INVERSE SCATTERING PROBLEM ON THE AXIS FOR THE TRIANGULAR 2×2 MATRIX POTENTIAL WITH A VIRTUAL LEVEL

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To the memory of A. Ya. Povzner (27.06.1915–22.04.2008), a remarkable person and an outstanding mathematician, who devoted himself to science till the end of his days[*]

ABSTRACT. The characteristic properties of scattering data for the Schrödinger operator on the axis with a triangular 2×2 matrix potential are obtained under the simple or multiple virtual levels being possibly present. Under a multiple virtual level, a pole for the reflection coefficient at $k = 0$ is possible. For this case, the modified Parseval equality is constructed.

0. INTRODUCTION

The scattering theory in mathematics (and even more so in physics) is really unbounded. We restrict ourselves to mentioning only some of the well-known monographs [1], [2], [4], [5], [8], [12], [13], [14], [15], [17], [21], [22]. Here is a quotation from [21, Remarks]: 'The earliest treatment of continuum eigenfunction expansions associated with partial differential operators is due to A. Ya. Povzner [19]. The connection with scattering theory was first emphasized in another his work [20].'

Initially, a complete solution of the inverse scattering problem (ISP) on the axis which allows for a presence of a virtual level (VL) for the Schrödinger equation

$$(1) \qquad -Y'' + V(x)Y = k^2 Y, \quad -\infty < x < \infty,$$

in the case of a real scalar potential having the first moment has been given in [14]. In [23]–[25] the authors solve the ISP on the axis for the equation (1) with a triangular 2×2 matrix potential with no VL[1] via reducing this ISP to the Marchenko equation, and establish necessary and sufficient conditions for a given collection of values to be the scattering data (SD) for a problem as above. Both in [23]–[25] and in the present work we make it implicit that the upper triangular 2×2 matrix potential $V(x)$ has the second moment on the axis and real diagonal elements:

$$(2) \qquad \{(1 + x^2)|V(x)|\} \in L^1(-\infty, +\infty), \quad \operatorname{Im} v_{ll}(x) = 0, \quad l = 1, 2; \quad v_{21}(x) \equiv 0.$$

The case when a VL exists (simple or multiple) for $k = 0$ appears to be essentially more difficult. This work is devoted to solving ISP with a VL being present.

[1]i. e., (1) for $k = 0$ has no bounded non-trivial solutions on the axis.

1. NOTATION AND DEFINITIONS

We set

$$\eta(x) \equiv \begin{cases} 1, & 0 < x < \infty, \\ 0, & -\infty < x < 0, \end{cases} \qquad \text{(the Heaviside function)};$$

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

No special symbol will be reserved for the integrals in the sense of the Cauchy principal value. Fourier integrals of functions from $L^2(-\infty, \infty)$, $L^2(a, \infty)$, or $L^2(-\infty, a)$ are treated implicitly in the sense of convergence in a corresponding L^2 space. In particular, $\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{ikx}}{k} dk = \frac{i}{2} \operatorname{sign} x$.

Indices of matrix elements are positioned as subscripts, the indices $\pm$ and 0 could be both subscripts and superscripts. Matrices are denoted by capital letters, while the matrix elements by the corresponding small letters.

We also use the matrix Wronski determinant,

$$W\{G(x), H(x)\} \equiv G(x)H'(x) - G'(x)H(x), \quad g[-1] \equiv \lim_{k \to 0}\{kg(k)\}.$$

In addition to (1), we consider also the tilda- ($\sim$)-equation:

(3) $$\widetilde{l}[\widetilde{Z}] \equiv -\widetilde{Z}'' + \widetilde{Z}V(x) = k^2\widetilde{Z}, \quad -\infty < x < \infty.$$

The solutions $E_\pm(x, k)$, $\widetilde{E}_\pm(x, k)$ of (1), (3) with asymptotics

(4) $$E_\pm(x, k) \sim e^{\pm ikx}I, \quad \widetilde{E}_\pm(x, k) \sim e^{\pm ikx}I, \quad x \to \pm\infty, \quad \operatorname{Im} k \geq 0,$$

are called the Jost solutions. These can be written in the form [11] (see also [14], [12], [1])

(5)
$$E_\pm(x, k) = Ie^{\pm ikx} \pm \int\limits_x^{\pm\infty} K_\pm(x, t)e^{\pm ikt}dt,$$
$$\widetilde{E}_\pm(x, k) = Ie^{\pm ikx} \pm \int\limits_x^{\pm\infty} \widetilde{K}_\pm(x, t)e^{\pm ikt}dt, \quad \operatorname{Im} k \geq 0,$$

in terms of transformation operators[2], where

(6) $$V(x) = \mp 2dK_\pm(x, x)/dx = \mp 2d\widetilde{K}_\pm(x, x)/dx.$$

In addition to the Jost solutions (4), we will need the solutions

(7) $$E_\pm^\wedge(x, k) \sim e^{\mp ikx}I, \quad \widetilde{E}_\pm^\wedge(x, k) \sim e^{\mp ikx}I, \quad x \to \pm\infty, \quad \operatorname{Im} k \geq 0, \quad k \neq 0,$$

which form fundamental systems together with $E_\pm(x, k)$ and, respectively, with $\widetilde{E}_\pm(x, k)$. Matrix solutions $E_\pm^\wedge(x, k)$ have been constructed and investigated in [1], and the solutions $\widetilde{E}_\pm^\wedge(x, k)$ can be constructed in a similar way. However, unlike the Jost solutions, the solutions (7) are not determined by their asymptotics unambiguously for $\operatorname{Im} k > 0$. On the other hand, once one of the solutions (7) is fixed, let it be $E_+^\wedge(x, k)$, then the corresponding solution $\widetilde{E}_+^\wedge(x, k)$ is determined uniquely under the additional assumption,

(8) $$W\{\widetilde{E}_+^\wedge(x, k), E_+^\wedge(x, k)\} \equiv \widetilde{E}_+^\wedge(x, k)\frac{d}{dx}E_+^\wedge(x, k) - \frac{d}{dx}\widetilde{E}_+^\wedge(x, k)E_+^\wedge(x, k) = 0,$$
$$\operatorname{Im} k \geq 0, \quad k \neq 0.$$

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[2]Note that transformation operators with integration from 0 to x in the general case were introduced by A. Ya. Povzner [18]. Another name for transformation operators is transmutation operators, or transmutators.

Given an arbitrary $\varepsilon > 0$, the solutions (7) can be chosen analytic in k for $|k| > \varepsilon$, $\operatorname{Im} k > 0$, and we will assume they are chosen exactly this way. (See also [13] for the scalar case.)

With real $k \neq 0$, the pairs of functions $E_+(x, \pm k)$ or $E_-(x, \pm k)$, together with $\widetilde{E}_+(x, \pm k)$ or $\widetilde{E}_-(x, \pm k)$, form fundamental systems of solutions for (1) or (3), respectively. Their Wronski determinants are independent of x, and (see, e.g., [3])

$$E_+(x, k) = E_-(x, -k)A(k) + E_-(x, k)B(k),$$
$$E_-(x, k) = E_+(x, -k)C(k) + E_+(x, k)D(k),$$
(9)
$$\widetilde{E}_+(x, k) = C(k)\widetilde{E}_-(x, -k) - D(-k)\widetilde{E}_-(x, k),$$
$$\widetilde{E}_-(x, k) = A(k)\widetilde{E}_+(x, -k) - B(-k)\widetilde{E}_+(x, k),$$

where

$$A(k) = \frac{1}{2ik}W\{\widetilde{E}_-(x, k), E_+(x, k)\},$$

$$C(k) = -\frac{1}{2ik}W\{\widetilde{E}_+(x, k), E_-(x, k)\},$$
(10)
$$B(k) = -\frac{1}{2ik}W\{\widetilde{E}_-(x, -k), E_+(x, k)\},$$

$$D(k) = \frac{1}{2ik}W\{\widetilde{E}_+(x, -k), E_-(x, k)\}, \quad k \in \mathbb{R}\setminus\{0\}.$$

The values

(11)
$$R^+(k) \equiv D(k)C(k)^{-1} = -A(k)^{-1}B(-k),$$
$$R^-(k) \equiv B(k)A(k)^{-1} = -C(k)^{-1}D(-k)$$

are called right (respectively, left) reflection coefficients. A relation between them is given by

(12) $R^-(k) = -A(-k)R^+(-k)A(k)^{-1} = -C(k)^{-1}R^+(-k)C(-k), \quad k \in \mathbb{R}.$

As a consequence of the well-known relations

(13)
$$A(-k)C(k) = I - B(k)D(k), \quad C(-k)A(k) = I - D(k)B(k),$$
$$B(-k)C(k) + A(k)D(k) = D(-k)A(k) + C(k)B(k) = 0,$$

which are themselves due to (9), (10), one has (see [14], [3])

(14) $(I - R^-(-k)R^-(k))^{-1} = A(k)C(-k), \quad (I - R^+(-k)R^+(k))^{-1} = C(k)A(-k).$

The eigenvalues k_j^2 of the problem (1), $j = \overline{1, p}$, coincide with the collection of eigenvalues for scalar scattering problems in the case of real diagonal elements $v_{ll}(x)$ of the matrix potential $V(x)$, as they are roots of the equation $\det A(k) = a_{11}(k)a_{22}(k) = 0$, $\operatorname{Im} k > 0$. Hence there are only finitely many eigenvalues, and $k_j^2 < 0$, $\operatorname{Im} k_j > 0$. Note that $a_{ll}(k) = c_{ll}(k)$ and $\det A(k) = \det C(k)$.

