Methods of Functional Analysis and Topology
Vol. 16 (2010), no. 1, pp. 17–27

OPERATORS DEFINED ON L_1 WHICH "NOWHERE" ATTAIN THEIR NORM

I. V. KRASIKOVA, V. V. MYKHAYLYUK, AND M. M. POPOV

Abstract. Let E be either ℓ_1 of L_1. We consider E-unattainable continuous linear operators T from L_1 to a Banach space Y, i.e., those operators which do not attain their norms on any subspace of L_1 isometric to E. It is not hard to see that if $T : L_1 \to Y$ is ℓ_1-unattainable then it is also L_1-unattainable. We find some equivalent conditions for an operator to be ℓ_1-unattainable and construct two operators, first ℓ_1-unattainable and second L_1-unattainable but not ℓ_1-unattainable. Some open problems remain unsolved.

1. Preliminaries

Concerning standard definitions and notation we follow mainly [9] and [10]. By $\mathcal{L}(X, Y)$ we denote the space of all continuous linear operators acting from a Banach space X to a Banach space Y. The symbol $\mathcal{L}(X)$ is used for $\mathcal{L}(X, X)$. The closed linear span of a sequence (x_n) in a Banach space X is denoted by $[x_n]$. If (Ω, Σ, μ) is a measure space and $x \in L_1(\mu)$ then by $\operatorname{supp} x$ we denote the support $\{\omega \in \Omega : \ x(\omega) \neq 0\}$ of x which is defined as a set, up to a measure null subset. Besides, for $A \in \Sigma^+$ (i.e., for $A \in \Sigma, \ \mu(A) > 0$), the symbol $L_1(A)$ is reserved for the subspace $\{x \in L_1(\mu) : \ \operatorname{supp} x \subseteq A\}$ of $L_1(\mu)$ and $L_1^+(A)$ is the positive cone of this subspace $\{x \in L_1(A) : \ x \geq 0\}$ (note that $x \leq y$ means that $x(\omega) \leq y(\omega)$ for almost all $\omega \in \Omega$). For $A, B, C \in \Sigma$ by $C = A \sqcup B$ we mean that both $C = A \cup B$ and $A \cap B = \emptyset$ hold. Analogously, for $x, y, z \in L_1(\mu)$ the equality $x = y \sqcup z$ means that $x = y + z$ and $\operatorname{supp} y \cap \operatorname{supp} z = \emptyset$.

If Y is a Banach space, $T \in \mathcal{L}\big(L_1(\mu), Y\big)$ and $A \in \Sigma^+$ then by T_A we denote the restriction of T to the subspace $L_1(A)$. The positive and the negative parts of an element $x \in L_1(\mu)$ are defined as $x^+(\omega) = x(\omega)$ when $x(\omega) \geq 0$ and $x^+(\omega) = 0$ when $x(\omega) < 0$ and $x^- = x^+ - x$. The characteristic function of a set $A \in \Sigma$ is denoted by $\mathbf{1}_A$. A sequence (x_n) in $L_1(\mu)$ is called *disjoint* provided $\operatorname{supp} x_i \cap \operatorname{supp} x_j = \emptyset$ for $i \neq j$.

By $\mathcal{B}$ we denote the Borel σ-field on $[0, 1]$ and by λ the Lebesgue measure on $\mathcal{B}$.

Let X and Y be Banach spaces over the reals with X infinite dimensional. We say that an operator $T \in \mathcal{L}(X, Y)$

- *attains its norm at an element* $x \in X \setminus \{0\}$ if $\|Tx\| = \|T\|\|x\|$;
- *attains its norm* provided that it attains its norm at some $x \in X$;
- *attains its norm on a subspace* $X_1 \subseteq X$ if the restriction $T|_{X_1} \in \mathcal{L}(X_1, Y)$ of T to X_1 attains its norm;
- *is E-unattainable* if does not attain its norms on any subspace of L_1 isometric to E[1];
- *nowhere attains its norm* if T does not attain its norm on any infinite dimensional subspace $X_1 \subseteq X$.

The set of all operators from $\mathcal{L}(X, Y)$ attaining their norm is denoted by $\mathcal{NA}(X, Y)$.

2000 *Mathematics Subject Classification.* Primary 47B38; Secondary 46B04.
Key words and phrases. Norm attaining operator, the space L_1.
V. V. Mykhaylyuk and M. M. Popov, Partially supported by Ukr. Derzh. Tema N 0103Y001103.

[1] We consider the cases $E = \ell_1$ and $E = L_1$ only.

The famous Bishop-Phelps theorem (1961) [2] asserts that for any Banach space X the set $\mathcal{NA}(X,\mathbb{R})$ of all norm attaining functionals $f \in X^*$ is dense in X^*. As it was shown by Lindenstrauss (1963) [8], this theorem is not longer true for operators. Among positive results in this direction it is ought to mention Bourgain's theorem on the denseness of $\mathcal{NA}(X,Y)$ in $\mathcal{L}(X,Y)$ in the case when X has the Radon-Nikodým property. The corresponding sets are dense in $\mathcal{L}(L_1)$ [4] and $\mathcal{L}\big(C[0,1]\big)$ [5], but is not dense in $\mathcal{L}\big(L_1, C[0,1]\big)$ [13]. We refer the reader to [1] for more details.

