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ON *-REPRESENTATIONS OF THE PERTURBATION OF TWISTED CCR

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Abstract. A classification of irreducible *-representations of a certain deformation of twisted canonical commutation relations is given.

1. Introduction

In this note we study representations of a $*$-algebra A_α defined by generators $a_i,\ a_i^*$, $i = 1, \dots, d$, satisfying the commutation relations of the following form:

$$\begin{aligned} a_i^* a_i &= 1 + \alpha_i^2 a_i a_i^* - \sum_{j<i} (1-\alpha_j^2) a_j a_j^*, \\ a_i^* a_j &= \alpha_i a_j a_i^*, \quad i < j, \\ a_j a_i &= \alpha_i a_i a_j, \quad i, j = 1, \dots, d, \quad i < j, \end{aligned} \tag{1}$$

where we additionally suppose that $\alpha_i^2 = \mu^{n_i}$, $0 < \mu < 1$, $n_i \in \mathbb{N}$, $i = 1, \dots, d$. When $n_i = 2$, $i = 1, \dots, d$, we get the twisted canonical commutation relations (TCCR) constructed and studied by W. Pusz and S. L. Woronowicz, see [5]. These relations also belong to the class of generalized canonical commutation relations (GCCR), defined in [3].

The aim of this paper is to study irreducible representations of A_α by, possibly unbounded, Hilbert space operators. Note that representations of TCCR were classified in [5]. The description of bounded representations of GCCR was obtained in [3]. In [2] the authors proved that the Fock representation of the universal enveloping C^*-algebra generated by GCCR is faithful.

To deal with the unbounded representations one has firstly to give a precise definition of a family of unbounded operators satisfying relations (1). To do so, let us perform some formal manipulations with generators and relations.

Construct the polar decompositions of a_i^*, $a_i^* = U_i C_i$, where $C_i^2 = a_i a_i^*$, U_i is a partial isometry and $\ker U_i = \ker C_i = \ker a_i^*$. Then the commutation relations (1) take the following form:

$$\begin{aligned} C_i^2 U_i^* &= U_i^* \Big(1 + \alpha_i^2 C_i^2 - \sum_{j<i} (1-\alpha_j^2) C_j^2\Big), \\ C_i^2 U_j^* &= \alpha_j^2 U_j^* C_i^2, \quad j < i, \\ C_i^2 U_j^* &= U_j^* C_i^2, \quad j > i, \end{aligned} \tag{2}$$

$$C_i C_j = C_j C_i, \quad U_j U_i = U_i U_j, \quad U_j^* U_i = U_i U_j^*, \quad i \neq j. \tag{3}$$

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Consider the functions

$$F_j(x_1,\dots,x_d) = \Big(x_1,\dots,x_{j-1}, 1+\alpha_j^2 x_j - \sum_{k<j}(1-\alpha_k^2)x_k,$$
$$\alpha_j^2 x_{j+1},\dots,\alpha_j^2 x_d\Big),\quad j=1,\dots,d.$$

Then, in a compact form, relations (2) can be written as follows:

$$(C_1^2,\dots,C_d^2)U_j^* = U_j^* F_j(C_1^2,\dots,C_d^2),\quad j=1,\dots,d.$$

Note that, in the bounded case, the relations (1) and (2, 3) are equivalent.

Definition 1. (see [4]). *Let a family of self-adjoint operators $\mathcal{C}=\{C_i^2,\ i=1,\dots,d\}$ commute on a dense invariant domain of analytic vectors. We say that the family $\mathcal{C}$ and partial isometries $\{U_i,\ i=1,\dots,d\}$ satisfy relations (2) if for any Borel set $\Delta\subset\mathbb{R}^d$ and any $j=1,\dots,d$ one has*

$$E_{\mathcal{C}}(\Delta)U_j^* = U_j^* E_{\mathcal{C}}(F_j^{-1}(\Delta)),$$

where $E_{\mathcal{C}}(\cdot)$ is the joint resolution of identity of the family $\mathcal{C}$.