We call the polynomials

$$Z_j^+(t) = -ie^{-ik_jt}\operatorname{Res}_{k_j}\{W^+(k)C(k)^{-1}e^{ikt}\},$$
(15)
$$Z_j^-(t) = -ie^{ik_jt}\operatorname{Res}_{k_j}\{W^-(k)A(k)^{-1}e^{-ikt}\}, \quad j = \overline{1, p}, \quad t \in \mathbb{R},$$
$$\widetilde{Z}_j^+(t) = -ie^{-ik_jt}\operatorname{Res}_{k_j}\{A^{-1}(k)\widetilde{W}^+(k)e^{ikt}\},$$

where

$$W^\pm(k) = \pm\frac{1}{2ik}W\{\widetilde{E}_\pm^\wedge(x, k), E_\mp(x, k)\},$$
(16)
$$\widetilde{W}^+(k) = -\frac{1}{2ik}W\{\widetilde{E}_-(x, k), E_+^\wedge(x, k)\},$$

respectively, the right and the left normalizing polynomials (compare to the scalar case [13], [7]). The normalizing polynomials do not depend on the choice of $\widetilde{E}^{\wedge}_{\pm}$ in the expression (16) for $W^{\pm}(k)$.

In the upper half-plane, similarly to (9), one has the following representation:

$$(17) \qquad E_-(x,k) = E_+(x,k)W^+(k) + E^{\wedge}_+(x,k)C(k), \quad \operatorname{Im} k > 0.$$

Definition 1. The problem (1) of the considered form will be said to have a multiplicity two (respectively, simple, i.e., multiplicity one) VL for $k = 0$ if $\det\{kA(k)\}$ has a multiplicity two (respectively, simple) root at $k = 0$. (Recall that the root $k = 0$ of an arbitrary scalar continuously differentiable function $\varphi(k)$ is called simple if $\varphi(0) = 0$ but $\varphi'(0) \neq 0$ and is said to have multiplicity two if $\varphi(k) = \varphi_1(k)\varphi_2(k)$, with each function $\varphi_l(k)$ being continuously differentiable and having a simple root at $k = 0$.)

Note that $ka_{ll}(k)$ can have root of multiplicity at most one at $k = 0$ because $|a_{ll}(k)| \geq 1$ for $k \in \mathbb{R}$ and $d\{ka_{ll}(k)\}/dk$ are continuous.

This can be rephrased by saying that the problem (1) possesses a multiplicity two VL for $k = 0$ if every scalar equation of the form (1) with potentials $v_{11}(x)$ and $v_{22}(x)$ has a VL. The problem (1) has a simple VL if just one of the potentials $v_{ll}(x)$ has a VL.

Also note that $\operatorname{def}\{kA(k)\}_{k=0} = 1$ in the case of simple VL, and it is 1 or 2 in the case of multiplicity two VL.

Definition 2. A scattering data is a collection of values

$$(18) \qquad \{\ R^+(k),\ k \in \mathbb{R};\ k_j^2 < 0,\ Z_j^+(t),\ j = 1, \ldots, p < \infty\ \},$$

where $R^+(k)$ is a matrix reflection coefficient, $Z_j^+(t)$ matrix normalizing polynomials, $k_j^2 < 0$ the discrete spectrum of the problem (1). $R^+(k), Z_j^+(t)$ are upper triangular 2×2 matrices, as well as the potential $V(x)$. These are right SD; the left SD will be indexed by '-'. If $R^+(k)$ is the right matrix reflection coefficient of a problem of the form (1), then its diagonal elements $r_{ll}^+(k)$ $(l = 1, 2)$ are right reflection coefficients for scalar problems of the form (1) with potentials $v_{ll}(x)$ $(l = 1, 2)$, respectively. Therefore, $r_{11}^+(k)$ and $r_{22}^+(k)$ possess all the properties of scalar reflection coefficients (see the monograph [14]). In particular, they are continuous on the axis, and the potential $v_{ll}(x)$ $(l = 1$ or $2)$ determines a problem with no VL if and only if $r_{ll}^+(0) = -1$, $l = 1, 2$. On the contrary, the potential $v_{ll}(x)$ makes sure a VL is present if and only if $-1 < r_{ll}^+(0) < 1$.

2. THE CASE OF MULTIPLICITY TWO VIRTUAL LEVEL

Theorem 1. *The collection of values (18) is the right SD for the problem (1) with an upper triangular 2×2 matrix potential, which is real on the diagonal, has the second moment on the axis and determines a multiplicity two VL if and only if the following conditions 1–6 are satisfied:*

  1)

$$(19) \qquad R^+(k) = O(k^{-1}), \quad dR^+(k)/dk = o(k^{-1}) \quad as \quad k \to \pm\infty.$$

   The functions

$$(20) \qquad \rho_{jl}(k) \equiv r_{jl}(k)k^{l-j}, \quad 1 \leq j \leq l \leq 2,$$

   are continuously differentiable at all $k \in \mathbb{R}$;

$$(21) \qquad \overline{r_{ll}^+(k)} = r_{ll}^+(-k), \quad |r_{ll}^+(k)| < 1 - \frac{C_l k^2}{1 + k^2}, \quad where \quad C_l > 0, \quad l = 1, 2.$$

2) *Set* $\gamma^+ = \lim\limits_{k \to 0} k r_{12}^+(k) \equiv \rho_{12}(0)$,

$$(22) \qquad R^{\gamma+}(k) = R^+(k) - \gamma^+ k^{-1} J, \quad F_R^{\gamma+}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R^{\gamma+}(k) e^{ikx} dk.$$

Then the function

$$(23) \qquad F_R^+(x) = F_R^{\gamma+}(x) - i\gamma^+ \eta(-x) J$$

is absolutely continuous, and for every $a > -\infty$ *one has*

$$(24) \qquad \left\{ (1+x^2) \left| \frac{d}{dx} F_R^+(x) \right| \right\} \in L^1(a, +\infty).$$

(Note that $F_R^+(x) \neq \frac{1}{2\pi} \int_{-\infty}^{\infty} R^+(k) e^{ikx} dk$ for $\gamma^+ \neq 0$.)

3) *The functions* $c_{ll}^0(z) \equiv a_{ll}^0(z)$, $l = 1, 2$, *given by*

$$(25) \qquad c_{ll}^0(z) \equiv a_{ll}^0(z) := e^{-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1-|r_{ll}^+(k)|^2)}{k-z} dk}, \quad \operatorname{Im} z > 0,$$

are continuously differentiable in the closed upper half-plane (being defined for z
with $\operatorname{Im} z = 0$ *by continuity).*

4) *Set, in view of (12),*

$$(26) \qquad R_0^{\gamma-}(k) \equiv -C_0(k)^{-1} R^+(-k) C_0(-k) - \gamma^- k^{-1} J,$$

where the diagonal elements $c_{ll}^0(k)$ *of the matrix* $C_0(k)$ *are defined by condition 3 of this theorem,* $c_{21}^0 \equiv 0$, $c_{12}^0(k) \equiv c_{12}^0(k+i0)$ *for* $k \in \mathbb{R} \backslash \{0\}$,

$$(27) \qquad z c_{12}^0(z) \equiv \left\{ \psi_0^+(z) - \psi_0^+(0) - \frac{\gamma^+}{\sqrt{1 - r_{22}^+(0)^2}} \sqrt{\frac{1 - r_{11}^+(0)}{1 + r_{11}^+(0)}} \right\} a_{11}^0(z), \quad \operatorname{Im} z > 0,$$

$$(28) \qquad \psi_0^{\pm}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{h^0(k)}{k - z} dk, \quad \pm \operatorname{Im} z > 0,$$

$$(29) \qquad h^0(k) = k a_{11}^0(-k) a_{22}^0(k) \{ r_{11}^+(-k) r_{12}^+(k) + r_{12}^+(-k) r_{22}^+(k) \},$$

$$(30) \qquad \gamma^- = \gamma^+ \prod_{l=1}^{2} \{1 - r_{ll}^+(0)\}^{1/2} \{1 + r_{ll}^+(0)\}^{-1/2}.$$

Then the function

$$(31) \qquad F_{R_0}^-(x) = F_{R_0}^{\gamma-}(x) - i\gamma^- \eta(x) J, \quad where \quad F_{R_0}^{\gamma-}(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} R_0^{\gamma-}(k) e^{-ikx} dk,$$

is absolutely continuous, and for every $a < +\infty$ *one has*

$$(32) \qquad \left\{ (1+x^2) \left| \frac{d}{dx} F_{R_0}^-(x) \right| \right\} \in L^1(-\infty, a).$$

5) $\deg Z_j^+(t) \leq \sum\limits_{l=1}^{2} \operatorname{sign} z_{ll}^{[j]+} - 1$, $j = \overline{1, p}$, *the elements* $z_{ll}^{[j]+}$ *are non-negative and constant.*

6) $\operatorname{rg} Z_j^+(t) = \operatorname{rg} \operatorname{diag} Z_j^+(t) = \operatorname{rg} \operatorname{diag} Z_j^+(0)$, $j = \overline{1, p}$.

