tag{3.62}$$

It follows from (2.10)–(2.12) that $y_+\in\mathfrak{N}_\lambda(L_0),\ y_-\in\mathfrak{N}_z(L_0)$ and

$$\Gamma_0y_+=h_0,\quad \Gamma_1y_+=M_+(\lambda)h_0;\quad P_1\Gamma_0y_-=h_1,\quad (\Gamma_1+iP_2\Gamma_0)y_-=M_-(z)h_1. \tag{3.63}$$

This and (3.13) imply that

$$y_+^{(2)}(0,h_0,\lambda)=\hat{h},\quad y_+^{(1)}(0,h_0,\lambda)=-m(\lambda)\hat{h}-M_{2+}(\lambda)h_0',\quad h_0=\{\hat{h},h_0'\}\in H^n\oplus\mathcal{H}_0',$$
$$y_-^{(2)}(0,h_1,z)=\hat{g},\quad y_-^{(1)}(0,h_1,z)=-m(z)\hat{g}-M_{2-}(z)h_1',\quad h_1=\{\hat{g},h_1'\}\in H^n\oplus\mathcal{H}_1'.$$

Therefore y_+ and y_- are solutions of (3.4) with the initial data $\widetilde{y}_+(0,h_0,\lambda)=\widetilde{S}_+(0,\lambda)h_0$, $\widetilde{y}_-(0,h_1,z)=\widetilde{S}_-(0,z)h_1$, so that $y_+=S_+(t,\lambda)h_0,\ y_-=S_-(t,z)h_1$ and by (3.62)

$$S_+(t,\lambda)h_0=(\gamma_+(\lambda)h_0)(t),\quad h_0\in\mathcal{H}_0;\quad S_-(t,z)h_1=(\gamma_-(z)h_1)(t),\quad h_1\in\mathcal{H}_1. \tag{3.64}$$

This and (3.56) show that the operator-functions v_0 and $u_\pm$ are solutions of (3.4), belonging to $L_2'[\cdot,H]$. Moreover in view of (3.63) for all $\hat{h}\in H^n,\ h_0'\in\mathcal{H}_0'$ and $h_1'\in\mathcal{H}_0'$

$$\Gamma_0'(v_0(t,\lambda)\hat{h})=0,\quad \Gamma_0'(u_+(t,\lambda)h_0')=h_0', \tag{3.65}$$

$$\Gamma_1'(v_0(t,\lambda)\hat{h})=M_{3+}(\lambda)\hat{h},\quad \Gamma_1'(u_+(t,\lambda)h_0')=M_{4+}(\lambda)h_0', \tag{3.66}$$

$$P_1'\Gamma_0'(v_0(t,z)\hat{h})=0,\qquad P_1'\Gamma_0'(u_-(t,z)h_1')=h_1', \tag{3.67}$$

$$(\Gamma_1'+iP_2'\Gamma_0')(v_0(t,z)\hat{h})=M_{3-}(z)\hat{h},\quad (\Gamma_1'+iP_2'\Gamma_0')(u_-(t,z)h_1')=M_{4-}(z)h_1'. \tag{3.68}$$

Now combining (3.58), (3.59), (3.65) and (3.67), we arrive at (3.44)–(3.47).

The uniqueness of the operator-functions v_0 and $u_\pm$ follows from the inclusion $0 \in \rho(\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(L_0)) \cap \rho(P_1\Gamma_0 \upharpoonright \mathfrak{N}_z(L_0))$. Finally the holomorphy of these functions is a consequence of the relations (3.60) and (3.61).

3) Formulas (3.48)–(3.50) are immediately implied by (3.64) and (3.56), (3.57).

The statement 4) follows from the statement 2) and formulas (3.60) and (3.61), where by the first equality in (2.17) $M_{2-}(z) = M_{3+}^*(\overline{z}),\ z \in \mathbb{C}_-$.

5) Let $\gamma(\cdot)$ and $M(\cdot)$ be operator functions defined by the representations (2.13) and (2.14), (2.15). Then $\gamma_1(\lambda) = \gamma(\lambda) \upharpoonright H^n$, $m(\lambda) = P_{H^n} M(\lambda) \upharpoonright H^n$ and in view of Proposition 2.5 $m(\cdot)$ is a uniformly strict Nevanlinna function obeying the identity

$$m(\mu) - m^*(\lambda) = (\mu - \overline{\lambda})\, \gamma_1^*(\lambda)\gamma_1(\mu), \quad \mu, \lambda \in \mathbb{C}_+. \tag{3.69}$$

Let us show that

$$\gamma_1^*(\lambda) f = \int_0^b v_0^*(t,\lambda) f(t)\, dt := \lim_{\eta \uparrow b} \int_0^\eta v_0^*(t,\lambda) f(t)\, dt, \quad f = f(t) \in \mathfrak{H}. \tag{3.70}$$

