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ON THE HYPERSPACE OF MAX-MIN CONVEX COMPACT SETS

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Abstract. A subset A of $\mathbb{R}^n$ is said to be max-min convex if, for any $x, y \in A$ and any $t \in \mathbb{R}$, we have $x \oplus t \otimes y \in A$ (here $\oplus$ stands for the coordinatewise maximum of two elements in $\mathbb{R}^n$ and $t \otimes (y_1, \dots, y_n) = (\min\{t, y_1\}, \dots, \min\{t, y_n\})$). It is proved that the hyperspace of compact max-min convex sets in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the punctured Hilbert cube. This is a counterpart of the result by Nadler, Quinn and Stavrokas proved for the hyperspace of compact convex sets.

We also investigate the maps of the hyperspaces of compact max-min convex sets induced by the projection maps of Euclidean spaces. It is proved that this map is a Hilbert cube manifold bundle.

1. Introduction

Nadler, Quinn and Stavrokas proved [1] that the hyperspace of compact convex sets in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the punctured Hilbert cube $Q \setminus \{*\}$. Analogous results were proved also for related structures, in particular, for the hyperspaces of compact convex bodies of constant width [2] and compact max-plus convex sets [3]. In this note we prove a counterpart of the mentioned result for the so-called max-min convex sets (see the definition below). The proofs are based on the technique of infinite-dimensional topology, in particular, Toruńczyk's characterization theorem for the Hilbert cube manifolds. We also prove a parametric version of this result showing that the map of the corresponding hyperspaces induced by the projection map of the cubes in Euclidean spaces is a trivial bundle with a fiber Q. This result has its counterpart in the case of max-plus convex sets (see [3]) and for convex sets (see [4]) and this witnesses for the assumption that the presence of convexity-like structure "enhances" the topology of the (maps of) hyperspaces.

We endow the space $\mathbb{R}^n$ with the ℓ_∞-metric (or max-metric): for every $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, let $d(x, y) := \max\{|x_i - y_i| \mid i = 1, \dots, n\}$ i.e., we consider the space $\mathbb{R}^n_\infty$. The Hausdorff distance between any nonempty compact sets A and B is evaluated with respect to this metric

$$d_H(A, B) := \inf\{\varepsilon > 0 \mid O_\varepsilon(A) \supset B \text{ and } O_\varepsilon(B) \supset A\},$$

where $O_\varepsilon(A) := \{a \mid d(a, A) < \varepsilon\}$ denotes the ε-neighborhood of a set A in $\mathbb{R}^n_\infty$.

We will use the following extended real line: $\mathbb{R}_{\min} = \mathbb{R} \cup \{\infty\}$.

We will consider the following operations on the set $\mathbb{R}^n$:

"addition": for every $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n_\infty$ let

$$x \oplus y := (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\});$$

"multiplication" by a number: for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n_\infty$ and for all $\alpha \in \mathbb{R}_{\min}$ let

$$\alpha \otimes x := (\min\{\alpha, x_1\}, \dots, \min\{\alpha, x_n\}\}).$$

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In particular, if $\alpha,\beta \in \mathbb{R}_{\min}$, then $\alpha \otimes \beta = \min\{\alpha,\beta\}$.

The triple $(\mathbb{R}_{\min}, \oplus, \otimes)$ is of great importance in the so-called idempotent mathematics, i.e. a part of mathematics dealing with the idempotent operations; see, e.g., the survey paper [5]. The operations $\oplus$ and $\otimes$ are also used in the theory of pseudo-additive measures, in particular, they are involved in the definition of the Sugeno integral (see, e.g., [6]).

For the sake of simplicity, we accept the following convention: first the operation $\otimes$ is performed and then the operation $\oplus$.

Let $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n), z = (z_1,\dots,z_n) \in \mathbb{R}^n_\infty$, $\alpha,\beta \in \mathbb{R}_{\min}$. Denote $(x)_* = \min\{x_1,\dots,x_n\}$, $(x)^* = \max\{x_1,\dots,x_n\}$. The defined operations satisfy the following properties:

(1) $x \oplus y = y \oplus x$;
(2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(3) $\alpha \otimes (\beta \otimes x) = (\alpha \otimes \beta) \otimes x$;
(4) $\alpha \otimes (x \oplus y) = \alpha \otimes x \oplus \alpha \otimes y$;
(5) $x \oplus (\alpha \otimes x) = x$;
(6) if $\alpha \otimes \beta = \beta$, then $\alpha \otimes x \oplus \beta \otimes y = \beta \otimes (\alpha \otimes x \oplus y)$;
(7) if $\alpha \le (x)_*$, then $\alpha \otimes x = (\alpha,\dots,\alpha)$;
(8) if $\alpha \ge (x)^*$, then $\alpha \otimes x = x$.

We consider the following *partial order* relation in the space $\mathbb{R}^n_\infty$.

Definition 1.1. We say that $x \le y$, whenever $x \oplus y = y$. If $x \le y$ and $x \ne y$, we say that $x < y$.

We have
(9) if $x \le y$ and $y \le z$, then $x \le z$;
(10) for every $x,y \in \mathbb{R}^n_\infty$ there exists $z = x \oplus y$ such that $x \le z$ $y \le z$.

The proof of the following statement is elementary.

Proposition 1.2. *Let $a,b,c,d \in \mathbb{R}$. If $|a-c| < \varepsilon$ and $|b-d| < \varepsilon$, then*

$$|\max\{a,b\} - \max\{c,d\}| < \varepsilon \quad \textit{and} \quad |\min\{a,b\} - \min\{c,d\}| < \varepsilon.$$

Corollary 1.3. *The operations $\oplus$ and $\otimes$ are continuous:*
for every $a,b,c,d \in \mathbb{R}^n_\infty$ such that $d(a,c) < \varepsilon$ and $d(b,d) < \varepsilon$ we have $d(a \oplus b, c \oplus d) < \varepsilon$;
for every $a,c, \in \mathbb{R}^n_\infty$ and for every $\alpha,\beta \in \mathbb{R}_{\min}$ such that $d(a,c) < \varepsilon$ and $|\alpha-\beta| < \varepsilon$ we have $d(\alpha \otimes a, \beta \otimes b) < \varepsilon$.