Definition 2. *Let families $\mathcal{C}$ and $\{U_i,\ i=1,\dots,d\}$ satisfy the conditions of the definition above and $\ker U_i=\ker C_i$, $i=1,\dots,d$, then we say that the family of operators $a_i^*=U_iC_i$, $i=1,\dots,d$ is an unbounded representation of relations (1).*

2. Representations of $\mathcal{A}_\alpha$

In this section we will use a dynamical system method developed in a series of papers by Yu. Samoilenko, V. Ostrovkyi, L. Turowska, E. Vaisleb et al., see [4] and the references therein.

Our considerations will be based on an analysis of the spectrum of C_1^2 in the irreducible representation. Since

$$C_1^2U_1^* = U_1^*(1+\alpha_1^2C_1^2),\quad C_1^2U_j^* = U_j^*C_1^2, C_1C_j=C_jC_1,\quad j\ge 2$$

in an irreducible representation of (2,3), the spectrum of C_1^2 is coincides with the positive orbit of the dynamical system $(f_1,\ \mathbb{R})$, where $f_1(t)=1+\alpha_1^2t$, see [4]. Such orbits can be subdivided onto the following three types:

(1) Fock orbit, $O_F=\{\frac{1-\alpha_1^{2n}}{1-\alpha_1^2},\ n\in\mathbb{Z}_+\}$;
(2) fixed point $O_{fix}=\{\frac{1}{1-\alpha_1^2}\}$;
(3) unbounded orbits, labeled by $x_1\in\tau_{y_1}=(1+\alpha_1^2y_1,y_1]$, $y_1>\frac{1}{1-\alpha_1^2}$ is fixed, $O_{x_1}=\{\frac{1-\alpha_1^{2n}}{1-\alpha_1^2}+\alpha_1^{2n}x_1,\ n\in\mathbb{Z}\}$.

In the following propositions we give a description of irreducible representations of A_α when the spectrum of C_1^2 is assumed to coincide with one of the orbits above.

We start with the most simple case.

Proposition 1. *Let in irreducible representation of A_α one has $\sigma(C_1^2)=\overline{O_F}$, then, up to a unitary equivalence, $\mathcal{H}=l_2(\mathbb{Z}_+)\otimes\mathcal{K}$ and*

$$C_1^2=d(f_1)\otimes\mathbf{1},\quad U_1^*=S\otimes\mathbf{1},$$
$$C_i^2=d(\alpha_1^2)\otimes\widehat{C}_i^2,\quad U_i^*=\mathbf{1}\otimes\widehat{U}_i^*,\quad i=2,\dots,d,$$

where, for the standard basis of $l_2(\mathbb{Z}_+)$ denoted by $\{e_n,\ n\in\mathbb{Z}_+\}$, one has

$$d(f_1)e_n=f_1^n(0)e_n=\frac{1-\alpha_1^{2n}}{1-\alpha_1^2}e_n,\quad d(\alpha_1^2)e_n=\alpha_1^{2n}e_n,\quad n\in\mathbb{Z}_+$$

and the family of the operators $\{\widehat{C}_i,\ \widehat{U}_i,\ i=2,\dots,d\}$ is irreducible on $\mathcal{K}$ and satisfies the relations (2,3) with $d-1$ generators.

Proof. The proof is analogous to the proof of the propositions below and the most trivial among them, so we omit it here. ☐

Let us now suppose that $\sigma(C_1^2) = \{\frac{1}{1-\alpha_1^2}\}$ and $d > 3$. Fix $y_2 > 0$, put $\sigma_{y_2} = (\mu^l y_2, y_2]$, where $l = \mathbf{GCD}(n_1, n_2)$, $\alpha_i^2 = \mu^{n_i}$, $i = 1, 2$. Let also $n_i = lk_i$, $i = 1, 2$ and $l = n_1 m_1 + n_2 m_2$.