Example 1. Set in (1) $V(x) = v(x)J$. In this simplest case it suffices to require that $V(x)$ has only the first moment: $(1+|x|)|V(x)| \in L^1(-\infty, \infty.)$ This problem determines a multiplicity two VL, while a discrete spectrum is absent. It is easy to see that in this case one has

$$K^{\pm}(x,t) = \widetilde{K}^{\pm}(x,t) = \pm\frac{1}{2}J\eta(\pm t \mp x)\int\limits_{\frac{x+t}{2}}^{\pm\infty} v(s)\,ds,$$

$$E^{\pm}(x,k) = \widetilde{E}^{\pm}(x,k) = e^{\pm ikx}I \pm \int\limits_{x}^{\pm\infty} K^{\pm}(x,t)e^{\pm ikt}dt,$$

$$A(k) = C(k) = I - \frac{\gamma^{\pm}}{k}J, \quad \pm R^{\pm}(\pm k) = D(k) = -B(-k) = \frac{1}{2ik}J\int\limits_{-\infty}^{\infty} v(t)e^{-2ikt}dt.$$

Thus $kR^+(k)$ here is a Fourier transform of $v(x)$, which has the first moment on the axis, and a solution of the inverse problem is now given by

$$V(x) = \frac{2i}{\pi}\int\limits_{-\infty}^{\infty} kR^+(k)e^{2ikx}dk.$$

Let us derive $V(x)$ from $R^+(k)$ under the general scheme. Under our notation,

$$\gamma^{\pm} = \frac{1}{2i}\int\limits_{-\infty}^{\infty} v(t)\,dt, \quad R^{\gamma+}(k) = \frac{1}{2ik}J\int\limits_{-\infty}^{\infty} v(t)(e^{-2ikt}-1)\,dt,$$

$$F^{\gamma+}(x) = -\frac{1}{2}J\int\limits_{x/2}^{(\operatorname{sign} x)\infty} v(t)\,dt, \quad F^+(x) = -\frac{1}{2}J\int\limits_{x/2}^{\infty} v(t)\,dt, \quad x \in \mathbb{R}.$$

The Marchenko equation in our case acquires the form $K^+(x,y) + F^+(x+y) = 0$, since the product of matrices $K^+(x,t)F^+(t+y)$ here is identically zero. Thus $V(x) = -2dK^+(x,x)/dx = v(x)J$ according to the general theory.

Proof of Theorem 1.

The 'only if' part for item 1 of Theorem 1. The asymptotic estimates (19) as $k \to \pm\infty$ are established in the same way as this has been done under absence of virtual level in [25, Theorems 1 and 2, Lemma 1]. The properties of $r_{ll}^+(k)$ (21) are direct consequences of the properties of $r^+(k)$ in the scalar problem [14]. The strict inequality in (21) is due to $|r^+(0)| < 1$ under presence of a VL (see [14], and also [10]).

Lemma 1. *The functions $\rho_{jl}(k)$ (20), $-\infty < k < \infty$, are continuously differentiable on the axis.*

Proof of Lemma 1 outside of a neighborhood of $k = 0$ coincides to that of Lemma 1 from [25] under absence of a VL.

Consider $r_{ll}^+(k)$ in a neighborhood of $k = 0$, which is equivalent to considering $r^+(k) = -\frac{b(-k)}{a(k)}$ for a scalar problem. We are about to apply the representations for $a(k)$, $b(k)$ under presence of a VL [10]

$$(33) \quad a(k) = \frac{1}{2}\left\{1 + \int\limits_{0}^{\infty}\varphi^+(t)e^{ikt}dt\right\}e_-(0,k) - \frac{1}{2}\left\{-1 + \int\limits_{-\infty}^{0}\varphi^-(t)e^{-ikt}dt\right\}e_+(0,k),$$

$$(34) \qquad b(k) = \frac{1}{2}\left\{ 1 - \int_{-\infty}^{0} \varphi^-(t)e^{ikt}dt \right\}e_+(0,k) + \frac{1}{2}\left\{ 1 + \int_{0}^{\infty} \varphi^+(t)e^{-ikt}dt \right\}e_-(0,k),$$

where $\varphi^{\pm}(z) \in L^1(0,\pm\infty)$ are bounded under existence of the first moment for $v(x)$, and satisfy the Marchenko equations

$$(35) \qquad \varphi^{\pm}(z) \mp \int_{0}^{\pm\infty} \varphi^{\pm}(t)F_R^{\pm}(t+z)\,dt = \pm F^{\pm}(z)$$

(which in the special case $e^{\pm}(0,0) = 0$ turn out to be homogeneous), $|F_R^{\pm}(x)| \le C\sigma^{\pm}(\frac{x}{2})$, $\sigma^{\pm}(x) \equiv \pm \int_{x}^{\pm\infty} |v(t)|dt$ [14, p. 195].

Let us demonstrate that *if the potential has the n-th moment (in our case $n = 2$), then $\varphi^{\pm}(z)$ have $(n-1)$-th moment (the first moment in our case).*

We multiply (35) by z^{n-1} and then integrate to obtain

$$\pm \int_{0}^{\pm\infty} |\varphi^{\pm}(z)z^{n-1}|dz \le \int_{0}^{\pm\infty} |\varphi^{\pm}(t)|dt \int_{t}^{\pm\infty} |F_R^{\pm}(z)z^{n-1}|dz + \int_{0}^{\pm\infty} |F_R^{\pm}(z)z^{n-1}|dz < \infty.$$

The existence of the moment for $\varphi^{\pm}(z)$ is now established. Therefore one may differentiate the expressions for $a(k)$ (33) and $b(k)$ (34) in k, with $k = 0$ included, under the integral. Even more, $da(k)/dk$, $db(k)/dk$ turn out to be continuous in k, with $k = 0$ included. This implies continuous differentiability for $r^+(k)$ and $r_{ll}^+(k)$ on the axis $-\infty < k < \infty$ since $|a(k)|^2 = 1 + |b(k)|^2 \ne 0$, $k \in \mathbb{R}$. Now we demonstrate continuous differentiability for $\rho_{12}(k)$ (20). It follows from (11) that

$$(36) \quad r_{12}^+(k) = \{d_{12}(k) - c_{12}(k)r_{11}^+(k)\}a_{22}(k)^{-1} = -\{b_{12}(-k) + a_{12}(k)r_{22}^+(k)\}a_{11}(k)^{-1}.$$

Therefore, continuous differentiability for $kr_{12}^+(k)$ follows from the differentiability properties of $a_{11}(k) \ne 0$ and $r_{22}^+(k)$ proved above, together with continuous differentiability in k for $ka_{12}(k)$ and $kb_{12}(-k)$, which is itself due to the representation

$$(37) \qquad \begin{aligned} A(k) &= I - \frac{1}{2ik}\left\{ \int_{-\infty}^{\infty} V(x)\,dx + \int_{-\infty}^{0} A_1(t)e^{-ikt}dt \right\}, \\ B(k) &= \frac{1}{2ik} \int_{-\infty}^{\infty} B_1(t)e^{-ikt}dt, \end{aligned}$$

where $(1 + |t|)|A_1(t)| \in L^1(-\infty, 0)$, $(1 + |t|)|B_1(t)| \in L^1(-\infty, \infty)$. This representation is given by Lemma 6 from [24], which is an analog of Lemma 3.5.1 from [14] applied to the case of matrix potentials having the second moment on the axis. This proves Lemma 1. $\qquad \square$

Hence the 'only if' part of Theorem 1 concerning all the conditions of item 1 of the theorem is proved.

The 'only if' part for item 2 of Theorem 1. Note that $r_{12}^{\gamma+}(k) = r_{12}^+(k) - \gamma^+ k^{-1} = [\rho_{12}(k) - \rho_{12}(0)]k^{-1}$ is continuous due to the properties of $\rho_{12}(k)$, hence $R^{\gamma+}(k) \in L^2(-\infty, \infty)$ by item 1. Therefore there exists $F_R^{\gamma+}(x) \in L^2(-\infty, \infty)$.

Lemma 2. *Set*

$$(38) \qquad F^+(x) = F_R^+(x) + \sum_{j=1}^{p} Z_j^+(x)e^{ik_jx}, \quad \operatorname{Im} k > 0,$$

where $F_R^+(x)$ is defined by (23) and $Z_j(x)$ by (15). Then $F^+(x)$ (38) satisfies the Marchenko equation

$$(39) \qquad K^+(x,y) + F^+(x+y) + \int_x^\infty K^+(x,t)F^+(t+y)\,dt = 0.$$

Hence $F^+(x)$ and $F_R^+(x)$ are absolutely continuous and $dF^+(x)/dx$, $dF_R^+(x)/dx$ have the second moment on (a,∞) for all $a > 0$.