First assume that $f(t) = 0,\ t \in (\eta, b)$ for some $\eta \in (0,b)$. Then by (3.48)

$$\begin{aligned}(\gamma_1(\lambda)\hat{h}, f)_{\mathfrak{H}} &= \int_0^b (v_0(t,\lambda)\hat{h}, f(t))\, dt \\ &= \int_0^b (\hat{h}, v_0^*(t,\lambda) f(t))\, dt = (\hat{h}, \int_0^b v_0^*(t,\lambda) f(t)\, dt)_{H^n}, \quad \hat{h} \in H^n,\end{aligned}$$

which gives (3.70) for finite functions $f \in \mathfrak{H}$. Next suppose that $f \in \mathfrak{H}$ is an arbitrary function and let $f_\eta := \chi_{[0,\eta]} f$ (here $\chi_{[0,\eta]}$ is an indicator of the segment $[0,\eta]$, $\eta < b$). Then $\lim_{\eta\uparrow b} ||f_\eta - f||_{\mathfrak{H}} = 0$ and therefore

$$\gamma_1^*(\lambda) f = \lim_{\eta\uparrow b} \gamma_1^*(\lambda) f_\eta = \lim_{\eta\uparrow b} \int_0^\eta v_0^*(t,\lambda) f(t)\, dt = \int_0^b v_0^*(t,\lambda) f(t)\, dt.$$

Now using (3.48) and taking (3.70) into account, one obtains

$$\gamma_1^*(\lambda)\gamma_1(\mu)\hat{h} = \lim_{\eta\uparrow b} \int_0^\eta (v_0^*(t,\lambda) v_0(t,\mu)\hat{h})\, dt = \lim_{\eta\uparrow b} \left(\int_0^\eta v_0^*(t,\lambda) v_0(t,\mu)\, dt \right) \hat{h}, \quad \hat{h} \in H^n.$$

This and (3.69) lead to the relations (3.54) and (3.55). □

In the case of a decomposing boundary triplet the statements of Theorem 3.12 can be rather simplified. Namely the following corollary is obvious.

Corollary 3.13. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a decomposing boundary triplet* (3.13) *for L (that is $\mathcal{H}_0' = \mathcal{H}_1' := \mathcal{H}'$), let $A_0 = L \upharpoonright \operatorname{Ker} \Gamma_0$ and let*

$$\gamma(\lambda) = (\gamma_1(\lambda)\ \ \gamma_2(\lambda)) : H^n \oplus \mathcal{H}' \to \mathfrak{H}, \quad \lambda \in \rho(A_0), \tag{3.71}$$

$$M(\lambda) = \begin{pmatrix} m(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}' \to H^n \oplus \mathcal{H}', \quad \lambda \in \rho(A_0) \tag{3.72}$$

be the corresponding γ-field and Weyl function (see Remark 2.6). Then

1) A_0 is a selfadjoint extension with the domain (3.43)*;*

2) for every $\lambda \in \rho(A_0)$ there exists the unique operator-functions $v_0(\cdot,\lambda) \in L_2'[H^n, H]$ and $u(\cdot,\lambda) \in L_2'[\mathcal{H}', H]$, satisfying (3.4) *and the boundary conditions*

$$v_0^{(2)}(0,\lambda) = I_{H^n}, \quad \Gamma_0'(v_0(t,\lambda)\hat{h}) = 0, \quad \hat{h} \in H^n, \quad \lambda \in \rho(A_0), \tag{3.73}$$

$$u^{(2)}(0,\lambda) = 0, \quad \Gamma_0'(u(t,\lambda)h') = h', \quad h' \in \mathcal{H}', \quad \lambda \in \rho(A_0); \tag{3.74}$$

3) the operator-functions $\gamma_1(\lambda) (\in [H^n, \mathfrak{H}])$ and $\gamma_2(\lambda) (\in [\mathcal{H}', \mathfrak{H}])$ satisfy the relations

$$(\gamma_1(\lambda)\hat{h})(t) = v_0(t,\lambda)\hat{h},\ \ (\gamma_2(\lambda)h')(t) = u(t,\lambda)h',\ \ \hat{h} \in H^n,\ \ h' \in \mathcal{H}',\ \lambda \in \rho(A_0); \tag{3.75}$$