Therefore, we obtain, in particular, another interesting property:
(11) if $d(a,b) < \varepsilon$ and $d(a,c) < \varepsilon$, then $d(a, b \oplus c) < \varepsilon$ and $d(a, \alpha \otimes b \oplus \beta \otimes c) < \varepsilon$.

2. Preliminaries

By ANR we denote the class of absolute neighborhood retracts for the class of metric spaces.

We will need the following result on the ANR spaces due to H. Toruńczyk [7].

Theorem 2.1. *A separable metrizable space is an ANR if it has a base $\mathcal{B}$ of topology with the following property: every finite intersection of elements of $\mathcal{B}$ is contractible.*

We say that a metric space X satisfies the *discrete approximation property* (DAP) if for every continuous function $\varepsilon\colon X \to (0,\infty)$ there exist continuous maps $f_1, f_2\colon X \to X$ such that $d(f_i(x), x) < \varepsilon(x)$, for every $x \in X$, $i = 1,2$, and $f_1(X) \cap f_2(X) = \emptyset$.

A *Q-manifold* is a separable metrizable space locally homeomorphic to the Hilbert cube Q. The following is a characterization theorem for Q-manifolds.

Theorem 2.2 (Toruńczyk [7]). *A locally compact ANR X is a Q-manifold if and only if X satisfies the DAP.*

A *trivial Q-manifold bundle* is a map $f\colon X \to Y$ which is homeomorphic to the projection $Y \times M \to Y$ onto the first factor, where M is a Q-manifold.

A map $f\colon X \to Y$ of compact metric spaces is said to satisfy the *fibrewise disjoint approximation property* if, for every $\varepsilon > 0$, there exist maps $g_1, g_2\colon X \to X$ such that

(1) $fg_1 = fg_2 = f$;
(2) $d(1_X, g_i) < \varepsilon$, $i = 1, 2$;
(3) $g_1(X) \cap g_2(X) = \emptyset$.

Recall that a map $f\colon X \to Y$ is called *soft* [8] provided that for every commutative diagram

(1)
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd}$$

such that Z is a paracompact space and A is a closed subset of Z there exists a map $\Phi\colon Z \to X$ such that $f\Phi = \psi$ and $\Phi|A = \varphi$.

The following result is proved in [9].

Theorem 2.3 (Toruńczyk-West characterization theorem for Q-manifold bundles)**.** *A map $f\colon X \to Y$ of compact metric ANR-spaces is a trivial Q-bundle if f is soft and f satisfies the fibrewise disjoint approximation property.*

A Q-manifold X is $[0,1)$-*stable* if X is homeomorphic to $X \times [0,1)$. The following result of R. Y. T. Wong [10] is often used in infinite-dimensional topology: a Q-manifold X is $[0,1)$-stable if and only if there is a proper homotopy $H\colon X \times [0,1) \to X$ such that $H(x,0) = x$ for every $x \in X$ (recall that a map is *proper* if the preimage of every compact set is compact).

We will need also a classification theorem for the $[0,1)$-stable Q-manifolds (see [11, Theorem 21.2]): two $[0,1)$-stable Q-manifolds are homeomorphic if and only if they are homotopy equivalent.

The following notion is introduced in [12]. A *c-structure* on a topological space X is an assignment to every nonempty finite subset A of X a contractible subspace $F(A)$ of X such that $F(A) \subset F(A')$ whenever $A \subset A'$. A pair (X, F), where F is a c-structure on X is called a *c-space*. A subset E of X is called an *F-set* if $F(A) \subset E$ for any finite $A \subset E$. A metric space (X, d) is said to be a *metric l.c.-space* if all the open balls are F-sets and all open r-neighborhoods of F-sets are also F-sets.

The following is generalization of the Michael Selection Theorem; see [13] for the proof. Recall that a multivalued map $F\colon X \to Y$ of topological spaces is called *lower semicontinuous* if, for any open subset U of Y, the set $\{x \in X \mid F(x) \cap U \neq \emptyset\}$ is open in X. A *selection* of a multivalued map $F\colon X \to Y$ is a (single-valued) map $f\colon X \to Y$ such that $f(x) \in F(x)$, for every $x \in X$.

Theorem 2.4. *Let (X, d, F) be a complete metric l.c.-space. Then any lower semicontinuous multivalued map $T\colon Y \to X$ of a paracompact space Y whose values are nonempty closed F-sets has a continuous selection.*

3. Ray and segment

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be such that $x_1 \le x_2 \le \dots \le x_n$. Then the *ray* $[-\infty, +\infty] \otimes x$ is defined as follows:

$$\begin{aligned} [-\infty, +\infty] \otimes x = &\{(\alpha, \dots, \alpha) | \alpha \in [-\infty, x_1]\} \cup \{(x_1, \alpha, \dots,) \\ & |\alpha \in [x_1, x_2]\} \cup \dots \cup \{(x_1, \dots, x_{n-1}, \alpha) \mid \alpha \in [x_{n-1}, x_n]\}. \end{aligned}$$

Definition 3.1. The *segment* between $x, y \in \mathbb{R}^n$ is the set

$$[x, y] := \{\alpha \otimes x \oplus \beta \otimes y | \alpha \oplus \beta = +\infty\}.$$

The equality $\alpha \oplus \beta = +\infty$ means that (at least) one of the numbers α or β equals $+\infty$ and therefore either $\alpha \otimes x = x$ or $\beta \otimes y = y$. If $\alpha = \beta = +\infty$, then $\alpha \otimes x \oplus \beta \otimes y = x \oplus y \in [x, y]$.

