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HEREDITARY PROPERTIES OF HYPERSPACES

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Abstract. In this paper, we investigate hereditary properties of hyperspaces. Our basic cardinals are the Suslin hereditary number, the hereditary π-weight, the Shanin hereditary number, the hereditary density, the hereditary cellularity. We prove that the hereditary cellularity, the hereditary π-weight, the Shanin hereditary number, the hereditary density, the hereditary cellularity for any Eberlein compact and any Danto space and their hyperspaces coincide.

1. Introduction

For a topological T_1-space X we denote

$$\exp X = \{F : F \subset X,\ F \neq \varnothing,\ F \text{ is a closed subset of } X\}.$$

Consider the family $\mathcal{B}$ of all sets in the form of

$$O\langle U_1, \dots, U_n\rangle = \Big\{F \in \exp X : F \subset \bigcup_{i=1}^{n} U_i,\ F \cap U_i \neq \varnothing,\ i = 1, 2, \dots, n\Big\},$$

where $U_1, U_2, \dots, U_n$ are arbitrary open sets in X. The family $\mathcal{B}$ generates a topology on the set $\exp X$. This topology is called the Vietoris topology. The set $\exp X$ with the Vietoris topology is called the exponential space or the hyperspace of the space X.

Let X be a topological T_1-space. We denote by $\exp_n X$ the family of all non-empty closed subsets of the space X of the cardinality not greater than cardinal number n, i.e., $\exp_n X = \{F \in \exp X : |F| \leq n\}$. Put $\exp_c X = \{F \in \exp X :\ F \text{ is a compact in } X\}$. It is clear that

$$\exp_n X \subset \exp_\omega X \subset \exp_c X \subset \exp X.$$

It is not difficult to see that $\exp_\omega X$ is everywhere dense in $\exp X$, hence, $\exp_c X$ is also everywhere dense in $\exp X$ [1].

A cardinal number $\tau > \aleph_0$ is said to be *a caliber* [2] of the space X if for any family $\mu = \{U_\alpha : \alpha \in A\}$ of non-empty open in X sets such that $|A| = \tau$, there exists $B \subset A$ such that $|B| = \tau$ and the family $\cap\{U_\alpha : \alpha \in B\} \neq \varnothing$.

Put $k(X) = \{\tau : \tau \text{ is a caliber for } X\}$.

A cardinal number $\tau > \aleph_0$ is said to be *a precaliber* [2] of the space X if for any family $\mu = \{U_\alpha : \alpha \in A\}$ of non-empty open in X sets such that $|A| = \tau$, there exists $B \subset A$ such that $|B| = \tau$ and the family $\{U_\alpha : \alpha \in B\}$ is centered.

Put $pk(X) = \{\tau : \tau \text{ is a precaliber for } X\}$.

The cardinal number $\min\{\tau : \tau^+ \text{ is a caliber of } X\}$ is called *the Shanin number* of the space X. This cardinal number is denoted $sh(X)$.

Further, the cardinal number $psh(X) = \min\{\tau : \tau^+ \text{ is a precaliber of } X\}$ is called *the preshanin number*.

In 1996, V. Fedorchuk proved the following.

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Theorem 1.1. ([3]). *Let X be an infinite compact. Then*

$$c(\exp X) = \sup\left\{c(X^n) : n \in \mathbb{N}\right\}.$$

The following result was proved in [4].

Theorem 1.2. ([4]). *Let X be an infinite T_1-space. Then*
1) $\pi\chi(X) = \pi\chi(\exp_n X) = \pi\chi(\exp_\omega X) \le \pi\chi(\exp_c X) \le \pi\chi(\exp X)$,
2) $\varphi(X) = \varphi(\exp_n X) = \varphi(\exp_\omega X) = \varphi(\exp_c X) = \varphi(\exp X)$,
where φ is one of the following cardinal function: wd, πw, k, pk, psh.

Let us recall the following definition [5]. If φ is any cardinal invariant, then the new cardinal invariant defined by the formula

$$h\varphi(X) = \sup\{\varphi(Y) : Y \subset X\}$$

is denoted $h\varphi$.

The invariants $hc(X)$, $hd(X)$, $h\pi w(X)$, $hsh(X)$ mean the Suslin hereditary number (or the hereditary cellularity), the hereditary density, the hereditary π-weight, and the Shanin hereditary number, respectively. The spread [5] $s(X)$ of the space X is the least infinite cardinal τ such that the power of a discrete subspace of X doesn't exceed τ, i.e., $s(X) = \sup\{\tau : \tau = |Y|, Y \subset X, Y \text{ is discrete}\}$. It is easy to see that the Suslin hereditary number $hc(X)$ of the space X coincides with its spread $s(X)$.

We need the following result of B. E. Shapirovski.

Theorem 1.3. ([6]). *Let X be a compact. Then*

$$hd(X) = h\pi w(X) = hsh(X).$$

2. Basic results

We begin this part with the following.