4) the operator functions $m(\lambda)(\in [H^n])$, $M_2(\lambda)(\in [\mathcal{H}', H^n])$ and $M_3(\lambda)(\in [H^n, \mathcal{H}'])$, $\lambda \in \rho(A_0)$ are uniquely defined by the relations

$$v_0(t,\lambda) := -c(t,\lambda)m(\lambda) + s(t,\lambda) \in L_2'[H^n, H]; \quad \Gamma_0'(v_0(t,\lambda)\hat{h}) = 0, \quad \hat{h} \in H^n, \tag{3.76}$$

$$u(t,\lambda) := -c(t,\lambda)M_2(\lambda) \in L_2'[\mathcal{H}', H]; \quad \Gamma_0'(u(t,\lambda)h') = h', \quad h' \in \mathcal{H}', \tag{3.77}$$

$$M_3(\lambda) = M_2^*(\overline{\lambda}), \quad \lambda \in \rho(A_0); \tag{3.78}$$

5) $m(\lambda)$ is a uniformly strict Nevanlinna function obeying (3.54) *for all $\mu, \lambda \in \rho(A_0)$.*

Definition 3.14. The operator function $m(\cdot)$, defined in Theorem 3.12, will be called a m-function, corresponding to the extension A_0.

In the next proposition we show that the m-function $m(\cdot)$ coincides with the Weyl function for a D-triplet of some symmetric extension $\hat{A} \in \mathrm{Ex}_{L_0}$.

Proposition 3.15. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet* (3.13) *for L. Then*

1) the operator $\hat{A} := L \restriction \mathcal{D}(\hat{A})$ with the domain $\mathcal{D}(\hat{A}) := \{y \in \mathcal{D} : \widetilde{y}(0) = 0,\ \Gamma_0'y = 0\}$ is a closed symmetric extension of L_0 and $\hat{A}^* = L \restriction \mathcal{D}(\hat{A}^*)$, where

$$\mathcal{D}(\hat{A}^*) = \{y \in \mathcal{D} : P_1'\Gamma_0'y = 0\}; \tag{3.79}$$

2) the maps $\hat{\Gamma}_0 : \mathcal{D}(\hat{A}^*) \to H^n \oplus \mathcal{H}_2'$ and $\hat{\Gamma}_1 : \mathcal{D}(\hat{A}^*) \to H^n$, defined via

$$\hat{\Gamma}_0 y = \{y^{(2)}(0),\, \Gamma_0'y\}\,(\in H^n \oplus \mathcal{H}_2'), \quad \hat{\Gamma}_1 y = -y^{(1)}(0)\,(\in H^n), \quad y \in \mathcal{D}(\hat{A}^*) \tag{3.80}$$

form a D-triplet $\hat{\Pi} = \{(H^n \oplus \mathcal{H}_2') \oplus H^n, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ for $\hat{A}^*$;

3) the extension $\hat{A}^* \restriction \mathrm{Ker}\,\hat{\Gamma}_0$ coincides with A_0 (see Theorem 3.12, 1)) and the corresponding γ-fields (2.13), (2.10) *and Weyl function* (2.14), (2.15) *for $\hat{\Pi}$ are*

$$(\hat{\gamma}(\lambda)\hat{h})(t) = v_0(t,\lambda)\hat{h},\ \ (\hat{\gamma}_-(z)\hat{h})(t) = v_0(t,z)\hat{h},\ \ \hat{h} \in H^n,\ \ \lambda \in \mathbb{C}_+,\ \ z \in \mathbb{C}_-, \tag{3.81}$$

$$\hat{M}(\lambda) = m(\lambda), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{3.82}$$

Proof. 1) It is easily seen that $\hat{A} = \widetilde{A}_\theta$ with $\theta = \{0\} \oplus (\{0\} \oplus \mathcal{H}_1')$ (see (2.8)). Therefore by Proposition 2.4, 2) $\hat{A}^* = \widetilde{A}_{\theta^\times}$, where $\theta^\times = (H^n \oplus \mathcal{H}_2') \oplus (H^n \oplus \mathcal{H}_1')$. This yields (3.79).

The statement 2) is obvious. The equalities (3.81) and (3.82) are implied by (3.48) and (3.41) respectively. □

Corollary 3.16. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a decomposing boundary triplet* (3.13) *for L (with $\mathcal{H}_0' = \mathcal{H}_1' := \mathcal{H}'$) and let $\hat{A}$ be an extension, defined in Proposition 3.15. Then*

1) $\mathcal{D}(\hat{A}^*) = \{y \in \mathcal{D} : \Gamma_0'y = 0\}$ and the collection $\hat{\Pi} = \{H^n, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ with

$$\hat{\Gamma}_0 y = y^{(2)}(0), \quad \hat{\Gamma}_1 y = -y^{(1)}(0), \quad y \in \mathcal{D}(\hat{A}^*) \tag{3.83}$$

is a boundary triplet for $\hat{A}^*$;