Let $\alpha = +\infty$. Denote $t(x, y, \beta) := x \oplus \beta \otimes y$. We have

1. $t(x, y, \beta) = x \oplus (\beta \otimes y) = x \oplus (\beta \otimes x) \oplus (\beta \otimes y) = x \oplus \beta \otimes (x \oplus y) = t(x, x \oplus y, \beta)$;
2. $t(x, y, \beta) = x$ if $\beta \leq x_*$;
3. $t(x, y, \beta) = x \oplus y$ if $\beta \geq (x \oplus y)^*$.

Therefore, $\tau(\beta) = t(x, y, \beta)$, $\beta \in [x_*, (x \oplus y)^*]$ is the segment that connects the points x and $x \oplus y$ and we have

$$[x, x \oplus y] = \{\tau(\beta) \mid \beta \in (-\infty, +\infty)\} = \{\tau(\beta) \mid \beta \in [x_*, (x \oplus y)^*]\}.$$

Let $\tau(\beta) = (\tau_i(\beta))$, $x = (x_i)$, $y = (y_i)$, $i = 1, 2, \ldots, n$. Coordinatewise

$$\begin{aligned} &\tau_i(\beta) \equiv x_i, && \text{if } x_i \geq y_i; \\ &\tau_i(\beta) = \begin{cases} x_i & \text{if } \beta \leq x_i, \\ \beta & \text{if } x_i < \beta < y_i, \\ y_i & \text{if } \beta \geq y_i, \end{cases} && \text{if } x_i < y_i. \end{aligned} \tag{2}$$

Remark 3.2. Clearly, if $\tilde{x}_i$ $\tilde{y}_i$ are numbers such that $|x_i - \tilde{x}_i| < \varepsilon$ and $|y_i - \tilde{y}_i| < \varepsilon$, then for every β we have $|\tau_i(\beta) - \tilde{\tau}_i(\beta)| < \varepsilon$, i.e the distance between the corresponding graphs is less then ε.

Similarly, if $\beta = +\infty$, then

$$[y, x \oplus y] = \{\theta(\alpha) \mid \alpha \in [(y)_*, (x \oplus y)^*]\},$$

where $\theta(\alpha) = t(y, x, \alpha) = y \oplus \alpha \otimes (x \oplus y)$.

Thus $[x, y] = [x, x \oplus y] \cup [x \oplus y, y]$ and $[x, x \oplus y] \cap [x \oplus y, y] = \{x \oplus y\}$.

Remark 3.3. The segment $\tau(\beta)$ (respectively $\theta(\alpha)$) is not an injective function of β; if, for some $i = 1, \ldots, n-1$, we have $y_i < x_{i+1}$, then, for any $\beta \in [y_i, x_{i+1}]$, the value $\tau(\beta)$ is constant.

Remark 3.4. It is obvious that if $|\beta_1 - \beta_2| < \varepsilon$, then $d(\tau(\beta_1), \tau(\beta_2)) < \varepsilon$.

The following statement is obvious.

Proposition 3.5. *If the intersection of two segments is nonempty, then it is either a segment or a point.*

Remark 3.6. Note that, if $[a, b] \cap [c, d] = [e, f]$, then it is possible that all the points a, b, c, d, e, f are distinct.

Proposition 3.7. *If $d(a, c) < \varepsilon$ and $d(b, d) < \varepsilon$, then the Hausdorff distance between the segments $[a, b]$ and $[c, d]$ is $< \varepsilon$.*

Proposition 3.8. *On the segment*

$$[x, y] = \{\tau(\beta) | \beta \in [-\infty, +\infty]\} \cup \{\theta(\alpha) | \alpha \in [-\infty, +\infty]\}$$

consider two points $\tau(\beta_1), \tau(\beta_2)$ $(\theta(\alpha_1), \theta(\alpha_2))$, such that $|\beta_1 - \beta_2| < \varepsilon$ $(|\alpha_1 - \alpha_2| < \varepsilon)$. Then also $d(\tau(\beta_1), \tau(\beta_2)) < \varepsilon$ $(d(\theta(\alpha_1), \theta(\alpha_2)) < \varepsilon)$.

4. MAX-MIN CONVEXITY

Definition 4.1. A subset $A \subseteq \mathbb{R}^n$ is called *max-min convex*, if, for any elements $x, y \in A$ and every $\alpha, \beta \in \mathbb{R}_{\min}$, such that $\max\{\alpha, \beta\} = +\infty$, we have $\alpha \otimes x \oplus \beta \otimes y \in A$.

Clearly, every segment is max-min convex and the set A is max-min convex if and only if, for every $x, y \in A$ we have $[x, y] \subset A$.

From Proposition 3.5 it follows:

Proposition 4.2. *The intersection of arbitrary nonempty family of max-min convex sets is a max-min convex set.*

It is easy to see that, in general, the max-min convexity is preserved only by translations in the direction of the vector $(a, \dots, a)$, $a \in \mathbb{R}$.

Proposition 4.3. *Any sequence $\{A_n\}$ of max-min-convex sets converges to a max-min-convex set.*

The following statement is obvious.

Proposition 4.4. *Every bounded max-min convex closed set A contains the maximal element $a = \max(A)$: $a \geq b$ for every $b \in A$. The maximal element continuously depends on A.*

For $\max A = a = (a_i)$, we have $a_i = \max\{x_i \mid (x_i) \in A\}$. Let $\min A = b = (b_i)$ $b_i = \min\{x_i \mid (x_i) \in A\}$. Clearly, $b \leq x$ for all $x \in A$, therefore one can say that b is the minimal element of the set A. Generally speaking, $\min A \notin A$. The minimal element also continuously depends on A.

Let $\operatorname{diam} A = \max\{d(x, y) \mid x, y \in A\}$ be the *diameter* of a set A. Clearly,

$$\operatorname{diam} A = \max\{|x_i - y_i| \mid (x_i), (y_i) \in A,\ i = 1, \dots, n\} = d(a, b).$$

The diameter $\operatorname{diam}(A)$ of A is a continuous function of A. If A is compact and nonempty, then $\operatorname{diam} A = d(a, b)$, for some $a, b \in A$.