Proposition 2. *If $d > 2$ and $\sigma(C_1^2) = \{\frac{1}{1-\alpha_1^2}\}$ and $C_2^2 \neq 0$ in the irreducible representation, then, up to a unitary equivalence, $\mathcal{H} = l_2(\mathbb{Z}) \otimes \bigotimes_{i=3}^d l_2(\mathbb{Z}_+)$ and*

$$C_1^2 = \frac{1}{1-\alpha_1^2}\mathbf{1} \otimes \bigotimes_{2<k\le d} \mathbf{1},$$

$$U_j^* = e^{i\phi_j} E^{k_j} \otimes \bigotimes_{2<k\le d}^{d} \mathbf{1}, \quad j = 1, 2, \quad m_1\phi_1 + m_2\phi_2 = 0, \quad \mod 2\pi,$$

$$C_2^2 = x_2 D(\mu^l) \otimes \bigotimes_{2<k\le d} \mathbf{1}, \quad x_2 \in \sigma_{y_2},$$

$$C_i^2 = D(\mu^l) \otimes \bigotimes_{2<k<i} \widehat{d}(\alpha_k^2) \otimes \widehat{d}(h_i(0, x_2)) \otimes \bigotimes_{i<k\le d} \mathbf{1}, \quad i = 3, \dots, d,$$

$$U_i^* = \mathbf{1} \otimes \bigotimes_{2<k<i} \mathbf{1} \otimes \widehat{S} \otimes \bigotimes_{i<k\le d} \mathbf{1}, \quad i = 3, \dots, d,$$

and

$$D(\mu^l), \quad E\colon l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \quad D(\mu^l)e_n = \mu^{nl} e_n, \quad Ee_n = e_{n+1}, \quad n \in \mathbb{Z},$$

where $\{e_n,\ n \in \mathbb{Z}\}$ is the standard basis of $l_2(\mathbb{Z})$;

$$h_i(t, x_2) = -(1-\alpha_2^2)x_2 + \alpha_i^2 t,$$

$$\widehat{d}(h_i(0, x_2)), \quad \widehat{S}, \quad \widehat{d}(\lambda)\colon l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \quad \widehat{S}e_0 = 0, \quad \widehat{S}e_{-n} = e_{-n+1}, \quad n \ge 1,$$

$$\widehat{d}(h_i(0, x_2))e_{-n} = h_i^{-n}(0, x_2)e_{-n}, \quad \widehat{d}(\lambda)e_{-n} = \lambda^{-n}e_{-n}, \quad n \in \mathbb{Z}_+,$$

where $\{e_{-n},\ n \in \mathbb{Z}_+\}$ is the standard basis of $l_2(\mathbb{Z}_-)$.

Proof. Since $C_1^2 = \frac{1}{1-\alpha_1^2}\mathbf{1}$ and $\ker U_1 = \ker U_1^* = \{0\}$, U_1 is a unitary operator. Furthermore, one has

$$C_i^2 U_i^* = U_i^* \Big(\alpha_i^2 C_i^2 - \sum_{1<j<i} (1-\alpha_j^2) C_j^2\Big), \quad i \ge 2.$$

In particular, $C_2^2 U_i^* = \alpha_i^2 U_i^* C_2^2$, $i = 1, 2$. Since $C_2^2 U_j = U_j C_2^2$, $C_2^2 C_j^2 = C_j^2 C_2^2$, $j > 2$, the spectrum of C_2^2 is concentrated on the positive orbit of the mapping $t \mapsto \mu^l t$, $l = \mathbf{GCD}(n_1, n_2)$. If $C_2^2 \neq 0$, then $\sigma(C_2^2) = \{\mu^{nl} x_2,\ n \in \mathbb{Z}\}$ for some $x_2 \in \sigma_{y_2}$ and all eigenvalues have the same multiplicities, see [4]. Then one can choose a basis in the representation space $\mathcal{H}$ so that $\mathcal{H} \simeq l_2(\mathbb{Z}) \otimes \mathcal{K}_1$ and

$$C_2^2 = x_2 D(\mu^l) \otimes \mathbf{1}.$$

Let $l = n_1 m_1 + n_2 m_2$, $m_1, m_2 \in \mathbb{Z}$, put $U := U_1^{m_1} U_2^{m_2}$, then

$$C_2^2 U^* = \mu^l U^* C_2^2,$$

and using unitary equivalence one can get $U^* = E \otimes \mathbf{1}$. Then the relations

$$C_2^2 U_i^* = \mu^{n_2} U_i^* C_2^2, \quad UU_i = U_i U, \quad i = 1, 2,$$

imply that $U_i^* = E^{k_i} \otimes \widetilde{U}_i^*$, where $n_i = lk_i$, $i = 1, 2$, and $\widetilde{U}_1, \widetilde{U}_2$ are unitaries. Analogously, from

$$C_2^2 C_j^2 = C_j^2 C_2^2, \quad C_j^2 U^* = \mu^l U^* C_j^2, \quad UU_j = U_j U, \quad C_2^2 U_j = U_j C_2^2, \quad j > 2,$$

we have $C_j^2 = D(\mu^l) \otimes \widetilde{C}_j^2$ and $U_j = \mathbf{1} \otimes \widetilde{U}_j$, $j > 2$.