2) the (selfadjoint) extension $A_0 := \hat{A}^* \restriction \mathrm{Ker}\,\hat{\Gamma}_0$ is defined by (3.43) *and the corresponding γ-field and Weyl function for $\hat{\Pi}$ are*

$$(\hat{\gamma}(\lambda)\hat{h})(t) = v_0(t,\lambda)\hat{h}, \quad \hat{h} \in H^n; \quad \hat{M}(\lambda) = m(\lambda), \quad \lambda \in \rho(A_0). \tag{3.84}$$

Remark 3.17. 1) For a scalar differential operator ($\dim H = 1$) with equal deficiency indices $n_+ = n_-$ the m-function $m(\cdot)$ coincides with the classical characteristic (Titchmarsh-Weyl) function for decomposing boundary conditions (see for instance [15, 20, 5]). In the case $\dim H \leq \infty$ the characteristic function was defined by the relation (3.76) in [7] for regular expressions of the second order and in [11] for quasi-regular expressions on the half-line $[0,\infty)$, i.e, for expressions (3.1) such that $\int_0^\infty ||c(t,\lambda)||^2\,dt < \infty$ and $\int_0^\infty ||s(t,\lambda)||^2\,dt < \infty,\ \lambda \in \mathbb{C}$ (clearly in this case $n_{\infty+} = n_{\infty-}$).

2) Assume that the expression $l[y]$ is quasiregular and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a decomposing boundary triplet (3.13) for L. Define (holomorphic) operator functions $N_j(\cdot) : \rho(A_0) \to [H^n, \mathcal{H}']$ and $S_j(\cdot) : \rho(A_0) \to [H^n, \mathcal{H}']$ by

$$N_j(\lambda)\hat{h} := \Gamma_j'(c(t,\lambda)\hat{h}), \quad S_j(\lambda)\hat{h} := \Gamma_j'(s(t,\lambda)\hat{h}), \quad \hat{h} \in H^n, \quad \lambda \in \rho(A_0), \quad j \in \{0,1\}.$$

It is not difficult to prove that $0 \in \rho(N_0(\lambda))$ and the entries of the matrix (3.72) are

$$m(\lambda) = N_0^{-1}(\lambda)S_0(\lambda), \quad M_2(\lambda) = -N_0^{-1}(\lambda), \quad M_3(\lambda) = -N_0^{-1*}(\overline{\lambda}),$$
$$M_4(\lambda) = N_1(\lambda)N_0^{-1}(\lambda).$$

For a scalar operator of the second order these equalities were obtained in [2].

3) In the scalar case the identity (3.54) is well known (see [15, 20]). In [7] this identity was proved for a regular expression $l_H[y]$ of the second order with $\dim H \le \infty$.

4. Boundary conditions and spectrum of proper extensions

4.1. **Boundary conditions for proper extensions.** Let H, $\mathcal{H}_j'$ be Hilbert spaces and let $\mathcal{H}_j := H^n \oplus \mathcal{H}_j'$, $j \in \{0,1\}$. Then according to Section 2.2 the equalities

$$\theta = \{(\hat{C}_0, \hat{C}_1); \mathcal{H}_0, \mathcal{H}_1; \mathcal{K}\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : \hat{C}_0 h_0 + \hat{C}_1 h_1 = 0\}, \tag{4.1}$$

$$\theta^\times = \{(\hat{C}_{0\times}, \hat{C}_{1\times}); \mathcal{H}_0, \mathcal{H}_1; \mathcal{K}_\times\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : \hat{C}_{0\times} h_0 + \hat{C}_{1\times} h_1 = 0\} \tag{4.2}$$

make it possible to identify a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and the corresponding linear relation $\theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ (see (2.3)) with admissible operator pairs $\hat{C}_j \in [\mathcal{H}_j, \mathcal{K}]$ and $\hat{C}_{j\times} \in [\mathcal{H}_j, \mathcal{K}_\times]$ ($j \in \{0,1\}$) respectively (recall that we do not distinguish equivalent operator pairs). Next assume that

$$\hat{C}_0 = (C_2 \;\; C_0') : H^n \oplus \mathcal{H}_0' \to \mathcal{K}, \quad \hat{C}_1 = (-C_1 \;\; C_1') : H^n \oplus \mathcal{H}_1' \to \mathcal{K}, \tag{4.3}$$

$$\hat{C}_{0\times} = (C_{2\times} \;\; C_{0\times}') : H^n \oplus \mathcal{H}_0' \to \mathcal{K}_\times, \quad \hat{C}_{1\times} = (-C_{1\times} \;\; C_{1\times}') : H^n \oplus \mathcal{H}_1' \to \mathcal{K}_\times \tag{4.4}$$

are the block-matrix representations of $\hat{C}_j$ and $\hat{C}_{j\times}$.Then admissibility of the pair $\hat{C}_0, \hat{C}_1$ means that for every $h \in \mathcal{K}$ there exist $\hat{h}_1, \hat{h}_2 \in H^n$, $h_0' \in \mathcal{H}_0'$ and $h_1' \in \mathcal{H}_1'$ such that

$$C_1\hat{h}_1 + C_2\hat{h}_2 + C_0'h_0' + C_1'h_1' = h.$$

Observe also that in view of (4.1) a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ can be identified with an operator quadruple C_1, C_2, C_0', C_1', which forms the operator pair $\hat{C}_0, \hat{C}_1$ via (4.3).