Proof of Lemma 2. The equation (39) for the diagonal elements $k_{ll}^+(x,y)$ and $f_{ll}^+(x)$ is known [14]. Similarly to [14, § 3.5], we use the matrix relation

$$(40) \qquad E_-(x,k)\{C(k)^{-1} - I\} = E^+(x,-k) + E^+(x,k)R^+(k) - E_-(x,k)$$

to extract the scalar relations corresponding to the indices 1 2

$$
(41) \quad
\begin{aligned}
& -e_{11}^-(x,k)\frac{c_{12}(k)}{c_{11}(k)c_{22}(k)} + e_{12}^-(x,k)\Big(\frac{1}{c_{22}(k)} - 1\Big) \\
&\qquad = e_{12}^+(x,-k) - e_{12}^-(x,k) + e_{11}^+(x,k)[r_{12}^{\gamma+}(k) + \gamma^+ k^{-1}] + e_{12}^+(x,k)r_{22}^+(k).
\end{aligned}
$$

Let us multiply this relation by $\frac{1}{2\pi}e^{iky}$ and then integrate in k from $-\infty$ to $+\infty$. We are going to apply the Fourier inversion formulas and representations for the Jost functions in terms of transformation operators. Using this techniques, we arrange a contour integration of the l.h.s. along the semicircles of radii N and ε in the upper half-plane, and the segments $(-N, -\varepsilon)$, (ε, N) ($\varepsilon \to 0$, $N \to \infty$). This procedure, which takes into account (15)–(17), allows one to deduce for $y > x$ that

$$
(42)
\begin{aligned}
& -e_{11}^-(x,0)\frac{ic_{12}[-1]}{2c_{11}(0)c_{22}(0)} = k_{12}^+(x,y) + \int_x^\infty k_{12}^+(x,t)f_{22}^+(t+y)\,dt + f_{12}^{\gamma+}(x+y) \\
&\qquad + \int_x^\infty k_{11}^+(x,t)f_{12}^{\gamma+}(t+y)\,dt + \frac{i\gamma^+}{2}\Big\{\operatorname{sign}(x+y) + \int_x^\infty k_{11}^+(x,t)\operatorname{sign}(t+y)\,dt\Big\},
\end{aligned}
$$

where $f_{12}^{\gamma+}(t)$ is the matrix element of $F^{\gamma+}(t) \equiv F_R^{\gamma+}(t) + \sum_{j=1}^p Z_j^+(t)e^{ik_j t}$. Now multiply (41) by k. With $k \to 0$, we get for $-\infty < x < \infty$

$$(43) \qquad -e_{11}^-(x,0)c_{11}(0)^{-1}c_{22}(0)^{-1}c_{12}[-1] = \gamma^+ e_{11}^+(x,0),$$

which corresponds to the VL for the potential $v_{11}(x)$.

Now we multiply (43) by $\frac{i}{2}$ and subtract this from (42) to obtain (39) with $F^+(x)$ (38). This completes the proof of Lemma 2. $\square$

Hence the 'only if' part of Theorem 1 concerning its condition 2 is proved.

The 'only if' part for item 3 of Theorem 1. The representations (25) for $a_{ll}(z)$ in terms of the reflection coefficients $r_{ll}^+(k)$ as in the direct problem (the Wronski determinants divided by $2ik$) are well known [14]. Those are continuously differentiable, as one can observe from the proof of Lemma 1, which was to be proved.

Remark 1. *As an additional explanation, we point out that, under some discrete spectrum being present, the expressions for $a_{ll}(z) \equiv c_{ll}(z)$ differ from those for $a_{ll}^0(z) \equiv c_{ll}^0(z)$ (25) by a product of linear-fractional terms $c_{ll}(z) \equiv a_{ll}(z) = c_{ll}^0(z)\prod_{j=1}^p[(z - k_j)/(z + k_j)]^{\operatorname{sign} z_{ll}^{[j]+}}$, $\operatorname{Im} z > 0$. This does not affect simultaneous smoothness of $c_{ll}(z)$ and $c_{ll}^0(z)$ for $\operatorname{Im} z \geq 0$. Furthermore, it is obvious that $c_{ll}(0) = (-1)^{p_l}c_{ll}^0(0)$, $a_{ll}(0) = (-1)^{p_l}a_{ll}^0(0)$, where $p_l = \sum_{j=1}^p \operatorname{sign} z_{ll}^{[j]+}$.*

The 'only if' part for item 4 of Theorem 1. Assume first that the problem (1), hence also the SD (18) has no discrete spectrum. Use the properties of $a_{ll}^0(k)$ and $R^+(k)$ (item 1 of Theorem 1) to deduce that $h^0(k) = O(k^{-1})$, $dh^0(k)/dk = o(k^{-1})$ as $k \to \pm\infty$, and these are continuous on the entire k-axis.

Thus (27), (28), (29) imply that $zc_{12}^0(z)$ (in a pair with $-za_{12}^0(-z)$) gives a bounded solution of the Riemann-Hilbert problem in the half-plane (with the factorized coefficient $\frac{a_{11}^0(k)}{a_{22}^0(-k)}$)

$$(44) \qquad \frac{kc_{12}^0(k)}{a_{11}^0(k)} - \psi_0^+(k) = -\frac{ka_{12}^0(-k)}{a_{22}^0(-k)} - \psi_0^-(k) \equiv \text{const.}$$

(It is implicit here that

$$(45) \qquad \psi_0^+(k) - \psi_0^-(k) = h^0(k)$$

due to the Plemelj-Sokhotski formula.)

The constant in (44) can be computed via passage to a limit as $k \to 0$ using

$$(46) \qquad \text{const} = a_{11}^0(0)^{-1}c_{12}^0[-1] - \psi_0^+(0) = a_{12}^0[-1]a_{22}^0(0)^{-1} - \psi_0^-(0)$$

and the subsequent application of the direct problem. Namely, (36) implies

$$(47) \qquad c_{12}^0[-1] = -\gamma^+ a_{22}^0(0)\{1 + r_{11}^+(0)\}^{-1}, \quad a_{12}^0[-1] = -\gamma^+ a_{11}^0(0)\{1 + r_{22}^+(0)\}^{-1}.$$

Also, by (25) one has

$$(48) \qquad a_{ll}^0(0) \equiv c_{ll}^0(0) \equiv (1 - |r_{ll}^+(0)|^2)^{-\frac{1}{2}}$$

due to the Plemelj-Sokhotski formulas. Since the integrand in (25) is odd in k at $z = 0$, we deduce that const has the same value in (44) and in (27). Therefore, $zc_{12}^0(z)$ (27) is the only bounded solution of the problem (44) (with const being fixed). On the other hand, the direct scattering problem implies that the matrix elements $c_{12}^0(k)$ and $a_{12}^0(-k)$ satisfy the same equation (44). Hence $c_{12}^0(z)$ in (27), together with $c_{ll}^0(z)$ ($l = 1, 2$) form the matrix $C_0(k) = C(k)$ (10) derived from the Wronski determinant divided by $2ik$ for the Jost solutions.

Thus the left reflection coefficient of the direct problem $R_0^-(k)$ (11) can be expressed via $R^+(k)$ by (12), where $C(k) = C_0(k)$ is given by (25) and (27). Then the function $F_{R_0}^-(x)$ (31) possesses the properties indicated in item 4, with (31) included, similarly to the function $F_R^+(x)$ (23), (24).

Let us verify (30). It follows from the direct problem, similarly to (47), that

$$(49) \quad \gamma^- \equiv r_{12}^{[0]-}[-1] = -c_{12}^0[-1]\{1 + r_{22}^{[0]-}(0)\}a_{11}^0(0)^{-1} = -a_{12}^0[-1]\{1 + r_{11}^{[0]-}(0)\}a_{22}^0(0)^{-1}.$$

This, together with (47), leads to (30), in view of $r_{ll}^{[0]-}(0) = -r_{ll}^+(0)$.

The 'only if' part for item 4 of Theorem 1 is already proved under absence of discrete spectrum. If some discrete spectrum is present, then the 'only if' part for item 4 of Theorem 1 is established just as in [25, Lemma 2]; this has been done by the authors under absence of a VL, via an application of the consecutive elimination of eigenvalues method[3].

The 'only if' part for items 5 and 6 of Theorem 1 under some discrete spectrum being present can be proved in the same way as it has been done in [23]–[25] under absence of VL.

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[3]Note that Theorem 1 is proved in [25] in two modifications, labeled by conditions 4 or 4a (both without a VL). Condition 4 of Theorem 1 in the present work is an analog of condition 4a from [25]. The 'only if' part for condition 4a in [25] was established after proving the 'only if' part for condition 4 in [25] by the method of consecutive elimination of eigenvalues. The latter condition was formulated with discrete spectrum being used explicitly.

Let us prove *the 'only if' part for conditions 1–4 of Theorem 1* under absence of discrete spectrum[4]. As a consequence of conditions 1–2, in view of Lemma 2, we have the equation (39), which has for every x a single solution $K^+(x, y)$, similarly to [14]. This solution is a kernel of the transformation operator (5) for solutions $E^+(x, k)$ of equations of the form (1) with a potential

$$(50) \qquad V(x) = V^+(x) = -2dK^+(x, x)/dx,$$

having the second moment for all $a < x < +\infty$ and real diagonal elements $v_{ll}^+(x)$. In a similar way, conditions 1, 3, 4 imply the equation

$$(51) \qquad K^-(x, y) + F_R^-(x + y) + \int_{-\infty}^{x} K^-(x, t) F_R^-(t + y) dt = 0,$$

with $F_R^-(x)$ (31). This equation is uniquely solvable with respect to $K^-(x, y)$ at every x, where $K^-(x, y)$ appears to be a kernel of the transformation operator (5) for solutions of an equation of the form (1) with a potential

$$(52) \qquad V(x) = V^-(x) = 2dK^-(x, x)/dx.$$

It remains to prove an important fact that

$$(53) \qquad V^+(x) = V^-(x), \quad \text{for} \quad -\infty < x < \infty.$$

For that, similarly to the case of scalar problem [14], [12], we introduce a matrix valued function $H^-(x, k)$

$$(54) \qquad H^-(x, k) = \{E^+(x, -k) + E^+(x, k) R^+(k)\} C(k).$$

It is clear from (54) that $H^-(x, k)$ is a solution of (1) with $V(x) = V^+(x)$.

Let us prove that

$$(55) \qquad H^-(x, k) = E^-(x, k),$$

which thus satisfies (1) also with $V(x) = V^-(x)$, hence satisfies (53) as well. Observe that in the scalar case the ISP with a real potential on the axis was solved [14], and it was also demonstrated that

$$(56) \qquad h_{ll}^-(x, k) = e_{ll}^-(x, k)$$

(see [14, proof of Theorem 3.5.1], [12, (6.5.17)]). In view of this, it suffices to prove that

$$(57) \qquad h_{12}^-(x, k) = e_{12}^-(x, k).$$

For doing this, we prove the following properties of $H^-(x, k)$, which are deducible from conditions 1–4 of Theorem 1.