Proposition 4.5. *Let $\varepsilon > 0$. The ε-neighborhood (with respect to the metric l_∞) $O_\varepsilon(I)$ of an arbitrary segment $I = [x, y]$ is max-min convex.*

Proof. Let $a', b' \in O_\varepsilon(I)$. Then there exist $a, b \in I$ such that $a' \in O_\varepsilon(a)$ and $b' \in O_\varepsilon(b)$, i.e. $|a'_i - a_i| < \varepsilon$ and $|b'_i - b_i| < \varepsilon$ for all $i = 1, \dots, n$. It remains only to take into account Remark 3.2. □

Corollary 4.6. *The closure $\bar{O}_\varepsilon(I)$ of the ε-neighborhood $O_\varepsilon(I)$ of an arbitrary segment I is max-min convex.*

Definition 4.7. Let $I \subset \{1, \dots, n\}$. The *$\varepsilon$-neighborhood of a set A over the set of indices I* is the set $O_{I,\varepsilon}(A)$ consisting of the points $(b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ satisfying the property: there exists a point $(a_1, a_2, \dots, a_n) \in A$ such that $|b_i - a_i| < \varepsilon$ whenever $i \in I$ and $b_i = a_i$ otherwise.

Evidently, $O_{\{1,2,\dots,n\},\varepsilon}(A) = O_\varepsilon(A)$.

The following is a simple consequence of what was proved above.

Proposition 4.8. *Let a set A be max-min convex and let I be an arbitrary set of indices. Then the ε-neighborhood of A on the set of indices I is also min-max convex. In particular, the ε-neighborhood of any min-max convex set is max-min convex.*

The closure of the ε-neighborhood of the set A on the set of indices I is max-min convex and, in particular, the closure $\bar{O}_\varepsilon(A)$ of the ε-neighborhood of the max-min convex set A is max-min convex.

Proposition 4.9. *Let A and B be max-min convex sets and $\alpha, \beta \in \mathbb{R}_{\min}$ be such that $\alpha \oplus \beta = +\infty$. Then the set*

$$C = \alpha \otimes A \oplus \beta \otimes b := \{\alpha \otimes a \oplus \beta \otimes b \mid a \in A,\ b \in B\}$$

is also max-min convex.

Proof. Let $c_1, c_2 \in C$, $c_1 = \alpha \otimes a_1 \oplus \beta \otimes b_1$, $c_2 = \alpha \otimes a_2 \oplus \beta \otimes b_2$. Then, for arbitrary α_1, β_1 with $\alpha_1 \oplus \beta_1 = +\infty$, we have

$$\begin{aligned}\alpha_1 \otimes c_1 \oplus \beta_1 \otimes c_2 =& \alpha_1 \otimes (\alpha \otimes a_1 \oplus \beta \otimes b_1) \oplus \beta_1 \otimes (\alpha \otimes a_2 \oplus \beta \otimes b_2)\\ =& \alpha \otimes (\alpha_1 \otimes a_1 \oplus \beta_1 \otimes a_2) \oplus \beta \otimes (\alpha_1 \otimes b_1 \oplus \beta \otimes b_2) \in C.\end{aligned}$$

□

Definition 4.10. For every set $A \subset \mathbb{R}^n$, the set $\mathrm{mmc}(A) := \bigcup [x, y]$, where $x, y \in A$ is called the *max-min convex hull* of A.

Proposition 4.11. *Let A and B be max-min convex sets. Then the set*

$$C = \mathrm{mmc}(A \cup B) := \{\alpha \otimes a \oplus \beta \otimes b \mid a \in A,\ b \in B, \alpha \oplus \beta = +\infty\}$$

is also max-min convex.

Proof. Let $a, a' \in A$, $b, b' \in B$, where the sets A, B are max-min convex. Let $c \in [a, b]$, $c' \in [a', b']$, $c'' \in [c, c']$. Show that then there exist $a'' \in [a, a']$, $b'' \in [b, b']$ such that $c'' \in [a'', b'']$.

We have $c = \alpha \otimes a \oplus \beta \otimes b$, $c' = \alpha' \otimes a' \oplus \beta' \otimes b'$ and let $c'' = \alpha'' \otimes c \oplus \beta'' \otimes c'$ ($\alpha \oplus \beta = +\infty$, $\alpha' \oplus \beta' = +\infty$, $\alpha'' \oplus \beta'' = +\infty$). Then

$$\begin{aligned}c'' =& \alpha'' \otimes (\alpha \otimes a \oplus \beta \otimes b) \oplus \beta'' \otimes (\alpha' \otimes a' \oplus \beta' \otimes b')\\ =& (\alpha'' \otimes \alpha \otimes a) \oplus (\alpha'' \otimes \beta \otimes b) \oplus (\beta'' \otimes \alpha' \otimes a') \oplus (\beta'' \otimes \beta' \otimes b')\\ =& ((\alpha'' \otimes \alpha \otimes a) \oplus (\beta'' \otimes \alpha' \otimes a')) \oplus ((\alpha'' \otimes \beta \otimes b) \oplus (\beta'' \otimes \beta' \otimes b')).\end{aligned}$$

Up to changing notations, one may assume that $\alpha'' = +\infty$ i $\alpha = +\infty$. Let $\beta = \beta \oplus \beta'' \otimes \beta'$, (or $\beta'' \otimes \beta' = \beta \oplus \beta'' \otimes \beta'$). Then

$$c'' = (\infty) \otimes ((\infty) \otimes a \oplus \beta'' \otimes \alpha' \otimes a') \oplus (\beta \otimes ((\infty) \otimes b) \oplus (\beta'' \otimes \beta' \otimes b'))$$

(or $c'' = (\infty) \otimes ((\infty) \otimes a \oplus \beta'' \otimes \alpha' \otimes a') \oplus (\beta'' \otimes \beta' \otimes (\beta \otimes b) \oplus ((\infty) \otimes b')))$. It remains to put

$$a'' = a \oplus \beta'' \otimes \alpha' \otimes a',\ b'' = b \oplus (\beta'' \otimes \beta' \otimes b')$$