Some related facts on the structure of the set of those elements $x \in L_1$ at which a given operator $T \in \mathcal{L}(L_1, Y)$ attains its norm are given in [11].

2. Introduction

It is an easy exercise to construct an operator $T \in \mathcal{L}(L_1)$ that does not attain its norm. Let $[0,1] = \bigsqcup\limits_{n=1}^{\infty} A_n$ be any decomposition, $A_n \in \mathcal{B}^+$ and $\alpha_n \uparrow 1$ be a sequence of positive numbers. Then the operator $T : L_1 \to L_1$ given by

$$Tx = \sum_{n=1}^{\infty} \Big(\frac{\alpha_n}{\lambda(A_n)} \int\limits_{A_n} x \, d\lambda \Big) \mathbf{1}_{A_n}, \quad x \in L_1,$$

does not attain its norm. Indeed, since

$$\|Tx\| \le \sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda(A_n)} \int\limits_{A_n} |x| d\lambda \, \|\mathbf{1}_{A_n}\| < \sum_{n=1}^{\infty} \int\limits_{A_n} |x| d\lambda = \|x\| \tag{2.1}$$

for each $x \in L_1$, one has that $\|T\| \le 1$. On the other hand, $\|T\mathbf{1}_{A_m}\| = \alpha_m \|\mathbf{1}_{A_m}\|$ for any $m \in \mathbb{N}$ and, hence, $\|T\| = 1$. The same strict inequality (2.1) yields that T does not attain its norm at any element. Nevertheless, the restriction T_{A_n} to any subspace $L_1(A_n)$, $n \in \mathbb{N}$, attains its norm at every element $x \in L_1^+(A_n)$.

Consider the following question.

Problem 2.1. *Let X and Y be Banach spaces with X infinite dimensional. Does there exist an operator $T \in \mathcal{L}(X,Y)$ which nowhere attains its norm? What if $X = L_1$?*

The following example due to M. Ostrovskii (private communication) gives a positive answer to this problem for classical sequence spaces.

Example 2.2 (M. Ostrovskii)**.** *Let $E = \ell_p$ with $1 \le p < \infty$, or $E = c_0$ and $(\alpha_n)_1^\infty$ be a sequence of scalars such that $0 < \alpha_n \uparrow 1$. Then the operator $T \in \mathcal{L}(E)$, given by*

$$T(\xi_1, \ldots, \xi_n, \ldots) = (\alpha_1 \xi_1, \ldots, \alpha_n \xi_n, \ldots),$$

nowhere attains its norm.

Indeed, since $\|Tx\| < \|x\|$ for each $x \in E \setminus \{0\}$, it is enough to prove that $\|T|_X\| = 1$ for each infinite dimensional subspace $X \subseteq E$. Given such an X, for every $n \in \mathbb{N}$ by E_n we denote the set of all vectors from E with zero coordinates from the first up to n-th. Since E_n has finite codimension in E and X is an infinite dimensional subspace of E, we obtain that $X \cap E_n \ne \{0\}$ for each n. Now fix any $\varepsilon > 0$ and pick an n such that $\alpha_n > 1 - \varepsilon$, and choose $x = (0, \ldots, 0, \xi_n, \xi_{n+1}, \ldots) \in X \cap E_n \ne \{0\}$ with $\|x\| = 1$. Then

$$\|Tx\| = \big\|(0, \ldots, 0, \alpha_n \xi_n, \alpha_{n+1} \xi_{n+1}, \ldots)\big\| \ge \alpha_n \|x\| > 1 - \varepsilon.$$

By arbitrariness of ε, one gets $\|T|_X\| = 1$.

As the proof shows, this example remains correct for the case of any sequence space E for which the inequality

$$\big\|(\xi_1, \ldots, \xi_n, \ldots)\big\| > \big\|(\alpha_1 \xi_1, \ldots, \alpha_n \xi_n, \ldots)\big\|$$

holds for every nonzero vector $(\xi_1, \ldots, \xi_n, \ldots) \in E$. Note also that the space ℓ_∞ does not have this property and the example is not longer valid, because the corresponding operator attains its norm at $(1, 1, \ldots)$.

Remark also that the idea of Example 2.2 cannot be applied to operators from $\mathcal{L}(X, Y)$ if X has the Daugavet property, e.g., $X = L_1$. Recall that a Banach space X is said to have the Daugavet property (DP, in short) if $\|Id + K\| = 1 + \|K\|$ for every rank one (equivalently, every weakly compact) operator $K \in \mathcal{L}(X)$ where Id is the identity of X [7].

Indeed, the main point of Example 2.2 is that $T \in \mathcal{L}(X, Y)$ possesses the properties: $\|T\| = 1$ and

for every $\varepsilon > 0$ there exists a finite codimensional subspace $X_0 \subseteq X$ with

$$\inf\big\{\|Tx\| : \ x \in S_{X_0}\big\} > 1 - \varepsilon. \tag{2.2}$$

The following statement shows that if X has the DP then such an operator must be an isometric embedding and hence attains its norm.