One can verify directly that the family $\{C_i^2,\ U_i,\ i = 1, \dots, d\}$ is irreducible iff the family $\{\widetilde{C}_i^2,\ i > 2,\ \widetilde{U}_i,\ i = 1, \dots, d\}$ is irreducible and the second family determines the first one up to a unitary equivalence.

Let us now rewrite the relations (2,3) in terms of the operators $\widetilde{C}_i^2$, $\widetilde{U}_i$. It is easy to show that (2,3) are equivalent to

$$\widetilde{C}_j^2 \widetilde{U}_i^* = \widetilde{U}_i^* \widetilde{C}_j^2, \quad \widetilde{U}_i \widetilde{U}_j = \widetilde{U}_j \widetilde{U}_i, \quad i = 1, 2, \quad j > 2 \tag{4}$$

and

$$\begin{aligned} &\widetilde{C}_i^2 \widetilde{U}_i^* = \widetilde{U}_i^* \Big(-(1-\alpha_2^2) x_2 + \alpha_i^2 \widetilde{C}_i^2 - \sum_{3 \le j < i} (1 - \alpha_j^2) \widetilde{C}_j^2 \Big), \quad i = 3, \dots, d, \\ &\widetilde{C}_i^2 \widetilde{U}_j^* = \alpha_j^2 \widetilde{U}_j^* \widetilde{C}_i^2, \quad i > j, \quad \widetilde{C}_i^2 \widetilde{U}_j^* = \widetilde{U}_j^* \widetilde{C}_i^2, \quad i < j, \\ &\widetilde{U}_i \widetilde{U}_j^* = \widetilde{U}_j^* \widetilde{U}_i, \quad \widetilde{U}_i \widetilde{U}_j = \widetilde{U}_j \widetilde{U}_i, \quad \widetilde{C}_i \widetilde{C}_j = \widetilde{C}_j \widetilde{C}_i, \quad i \neq j. \end{aligned} \tag{5}$$

Since $\widetilde{U}_i$, $i = 1, 2$ are unitaries, the Schur lemma and relations (4) imply that $\widetilde{U}_i = e^{\imath \phi_i} \mathbf{1}$, $i = 1, 2$, $\phi_1 m_1 + \phi_2 m_2 = 0 \mod 2\pi$.

Furthermore, since

$$\widetilde{C}_3^2 \widetilde{U}_3^* = \widetilde{U}_3^* \big(-(1-\alpha_2^2) x_2 + \alpha_3^2 \widetilde{C}_3^2 \big), \quad \widetilde{C}_3^2 \widetilde{U}_j^* = \widetilde{U}_j^* \widetilde{C}_3^2, \quad j > 3,$$

in the irreducible representation, the spectrum of $\widetilde{C}_3^2$ is concentrated on the positive orbit of the mapping

$$h_3(t, x_2) = -(1 - \alpha_2^2) x_2 + \alpha_3^2 t.$$

For this mapping we have the unique positive orbit, the anti-Fock one

$$\sigma(\widetilde{C}_3^2) = \overline{\{h_3^{-n}(0, x_2),\ n \in \mathbb{Z}_+\}}$$

and, as above, all eigenvalues have the same multiplicities. Then $\mathcal{K}_1 = l_2(\mathbb{Z}_-) \otimes \mathcal{K}_2$ and, up to a unitary equivalence,

$$\widetilde{C}_3^2 = \widehat{d}(h_3(0, x_2)) \otimes \mathbf{1}, \quad \widetilde{U}_3^* = \widehat{S} \otimes \mathbf{1}$$

and the relations (5) imply that

$$\widetilde{C}_j^2 = \widehat{d}(\alpha_3^2) \otimes \widehat{C}_j^2, \quad \widetilde{U}_j^* = \mathbf{1} \otimes \widehat{U}_j^*, \quad j > 3,$$

where the family $\{\widehat{C}_j,\ \widehat{U}_j,\ j > 3\}$ should be irreducible and satisfy the relations (5) with $d - 3$ generators. Finally, note that the family $\{\widehat{C}_j,\ \widehat{U}_j,\ j > 3\}$ determines the family $\{\widetilde{C}_j,\ \widetilde{U}_j,\ j > 2\}$ up to a unitary equivalence. Then the evident induction on the number of generators completes the proof. $\square$