(or $b'' = (\beta \otimes b) \oplus b'$). □

5. The hyperspace of max-min convex sets

Denote by $\mathrm{mmcc}\,\mathbb{R}^n$ the set of all nonempty max-min convex compact subsets of $\mathbb{R}^n$. Proposition 4.3 implies

Corollary 5.1. *The hyperspace* $\mathrm{mmcc}\,\mathbb{R}^n$ *of compact max-min-convex sets is a closed subspace of the hyperspace* $\exp \mathbb{R}^n$ *of all compact sets.*

One can define the operations $\oplus$ and $\otimes$ on the hyperspace $\mathrm{mmcc}\,\mathbb{R}^n$ in a natural way

$$A \oplus B := \{a \oplus b \mid a \in A, b \in B\},\ \alpha \otimes A := \{\alpha \otimes a \mid a \in A\},$$

for every $A, B \in \mathrm{mmcc}\,\mathbb{R}^n$ and for every $\alpha \in \mathbb{R}_{\min}$. It follows from Proposition 4.9 that these operations are well-defined, i.e. the resulting sets belong to the space $\mathrm{mmcc}\,\mathbb{R}^n$.

Corollary 1.3 can be formulated also for the space $\mathrm{mmcc}\,\mathbb{R}^n$:

Corollary 5.2. *The operations $\oplus$ and $\otimes$ are continuous on the space* $\mathrm{mmcc}\,\mathbb{R}^n$*:*

1) *for every* $A, B, C, D \in \operatorname{mmcc}\mathbb{R}^n$ *such that* $d_H(A, C) < \varepsilon$ *and* $d_H(B, D) < \varepsilon$, *we have* $d_H(A \oplus B, C \oplus D) < \varepsilon$;
2) *for every* $A, C, \in \operatorname{mmcc}\mathbb{R}^n$ *and for every* $\alpha, \beta \in \mathbb{R}_{\min}$ *such that* $d_H(A, C) < \varepsilon$ *and* $|\alpha - \beta| < \varepsilon$, *we have* $d_H(\alpha \otimes A, \beta \otimes B) < \varepsilon$.

Let $H_r(A)$ be the r-neighborhood of the element A in the space $\operatorname{mmcc}\mathbb{R}^n$ with respect to the Hausdorff metric d_H generated by the l_∞-metric.

If $A, B \in \operatorname{mmcc}\mathbb{R}^n$, one can define the segment as follows:

$$[\,A, B\,] := \{\alpha \otimes A \oplus \beta \otimes B \mid \alpha \oplus \beta = +\infty\}.$$

One can naturally introduce the following notion (cf. 4.1).

Definition 5.3. A set $K \subset \operatorname{mmcc}(\mathbb{R}^n)$ is called *max-min convex*, if for every $A, B \in K$ and every α, β with $\alpha \oplus \beta = +\infty$ we have $\alpha \otimes A \oplus \beta \otimes B \in K$.

Proposition 5.4. *For arbitrary* $B, C \in H_\varepsilon(A)$ *and* $\alpha \oplus \beta = +\infty$, *we have* $\alpha \otimes B \oplus \beta \otimes C \in H_\varepsilon(A)$, *i.e. the* ε*-neighborhood of an arbitrary* $A \in \operatorname{mmcc}\mathbb{R}^n$ *is max-min convex.*

Proof. Consider an arbitrary element $(\alpha \otimes b) \oplus (\beta \otimes c) \in (\alpha \otimes B) \oplus (\beta \otimes C)$, $b \in B$, $c \in C$. Then there exist elements $a, a' \in A$ such that $d(a, b) < \varepsilon$ and $d(a', c) < \varepsilon$. By Proposition 3.7, we have $d_H([a, a'], [b, c]) < \varepsilon$, i.e. $d_H(\alpha \otimes b \oplus \beta \otimes C,\, A)$, which finishes the proof. ☐

Theorem 5.5. *Every max-min convex subset* $K \subset \operatorname{mmcc}(\mathbb{R}^n)$ *is contractible.*

Proof. For an arbitrary max-min-convex subset $K \subset \operatorname{mmcc}(\mathbb{R}^n)$ and arbitrary $A_0 \in K$, let $K \oplus \{A_0\} = \{B \oplus A_0 \mid B \in K\}$. The obtained set is also max-min convex. Indeed, let $C = B \oplus A_0$, $C' = B' \oplus A_0$ and $\alpha \oplus \beta = +\infty$. Then

$$\begin{aligned}\alpha \otimes C \oplus \beta \otimes C' =&(\alpha \otimes (B \oplus A_0)) \oplus (\beta \otimes (B' \oplus A_0))\\ =&((\alpha \otimes B) \oplus (\beta \otimes B')) \oplus ((\alpha \oplus \beta) \otimes A_0)\\ =&((\alpha \otimes B) \oplus (\beta \otimes B')) \oplus A_0.\end{aligned}$$

The map $\varphi\colon K \to K \oplus \{A_0\}$, $\varphi(B) = B \oplus A_0$, $B \in K$, is homotopic to the identity map, the homotopy is $H_1\colon K \times [t_*, t^*] \to K$, $H_1(B, t) = B \oplus (t \otimes A_0)$, where $t_* = \min\{x_i \mid (x_1, \dots, x_n) \in \cup_{A \in K} A,\ i = 1, \dots, n\}$, $t^* = \max\{x_i \mid (x_1, \dots, x_n) \in \cup_{A \in K} A,\ i = 1, \dots, n\}$. Let us prove the continuity of the map H_1 at the point (B_0, t_0). Indeed, if we take $d_H(B, B_0) < \frac{\varepsilon}{2}$ and $|t - t_0| < \frac{\varepsilon}{2}$, then we have

$$\begin{aligned}d_H(B \oplus (t \otimes A_0), B_0 \oplus (t_0 \otimes A_0)) \leq& d_H(B \oplus (t \otimes A_0), B \oplus (t_0 \otimes A_0))\\ &+ d_H(B \oplus (t_0 \otimes A_0), B_0 \oplus (t_0 \otimes A_0)) < \varepsilon.\end{aligned}$$