I). $H^-(x, k)$ admits an analytic continuation to the half-plane $\operatorname{Im} z > 0$, and there with $z \to \infty$ one has

$$(58) \qquad |H^-(x, z) - e^{-ixz} I| = O\left(|z|^{-1} e^{x \operatorname{Im} z}\right).$$

II). $zH^-(x, z)$ is continuous in the closed upper half-plane $\operatorname{Im} z \geq 0$ and there $zH^-(x, z) \to 0$ as $z \to 0$ uniformly in x.

III).

$$(59) \qquad \{H^-(x, k) - e^{-ikx} I\} \in L^2(-\infty, \infty; dk).$$

The principal distinction here from the scalar case [14], [12], [8] is a possible singularity of order k^{-1} as $k \to 0$ for the elements $r_{12}^+(k)$, $a_{12}(k)$, $c_{12}(k)$ (cf. [7]).

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[4]Since in this special case the values labeled by 0 coincide to those without index 0, the index 0 will be omitted throughout this proof to simplify notation.

Set

$$(60) \qquad G^+(x,y) \equiv F^{\gamma+}(x+y) + \int_x^\infty K^+(x,t)F^{\gamma+}(t+y)dt.$$

Then

$$(61) \qquad \int_{-\infty}^\infty G^+(x,y)e^{-iky}dy = E^+(x,k)R^{\gamma+}(k).$$

Now (39) and (22) imply (23) for all $x \in \mathbb{R}$, taking into account that $K^+(x,y) = 0$ for $y < x$

$$(62)$$
$$H^-(x,k) = \left\{e^{-ikx}I + \int_{-\infty}^x G^+(x,y)e^{-iky}dy\right\}C(k) + \gamma^+k^{-1}Jc_{22}(k)$$

$$\times \left\{e_{11}^+(x,k) + \eta(-x)\left[-2i\sin kx + \int_x^{-x} k_{11}^+(x,t)(e^{-ikx} - e^{ikt})dt\right]\right\}.$$

This formula implies property I) for $H^-(x,z)$ by virtue of items 1 and 3, together with (27)–(29) in the formulation of Theorem 1, whence

$$|C(z) - I| = O(|z|^{-1}) \quad \text{for } |z| \to \infty, \quad \text{Im } z \geq 0,$$

using the Plemelj-Sokhotski and Plemelj-Privalov theorems [16, § 18], [9, § 5], and the representation (37) for the diagonal elements within the direct problem. We deduce from (62) via integration by parts which incorporates the jump of $G^+(x,y)$ in y at $y = -x$, that for $x > 0$

$$H^-(x,z) - e^{-ixz}I = e^{-izx}\{C(z) - I\} + \left\{\frac{e^{-izy}}{-iz}G^+(x,y)\,|_{y=-\infty}^x\right.$$

$$- \left[\int_{-\infty}^{-x-0} + \int_{-x+0}^x\right]G_y^{+\prime}(x,y)\frac{e^{-izy}}{-iz}dy - G(x,y)\,|_{y=-x-0}^{y=-x+0}\, e^{izx}\right\}C(k)$$

$$+ \gamma^+z^{-1}Jc_{22}(z)e_{11}^+(x,z),$$

whence the required estimate.

On the contrary, if $x < 0$, then the jump of $G^+(x,y)$ is outside the integration interval. However, this case requires an additional estimate for the term $\gamma^+z^{-1}Jc_{22}(z)\{\ldots\}$ in the r.h.s. of (62), taking into account that $\eta(-x) = 1$ for $x < 0$. Let us estimate the expression in the above braces $\{\ldots\}$

$$\{\ldots\} = e_{11}^+(x,z) - 2i\sin zx + \int_x^{-x} k_{11}^+(x,t)(e^{-izx} - e^{izt})dt$$

$$= \int_{-x}^\infty k_{11}^+(x,t)e^{izt}dt + e^{-izx} + e^{-izx}\int_x^{-x} k_{11}^+(x,t)dt,$$

whence the required exponential estimate for $x < 0$: $|\{\ldots\}| \leq e^{-izx}\text{const}$, where const depends on $x < 0$, but not on z. This assures the validity of property I for the matrix valued function $H^-(x,z)$. Note that for the scalar equation, hence for the diagonal elements $h_{ll}^-(x,z)$, property I was established in [14] in the proof of theorem 3.5.1.

Let us prove property II. We use the formulas (27)–(29) for $c_{12}(k)$, together with (48), to deduce from (62) via multiplying by k as $k \to 0$, in view of the relation

(63) $$f_{ll}^{\gamma+}(x) = f_{ll}^{+}(x) \quad (l = 1, 2)$$

that

(64) $$h_{12}^{-}(x, [-1]) \equiv \lim_{k \to 0}\{kh_{12}^{-}(x, k)\} = \Big\{1 + \int\limits_{-\infty}^{x} g_{11}^{+}(x, y)dy\Big\}c_{12}[-1] + \gamma^{+}c_{22}(0)e_{11}^{+}(x, 0).$$

On the other hand, (56) and (62) imply

(65) $$e_{11}^{-}(x, 0) = h_{11}^{-}(x, 0) = \Big\{1 + \int\limits_{-\infty}^{x} g_{11}^{+}(x, y)dy\Big\}c_{11}(0).$$

Hence (64) yields

$$h_{12}^{-}(x, [-1]) = \gamma^{+}[1 - r_{22}^{+}(0)^2]^{-1/2}\{e_{11}^{+}(x, 0) - [1 + r_{11}^{+}(0)]^{-1}e_{11}^{-}(x, 0)\} = 0,$$

by virtue of (43), (47), (48). That is, for Im $z \geq 0$ one has

(66) $$\lim_{z \to 0}\{zh_{12}^{-}(x, z)\} = 0.$$

This already implies property II.

Let us prove property III. Due to (56), it suffices to establish that $h_{12}^{-}(x, k) \in L^2(-\infty, \infty; dk)$, where, in view of (54),

(67) $$\begin{aligned} h_{12}^{-}(x, k) &= \{e_{11}^{+}(x, -k) + e_{11}^{+}(x, k)r_{11}^{+}(k)\}c_{12}(k) \\ &+ \{e_{12}^{+}(x, -k) + e_{11}^{+}(x, k)r_{12}^{+}(k) + e_{12}^{+}(x, k)r_{22}^{+}(k)\}c_{22}(k). \end{aligned}$$

We use asymptotics of the terms as $k \to \pm\infty$, and, in particular, $e_{12}^{+}(x, -k) = \int\limits_{x}^{\infty} k_{12}^{+}(x, t)e^{-ikt}dt \in L^2(-\infty, \infty; dk)$, to observe that it remains to demonstrate that $h_{12}^{-}(x, k) \in L^2$ in a neighborhood of $k = 0$. Note that by (66), the singularities in (67) of order k^{-1} as $k \to 0$ annihilate each other (they appear due to $c_{12}(k)$ and $r_{12}^{+}(k)$). This implies that

(68) $$\begin{aligned} h_{12}^{-}(x, k) &\equiv \{x, k\}c_{12}(k) - \{x, 0\}\frac{c_{12}[-1]}{k} + \frac{\gamma^{+}}{k}[e_{11}^{+}(x, k)c_{22}(k) \\ &- e_{11}^{+}(x, 0)c_{22}(0)] + O(1) \quad \text{for} \quad k \to 0, \end{aligned}$$

where $\{x, k\}$ here stands for the first braces in (67) and, in particular, $\{x, 0\} = e_{11}^{+}(x, 0)[1 + r_{11}^{+}(0)]$. We have

(69) $$\begin{aligned} h_{12}^{-}(x, k) &= [\{x, k\} - \{x, 0\}]c_{12}(k) + \{x, 0\}\Big[c_{12}(k) - \frac{c_{12}[-1]}{k}\Big] \\ &+ \frac{\gamma^{+}}{k}[e_{11}^{+}(x, k)c_{22}(k) - e_{11}^{+}(x, 0)c_{22}(0)] + O(1). \end{aligned}$$

It is easy to see that the first and the third terms are bounded in k in a neighborhood of zero, which is, in particular, due to, continuous differentiability in k of $r_{ll}^{+}(k)$ and $c_{ll}(k) \equiv a_{ll}(k)$. It remains to observe that the second term in (69) allows by virtue of (27)–(29) an estimate

(70) $$|c_{12}(k) - c_{12}[-1]k^{-1}| = |k^{-1}\{kc_{12}(k) - c_{12}[-1]\}| = O(|k|^{-\varepsilon})$$

with $\varepsilon > 0$ being arbitrarily small, as $h^0(k)$ in (28) is continuously differentiable by (29) and conditions of item 1 of Theorem 1. Therefore, $c_{12}(k)$ (27) belongs to the Hölder

class with exponential $\mu = 1 - \varepsilon$ by the Plemelj-Privalov theorem mentioned above [16], [9]. This already implies the estimate (70), which completes the proof of property III for $H^-(x, k)$ (59).