Example 2.1. *Let X^{**} be a compact of two Alexandrov arrows. Then*
1) $hd(\exp X^{**}) \ne hd(X^{**})$;
2) $h\pi w(\exp X^{**}) \ne h\pi w(X^{**})$;
3) $hsh(\exp X^{**}) \ne hsh(X^{**})$;
4) $hc(\exp X^{**}) \ne hc(X^{**})$;
5) $s(\exp X^{**}) \ne s(X^{**})$.

Indeed, consider the space "one arrow" $X^* = [0, 1)$, the base of which is formed by subsets of the form $[\alpha, \beta)$ where $0 \le \alpha < 1$, $\alpha < \beta \le 1$. Consider, in $\exp_2^0(X^*)$, the following set:

$$Y = \left\{F_t = \{t, 1-t\} : 0 < t < \frac{1}{2}\right\}.$$

Let us show that Y is a discrete set of the power continuum. Let $OF_t = \langle O_1^t, O_2^t\rangle$ where $O_1^t = \left[t, \frac{1}{2}\right)$, $O_2^t = [1-t, 1)$. We show that $OF_t \cap Y = F_t$. In fact, let $F_{t'} \subset OF_t$. Since $t' < \frac{1}{2}$, then $t' \in O_1^t$ therefore $t' > t$. But $t' \in O_1^t$ implies that $1 - t' \in O_2^t$, so $1 - t' > 1 - t$, which yields $-t' > -t$ or $t' < t$. The obtained contradiction proves that $OF_t \cap Y = F_t$, hence Y is a discrete set of the power continuum c.

By definition of the spread, $s(\exp_2^0 X^*) = c$. It is known that the space X^* is topologically embedded in X^{**}, which implies $s(\exp_2^0 X^{**}) = c$, hence $hd(\exp_2^0 X^{**}) = c$.

We have from Theorem 1.3 that $hd(X) = h\pi w(X) = hsh(X)$ for any compact X, what is more $hc(X) = s(X)$. So, we obtain

$$\begin{aligned} c = hd(\exp X^{**}) = h\pi w(\exp X^{**}) = hsh(\exp X^{**}) \ne hd(X^{**}) \\ = h\pi w(X^{**}) = hsh(X^{**}) \le \aleph_0 \end{aligned}$$

and

$$c = s(\exp X^{**}) = hx(\exp X^{**}) \ne s(X^{**}) = hc(X^{**}) = \aleph_0.$$

Proposition 2.1. *Let X be an infinite compact. Then*
1) $hd(\exp X) \leq 2^{hd(X)}$;
2) $h\pi w(\exp X) \leq 2^{h\pi w(X)}$;
3) $hsh(\exp X) \leq 2^{hsh(X)}$.

Proof. B. E. Shapirovski [7] showed that if X is a regular space of pointwise countable type, then $w(X) \leq 2^{c(X)\cdot t(X)}$. Since $t(X) \leq hd(X)$ and $c(X) \leq hd(X)$, we have that $hd(\exp X) \leq w(\exp X) = w(X) \leq 2^{c(X)\cdot t(X)} \leq 2^{hd(X)}$. Relations 2) and 3) follow from the equality $hdX() = h\pi w(X) = hsh(X)$ for any compact X. □

Remark 2.1. *Proposition 2.1 is valid for spaces* $\exp_n X$, $\exp_\omega X$, *and* $\exp_c X$.

Theorem 2.1. *Let X be an infinite compact such that $C_p(X)$ is a Lindelöf Σ-space. Then*
1) $c(\exp X) \leq c(X)$,
2) $hc(\exp X) = hc(X)$,
3) $hd(\exp X) = hd(X)$,
4) $h\pi w(\exp X) = h\pi w(X)$,
5) $hsh(\exp X) = hsh(X)$.

Proof. 1) By the Argiros-Negrepontis theorem [8], if X is a compact and $C_p(X)$ is a Lindelöf Σ-space, then $c(X) = w(X)$. Hence we have

$$c(\exp X) = w(\exp X) = w(X) = c(X).$$

2) $hc(X) \leq hc(\exp X)$ because X id a subspace of X, and it is evident that $hc(\exp X) \leq w(\exp X) = w(X) = c(X) \leq hc(X)$. So, we have $hc(X) = hc(\exp X)$.

3) It is clear that $hd(X) \leq w(X)$ and $c(X) \leq hd(X)$. It is known that $hd(X) \leq hd(\exp X)$. Let us prove the converse inequality $hd(X) \geq hd(\exp X)$. In fact, $hd(\exp X) \leq w(\exp X) = w(X) = c(X) \leq hd(X)$. Therefore, $hd(\exp X) = hd(X)$.

4) Relations 4) and 5) follow immediately from the equality $hsh(X) = h\pi w(X) = hd(X)$ for any compact X. □

Remark 2.2. *Theorem 2.1 is valid for the spaces* $\exp_n X$, $\exp_\omega X$, $\exp_c X$.

Let us recall that a compact F is called *an Eberlein compact* if there exists a compact X such that F is homeomorphic to the subspace $C_p(X)$.

Since the class of the Eberlein compacts is contained in the class of compacts for which $C_p(X)$ is a Lindelöf space, we obtain from Theorem 2.1 the following.