The homotopy

$$H_2\colon K \oplus \{A_0\} \times [t_*, t^*] \to K \oplus \{A_0\},$$

defined by the formula $H_2(B, t) = A_0 \oplus (t \otimes B)$ is also continuous. Indeed, for arbitrary $B, B' \in \operatorname{mmcc}\mathbb{R}^n$ such that $d_H(B, B') < \varepsilon$ and for arbitrary $t, t' \in [t_*, t^*]$, $|t - t'| < \varepsilon$, we have

$$d_H(H_2(B, t), H_2(B', t')) = d_H(A_0 \oplus t \otimes B, A_0 \oplus t' \otimes B') < \varepsilon,$$

because $d_H(t \otimes B, t' \otimes B') < \varepsilon$ (see Corollary 5.2)

The homotopy H_2 connects the identity map and the constant map. Thus, $K \oplus \{A_0\}$ is contractible. ☐

Theorem 5.6. *The hyperspace* $\operatorname{mmcc}\mathbb{R}^n$, $n \geq 2$, *satisfies the disjoint approximation property (DAP): for an arbitrary continuous function* $\varepsilon\colon \operatorname{mmcc}\mathbb{R}^n \to (0, \infty)$ *there exist continuous maps* $f_1, f_2\colon \operatorname{mmcc}\mathbb{R}^n \to \operatorname{mmcc}\mathbb{R}^n$ *such that:*
(1) $d_H(f_i(A), A) < \varepsilon(A)$ *for every* $A \in \operatorname{mmcc}\mathbb{R}^n$;
(2) $f_1(\operatorname{mmcc}\mathbb{R}^n) \cap f_2(\operatorname{mmcc}\mathbb{R}^n) = \emptyset$.

Proof. Consider arbitrary $A \in \operatorname{mmcc}\mathbb{R}^n$ and let $\varepsilon = \varepsilon(A)$. Let $f_1(A) = \bar{O}_{\varepsilon(A)}(A)$ and $f_2(A) = \operatorname{mmc}(A \cup \{a_{\varepsilon(A)}\})$, where $a = (a_1, \dots, a_n) = \max(A)$ and $a_{\varepsilon(A)} = (a_1 + \varepsilon(A), \dots, a_n + \varepsilon(A))$.

It is easy to see that the maps

$$f_1, f_2 \colon \operatorname{mmcc}\mathbb{R}^n \to \operatorname{mmcc}\mathbb{R}^n$$

are continuous.

We are going to show that $f_1(\operatorname{mmcc}\mathbb{R}^n) \cap f_2(\operatorname{mmcc}\mathbb{R}^n) = \emptyset$.

Let $B = f_1(A) \in f_1(\operatorname{mmcc}\mathbb{R}^n)$. The point $\max B \in B$ belongs to some cube $V \subset B$ for which this point is maximal, namely, $V = \{x \in \mathbb{R}^n_\infty \mid d(x, \max A) \le \varepsilon(A)\}$ and $\max B = \max V$.

Now let $B' \in f_2(\operatorname{mmcc}\mathbb{R}^n)$. Show that its maximal point $b' = \max B'$ is not the maximal point of any cube lying in B'. Without loss of generality, we may assume that $b'_n = \max\{b'_i \mid i = 1, \dots, n\}$, $b'_1 = \min\{b'_i \mid i = 1, \dots, n\}$ and $b'_1 < b'_n$. Let us show that the last coordinate of all the other points of the set B' is strictly less than b'_n and the point b' cannot be the maximal point of any cube.

To this end, assume that there exists a point $\widetilde{b} \ne b'$, whose last coordinate equals b'_n and $\widetilde{b}_i \in [a'_i, b'_i]$, $i = 1, \dots, n-1$. Let $A' \in \operatorname{mmcc}\mathbb{R}^n$ be such that $B' = f_2(A') = \operatorname{mmc}(A' \cup \{a_{\varepsilon(A')}\})$, $a' = \max A'$, $b'_i = a'_i + \varepsilon(A')$, $i = 1, \dots, n$. Since $\widetilde{b} \notin A' \cup \{b'\}$, one can only assume that this point belongs to some segment $[x, b']$, where $x = (x_1, \dots, x_n) \in A'$. Since $x \le \widetilde{b}$, then there exists β_0 such that $\widetilde{b} = t(x, b', \beta_0) = x \oplus \beta_0 \otimes b'$. By formulas 2 for $i = 1$ and taking into account that $x_1 < b'_1$ we obtain $\beta_0 = \widetilde{b}_1$. However, then $\widetilde{b}_n = x_n \oplus \beta_0 \otimes b'_n = x_n \oplus b'_1 < b'_n$ and we obtain a required contradiction.

In the case $b'_1 = b'_2 = \dots = b'_n$ we have $B' = A' \cup \{(t, t, \dots, t) \mid t \in [a'_n, b'_n]\}$ and therefore the set B' contains no cube with the maximal point b'. □

Theorem 5.7. *The hyperspace* $\operatorname{mmcc}\mathbb{R}^n$*,* $n \ge 2$*, is homeomorphic to* $Q \setminus \{*\}$*.*

Proof. We apply Toruńczyk's characterization theorem.

First, we show that $\operatorname{mmcc}\mathbb{R}^n$ is an absolute retract. It follows from Theorem 5.5 and Proposition 5.4 that the hyperspace $\operatorname{mmcc}(\mathbb{R}^n)$ possesses a base satisfying the properties of Toruńczyk's theorem. Thus, $\operatorname{mmcc}(\mathbb{R}^n)$ is an AR.

That $\operatorname{mmcc}\mathbb{R}^n$ satisfies the DAP, is precisely Theorem 5.6. Now, it follows from Toruńczyk's characterization theorem that $\operatorname{mmcc}\mathbb{R}^n$ is a Q-manifold.