It remains only to consider the third type of orbits.

Proposition 3. *Let*

$$\sigma(C_1^2) = \overline{\{\frac{1 - \alpha_1^{2n}}{1 - \alpha_1^2} + \alpha_1^{2n} x_1,\ n \in \mathbb{Z}\}}$$

in an irreducible representation of A_α for some fixed $x_1 \in \tau_{y_1}$. Then, up to a unitary equivalence, the representation space is $\mathcal{H} = l_2(\mathbb{Z}) \otimes \bigotimes_{k=2}^d l_2(\mathbb{Z}_-)$ and

$$C_1^2 = D(f_1, x_1) \otimes \bigotimes_{2 \le k \le d} \mathbf{1}, \quad U_1^* = E \otimes \bigotimes_{2 \le k \le d} \mathbf{1},$$

$$C_i^2 = D(\alpha_1^2) \otimes \bigotimes_{2 \le k < i} \widehat{d}(\alpha_k^2) \otimes \widehat{d}(u_i(0, x_1)) \otimes \bigotimes_{i < k \le d} \mathbf{1},$$

$$U_i^* = \mathbf{1} \otimes \bigotimes_{2 \le k < i} \mathbf{1} \otimes \widehat{S} \otimes \bigotimes_{i < k \le d} \mathbf{1}, \quad i = 2, \dots, d,$$

where

$$D(f_1, x_1) \colon l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \quad D(f_1, x_1) e_n = \Big(\frac{1 - \alpha_1^{2n}}{1 - \alpha_1^2} + \alpha_1^{2n} x_1 \Big) e_n, \quad n \in \mathbb{Z}$$

and $u_i(t, x_1) = 1 - x_1 + \alpha_i^2 t$, $i = 2, \dots, d$,

$$\widehat{d}(u_i(0, x_1)) \colon l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \quad \widehat{d}(u_i(0, x_1)) e_{-n} = u_i^{-n}(0, x_1) e_{-n}, \quad n \in \mathbb{Z}_+.$$

Proof. As in the proof of Proposition 2, we will use induction on the number of generators. If $\sigma(C_1^2) = \overline{O_{x_1}}$, then, up to a unitary equivalence, $\mathcal{H} = l_2(\mathbb{Z}) \otimes \mathcal{K}_1$ and

$$C_1^2 = D(f_1, x_1) \otimes \mathbf{1}, \quad U_1^* = E \otimes \mathbf{1}.$$

The relations (2,3) imply that

$$C_i^2 = D(\alpha_1^2) \otimes \widetilde{C}_i^2, \quad U_i^* = \mathbf{1} \otimes \widetilde{U}_i^*, \quad i \ge 2,$$

where the family $\{\widetilde{C}_i,\ \widetilde{U}_i,\ i \ge 2\}$ is irreducible and determines $\{C_i,\ U_i,\ i \ge 1\}$ up to a unitary equivalence. Moreover, the following relations are satisfied:

$$\begin{aligned} &\widetilde{C}_i^2 \widetilde{U}_i^* = \widetilde{U}_i^* \Big(1 - x_1 + \alpha_i^2 \widetilde{C}_i^2 - \sum_{2 \le j \le i-1} (1 - \alpha_j^2) \widetilde{C}_j^2 \Big), \quad i = 2, \dots, d, \\ &\widetilde{C}_i^2 \widetilde{U}_j^* = \alpha_j^2 \widetilde{U}_j^* \widetilde{C}_i^2, \quad i > j, \quad \widetilde{C}_i^2 \widetilde{U}_j^* = \widetilde{U}_j^* \widetilde{C}_i^2, \quad i < j, \\ &\widetilde{U}_i \widetilde{U}_j^* = \widetilde{U}_j^* \widetilde{U}_i, \quad \widetilde{U}_i \widetilde{U}_j = \widetilde{U}_j \widetilde{U}_i, \quad \widetilde{C}_i \widetilde{C}_j = \widetilde{C}_j \widetilde{C}_i \quad i \ne j. \end{aligned} \tag{6}$$