Proposition 2.3. *Let X, Y be Banach spaces with X having the DP, $T \in \mathcal{L}(X, Y)$, $\|T\| = 1$, $0 < a < 1$, and $\inf\big\{\|Tx\| : \ x \in S_X\big\} < 1 - a$. Then for every finite codimensional subspace $X_0 \subseteq X$ one has that*

$$\inf\big\{\|Tx\| : \ x \in S_{X_0}\big\} < 1 - \frac{a}{2}. \tag{2.3}$$

Proof. Fix an $x_0 \in S_X$ with $\|Tx_0\| < 1 - a$ and an $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$. Given any finite codimensional subspace $X_0 \subseteq X$, we set $X_1 = X_0 \cap \ker x_0^*$ and $X_2 = \operatorname{lin}\{x_0, X_1\}$. Since X_2 is a finite codimensional subspace of a Banach space with the DP, it itself has the DP [7]. This implies that the natural projection $P : X_2 \to X_1$ defined by $Px = x - x_0^*(x)x_0$ has norm $1 + 1 = 2$. So, for every $\varepsilon > 0$ there exists an $u \in S_{X_2}$ with $\|Pu\| > 2 - \varepsilon$. Now, since $Pu/\|Pu\| \in S_{X_0}$, we have that

$$\begin{aligned}\inf\big\{\|Tx\| : \ x \in S_{X_0}\big\} &\le \left\|\frac{TPu}{\|Pu\|}\right\| < \frac{\|TPu\|}{2 - \varepsilon} = \frac{\big\|T\big(u - x_0^*(u)x_0\big)\big\|}{2 - \varepsilon} \\ &= \frac{\big\|Tu - x_0^*(u)Tx_0\big\|}{2 - \varepsilon} < \frac{1 + 1 - a}{2 - \varepsilon} = \frac{2 - a}{2 - \varepsilon}.\end{aligned}$$

By the arbitrariness of ε, the proposition is proved. □

Thus, it remains to observe that both conditions (2.2) and (2.3) imply that T is an isometric embedding.

This paper is devoted to some questions close to the second part of Problem 2.1 concerning L_1. More precisely, we investigate the following particular question.

Problem 2.4. *Do there exist a Banach space X and an E-unattainable operator $T \in \mathcal{L}(L_1, X)$ for $E = \ell_1$ or $E = L_1$?*

3. ℓ_1-unattainable operators

Observe that if Y is a Banach space and an operator $T \in \mathcal{L}(L_1, Y)$ is ℓ_1-unattainable then T does not attain its norm. Indeed, let T attain its norm at some element $x_1 \in L_1 \setminus \{0\}$. Without loss of generality we may assume that $\lambda([0,1] \setminus \operatorname{supp} x_1) > 0$, otherwise we decompose $x_1 = y \sqcup z$ with $y, z \neq 0$ and T must attain its norm at least on one of the elements y, z, which obviously satisfies the desired condition. Next we choose any disjoint sequence of nonzero elements $x_2, x_3, \ldots \in L_1([0,1] \setminus \operatorname{supp} x_1)$. Then T attains its norm on the subspace $[x_n]_{n=1}^\infty$ which is isometric to ℓ_1. The same argument shows that the following statement is true.

Proposition 3.1. *Let Y be a Banach space. If an operator $T \in \mathcal{L}(L_1, Y)$ is ℓ_1-unattainable then T is L_1-unattainable.*

Now we are going to choose more deep properties of ℓ_1-unattainable operators. According to Pełczyński [12], a subspace $X \subseteq L_1$ is isometric to ℓ_1 if and only if it is spanned by a disjoint sequence (x_n), $X = [x_n]$. Besides, we shall use the following simple observation: if Y is a Banach space and $T \in \mathcal{L}(\ell_1, Y)$, then $\|T\| = \sup\limits_n \|Te_n\|$, where (e_n) is the standard basis for ℓ_1. Thus, if (x_n) is a normalized disjoint sequence in L_1, $X = [x_n]$ + $T \in \mathcal{L}(L_1, Y)$, then $\|T|_X\| = \sup\limits_n \|Tx_n\|$.

Theorem 3.2. *Let Y be a Banach space and $T \in \mathcal{L}(L_1, Y)$ with $\|T\| = 1$. Then the following assertions are equivalent:*

(i) T is ℓ_1-unattainable;

(ii) for any normalized disjoint sequence (x_n) in L_1 one has

$$\sup_m \|Tx_m\| > \|Tx_n\| \quad \textit{for every} \quad n \in \mathbb{N};$$

(iii) for any normalized disjoint sequence (x_n) in L_1 one has

$$\|Tx_n\| < 1 \quad \textit{for every} \quad n \in \mathbb{N} \quad \textit{and} \quad \lim_{i\to\infty} \|Tx_i\| = 1;$$

(iv) for any normalized sequence (x_n) in L_1 with $\lim\limits_{n\to\infty} \lambda(\operatorname{supp} x_n) = 0$ one has

$$\|Tx_n\| < 1 \quad \textit{for every} \quad n \in \mathbb{N} \quad \textit{and} \quad \lim_{i\to\infty} \|Tx_i\| = 1.$$

Proof. Equivalence $(i) \Leftrightarrow (ii)$ follows from Pełczyński's theorem and the above remark. Besides, implications $(iv) \Rightarrow (iii) \Rightarrow (ii)$ are obvious.