Corollary 2.1. *For any Eberlein compact X, we have*
1) $c(X) \leq c(\exp X)$,
2) $hc(X) = hc(\exp X)$,
3) $hd(X) = hd(\exp X)$,
4) $h\pi w(X) = h\pi w(\exp X)$,
5) $hsh(X) = hsh(\exp X)$.

Let A_τ be the compactification by a point (in the sense of P. S. Alexandrov) of a discrete space of the power $\tau \geq \aleph_0$. Since A_τ is an Eberlein compact for any τ and $w(A_\tau) = \tau$ [8], we obtain the following.

Corollary 2.2. *We have always that*
1) $c(\exp A_\tau) \leq c(A_\tau)$,
2) $hc(\exp A_\tau) = hc(A_\tau)$,
3) $hd(\exp A_\tau) = hd(A_\tau)$,
4) $h\pi w(\exp A_\tau) = h\pi w(A_\tau)$,
5) $hsh(\exp A_\tau) = hsh(A_\tau)$.

Corollary 2.3. *Let X be a pseudocompact subset of a Banach space Y in the weak topology. Then*
1) $c(\exp X) \leq c(X)$,
2) $hc(\exp X) = hc(X)$,
3) $hd(\exp X) = hd(X)$,
4) $h\pi w(\exp X) = h\pi w(X)$,
5) $hsh(\exp X) = hsh(X)$.

Proof of this corollary is based on the fact that a pseudocompact subset of a Banach space in the weak topology is an Eberlein compact [8].

The Corson compacts [8] are compact subsets of the Σ-product of separable metrizable spaces (or, what is the same, compact subsets of the Σ-product of segments).

Proposition 2.2. *Let X be an infinite Corson compact such that $C_p(C_p(X))$ is the Lindelöf Σ-space. Then*
1) $c(\exp X) \leq c(X)$,
2) $hc(\exp X) = hc(X)$,
3) $hd(\exp X) = hd(X)$,
4) $h\pi w(\exp X) = h\pi w(X)$,
5) $hsh(\exp X) = hsh(X)$.

Proof of this proposition is based on the fact that if X is a Corson compact for which $C_p(C_p(X))$ is the Lindelöf Σ-space, then $C_p(X)$ is the Lindelöf Σ-space and hence $c(X) = w(X)$ [9].

Let τ be an infinite cardinal number, X be a topological space, and X' be its subspace.

The subspace X' is said to be *τ-monolithic* [10] in X if for any $A \subset X'$ such that $|A| \leq \tau$, $[A]_X$ is a compact of the weight $\leq \tau$.

We say that X *τ-suppresses* X' [10] if $\lambda \geq \tau$ and $A \subset X'$, $|A| \leq 2^\lambda$, imply that there exists $A' \subset X$ such that $[A'] \supset A$ and $|A'| \leq \lambda$.

A topological space X is called *a Danto space* [10] if for any infinite cardinal number τ there exists an everywhere dense in X subspace X' which is simultaneously
1) τ-monolithic in itself,
2) τ-suppressed by the space X.

Theorem 2.2. *Let X be an infinite Danto space. Then*
1) $\chi(\exp X) = \chi(X)$,
2) $t(\exp X) = t(X)$,
3) $hd(\exp X) = hd(X)$,
4) $h\pi w(\exp X) = h\pi w(X)$,
5) $hsh(\exp X) = hsh(X)$,
6) $hc(\exp X) = hc(X)$.

Proof. Let X be an infinite Danto space. Then it is compact and $w(X) = t(X)$ [11]. By compactness of the space X, $t(X) \leq hc(X)$ and $w(X) = w(\exp X)$. So:

1) $\chi(X) \leq \chi(\exp X) \leq w(\exp X) = w(X) = t(X) \leq \chi(X)$. We have from here that $\chi(X) = \chi(\exp X)$.

2) $t(X) \leq t(\exp X) \leq w(\exp X) = w(X) = t(X)$. This implies that $t(X) = t(\exp X)$.

3) $hd(X) \leq hd(\exp X) \leq w(\exp X) = w(X) = t(X) \leq hc(X) \leq hd(X)$, which implies $hd(X) = hd(\exp X)$.

Relations 4) and 5) follow immediately from the equality $hd(X) = h\pi w(X) = hsh(X)$ for any compact X.

6) $hc(X) \leq hc(\exp X) \leq w(\exp X) = w(X) = t(X) \leq hc(X)$. We have from here that $hc(X) = hc(\exp X)$. □

Corollary 2.4. *Let X be a diadic compact. Then*

1) $\chi(\exp X) = \chi(X)$,
2) $t(\exp X) = t(X)$,
3) $hd(\exp X) = hd(X)$,
4) $h\pi w(\exp X) = h\pi w(X)$,
5) $hsh(\exp X) = hsh(X)$,
6) $hc(\exp X) = hc(X)$.

Proof. Since any diadic compact is a Danto compact [11], relations 1), 2), 3), 4), 5), 6) follow immediately from Theorem 2.2. □

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