In particular, the spectrum of $\widetilde{C}_2^2$ is concentrated on the positive orbit of the mapping

$$u_2(t, x_1) = 1 - x_1 + \alpha_2^2 t,$$

since $x_1 > \frac{1}{1 - \alpha_1^2} > 1$ and $\alpha_2^2 < 1$, the unique positive orbit of $u_2(t, x_1)$ is the anti-Fock orbit. Then the proof is analogous to the final part of the proof of Proposition 2. □

To get a general description of representations of A_α, we have to combine the results of Propositions 1,2,3. Namely, let us construct three types of representations.

The *first* is the Fock one: $\mathcal{H} = \bigotimes_{k=1}^d l_2(\mathbb{Z}_+)$,

$$C_j^2 = \bigotimes_{k<j} d(\alpha_k^2) \otimes d(f_j) \otimes \bigotimes_{k>j} \mathbf{1}, \quad U_j^* = \bigotimes_{k<j} \mathbf{1} \otimes S \otimes \bigotimes_{k>j} \mathbf{1}, \quad j = 1, \dots, d.$$

The *second* type is the representations with first $i-1$ generators as in the Fock representation and with

$$\sigma(C_i^2) = \overline{\Big\{ \alpha_1^{2n_1} \cdots \alpha_{i-1}^{2n_{i-1}} \frac{1}{1 - \alpha_i^2},\ n_1, \dots, n_{i-1} \in \mathbb{Z}_+ \Big\}}.$$

Let firstly $i < d$, then fix any $t_i \in \mathbb{Z}_+$ such that $i + t_i \le d$. If $i + t_i < d$ put $s_i := i + t_i + 1$ and fix $y_{s_i} > 0$, $\sigma_{y_{s_i}} = (\mu^{l_{is_i}} y_{s_i}, y_{s_i}]$, where $l_{is_i} = \mathbf{GCD}(n_i, n_{s_i})$, $\alpha_i^2 = \mu^{n_i}$, $\alpha_{s_i}^2 = \mu^{n_{s_i}}$.

Let also $n_i = l_{is_i}k_i$, $n_{s_i} = l_{is_i}k_{s_i}$. Then construct the family of operators acting on the space

$$\mathcal{H} = \bigotimes_{k=1}^{i-1} l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}) \otimes \bigotimes_{k=s_i+1}^{d} l_2(\mathbb{Z}_-)$$

by the formulas

$$C_j^2 = \bigotimes_{k<j} d(\alpha_k^2) \otimes d(f_j) \otimes \bigotimes_{k>j, k\geq s_i} \mathbf{1}, \quad U_j^* = \bigotimes_{k<j} \mathbf{1} \otimes S \otimes \bigotimes_{k>j, k\geq s_i} \mathbf{1}, \quad j<i,$$

$$C_i^2 = \frac{1}{1-\alpha_1^2} \bigotimes_{k<i} d(\alpha_k^2) \otimes \bigotimes_{k\geq s_i} \mathbf{1}, \quad U_i^* = e^{\imath\phi_i} \bigotimes_{k<i} \mathbf{1} \otimes E^{k_i} \otimes \bigotimes_{k>s_i} \mathbf{1},$$

$$C_j^2 = 0, \quad U_j = 0, \quad i<j<s_i-1,$$

$$C_{s_i}^2 = \bigotimes_{k<i} d(\alpha_k^2) \otimes x_{s_i} D(\mu^{l_{is_i}}) \otimes \bigotimes_{k>s_i} \mathbf{1}, \quad U_{s_i}^* = e^{\imath\phi_{s_i}} \bigotimes_{k<i} \mathbf{1} \otimes E^{k_{s_i}} \otimes \bigotimes_{k>s_i} \mathbf{1},$$