$(ii) \Rightarrow (iii)$. Let (ii) holds for a given operator $T \in \mathcal{L}(L_1, Y)$. First we prove that T does not attain its norm, i.e., $\|Tx\| < 1$ for each $x \in S(L_1)$. Suppose to the contrary that $\|Tx\| = 1$ for some $x \in S(L_1)$. Choose any disjoint sequence $A_n \in \mathcal{B}^+$ with $\operatorname{supp} x = \bigsqcup\limits_{n=1}^{\infty} A_n$ and set $y_n = x \cdot \mathbf{1}_{A_n}$ for each $n \in \mathbb{N}$. Then from

$$1 = \Big\| \sum_{n=1}^{\infty} Ty_n \Big\| \le \sum_{n=1}^{\infty} \|Ty_n\| \le 1 \sum_{n=1}^{\infty} \|y_n\| = 1$$

we deduce that $\sum\limits_{n=1}^{\infty} \|Ty_n\| = 1 \sum\limits_{n=1}^{\infty} \|y_n\|$. But this easily implies that $\|Ty_n\| = 1\|y_n\|$ for each n. Thus, for the normalized disjoint sequence $x_n = y_n/\|y_n\|$ one has $\|Tx_n\| = 1$ for each n, which contradicts (ii).

Now we prove the second part of (iii). Suppose to the contrary that there exists a normalized disjoint sequence (u_i) in L_1 such that the equality $\lim\limits_{i\to\infty} \|Tu_i\| = 1$ does not hold. Then there are a number $\delta > 0$ and a normalized disjoint sequence (x_n) in L_1 such that $\|Tx_n\| < 1 - \delta$ for each $n \in \mathbb{N}$. Note that, without loss of generality, we may assume that $\bigsqcup\limits_{n=1}^{\infty} \operatorname{supp} x_n = [0, 1]$. Indeed, if $\lambda(A) > 0$ where $A = [0, 1] \setminus \bigsqcup\limits_{n=1}^{\infty} \operatorname{supp} x_n$ then we choose a $\gamma > 0$ so that $\|Tx_1'\| < 1 - \delta$ where

$$x_1' = \frac{x_1 + \gamma \mathbf{1}_A}{\|x_1 + \gamma \mathbf{1}_A\|}$$

and consider the sequence $x_1', x_2, x_3, \ldots$.

We pick $x \in S(L_1)$ so that $\|Tx\| \ge 1 - \delta$ and set $y_n = x \cdot \mathbf{1}_{\operatorname{supp} x_n}$ for each $n \in \mathbb{N}$. Since

$$\sum_{n=1}^{\infty} \|y_n\| \big(1 - \delta\big) = 1 - \delta \le \|Tx\| = \Big\| T \sum_{n=1}^{\infty} y_n \Big\| \le \sum_{n=1}^{\infty} \|Ty_n\|,$$

there is a number n_0 such that $\|Ty_0\| \geq \big(1-\delta\big)\|y_{n_0}\|$. Then putting $z_n = x_n$ for $n \neq n_0$ and $z_{n_0} = y_{n_0}/\|y_{n_0}\|$, for the normalized disjoint sequence (z_n) one obtains $\sup\limits_n \|Tz_n\| = \|Tz_{n_0}\|$ what contradicts (ii).

$(iii) \Rightarrow (iv)$. Suppose that (iii) fulfills, however there exist a $\delta > 0$ and a normalized sequence $x_n \in L_1$ such that $\|Tx_n\| \leq 1(1-\delta)$ for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \lambda(\operatorname{supp} x_n) = 0$. Without loss of generality, we may assume that the series $\sum\limits_{n=1}^{\infty} \lambda(\operatorname{supp} x_n)$ converges (otherwise we pass to a subsequence). Then we choose a subsequence (x_{n_k}) such that $\|u_k\| < \delta/3$ for each $k \in \mathbb{N}$ where $u_k = x_{n_k} \cdot \mathbf{1}_{\bigcup\limits_{i=k+1}^{\infty} \operatorname{supp} x_{n_i}}$. Now for each $k \in \mathbb{N}$ we set $y_k = x_{n_k} - u_k$ and $z_k = y_k/\|y_k\|$. Then for the normalized disjoint sequence (z_k) we have

$$\|Tz_k\| = \frac{\|Tx_{n_k} - Tu_k\|}{\|x_{n_k} - u_k\|} \leq \frac{\|Tx_{n_k}\| + \|Tu_k\|}{\|x_{n_k}\| - \|u_k\|} \leq \frac{1(1-\delta) + 1\,\frac{\delta}{3}}{1-\frac{\delta}{3}} = 1\,\frac{1-\frac{2\delta}{3}}{1-\frac{\delta}{3}}$$

for each k that contradicts (iii). □

For convenience of the notation, if for a given $T \in \mathcal{L}(L_1, Y)$ and an $A \in \mathcal{B}^+$ the restriction T_A is not an isomorphic embedding, we then set $\|T_A^{-1}\| = \infty$.