Now we are in a position to prove (55). Due to properties I–III we have for some $P^-(x, t) \in L^2(-\infty, x)$

$$(71) \qquad H^-(x, k) = e^{-ikx} I + \int_{-\infty}^{x} P^-(x, t) e^{-ikt} dt.$$

On the other hand, it follows from (54) and (12), (14) that

$$(72) \qquad E^+(x, k) A(k)^{-1} - e^{ikx} I = H^-(x, k) R^-(k) + H^-(x, -k) - e^{ikx} I.$$

Also, (72) implies in view of (71) that
(73)

$$E^+(x, k) A(k)^{-1} - e^{ikx} I = \int_{-\infty}^{x} P^-(x, t) e^{ikt} dt + e^{-ikx} R^-(k) + \int_{-\infty}^{x} P^-(x, t) e^{-ikt} dt R^-(k).$$

Let us multiply both sides of (73) by $\frac{1}{2\pi} e^{-iky}$ and integrate in dk for $y < x$ from $-\infty$ to $+\infty$. In the l.h.s. of (73), a contour integration along $(-N, -\varepsilon)$, (ε, N), and upper semicircles of radii ε and N, gives zeros for the diagonal elements as $\varepsilon \to 0$, $N \to \infty$, similarly to the scalar case [14]. Thus for $y < x$ we obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \{ E^+(x, k) A(k)^{-1} - e^{ikx} I \} e^{-iky} dk$$

$$(74) \qquad = \frac{1}{2\pi} J \int_{-\infty}^{\infty} \left\{ -\frac{a_{12}(k)}{c_{11}(k) c_{22}(k)} e_{11}^+(x, k) + \frac{1}{c_{22}(k)} e_{12}^+(x, k) \right\} e^{-iky} dk$$

$$= -\frac{i}{2} J \frac{a_{12}[-1]}{c_{11}(0) c_{22}(0)} e_{11}^+(x, 0) = -\frac{i}{2} J \frac{a_{12}[-1]}{c_{22}(0)} (1 + r_{11}^-(0)) e_{11}^-(x, 0)$$

$$= \frac{i}{2} \gamma^- e_{11}^-(x, 0) J = \frac{i}{2} \gamma^- J \left(1 + \int_{-\infty}^{x} p_{11}^-(x, t) dt \right).$$

It is implicit here that

$$(75) \qquad e_{11}^+(x, 0) c_{11}(0) = e_{11}^-(x, 0)(1 + r_{11}^-(0)),$$

which is derived from (43) in view of (47)–(49) and (30). On the other hand, an integration in the r.h.s. of (73) for $y < x$ and in the notation of (30), (31) gives

$$P^-(x, y) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ R^{\gamma^-}(k) + J \frac{\gamma^-}{k} \right\} e^{-ik(x+y)} dk$$

$$(76) \qquad + \int_{-\infty}^{x} P^-(x, t) \frac{dt}{2\pi} \int_{-\infty}^{\infty} \left\{ R^{\gamma^-}(k) + J \frac{\gamma^-}{k} \right\} e^{-ik(t+y)} dk$$

$$= P^-(x, y) + F^{\gamma^-}(x + y) - J \frac{i}{2} \gamma^- \operatorname{sign}(x + y)$$

$$+ \int_{-\infty}^{x} P^-(x, t) \left\{ F^{\gamma^-}(t + y) - J \frac{i}{2} \gamma^- \operatorname{sign}(t + y) \right\} dt.$$

Now we equate (74) and (76) to obtain

$$\frac{i}{2}\gamma^- J\Big(1 + \int_{-\infty}^{x} p_{11}^-(x,t)dt\Big) = P^-(x,y) + F^-(x+y)$$

$$+ \frac{i}{2}\gamma^- J + \int_{-\infty}^{x} P^-(x,t)\Big\{F^-(t+y) + J\frac{i}{2}\gamma^-\Big\}dt,$$

i.e., $P^-(x,y)$ satisfies the Marchenko equation

$$P^-(x,y) + F^-(x+y) + \int_{-\infty}^{x} P^-(x,t)F^-(t+y)dt = 0.$$

This means that $P^-(x,t)$ is a transformation operator such that $P^-(x,t) = K^-(x,y)$, hence (55) holds, and Theorem 1 is proved completely under absence of discrete spectrum.

Now consider the case when the data (18) contains finitely many $k_j^2 < 0, j = 1, \ldots, p < \infty$, and the corresponding matrix polynomials $Z_j^+(t)$. Then Theorem 1 can be proved, in view of conditions 5 and 6, by the subsequent adding the eigenvalues method, which is well-known for the scalar selfadjoint case (see, e.g., [12] or [4]). In our case the eigenvalues k_j^2 can be simple or multiplicity two, according to ranks of the normalizing polynomials $Z_j(t)$. For $p = 0$ the set $\{k_j^2, \quad Z_j(t)\}$ is empty, so that in this case Theorem 1 is proved, with the corresponding potential $V(x) = V_0(x)$ being uniquely determined by (50) using the solution $K^+(x,y) = K_0^+(x,y)$ of the equation (39).

Lemma 3. *Suppose that the assumptions of Theorem 1 are satisfied for a data of the form (18) with $j = 1, \ldots, p, p + 1$, then they are satisfied for a part of this data with $j = 1, \ldots, p$, where $k_{j+1}^2 < k_j^2 < 0$. Suppose also that for given p the ISP is uniquely solvable. Denote the corresponding potential by $V_p(x)$. Then the ISP with $p + 1$ instead of p in (18) is also uniquely solvable and*

(77) $$V_{p+1}(x) = V_p(x) - 2dB_p(x,x)/dx.$$

Here $B_p(x,y)$ is determined from the degenerate integral equation

(78) $$B_p(x,y) + F_p(x,y) + \int_{x}^{\infty} B_p(x,t)F_p(t,y)dt = 0 \quad (x < y)$$

where

(79)
$$F_p(x,y) = E_+^{<p>}(x,k_{p+1})Z_{p+1}^+(0)\widetilde{E}_+^{<p>}(y,k_{p+1})$$

$$- i\frac{d}{dk}\{E_+^{<p>}(x,k)Z_{p+1}^{+'}(0)\widetilde{E}_+^{<p>}(y,k)\}_{k=k_{p+1}}$$

and $E_+^{<p>}(x,k), \widetilde{E}_+^{<p>}(y,k)$ stand for the Jost solutions of the equation (1) with $V(x) = V_p(x)$.

An explicit solution of the equation (78) and its investigation has been done by the authors in [24] under absence of a VL. However, the argument and the result under some VL being present (both multiple and simple) remain intact, so we do not reproduce them here.

Let us note only that, under the procedure of adding eigenvalues, starting from $p = 0$ up to the given p, the right reflection coefficient $R^+(k)$ does not vary. In particular, the values $r_{11}^+(0)$ and $r_{22}^+(0)$ are the same for all p, hence the potentials $V_p(x)$ produced within this induction argument, starting from $p = 0$ up to given p have (or have not) a VL of the same multiplicity. This completes the proof of Theorem 1.   □

3. The case of multiplicity one virtual level

Theorem 2. *The values (18) are the right SD for the problem (1) with a matrix potential of the form considered in Theorem 1, but with precisely multiplicity one VL if and only if the following conditions 1–6 are satisfied:*

1) *All the claims in item 1 of Theorem 1 are valid with the refinement as follows. Looking at the inequalities (21) for $|r_{ll}^+(k)|$, one should choose l (either $l = 1$ or $l = 2$) such that for this l there exists a VL. The corresponding inequality should be left strict, and in another one (with the different l) one should replace '<' by '≤'.*

Besides that, in the case of a VL being present for the potential $v_{11}(x)$, one should have $dr_{12}^+(k)/dk \in C(\mathbb{R})$, and in the case of a VL for the potential $v_{22}(x)$ it should be $r_{12}^+(k) \in C(\mathbb{R}) \bigcap C^1(\mathbb{R}\backslash\{0\})$.

2) *All the claims in item 2 of Theorem 1 are valid with $\gamma^+ = 0$, hence also with $R^+(k) = R^{\gamma+}(k)$ and with $F_R^+(x) = F_R^{\gamma+}(x)$.*

3) *The claim in item 3 of Theorem 1 on continuous differentiability of $c_{ll}^0(z) \equiv a_{ll}^0(z)$ (25) for $\operatorname{Im} z \geq 0$ is valid for exactly that $l = 1$ or 2, for which a VL is present, and for another l these are functions $zc_{ll}^0(z) \equiv za_{ll}^0(z)$ which are continuously differentiable for $\operatorname{Im} z \geq 0$.*

4) *According to (12), set*

$$R_0^-(k) \equiv -C_0(k)^{-1}R^+(-k)C_0(-k). \tag{80}$$

Here the diagonal elements $c_{ll}^0(k)$ of the matrix $C_0(k)$ are determined by condition 3 of this theorem, $c_{21}^0 \equiv 0$, $c_{12}^0(k) = c_{12}^0(k + i0)$, $k \in \mathbb{R}\backslash\{0\}$. Also, if a VL corresponds to the potential $v_{11}(x)$ then

$$zc_{12}^0(z) = [\psi_0^+(z) - \psi_0^+(0) + h^0(0)]a_{11}^0(z), \quad \operatorname{Im} z > 0. \tag{81}$$

On the other hand, if a VL is present for the potential $v_{22}(x)$ then

$$zc_{12}^0(z) = [\psi_0^+(z) - \psi_0^+(0)]a_{11}^0(z), \quad \operatorname{Im} z > 0. \tag{82}$$

In both cases $\psi_0^\pm(z)$ and $h^0(k)$ are given by (28) and (29).

Then the function $F_{R_0}^-(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} R_0^-(k)e^{-ikx}dk$ is absolutely continuous, and for every $a < \infty$ one has (32).

5), 6) *The items 5 and 6 of Theorem 1 are to be reproduced literally.*

Proof of Theorem 2.