In the next theorem we describe the class Ex_{L_0} in terms of a D-triplet for L.

Theorem 4.1. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet* (3.13) *for L. Then: 1) the equalities*

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D} : C_1 y^{(1)}(0) + C_2 y^{(2)}(0) + C_0'\Gamma_0' y + C_1'\Gamma_1' y = 0\}, \quad \widetilde{A} = L \upharpoonright \mathcal{D}(\widetilde{A}) \tag{4.5}$$

establish a bijective correspondence between all proper extensions $\widetilde{A} \in \mathrm{Ex}_{L_0}$ and all admissible operator pairs $\theta = \{(\hat{C}_0, \hat{C}_1); \mathcal{H}_0, \mathcal{H}_1; \mathcal{K}\}$ defined by (4.3). *Moreover if an extension $\widetilde{A}$ is given by the equality* (4.5), *then the adjoint $\widetilde{A}^*$ is defined by the same equality with operators $C_{1\times}, C_{2\times}, C_{0\times}'$ and $C_{1\times}'$ taken from* (4.4).

2) the equalities

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D} : N_1 y^{(1)}(0) + N_2 y^{(2)}(0) = 0,\ N_0'\Gamma_0' y + N_1'\Gamma_1' y = 0\}, \quad \widetilde{A} = L \upharpoonright \mathcal{D}(\widetilde{A}) \tag{4.6}$$

give a bijective correspondence between all extensions $\widetilde{A} \in \mathrm{Dex}_{L_0}$ (see Definition 3.1) and all collections formed by admissible operator pairs $\theta_0 = \{(N_2, -N_1); H^n; \mathcal{K}\}$ and $\theta' = \{(N_0', N_1'); \mathcal{H}_0', \mathcal{H}_1'; \mathcal{K}'\}$ (see (2.1)*).*

3) the extension (4.5) *(respectively* (4.6)*) is maximal dissipative, maximal accumulative, maximal symmetric or selfadjoint if and only if the corresponding operator pair θ (both the pairs θ_0 and θ') belongs to the class* Dis, Ac, Sym *or* Self *respectively.*

Proof. The statement 1) is a direct consequence of Proposition 2.4, 2).

2) For every $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ put $\theta_0 := \theta \cap (H^n \oplus H^n)$ and $\theta' := \theta \cap (\mathcal{H}'_0 \oplus \mathcal{H}'_1)$. It follows from (3.16) that an extension $\widetilde{A} = \widetilde{A}_\theta$ defined by (2.8), satisfies the relations

$$\mathcal{D}(\widetilde{A}_\theta) \cap \mathcal{D}_1 = \{y \in \mathcal{D} : \{\Gamma_0 y, \Gamma_1 y\} \in \theta_0\}, \quad \mathcal{D}(\widetilde{A}_\theta) \cap \mathcal{D}_2 = \{y \in \mathcal{D} : \{\Gamma_0 y, \Gamma_1 y\} \in \theta'\}.$$

Therefore according to Definition 3.1

$$\widetilde{A}_\theta \in \mathrm{Dex}_{L_0} \iff \theta = \theta_0 \oplus \theta' \iff \mathcal{D}(\widetilde{A}_\theta) = \{y \in \mathcal{D} : \widetilde{y}(0) \in \theta_0,\ \Gamma' y \in \theta'\},$$

which yields the correspondence (4.6).