Theorem 3.3. *Let Y be a Banach space. Then an operator $T \in \mathcal{L}(L_1, Y)$ with $\|T\| = 1$ is ℓ_1-unattainable if and only if the following conditions hold:*

(a) *there exists a $\delta > 0$ such that T_A is an isomorphic embedding which does not attain its norm whenever $A \in \mathcal{B}^+$ and $\lambda(A) < \delta$;*

(b) *$\|T_A\| = 1$ and $\|T_A\|\|T_A^{-1}\| > 1$ for every $A \in \mathcal{B}^+$;*

(c) *$\lim\limits_{\lambda(A)\to 0} \|T_A\|\|T_A^{-1}\| = 1$.*

Proof. The "only if" part. (a). Suppose to the contrary that there exists a sequence of sets $A_n \in \mathcal{B}^+$ such that $\lambda(A_n) \leq 2^{-n}$ and the operators T_{A_n} are unbounded from below for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we pick $x_n \in S\big(L_1(A_n)\big)$ so that $\|Tx_n\| < 2^{-n}$. Then we choose a sequence of numbers $1 \leq k_1 < k_2 < \ldots$ so that for $y_n = x_{k_n} \cdot \mathbf{1}_{A_{k_n} \setminus A_{k_{n+1}}}$ we have $\big\|x_{k_n} - y_n\big\| < 2^{-n}$. Then the sequence $z_n = y_n/\|y_n\|$ is normalized and disjoint. Besides,

$$1 = \big\|x_{k_n}\big\| \geq \|y_n\| \geq 1 - \big\|x_{k_n} - y_n\big\| \geq 1 - \frac{1}{2^n}\,,$$

whence $\Big|\|y_n\| - 1\Big| < 2^{-n}$. Therefore

$$\big\|x_{k_n} - z_n\big\| \leq \big\|x_{k_n} - y_n\big\| + \|y_n - z_n\| < \frac{1}{2^n} + \|y_n\|\left|1 - \frac{1}{\|y_n\|}\right| < \frac{1}{2^{n-1}}\,.$$

Thus,

$$\|Tz_n\| \leq \big\|Tx_{k_n}\big\| + 1\big\|x_{k_n} - z_n\big\| < \frac{1}{2^{k_n}} + 1\,\frac{1}{2^{n-1}}\,,$$

and hence $\lim\limits_{n\to\infty} \|Tz_n\| = 0$, which contradicts Theorem 3.2.

(b). Given $A \in \mathcal{B}^+$, we choose any sequence $A_n \in \mathcal{B}^+$ with $A_n \subseteq A$ and $\lim\limits_{n\to\infty} \lambda(A_n) = 0$. Then putting $x_n = \mathbf{1}_{A_n}/\lambda(A_n)$ for each $n \in \mathbb{N}$, we obtain that $\|T_A\| \geq \|Tx_n\|$ for each n. By Theorem 3.2 (iv), $\|T_A\| \geq 1$, i.e., $\|T_A\| = 1$.

If we had $\big\|T_A\big\|\,\big\|T_A^{-1}\big\| = 1$ for a given $A \in \mathcal{B}^+$ then T_A would attain its norm at each element $x \in L_1(A)$, $x \neq 0$. Indeed,

$$\|T_A x\| \leq \|T_A\|\|x\| = \|T_A\|\|T_A^{-1}(T_A x)\| \leq \|T_A\|\|T_A^{-1}\|\|T_A x\| = \|T_A x\|,$$

whence $\|T_A x\| = \|T_A\|\|x\|$.

(c). By (a), T_A^{-1} exists and is bounded for each $A \in \mathcal{B}^+$. Without loss of generality, we assume that $1 = 1$. Suppose that (c) does not hold. Since $\|T_A^{-1}\| > 1$ for each $A \in \mathcal{B}^+$ by (b), there are a $\delta > 0$ and a sequence $A_n \in \mathcal{B}^+$ such that $\|T_A^{-1}\| > 1 + \delta$ for every $n \in \mathbb{N}$. Now pick a normalized sequence $x_n \in L_1(A_n)$ so that $\|Tx_n\| \le \frac{1}{1+\delta}$ for each $n \in \mathbb{N}$. This contradicts Theorem 3.2 (iv).