The 'only if' part. In item 1 making the inequality (21) non-strict for one of the values of l (either 1 or 2) is due to the fact that under the presence of a VL one has $|r_{ll}^+(0)| < 1$, while with a VL being absent it should be $r_{ll}^+(0) = -1$.

It follows from (36) that, under a VL being present only for $v_{11}(x)$, one has

$$r_{12}^+(k) = \{kd_{12}(k) - kc_{12}(k)r_{11}^+(k)\}\{kc_{22}(k)\}^{-1},$$

where all the products in braces are continuously differentiable, and $\{kc_{22}(k)\}|_{k=0} \neq 0$. Hence $r_{12}^+(k) \in C^1(\mathbb{R})$.

On the other hand, if a VL is present for $v_{22}(x)$ (only!), then $r_{12}^+(k) = \{d_{12}(k)+c_{12}(k)- c_{12}(k)[r_{11}^+(k) + 1]\}c_{22}(k)^{-1}$. Here $c_{22}(k)^{-1} \in C^1(\mathbb{R})$ by Lemma 1, $\{c_{12}(k)[r_{11}^+(k) + 1]\} \in C(\mathbb{R})$ since $r_{11}^+(0) + 1 = 0$, $r_{11}^+(k) \in C^1(\mathbb{R})$ and $kc_{12}(k) \in C^1(\mathbb{R})$.

Lemma 4. *The matrix valued functions $\{D(k) + C(k)\}$ and $\{A(k) + B(k)\}$ (see (10)) are continuous on the axis.*

Proof of Lemma 4. We have

$$
c_{12}(k) + d_{12}(k) = \frac{1}{2ik}[w\{e_{11}^+(x,-k) - e_{11}^+(x,k), e_{12}^-(x,k)\}
$$

$$
+ w\{\widetilde{e}_{12}^+(x,-k) - \widetilde{e}_{12}^+(x,k), e_{22}^-(x,k)\}] = -w\Big\{\frac{\sin kx}{k} + \int_x^\infty k_{11}^+(x,t)\frac{\sin kt}{k}dt, e_{12}^-(x,k)\Big\}
$$

$$
- w\Big\{\int_x^\infty \widetilde{k}_{12}^+(x,t)\frac{\sin kt}{k}dt, e_{22}^-(x,k)\Big\} \xrightarrow[k\to 0]{} -w\Big\{x + \int_x^\infty k_{11}^+(x,t)tdt, e_{12}^-(x,0)\Big\}
$$

$$
- w\Big\{\int_x^\infty \widetilde{k}_{12}^+(x,t)tdt, e_{22}^-(x,0)\Big\} = \text{const.}
$$

Similarly, one can establish the existence of $\lim\limits_{k\to 0}\{c_{ll}(k) + d_{ll}(k)\}$, hence continuity of the sum $C(k) + D(k)$ for $k = 0$, and therefore for $k \in \mathbb{R}$. In the same way, one can establish continuity for $A(k) + B(k)$. Lemma 4 is proved. □

Hence the 'only if' part for item 1 of Theorem 2 is proved. (Note that in the scalar case boundedness for the sum $a(k) + b(k) = O(1)$ as $k \to 0$ has been proved in [12, lemma 6.1.6].)

The 'only if' part for item 2 of Theorem 2 can be proved similarly to the 'only if' part for item 2 of Theorem 1. This requires an application of Lemma 2 with appropriate simplifications being introduced, because $\gamma^+ = 0$ (by virtue of item 1 of Theorem 2).

The 'only if' part for item 3 follows, just as in Theorem 1, from the known [14] representation (25) for $a_{ll}^0(k)$ in terms of $r_{ll}^+(k)$, and the proof of Lemma 1 (with a VL being present for a given l) or the proof of item 3 of Theorems 1 in [24]–[25] under absence of a VL for $v_{ll}(x)$ for a given l.

The 'only if' part for item 4 of Theorem 2 can be proved similarly to that for item 4 of Theorem 1 (with some simplifications). It should be taken into account that $|a_{11}^0(0)| < \infty$ with a VL being present for $v_{11}(x)$ and $|a_{11}^0(k)| \asymp |k|^{-1}$ ($k \to 0$) with a VL being present for $v_{22}(x)$.

The 'only if' part for items 5 and 6 of Theorem 2 can be proved in the same way as in [23]–[25].

The 'if' part for conditions 1–4 of Theorem 2 under absence of discrete spectrum can be proved similarly to that for conditions 1–4 of Theorem 1 (with some simplifications). If finitely many negative discrete levels are present, it can be proved just as Theorem 1, with conditions 5 and 6 being taken into account, using the method of subsequent adding simple or multiplicity two eigenvalues (see Lemma 3 and [24]). The proof of Theorem 2 is complete. □

4. THE PARSEVAL EQUALITY

If a VL is absent, hence also in the case when the reflection coefficient $R^+(k)$ has no pole, i.e., with $\gamma^+ = 0$, the Parseval equality, or the expansion of the Dirac δ-function for the system (1) has the form

$$
\delta(x - t)I = \frac{1}{2\pi}\int_{-\infty}^\infty E_+(x,k)A^{-1}(k)\widetilde{E}_-(t,k)dk
$$

(83)

$$
+ \sum_{j=1}^p \sum_{l=0}^1 \frac{d^l}{i^l dk^l}\Big\{E_+(x,k)\left(\widetilde{Z}_j^+\right)^{(l)}(0)\widetilde{E}_+(t,k)\Big\}_{k_j},
$$

see [23, Lemma 4]. It can be also rewritten in the form

$$(84) \qquad \int\limits_{-\infty}^{\infty} \Phi(x)\Psi(x)dx = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} E_-(\Phi,k)C^{-1}(k)\widetilde{E}_+(\Psi,k)dk$$

$$+ \sum_{j=1}^{p}\sum_{l=0}^{1} \frac{d^l}{i^l dk^l} \left\{ E_-(\Phi,k)\left(Z_j^+\right)^{(l)}(0)\widetilde{E}_+(\Psi,k) \right\}_{k=k_j}.$$

Here $\Phi(x)$ and $\Psi(x)$ are square 2×2-matrix valued functions with compact support, which are continuous in x:

$$(85) \qquad E_\pm(\Phi,k) = \int\limits_{-\infty}^{\infty} \Phi(x)E_\pm(x,k)dx, \quad \widetilde{E}_\pm(\Psi,k) = \int\limits_{-\infty}^{\infty} \widetilde{E}_\pm(x,k)\Psi(x)dx.$$

(A modern approach to the Dirac δ-function can be found in [2].)

Now it should be noted that even in the case of a multiple VL and $\gamma^+ = 0$, the relation

$$(86) \qquad F_R^+(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} R^+(k)e^{ikx}dx$$

is still valid by virtue of (22), (23). This, together with Lemma 2, allows one also in the matrix case with multiple VL and $\gamma^+ = 0$ considered in this paper, to apply the argument used to prove the Parseval equality (83), (84) in earlier works. Namely, the argumentation which was used in the scalar case in [14, Problem 4 to Chapter 3, § 5], and which was also used in the matrix case under absence of VL in [23]. This argumentation is based on an application of the Marchenko equation (39) with $F^+(x)$ of the form (38), where in $F_R^+(x)$ is given by (86). Thus the formulas (83), (84) are also valid under a multiple VL with $\gamma^+ = 0$.

On the other hand, the simplest Example 1 clearly indicates that for $\gamma^+ \neq 0$ the above relations (83), (84) become invalid. In particular, in addition to $\delta(x-t)$, an excessive term appears in the r.h.s. of (83). This term vanishes only with $\gamma^+ = 0$, i.e., under $\int\limits_{-\infty}^{\infty} v_{12}(x)dx = 0$ in the example mentioned above. In order to get again formulas for the expansion of the Dirac δ-function, starting from (83), (84), we replace the potential $V(x)$ in (1) by the potential

$$(87) \qquad\qquad V_\alpha(x) = V(x) + \alpha J\delta(x).$$

After that, we choose $\alpha = \alpha_0$ in such a way that the perturbed equation

$$(88) \qquad\qquad -Y'' + V_\alpha(x)Y = k^2 Y$$

determines $\gamma_{\alpha_0}^+ = 0$. To be rephrased, the equation (1) with $V_\alpha(x)$ instead of $V(x)$, determines the reflection coefficient $R_{\alpha_0}^+(k)$ with no pole at $k = 0$, in spite of a multiplicity two VL being present.

Remark 2. *The multiplicity of a VL for all α remains the same. Also, the discrete spectrum and its algebraic multiplicity does not depend on α.*

For the perturbed equation with the potential $V_\alpha(x)$ and $\gamma_{\alpha_0}^+ = 0$, the relations (83), (84) still hold if one replaces the solutions $E_\pm(x,k)$, $\widetilde{E}_\pm(x,k)$ of the non-perturbed equation (1), the matrices $A(k)$, $C(k)$, and other values involved therein, by the corresponding solutions of the perturbed equation (88), $E_{\alpha_0}^\pm(x,k)$, $\widetilde{E}_{\alpha_0}^\pm(x,k)$; the matrices $A(k)$, $C(k)$ are to be replaced by $A_{\alpha_0}(k)$, $C_{\alpha_0}(k)$, etc. Since $E_\alpha^\pm(x,k)$, $\widetilde{E}_\alpha^\pm(x,k)$ etc. are easily expressible in terms of $E_\pm(x,k)$, $\widetilde{E}_\pm(x,k)$ etc., we substitute these expressions to (83), (84)

instead of $E_\alpha^\pm(x,k)$, $\widetilde{E}_\alpha^\pm(x,k)$ to deduce the modified Parseval equality for the equation (1) in the case when the reflection coefficient $R^+(k)$ has a pole, that is, for $\gamma^+ \neq 0$.