3) Clearly a linear relation (operator pair) $\theta = \theta_0 \oplus \theta'$ belongs to one of the classes Dis, Ac, Sym or Self if and only if both the relations θ_0 and θ_1 belong to the same class. This and Proposition 2.4, 2) yield the required statement. □

Corollary 4.2. *1) A selfadjoint extension $\widetilde{A} \in \mathrm{Dex}_{L_0}$ exists if and only if $n_{b+} = n_{b-}$.*

2) Let $n_{b+} = n_{b-}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a decomposing boundary triplet (3.13) *for L (so that $\mathcal{H}'_0 = \mathcal{H}'_1 := \mathcal{H}'$). Then*

(i) the equalities (4.6) *establish a bijective correspondence between all selfadjoint extensions $\widetilde{A} \in \mathrm{Dex}_{L_0}$ and all collections formed by selfadjoint operator pairs (linear relations) $\theta_0 = \{(N_2, -N_1); H^n; \mathcal{K}\}$ and $\theta' = \{(N'_0, N'_1); \mathcal{H}'; \mathcal{K}'\}$;*

(ii) the relations

$$\mathcal{D}(\widetilde{A}_b) = \{y \in \mathcal{D} : \widetilde{y}(0) = 0,\ N'_0 \Gamma'_0 y + N'_1 \Gamma'_1 y = 0\}, \quad \widetilde{A}_b = L \upharpoonright \mathcal{D}(\widetilde{A}_b), \tag{4.7}$$

$$\mathcal{D}(\widetilde{A}^*_b) = \{y \in \mathcal{D} : N'_0 \Gamma'_0 y + N'_1 \Gamma'_1 y = 0\}, \quad \widetilde{A}^*_b = L \upharpoonright \mathcal{D}(\widetilde{A}^*_b) \tag{4.8}$$

give a bijective correspondence between all symmetric extensions $\widetilde{A}_b \in \mathrm{Sym}_{L_0,b}$ (see Definition 3.2) and their adjoint $\widetilde{A}^*_b$ on the one hand, and all selfadjoint operator pairs $\theta' = \{(N'_0, N'_1); \mathcal{H}'; \mathcal{K}'\}$ on the other hand. Moreover if the extension $\widetilde{A}_b \in \mathrm{Sym}_{L_0,b}$ is given by (4.7)*, then the equalities* (4.6) *establish a bijective correspondence between all proper extensions $\widetilde{A}$ of the operator $\widetilde{A}_b$ and all admissible operator pairs (linear relations) $\theta_0 = \{(N_2, -N_1); H^n; \mathcal{K}\}$.*

Proof. 1) Without loss of generality suppose that $n_{b-} \le n_{b+}$ and let Π be a decomposing D-triplet (3.13) for L (see Proposition 3.9). Then by Theorem 4.1 a selfadjoint extension $\widetilde{A} \in \mathrm{Dex}_{L_0}$ exists if and only if $\mathrm{Self}(\mathcal{H}'_0, \mathcal{H}'_1) \neq \emptyset$. According to [17] the last condition is equivalent to $\dim \mathcal{H}'_0 = \dim \mathcal{H}'_1$, which in view of (3.26) gives the required statement.

2) The statement (i) directly follows from Theorem 4.1. The statement (ii) is a consequence of the statement (i) and Theorem 4.1. □

It is not difficult to verify that an extension $\widetilde{A}_b \in \mathrm{Ex}_{L_0}$ satisfies (4.7) and (4.8) if and only if the relations

$$\mathcal{D}(\widetilde{A}_b) \subset \mathcal{D}_1, \quad \mathcal{D}(\widetilde{A}^*_b) = \mathcal{D}(\widetilde{A}_b) \dotplus (\mathcal{D}(\widetilde{A}^*_b) \cap \mathcal{D}_2) \tag{4.9}$$

are valid. Therefore Definition 3.2 is equivalent to the following one.

Definition 4.3. An extension $\widetilde{A}_b \in \mathrm{Ex}_{L_0}$ belongs to the class $\mathrm{Sym}_{L_0,b}$ if (4.9) holds.

Remark 4.4. 1) For a scalar expression $l[y]$ the relation (4.5) is in fact a system of scalar boundary conditions defining the extension $\widetilde{A}$. Similarly the two equations in (4.6) are equivalent to two systems of scalar boundary conditions at the ends of the interval Δ, which together form a decomposing system [5]. Note in this connection that the classical method of construction of extensions $\widetilde{A} \in \mathrm{Ex}_{L_0}$ by means of scalar boundary conditions [5, 20] is not convenient in the case $\dim H = \infty$, when the equality $\dim \mathcal{D}(\widetilde{A})/\mathcal{D}(L_0) = \infty$ is possible. At the same time in view of (3.28) and (3.29) one can consider the maps $\Gamma_j' : \mathcal{D} \to \mathcal{H}_j'$, $j \in \{0,1\}$ as operator boundary values at the point b (cf. Corollary 13.2.28 in [5]). Therefore Theorem 4.1 gives a description of extensions $\widetilde{A} \in \mathrm{Ex}_{L_0}$ (for the case $\dim H \le \infty$) in terms of operator boundary conditions at the ends of the interval $\Delta = [0, b\rangle$ (see equations in (4.5) and (4.6)).