The "if" part. It is enough to consider the case $1 = 1$. Let (x_n) be any normalized sequence with $\lim\limits_{n\to\infty} \lambda(A_n) = 0$ where $A_n = \operatorname{supp} x_n$. By the theorem assumptions, $\lim\limits_{n\to\infty} \|T_{A_n}^{-1}\| = 1$. Since $\|T_{A_n}^{-1}\| \ge \frac{\|x_n\|}{\|Tx_n\|}$, we have that $\|T_{A_n}^{-1}\|^{-1} \le \|Tx_n\| \le 1$ for each $n \in \mathbb{N}$. Thus, $\lim\limits_{n\to\infty} \|Tx_n\| = 1$. By Theorem 3.2, T is ℓ_1-unattainable. □

Consider the following example. We define an operator $T \in \mathcal{L}(L_1)$ by putting for each $x \in L_1$

$$Tx = x - \frac{1}{2}\int x d\lambda \cdot \mathbf{1},$$

where $\mathbf{1} = \mathbf{1}_{[0,1]}$. Observe that if $y = Tx$ then $\int y d\lambda - 1/2 \int x d\lambda$ and hence $x = y + \int y d\lambda \cdot \mathbf{1}$. Thus, T is an isomorphic embedding with $\|T^{-1}\| \le 2$. Obviously, $\|T_A\| = 3/2$ for each $A \in \mathcal{B}^+$. We show that the operators T_A do not attain their norm. Let $x \in S(L_1)$ and $\|Tx\| = 3/2$. Since $\left|\int x d\lambda\right| = 1$, we have that either $x \ge 0$ or $x \le 0$. Suppose that $x \ge 0$. We set $B = \{t \in [0,1] : \ x(t) \ge 1/2\}$. Then $\lambda(B) > 0$ and

$$\|Tx\| = \Big\|x - \frac{1}{2}\mathbf{1}\Big\| = \int\limits_B \Big(x(t) - \frac{1}{2}\Big) d\lambda(t) + \int\limits_{[0,1]\setminus B} \Big(\frac{1}{2} - x(t)\Big) d\lambda(t)$$
$$\le \|x\| - \frac{\lambda(B)}{2} + \frac{1 - \lambda(B)}{2} = \frac{3}{2} - \lambda(B),$$

a contradiction.

On the other hand, $\|T_A^{-1}\| \ge 1$ for every $A \in \mathcal{B}^+$, because $Tx = x$ for each $x \in L_1(A)$ with $\int x d\lambda = 0$. Thus, condition (c) from Theorem 3.3 does not hold.

This example shows that conditions (a) and (b) for an operator do not imply that this operator is ℓ_1-unattainable. Besides, it is not very hard to verify concerning our example that T fulfills conditions $(ii) - (iv)$ from Theorem 3.2 for the case when $x_n \ge 0$ for all n in these conditions. This shows that conditions $(ii) - (iv)$ in Theorem 3.2 cannot be stated for positive sequences only.

Theorem 3.4. *There exists an ℓ_1-unattainable operator $T \in \mathcal{L}(L_1)$.*

First we need the following auxiliary construction.

Lemma 3.5. *Given any sets $A, B \in \mathcal{B}$ with $[0,1] = A \sqcup B$ and $\lambda(A) = \lambda(B) = 1/2$, there exists an operator $T = T_{A,B} \in \mathcal{L}\big(L_1, L_1(A \times B)\big)$ with the following properties:*

1. *for each $C \in \mathcal{B}$ one has that $\|T\mathbf{1}_C\| = \lambda(C)$ if and only if either $C \subseteq A$ or $C \subseteq B$;*
2. *$\|T\| = 1$;*
3. *$\|Tx\| \ge 1 - 2\lambda(\operatorname{supp} x)$ for every $x \in L_1$.*

Proof. For each $x \in L_1$ we define a function $Tx \in L_1(A \times B)$ of two variables as follows

$$(Tx)(s,t) = 2x|_A(s) - 2x|_B(t)$$

(here by $x|_C$ we denote the restriction of x to a set $C \in \mathcal{B}$, i.e. $x|_C = x \cdot \mathbf{1}_C$).

(1) Fix any $C \in \mathcal{B}$. We set $C_A = C \cap A$ and $C_B = C \cap B$. Then one has

$$\|T\mathbf{1}_C\| = 2 \iint\limits_{A\times B} \big|\mathbf{1}_C(s) - \mathbf{1}_C(t)\big| dsdt = 2 \iint\limits_{A\times B} \big|\mathbf{1}_{C_A}(s) - \mathbf{1}_{C_B}(t)\big| dsdt. \tag{3.1}$$

On the other hand,

$$2\iint\limits_{A\times B}\big(\mathbf{1}_{C_A}(s)+\mathbf{1}_{C_B}(t)\big)dsdt = 2\lambda(B)\int\limits_A \mathbf{1}_{C_A}(s)ds + 2\lambda(A)\int\limits_B \mathbf{1}_{C_B}(t)\,dt = \lambda(C_A)+\lambda(C_B)=\lambda(C). \tag{3.2}$$

From (3.1) and (3.2) we conclude that $\|T\mathbf{1}_C\| = \lambda(C)$ if and only if

$$\iint\limits_{A\times B}\big|\mathbf{1}_{C_A}(s)-\mathbf{1}_{C_B}(t)\big|dsdt = \iint\limits_{A\times B}\big(\mathbf{1}_{C_A}(s)+\mathbf{1}_{C_B}(t)\big)dsdt,$$

what is possible if and only if

$$\big|\mathbf{1}_{C_A}(s)-\mathbf{1}_{C_B}(t)\big| = \mathbf{1}_{C_A}(s)+\mathbf{1}_{C_B}(t) \tag{3.3}$$

for almost all $(s,t)\in A\times B$. Since (3.3) does not hold if $(s,t)\in C_A\times C_B$, we obtain that $\lambda(C_A\times C_B)=\lambda(C_A)\times\lambda(C_B)=0$. Thus, either $\lambda(C_A)=0$ or $\lambda(C_B)=0$. Equivalently, either $C\subseteq A$ or $B\subseteq A$.