Lemma 5. *The equation* (88) *with a multiple VL determines* $\gamma_{\alpha_0}^+ = 0$ *when*

$$(89) \qquad \alpha_0 = \frac{c_{12}[-1]}{e_{11}^+(0,0)e_{22}^-(0,0)} \cdot \frac{1 + r_{11}^+(0)}{1 - r_{11}^+(0)}.$$

In this case, the Jost solutions of (88) *have the form*

$$(90) \qquad E_{\alpha_0}^-(x,k) = \begin{cases} E_-(x,k), & x < 0, \\ E_-(x,k) + \mu_0 k^{-1}\Big\{ E_+(x,k)A^{-1}(k)\widetilde{E}_-(0,k) \\ \qquad\qquad - E_-(x,k)C^{-1}(k)\widetilde{E}_+(0,k) \Big\} J, & x > 0, \end{cases}$$

where

$$(91) \qquad \mu_0 = \frac{-ic_{12}[-1]}{2e_{11}^+(0,0)} \cdot \frac{1 + r_{11}^+(0)}{1 - r_{11}^+(0)},$$

and the tilda-Jost solutions have the form

$$(92) \qquad \widetilde{E}_{\alpha_0}^+(x,k) = \begin{cases} \widetilde{E}_+(x,k), & x > 0, \\ \widetilde{E}_+(x,k) + \widetilde{\mu}_0 k^{-1} J\Big\{ E_-(0,k)C^{-1}(k)\widetilde{E}_+(x,k) \\ \qquad\qquad - E_+(0,k)A^{-1}(k)\widetilde{E}_-(x,k) \Big\}, & x < 0, \end{cases}$$

where

$$(93) \qquad \widetilde{\mu}_0 = -\mu_0 e_{11}^+(0,0)e_{22}^-(0,0)^{-1} = \frac{ic_{12}[-1]}{2e_{22}^-(0,0)} \cdot \frac{1 + r_{11}^+(0)}{1 - r_{11}^+(0)},$$

and α_0 *is the same as in* (89). *Besides that,*

$$(94) \qquad C_\alpha(k) = C(k) - \frac{\alpha}{2ik} J e_{11}^+(0,k)e_{22}^-(0,k),$$

$$(95) \qquad C_\alpha(k)^{-1} = C(k)^{-1} + \frac{\alpha}{2ik} \cdot \frac{e_{11}^+(0,k)e_{22}^-(0,k)}{c_{11}(k)c_{22}(k)} J.$$

Proof. Let us substitute $E_\alpha^-(x,k)$ to (88), and then integrate from $x = -0$ to $x = +0$. This procedure, which takes into account the fact that $E_\alpha^-(x,k)$ is continuous in x, yields

$$(96) \qquad - E_\alpha^{-\prime}(x,k)\Big|_{x=-0}^{x=+0} + \alpha J E_\alpha^-(0,k) = 0.$$

Therefore, the following system should be satisfied:

$$(97) \qquad \begin{cases} E_\alpha^-(+0,k) = E^-(0,k), \\ E_\alpha^{-\prime}(+0,k) = E^{-\prime}(0,k) + \alpha J E_\alpha^-(0,k). \end{cases}$$

One can verify, via a comparison to the ordinary construction for the kernel of resolvent (the Green function) of the problem (1), that the system (97) has the solution $E_\alpha^-(x,k)$ given by (90) with arbitrary α and μ without indices 0, subject to the only condition

$$(98) \qquad \alpha = 2i\mu e_{22}^-(0,0).$$

The values $\mu = \mu_0$ (91) and, by virtue of (98), also $\alpha = \alpha_0$, are deducible from $\gamma_{\alpha_0}^+ = 0$. The latter condition is equivalent to $r_{12}^+[-1] = 0$, or, by (36), to

$$(99) \qquad d_{12}^\alpha[-1] - c_{12}^\alpha[-1]r_{11}^+(0) = 0.$$

The tilda-solution $\widetilde{E}_\alpha^+(x,k)$ of the equation (3) with $V = V_\alpha(x)$ is found in a similar way from (92). Within the process, one should discard the indices '0' at α and $\widetilde{\mu}$, but to keep

the relation between $\widetilde{\mu}$ and α in view of (93) (with the indices '0' being also discarded) and (98).

Now (99), together with (10) and (90), implies that

$$0 = d_{12}[-1] - c_{12}[-1]r_{11}^+(0) + 2i\mu_0 e_{11}^+(0,0)\left[1 - r_{11}^+(0)\right].$$

This allows us to find μ_0 (91) via observing that $c_{12}[-1]+d_{12}[-1] = 0$ by Lemma 4. Then we use (91) to find α_0 (89) by virtue of (98). The formulas (94), (95) are established via a direct computation.

The value α_0 for the tilda-solution (92) is the same as (89), because the right SD of the problem (88) and the tilda-problem with the same potential coincide [23, Lemma 2]. The value $\widetilde{\mu}_0$ is derived from μ_0 (91) using (93). Lemma 5 is proved. $\qquad\square$

Now let us write down the Parseval equality (84) for the equation (88), (87) with $\alpha = \alpha_0$, hence with $\gamma_{\alpha_0}^+ = 0$. We express this Parseval equality in terms of transforms of the matrix valued functions $\Phi(x)$ and $\Psi(x)$ in solutions of the form (90) – (93). The latter transforms are in turn expressible through transforms in non-perturbed solutions of (1) and (3), where $\alpha = 0$. To simplify matters, we assume that there is no discrete spectrum. In fact, such spectrum can be very well added (eliminated) via subsequent adding (eliminating) eigenvalues (see [24], [25]).

Theorem 3. *The Parseval equality for transforms in solutions of the problems (1) and (3) for square 2×2 continuous matrix valued functions $\Phi(x)$ and $\Psi(x)$ with compact supports, in the case when the reflection coefficient $R^+(k)$ has a pole at $k = 0$ (hence a multiplicity two VL being present) and no discrete spectrum, can be written in the form:*

(100)

$$\int_{-\infty}^{\infty} \Phi(x)\Psi(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_-(\Phi,k)\left\{C^{-1}(k) + \frac{\alpha_0}{2ik}J\frac{e_{11}^+(0,k)e_{22}^-(0,k)}{c_{11}(k)c_{22}(k)}\right\}\widetilde{E}_+(\Psi,k)dk$$

$$+ \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} E_-(\Phi,k)\frac{dk}{k}c_{11}^{-1}(k)c_{22}^{-1}(k)\int_{-\infty}^{0}\left\{e_{22}^-(0,k)e_{22}^+(t,k) - e_{22}^+(0,k)e_{22}^-(t,k)\right\}\cdot J\Psi(t)dt$$

$$+ \frac{\widetilde{\mu}_0}{2\pi}\int_{-\infty}^{\infty} \frac{dk}{k}c_{11}^{-1}(k)c_{22}^{-1}(k)\int_{0}^{\infty}\Phi(x)J\left\{e_{11}^+(x,k)e_{11}^-(0,k) - e_{11}^-(x,k)e_{11}^+(0,k)\right\}\widetilde{E}_+(\Psi,k)dx.$$

Here $E^{\pm}(\Phi,k)$, $\widetilde{E}^{\pm}(\Psi,k)$ are determined by (85), and the values α_0, μ_0, $\widetilde{\mu}_0$ are given by (89), (91), (93). In this setting α_0, μ_0, $\widetilde{\mu}_0$ vanish for $\gamma^+ = 0$ (i.e., in the case of no pole for $R^+(k)$ at $k = 0$), when the relation (100) acquires the form (84) established in [23] under absence of a VL.

Proof of the theorem is based on Lemma 5 and an application of the Marchenko equation (39), (38), (23) (cf. [14, Problem 4 to Chapter 3, § 5]). The method of proving is described above, so we need not reproduce it again.

Example 2. Let us apply the general form of the Parseval equality (100) in the case $\gamma^+ \neq 0$ to the simplest example 1, where $V(x) = v(x)J$. In this case

$$E_-(x,k) = \widetilde{E}_-(x,k) = e^{-ikx}I - \frac{1}{2ik}J\int_{-\infty}^{x} v(s)\left[e^{-ikx} - e^{ik(x-2s)}\right]ds,$$

$$E_+(x,k) = \widetilde{E}_+(x,k) = e^{ikx}I + \frac{1}{2ik}J\int_{x}^{\infty} v(s)\left[e^{ik(2s-x)} - e^{ikx}\right]ds,$$

$$\alpha_0 = - \int\limits_{-\infty}^{\infty} v(s)ds = -2i\gamma^{\pm}, \quad C_{\alpha_0} = I, \quad C(k)^{-1} = I + \gamma^+ k^{-1} J.$$

Under the above setting the expansion of the Dirac δ-function acquires the form

$$\delta(x-t)I = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} E_-(x,k)\widetilde{E}_+(t,k)dk$$

$$+ \frac{\gamma^+}{2\pi} \int\limits_{-\infty}^{\infty} E_-(x,k)\frac{J}{k}\eta(-t)\left[\widetilde{E}_+(t,k) - \widetilde{E}_-(t,k)\right] dk$$

$$- \frac{\gamma^+}{2\pi} \int\limits_{-\infty}^{\infty} k^{-1}\eta(x)\left[E_+(x,k) - E_-(x,k)\right]J\widetilde{E}^+(t,k)dk,$$

which is deducible by a direct computation.

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