2) Description of all selfadjoint extensions $\widetilde{A} \in \mathrm{Ex}_{L_0}$ for a regular expression $l_H[y]$ ($\dim H \le \infty$) by means of the boundary triplet (3.27) and formulas (4.5), (4.6) was first obtained in [21]. In the paper [11] this result was extended to quasiregular expressions.

3) The statement 1) of Corollary 4.2 contains a criterion of existence of selfadjoint decomposing boundary conditions (or equivalently extensions $\widetilde{A} = \widetilde{A}^* \in \mathrm{Dex}_{L_0}$), while the description of such conditions is given in the statement 2). For regular and quasiregular expressions similar description was obtained in [21] and [11] respectively (see also [23]).

4.2. **Spectrum of proper extensions.** In the next theorem, which is a direct consequence of Proposition 4.1 in [18], we describe the spectrum of proper extensions $\widetilde{A} \in \mathrm{Ex}_{L_0}$ in terms of boundary conditions (4.5) and the Weyl functions $M_\pm(\cdot)$.

Theorem 4.5. *Let* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a decomposing D-triplet* (3.13) *for* L, *let* $M_\pm(\cdot)$ *be the corresponding Weyl functions* (3.41), (3.42) *and let* $\widetilde{A} \in \mathrm{Ex}_{L_0}$ *be an extension defined by* (4.5). *Moreover assume that* $\hat{C}_j \in [\mathcal{H}_j, \mathcal{K}]$, $j \in \{0.1\}$ *are operators* (4.3), $\widetilde{C}_1 := \hat{C}_0 \upharpoonright \mathcal{H}_1$, $\widetilde{C}_0 := (\hat{C}_1 \quad -i\, C_0' \upharpoonright \mathcal{H}_2') : \mathcal{H}_1 \oplus \mathcal{H}_2' \to \mathcal{K}$ *and let*

$$S_+(\lambda) := \hat{C}_0 + \hat{C}_1 M_+(\lambda), \quad \lambda \in \mathbb{C}_+, \quad S_-(\lambda) := \widetilde{C}_1 + \widetilde{C}_0 M_-(\lambda), \quad \lambda \in \mathbb{C}_-.$$

Then the following relations hold

$$\lambda \in \rho(\widetilde{A}) \Leftrightarrow 0 \in \rho(S_\pm(\lambda)), \quad \lambda \in \sigma_j(\widetilde{A}) \Leftrightarrow 0 \in \sigma_j(S_\pm(\lambda)), \;\; j \in \{p,c,r\}, \quad \lambda \in \mathbb{C}_\pm, \tag{4.10}$$

$$\lambda \in \hat{\rho}(\widetilde{A}) \iff 0 \in \hat{\rho}(S_\pm(\lambda)), \quad \lambda \in \mathbb{C}_\pm, \tag{4.11}$$

$$\overline{\mathcal{R}(\widetilde{A} - \lambda)} = \mathcal{R}(\widetilde{A} - \lambda) \iff \overline{\mathcal{R}(S_\pm(\lambda))} = \mathcal{R}(S_\pm(\lambda)), \quad \lambda \in \mathbb{C}_\pm, \tag{4.12}$$

$$\dim \mathrm{Ker}(\widetilde{A} - \lambda) = \dim \mathrm{Ker}\, S_\pm(\lambda), \;\; \mathrm{codim}\, \mathcal{R}(\widetilde{A} - \lambda) = \mathrm{codim}\, \mathcal{R}(S_\pm(\lambda)), \;\; \lambda \in \mathbb{C}_\pm. \tag{4.13}$$

In the case $\mathcal{H}_1 = \mathcal{H}_0 := \mathcal{H}$ the following corollary is directly implied by Theorem 4.5.

Corollary 4.6. *Assume that* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a decomposing boundary triplet* (3.13) *for* L, $A_0 (= A_0^*)$ *is the extension* (3.43) *and* $M(\cdot)$ *is the Weyl function* (3.72). *Moreover let an extension* $\widetilde{A} \in \mathrm{Ex}_{L_0}$ *be defined by* (4.5), *let* $\hat{C}_j \in [\mathcal{H}, \mathcal{K}]$, $j \in \{0.1\}$ *be operators* (4.3) *and let* $S(\lambda) := \hat{C}_0 + \hat{C}_1 M(\lambda)$, $\lambda \in \rho(A_0)$. *Then for every* $\lambda \in \rho(A_0)$ *the relations* (4.10)–(4.13) *hold with* $S(\lambda)$ *instead of* $S_\pm(\lambda)$.