(2) In view of (1), it is enough to show that $\|T\|\le 1$. For each $x\in L_1$ one has

$$\begin{aligned}\|Tx\| &\le 2\iint\limits_{A\times B}\big|x|_A(s)\big|\,ds\,dt + 2\iint\limits_{A\times B}\big|x|_B(t)\big|dsdt\\ &= 2\lambda(B)\int\limits_A |x(s)|ds + 2\lambda(A)\int\limits_B |x(t)|dt = \|x\|.\end{aligned}$$

(3) Given any $x\in S(L_1)$, we set $D=\operatorname{supp} x$, $A_1=A\cap D$ and $B_1=B\cap D$. Then

$$\begin{aligned}\|Tx\| &= \iint\limits_{A\times B}\big|2x|_A(s)-2x|_B(t)\big|dsdt\\ &\ge \iint\limits_{A_1\times(B\setminus B_1)}\big|2x|_A(s)-2x|_B(t)\big|dsdt + \iint\limits_{(A\setminus A_1)\times B_1}\big|2x|_A(s)-2x|_B(t)\big|dsdt\\ &= 2\iint\limits_{A_1\times(B\setminus B_1)}|x(s)|dsdt + 2\iint\limits_{(A\setminus A_1)\times B_1}|x(t)|dsdt\\ &= 2\big(\lambda(B)-\lambda(B_1)\big)\int\limits_{A_1}|x(s)|ds + 2\big(\lambda(A)-\lambda(A_1)\big)\int\limits_{B_1}|x(t)|dt\\ &\ge \big(1-2\lambda(D)\big)\Big(\int\limits_{A_1}|x(s)|ds + \int\limits_{B_1}|x(t)|dt\Big) = 1-2\lambda(D).\end{aligned}$$

□

Proof of Theorem 3.4. For each $n\in\mathbb{N}$ decompose $[0,1]=A_n\sqcup B_n$ with $A_n,B_n\in\mathcal{B}^+$ so that $\lambda(A_n)=\lambda(B_n)=1/2$ and

$$\lambda\Big(\bigcap_{k=1}^n C_k\Big)=2^{-n}\quad\text{for each}\quad n\quad\text{and}\quad C_k\in\{A_k,B_k\} \tag{$*$}$$

(for example, one can set $A_n=\{t\in[0,1]:\ r_n(t)=1\}$ where (r_n) is the Rademacher system on $[0,1]$). Then decompose $[0,1]=\bigsqcup_{n=1}^\infty D_n$ with $D_n\in\mathcal{B}^+$. For every $n\in\mathbb{N}$ let $T_{A_n,B_n}: L_1\to L_1(A_n\times B_n)$ be an operator having properties $(1)-(3)$ from Lemma 3.5. Let $J_n: L_1(A_n\times B_n)\to L_1(D_n)$ be any linear isometric embedding for each n. Then set

$T_n = J_n \circ T_{A_n,B_n}$ and observe that $T_n \in \mathcal{L}\big(L_1, L_1(D_n)\big)$ has properties (1) – (3) for each n as well. Finally we put $T = \sum\limits_{n=1}^{\infty} 2^{-n} T_n$. Obviously, $T \in \mathcal{L}(L_1)$ with $\|T\| \le 1$. Our goal is to show that T satisfies condition (iii) from Theorem 3.2. Let (x_i) be any normalized disjoint sequence in L_1. Then by definition of T and property (3) for T_n's we obtain

$$\|Tx_i\| = \sum_{n=1}^{\infty} 2^{-n} \|T_n x_i\| \ge 1 - 2\lambda(\operatorname{supp} x_i) \longrightarrow 1 \quad \text{as} \quad i \to \infty.$$

Hence $\|T\| = 1$ and $\lim\limits_{i\to\infty} \|Tx_i\| = \|T\|$. It remains to show that T does not attain its norm. Suppose to the contrary that T attains its norm. Then by Lemma 3.2 of [11], there exists a set $A \in \mathcal{B}^+$ such that T attains its norm on the positive cone $L_1^+(A)$. In particular, $\|T\mathbf{1}_A\| = \lambda(A)$. On the other hand, $\|T\mathbf{1}_A\| = \sum\limits_{n=1}^{\infty} 2^{-n}\|T_n \mathbf{1}_A\|$. We claim that $\|T_n\mathbf{1}_A\| = \lambda(A)$ for each n. Indeed, if we suppose that $\|T_{n_0}\mathbf{1}_A\| < \lambda(A)$ for some n_0 then, taking into account that $\|T_n\mathbf{1}_A\| \le \lambda(A)$ for each n, we would obtain that $\|T\mathbf{1}_A\| < \lambda(A)$. Thus, $\|T_n\mathbf{1}_A\| = \lambda(A)$ for each n is established. By condition (1) of Lemma 3.5, for each n we have that $A \subseteq C_n$ where $C_n \in \{A_n, B_n\}$. Thus, for each n one has that $A \subseteq \bigcap\limits_{k=1}^{n} C_k$ whence $\lambda(A) \le 2^{-n}$ by choice of the sets A_n, B_n. But this contradicts the condition $\lambda(A) > 0$. □