Define a map

$$H \colon \operatorname{mmcc}\mathbb{R}^n \times [0, \infty) \to \operatorname{mmcc}\mathbb{R}^n$$

by the formula: $H(A, t) = \bar{O}_t(A)$. Obviously, H is a continuous and proper map. It follows from classification theorem for $[0, 1)$-stable Q-manifolds cited in Section 2 that $\operatorname{mmcc}\mathbb{R}^n$ is homeomorphic to $Q \setminus \{*\}$. □

One can similarly prove the following result.

Theorem 5.8. *Let* $[a_i, b_i] \subset \mathbb{R}$*, where* $a_i < b_i$*,* $i = 1, \dots, n$*, where* $n \ge 2$*. The hyperspace* $\operatorname{mmcc}\left(\prod_{i=1}^n [a_i, b_i]\right)$ *is homeomorphic to* Q*.*

6. Projections

Let $\operatorname{pr} \colon \mathbb{R}^{m+k} \to \mathbb{R}^m$, $m + k = n$, denote the projection map,

$$\operatorname{pr}(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k}) = (x_1, \dots, x_m),$$

for every $(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k}) \in \mathbb{R}^{m+k}$. Clearly,

$$\operatorname{pr}(A) = \{\operatorname{pr}(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k}) \mid (x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k}) \in A\}.$$

We keep the same notation for the corresponding map of the hyperspaces, i.e. the map $A \mapsto \mathrm{pr}(A)\colon \operatorname{mmcc}\mathbb{R}^{m+k} \to \operatorname{mmcc}\mathbb{R}^m$.

Let us list the simplest properties of the projection operation.

1. Let $\mathrm{pr}(b) = a$, $b \in \mathbb{R}^{m+k}$, $a \in \mathbb{R}^m$. It is easy to see that $\mathrm{pr}(O_\varepsilon(\{b\})) = O_\varepsilon(\{a\})$.

2. The image of any segment is a segment, if $b, b' \in \mathbb{R}^{m+k}$ and $\mathrm{pr}(b) = a$, $\mathrm{pr}(b') = a'$, $a, a' \in \mathbb{R}^m$, then $\mathrm{pr}([b, b']) = [a, a']$. The fact easily follows from formulas (2).

3. The projection of any max-min convex set is again a max-min convex set.

Theorem 6.1. *The projection map* $\mathrm{pr}\colon \operatorname{mmcc}\mathbb{R}^{m+k} \to \operatorname{mmcc}\mathbb{R}^m$ *is open.*

Proof. Consider an arbitrary $B \in \operatorname{mmcc}\mathbb{R}^{m+k}$. Let a sequence $\{A_i\}$, where $A_i \in \operatorname{mmcc}\mathbb{R}^m$, converge to $A = \mathrm{pr}(B) \in \operatorname{mmcc}\mathbb{R}^m$. We are going to show that there exists a sequence $\{B_i\}$, where $B_i \in \operatorname{mmcc}\mathbb{R}^{m+k}$, that converges to B.

Choose an arbitrary i and let $d_H(A_i, A) = \varepsilon$. Then $A_i \subset \mathrm{pr}(O_\varepsilon(B))$ (this follows from property 1). Denote

$$B_i = \{(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k}) \in O_\varepsilon(B) \mid (x_1, \dots, x_m) \in A_i\}$$

and show that this set is max-min convex. Consider two arbitrary points $a, a' \in A_i$ and let $b, b' \in B_i \subset O_\varepsilon(B)$ be two arbitrary their preimages. We have $[b, b'] \subset O_\varepsilon(B)$ $\mathrm{pr}([b, b']) = [a, a'] \subset A_i$. Therefore, $[b, b'] \in B_i$. □

Define a c-structure F on the hyperspace $\operatorname{mmcc}(\mathbb{R}^n)$ as follows: for any finite subset $\{A_1, \dots, A_k\} \subset \operatorname{mmcc}\mathbb{R}^n$, let $F(\{A_1, \dots, A_k\}) = \{\oplus_{i=1}^k \lambda_i \otimes A_i \mid \oplus_{i=1}^k \lambda_i = \infty\}$. It is an immediate consequence of the previous statements that $\operatorname{mmcc}(\mathbb{R}^n)$ is a metric *l.c.*-space.

Theorem 6.2. *The map* $\mathrm{pr}\colon \operatorname{mmcc}\mathbb{R}^{m+k} \to \operatorname{mmcc}\mathbb{R}^m$ *is soft.*

Proof. Consider diagram (1), where $X = \operatorname{mmcc}\mathbb{R}^{m+k}$, $Y = \operatorname{mmcc}\mathbb{R}^m$, and $f = \mathrm{pr}$. Define a multivalued map $F\colon Z \to X$ as follows:

$$F(z) = \begin{cases} \{\varphi(z)\}, & \text{if } z \in A, \\ \mathrm{pr}^{-1}(\psi(z)), & \text{otherwise.} \end{cases}$$

It follows from the openness of the map pr (Theorem 6.1) that the map F is lower semicontinuous. Also, the map F has the property that every its fibre is an F-set with respect to the c-structure introduced above. It follows from the generalization of Michael selection theorem (Theorem 2.4) that the map pr has a continuous selection, Φ. Then, clearly, $\Phi|A = \varphi$ and $\mathrm{pr}\Phi = \psi$, i.e. the map pr is soft. □

Definition 6.3. Let $I \subset \{1, 2, \dots, n\}$ be a set of indices. A *pseudocone* with the vertex b and the coefficient $p > 0$, with respect to the set of indices I, is the set

$$K_{I,p}(b) = \{x = (x_i) \,|\, x_i + px_j \geq b_i + pb_j, \text{ for all } i, j \in I,\ i \neq j\}.$$

The set $K_{I,p}(b)$, together with every its point, contains all the points greater the or equal to it. Therefore, this set is max-min convex unbounded.

Let $n = m + k$, $m, k \geq 2$, and $I = \{m + 1, \dots, m + k\}$.