Corollary 4.7. *let under conditions of Corollary 4.2, 2)* $\theta' = \{(N_0', N_1'); \mathcal{H}'; \mathcal{K}'\}$ *be a selfadjoint operator pair, let* $\widetilde{A}_b \in \mathrm{Sym}_{L_0, b}$ *be the corresponding extension* (4.7) *and let* $A_{\theta'} \in \mathrm{Dex}_{L_0}$ *be a selfadjoint extension with the domain*

$$\mathcal{D}(A_{\theta'}) = \{y \in \mathcal{D} : y^{(2)}(0) = 0, \; N_0'\Gamma_0' y + N_1'\Gamma_1' y = 0\}. \tag{4.14}$$

Then: 1) for every $\lambda \in \rho(A_{\theta'})$ *there exists the unique operator function* $v_{\theta'}(\cdot, \lambda) \in L_2'[H^n, H]$ *satisfying the equation* (3.4) *and the boundary conditions*

$$v_{\theta'}^{(2)}(0, \lambda) = I_{H^n}, \tag{4.15}$$

$$(N_0'\Gamma_0' + N_1'\Gamma_1')(v_{\theta'}(t, \lambda)\hat{h}) = 0, \quad \hat{h} \in H^n, \quad \lambda \in \rho(A_{\theta'}); \tag{4.16}$$

2) if a proper extension $\widetilde{A}$ *of the operator* $\widetilde{A}_b$ *is given by* (4.6)*, then for every* $\lambda \in \rho(A_{\theta'})$ *the relations* (4.10)–(4.13) *hold with* $S_{\theta'}(\lambda) := N_1 v_{\theta'}^{(1)}(0,\lambda) + N_2$ *in place of* $S_\pm(\lambda)$*.*

Proof. Without loss of generality assume that $N_0' = \cos B,\ N_1' = \sin B$ (see (2.6)) and let

$$\Gamma_{0,b}' := N_0'\Gamma_0' + N_1'\Gamma_1', \quad \Gamma_{1,b}' := -N_1'\Gamma_0' + N_0'\Gamma_1'.$$

Moreover let $\Gamma_{j,b}$ be linear maps from $\mathcal{D}$ to $\mathcal{H}(= H^n \oplus \mathcal{H}')$, given by (3.13) with $\Gamma_{j,b}'$ instead of $\Gamma_j',\ j \in \{0,1\}$. Then the immediate checking shows that $\Pi_b := \{\mathcal{H}, \Gamma_{0,b}, \Gamma_{1,b}\}$ is a boundary triplet for L with $A_{0,b} := L \upharpoonright \operatorname{Ker}\Gamma_{0,b} = A_{\theta'}$. Now applying statement 2) of Corollary 3.13 to this triplet one obtains the statement 1). Moreover in view of (3.76) the corresponding m-function for the triplet Π_b is $m_b(\lambda) = -v_{\theta'}^{(1)}(0,\lambda),\ \lambda \in \rho(A_{\theta'})$.

Next application of Corollary 3.16 to the boundary triplet Π_b shows that the operators

$$\hat{\Gamma}_{0,b} y = y^{(2)}(0), \quad \hat{\Gamma}_{1,b} y = -y^{(1)}(0), \quad y \in \mathcal{D}(\widetilde{A}_b^*)$$

form a boundary triplet $\hat{\Pi}_b = \{H^n, \hat{\Gamma}_{0,b}, \hat{\Gamma}_{1,b}\}$ for $\widetilde{A}_b^*$. Moreover for this triplet $\hat{A}_{0,b} := \widetilde{A}_b^* \upharpoonright \operatorname{Ker}\hat{\Gamma}_{0,b} = A_{\theta'}$ and by (3.84) the corresponding Weyl function is

$$\hat{M}_b(\lambda) = m_b(\lambda) = -v_{\theta'}^{(1)}(0,\lambda), \quad \lambda \in \rho(A_{\theta'}).$$

Now assume that an extension $\widetilde{A}$ of the operator $\widetilde{A}_b$ is given by (4.6). Then in terms of the triplet $\hat{\Pi}_b$

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D}(\widetilde{A}_b^*) :\ N_2\hat{\Gamma}_{0,b}y - N_1\hat{\Gamma}_{1,b}y = 0\}, \quad \widetilde{A} = \widetilde{A}_b^* \upharpoonright \mathcal{D}(\widetilde{A}) \tag{4.17}$$

and application of Proposition 4.1 from [18] to the triplet $\hat{\Pi}_b$ and the extension (4.17) gives rise to the statement 2). □

Remark 4.8. The operator function $v_{\theta'}(t,\lambda)$ is a fundamental solution of the problem (3.4), (4.16) in the sense of the papers [22, 23], where some kinds of such solutions were constructed by the more complicated way. Observe also that for selfadjoint extensions $\widetilde{A}$ the statement 2) of Corollary 4.7 (with a rather different fundamental solution) was proved in [22] (see also [23]).

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