$$C_j^2 = \bigotimes_{k<i} d(\alpha_k^2) \otimes D(\mu^{l_{is_i}}) \otimes \bigotimes_{s_i<k<j} \widehat{d}(\alpha_k^2)\widehat{d}(h_j(0,x_{s_i})) \otimes \bigotimes_{k>j} \mathbf{1}, \quad j>s_i,$$

$$U_j^* = \bigotimes_{k<i} \mathbf{1} \otimes \bigotimes_{s_i\leq k<j} \mathbf{1} \otimes \widehat{S} \otimes \bigotimes_{k>j} \mathbf{1}, \quad j>s_i,$$

where $m_i\phi_i + m_{s_i}\phi_{s_i} = 0 \mod 2\pi$, $x_{s_i} \in \sigma_{y_{s_i}}$ is fixed, and

$$h_j(t, x_{s_i}) = -(1-\alpha_{s_i}^2)x_{s_i} + \alpha_j^2 t.$$

If $i = d$, then $\mathcal{H} = \bigotimes_{k=1}^{d-1} l_2(\mathbb{Z}_+)$ and

$$C_j^2 = \bigotimes_{k<j} d(\alpha_k^2) \otimes d(f_j) \otimes \bigotimes_{k>j} \mathbf{1}, \quad U_j^* = \bigotimes_{k<j} \mathbf{1} \otimes S \otimes \bigotimes_{k>j} \mathbf{1}, \quad j=1,\dots,d-1,$$

$$C_d^2 = \frac{1}{1-\alpha_d^2} \bigotimes_{k<d} d(\alpha_k^2), \quad U_d^* = e^{\imath\phi_d} \bigotimes_{k<d} \mathbf{1}.$$

In the *third* type representations, the generators C_j^2, U_j^*, $j = 1,\dots,i-1$, are as in the Fock representation and

$$\sigma(C_i^2) = \overline{\left\{\alpha_1^{2n_1}\cdots\alpha_{i-1}^{2n_{i-1}}\left(\frac{1-\alpha_i^{2n_i}}{1-\alpha_i^2} + \alpha_i^{2n_i}x_i\right),\ n_1,\dots,n_{i-1} \in \mathbb{Z}_+,\ n_i \in \mathbb{Z}\right\}},$$

where $x_i \in \tau_{y_i} = (1+\alpha_i^2 y_i, y_i]$, $y_i > \frac{1}{1-\alpha_i^2}$ is fixed.

In this case we have

$$\mathcal{H} = \bigotimes_{k<i} l_2(\mathbb{Z}_+) \otimes l_2(\mathbb{Z}) \otimes \bigotimes_{k>i} l_2(\mathbb{Z}_-)$$

and

$$C_j^2 = \bigotimes_{k<j} d(\alpha_k^2) \otimes d(f_j) \otimes \bigotimes_{k>j} \mathbf{1}, \quad U_j^* = \bigotimes_{k<j} \mathbf{1} \otimes S \otimes \bigotimes_{k>j} \mathbf{1}, \quad j<i,$$

$$C_i^2 = \bigotimes_{k<i} d(\alpha_k^2) \otimes D(f_i, x_i) \otimes \bigotimes_{k>i} \mathbf{1}, \quad U_i^* = \bigotimes_{k<i} \mathbf{1} \otimes E \otimes \bigotimes_{k>i} \mathbf{1},$$

$$C_j^2 = \bigotimes_{k<i} d(\alpha_k^2) \otimes D(\alpha_i^2) \otimes \bigotimes_{i<k<j} \widehat{d}(\alpha_k^2) \otimes \widehat{d}(u_j(0,x_i)) \otimes \bigotimes_{k>j} \mathbf{1}, \quad j>i,$$

$$U_j^* = \bigotimes_{k<j} \mathbf{1} \otimes \widehat{S} \otimes \bigotimes_{k>j} \mathbf{1}, \quad j>i,$$

where $u_j(t, x_i) = 1 - x_i + \alpha_j^2 t$.

Combining the results of Propositions 1,2,3 we get the following theorem.

Theorem 1. *Any irreducible representation of A_α belongs to one of the types described above. Representations corresponding to the different types or to the different parameters within the same type are non-equivalent.*

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