Theorem 6.4. *The projection operation* $\mathrm{pr}\colon \operatorname{mmcc}\mathbb{R}^{m+k} \to \operatorname{mmcc}\mathbb{R}^m$ *satisfies the fibrewise disjoint approximation property (FDAP): for every* $\varepsilon\colon \operatorname{mmcc}\mathbb{R}^{m+k} \to (0, \infty)$ *there exist maps* $f, g\colon \operatorname{mmcc}\mathbb{R}^n \to \operatorname{mmcc}\mathbb{R}^n$ *such that*

$$d_H(A, f(A)) \leq \varepsilon(A), \quad d_H(A, g(A)) \leq \varepsilon(A)$$

and $\mathrm{pr}(f(A)) = \mathrm{pr}(g(A)) = \mathrm{pr}(A)$ *for every* $A \in \operatorname{mmcc}\mathbb{R}^n$*, but* $f(\operatorname{mmcc}\mathbb{R}^n) \cap g(\operatorname{mmcc}\mathbb{R}^n) = \emptyset$.

Proof. Let $f\colon \operatorname{mmcc}\mathbb{R}^n \to \operatorname{mmcc}\mathbb{R}^n$, $f(A) = O_{I,\varepsilon}(A)$. Then $d_H(A, f(A)) \leq \varepsilon(A)$ and $\operatorname{pr}(f(A)) = \operatorname{pr}(A)$. Let $g\colon \operatorname{mmcc}\mathbb{R}^n \to \operatorname{mmcc}\mathbb{R}^n$ be defined by the formula

$$g(A) = O_{I,\varepsilon(A)}(A) \cap K_{I,\frac{\varepsilon(A)}{2(\operatorname{diam}(A)+\varepsilon(A))}}(c),$$

where $c = (b_i + \varepsilon(A))$. Then $d_H(A, g(A)) \leq \varepsilon(A)$ and $\operatorname{pr}(g(A)) = \operatorname{pr}(A)$. Thus, the maps f and g modify the set A only "fibrewisely". It is also clear that the maps f, g are continuous.

Let us show that $f(\operatorname{mmcc}\mathbb{R}^n) \cap g(\operatorname{mmcc}\mathbb{R}^n) = \emptyset$. Let $e \in A$ be such that, for some $i_0 > m$, we have $c_{i_0} = b_{i_0} = \{x_{i_0} \mid (x_i) \in A\}$. Consider over any point $\operatorname{pr}(c)$ the fibers $P_{\operatorname{pr}(c)}(f(A))$ and $P_{\operatorname{pr}(c)}(g(A))$. Without loss of generality, we may assume that $P_{\operatorname{pr}(c)}(f(A)), P_{\operatorname{pr}(c)}(g(A)) \in \operatorname{mmcc}(\mathbb{R}^k)$. The boundary of the first set contains the $(k-1)$-dimensional ball

$$\{(x_i) \mid x_{i_0} = c_{i_0} - \varepsilon(A)\} \cap O_{\varepsilon(A)}((c_{m+1}, \dots, c_{m+k})),$$

while this is not the case for the boundary of the second set. □

The following result is a consequence of Theorem 6.4 and the Toruńczyk-West Characterization theorem.

Theorem 6.5. *The map* $\operatorname{pr}\colon \operatorname{mmcc}\mathbb{R}^{n+k} \to \operatorname{mmcc}\mathbb{R}^n$*,* $k \geq 2$*, is a* Q*-manifold bundle.*

Remark 6.6. Note that the condition $k > 1$ is essential in Theorem 6.5. Indeed, if $k = 1$, then one can easily see that the preimages of the singletons are 2-dimensional.

Similarly to Theorem 6.5 one can prove the following result.

Theorem 6.7. *Let* $[a_i, b_i] \subset \mathbb{R}$*, where* $a_i < b_i$*,* $i = 1, \dots, n+k$*,* $n \geq 1$*,* $k \geq 2$*. The map* $\operatorname{pr}\colon \operatorname{mmcc}\left(\prod_{i=1}^{n+k}[a_i, b_i]\right) \to \operatorname{mmcc}\left(\prod_{i=1}^{n}[a_i, b_i]\right)$ *is a trivial* Q*-bundle.*

The notion of max-min convex subset can be naturally defined also for the Hilbert cube $Q = \prod_{i=1}^{\infty}[a_i, b_i]$, where $[a_i, b_i] \subset \mathbb{R}$, $a_i < b_i$, for all i.

Theorem 6.8. *Let* $[a_i, b_i] \subset \mathbb{R}$*, where* $a_i < b_i$*,* $i = 1, 2, \dots$*. Then the space*

$$\operatorname{mmcc}\left(\prod_{i=1}^{\infty}[a_i, b_i]\right)$$

is homeomorphic to the Hilbert cube.

Proof. We represent the product $\prod_{i=1}^{\infty}[a_i, b_i]$ as the inverse limit of the sequence

$$\left\{\prod_{i=1}^{n}[a_i, b_i], \operatorname{pr}_{mn}\right\},$$

where $\operatorname{pr}_{mn}\colon \prod_{i=1}^{m}[a_i, b_i] \to \prod_{i=1}^{n}[a_i, b_i]$ denotes the projection. Then one can easily verify that

$$\operatorname{mmcc}\Big(\prod_{i=1}^{\infty}[a_i, b_i]\Big) = \varprojlim \Big\{\operatorname{mmcc}\Big(\prod_{i=1}^{m}[a_i, b_i]\Big), \operatorname{pr}_{mn}\Big\}$$

and the result follows from Theorem 6.7. □

7. Remarks and open problems

L. Montejano [14] proved that the hyperspace $\mathrm{cc}(U)$ of compact convex sets contained in an open set U of $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the Q-manifold $U \times Q \times [0,1)$. We leave as an open question whether a counterpart of this result is true for the hyperspace of compact max-plus convex sets.

Let ℓ^2_+ denote the set of points in the separable Hilbert space ℓ^2 consisting of the points with nonnegative coordinates. The notion of max-min convex set can also be defined in the case of subsets in ℓ^2_+.

Question 7.1. Is the hyperspace of nonempty compact max-min convex sets in ℓ^2_+ homeomorphic to ℓ^2?

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