4. An L_1-unattainable operator, which is not ℓ_1-unattainable

Lemma 4.1. *Let X and Y be Banach spaces, $T \in \mathcal{L}(X,Y)$ and $x, y, z \in X$ satisfy $x = y + z$ and $\|x\| = \|y\| + \|z\|$. If*

$$\frac{\|Tx\|}{\|x\|} \ge \max\Big\{\frac{\|Ty\|}{\|y\|}, \frac{\|Tz\|}{\|z\|}\Big\}, \tag{4.1}$$

then the following equalities hold:

(i) $$\|Tx\| = \|Ty\| + \|Tz\|;$$

(ii) $$\frac{\|Tx\|}{\|x\|} = \frac{\|Ty\|}{\|y\|} = \frac{\|Tz\|}{\|z\|}.$$

Proof. (i). We set $\alpha = \frac{\|Tx\|}{\|x\|}$. Then by (4.1),

$$\alpha\|y\| + \alpha\|z\| = \alpha\|x\| = \|Tx\| \le \|Ty\| + \|Tz\| \le \alpha\|y\| + \alpha\|z\|,$$

which implies (i).

(ii). If we suppose that $\frac{\|Ty\|}{\|y\|} < \alpha$, then $\|Ty\| + \|Tz\| < \alpha\|y\| + \alpha\|z\|$, a contradiction. □

Note that (4.1) is valid if T attains its norm at x.

Theorem 4.2. *Let (Ω, Σ, μ) be a measure space, Y be a Banach space and an operator $T \in \mathcal{L}\big(L_1(\mu), Y\big)$ attains its norm at $x \in L_1^+(\mu)$. Then T atains its norm at any element $0 \ne y \in L_1^+(\operatorname{supp} x)$.*

Proof. It is enough to prove that T attains its norm at any element of some dense subset $M \subseteq L_1^+(\operatorname{supp} x)$, since if $y \in L_1^+(\operatorname{supp} x)$, $y_n \in M$ and $\lim\limits_{n\to\infty} y_n = y$ then

$$\frac{\|Ty\|}{\|y\|} = \lim_{n\to\infty} \frac{\|Ty_n\|}{\|y_n\|} = \|T\|.$$

For every $n \in \mathbb{N}$ we put $A_n = \{\omega \in \Omega : \ x(\omega) \geq 1/n\}$ and $M = \bigcup\limits_{n=1}^{\infty} L_\infty^+(A_n)$. Fix any $u \in M$, say, $u \in L_\infty(A_m)$. Then for

$$y = \frac{u}{m \, \|u\|_\infty}$$

one obtains that $0 \leq y \leq x$. Thus, for $x, y + z = x - y$ the assumptions of Lemma 4.1 are satisfied. By item (ii) of this lemma, T attains its norm at y, and hence, at u. It is enough to note that M is dense in $L_1^+(\operatorname{supp} x)$, because $\operatorname{supp} x = \bigcup\limits_{n=1}^{\infty} A_n$, up to a measure null set. □

The following statement clarifies Lemma 3.2 of [11].

Corollary 4.3. *Let (Ω, Σ, μ) be a measure space, Y be a Banach space and an operator $T \in \mathcal{L}\big(L_1(\mu), Y\big)$ attains its norm at $x \in L_1(\mu)$. Then T attains its norm at any element $0 \neq y \in L_1^+(\operatorname{supp} x^+) \cup L_1^+(\operatorname{supp} x^-)$.*

Proof. Using Lemma 4.1 (ii) for $y = x^+$ and $z = -x^-$, we obtain that T attains its norm at each element x^+, x^-. Then use Theorem 4.2 for x^+ and x^-. □

The functional $f(x) = \int\limits_0^{\frac{1}{2}} x d\lambda - \int\limits_{\frac{1}{2}}^{1} x d\lambda$, which attains its norm at $x = \mathbf{1}_{[0,1/2)} - \mathbf{1}_{[1/2,1]}$, however does not attain its norm at any element of the form $y = \mathbf{1}_A$, where $\lambda\big(A \cap [0, 1/2)\big) = \lambda(A)/2$ (one has that $f(y) = 0$ in this case), shows that the positivity condition on x in Theorem 4.2 is essential, and that of any x we cannot say more than Corollary 4.3 gives.

Recall that a Banach space Y is said to be *strictly convex* if for any elements $x \neq y$ of $S(Y)$ one has $\|x + y\| < 2$, or equivalently, if $S(Y)$ contains no segment.

Theorem 4.4. *Let Y be a strictly convex Banach space, $x \in L_1$, $A_1 = \operatorname{supp} x^+$ and $A_2 = \operatorname{supp} x^-$. Suppose that an operator $T \in \mathcal{L}(L_1, Y)$ attains its norm at x. Then T_{A_i} are rank one operators for $i = 1, 2$.*

Proof. We prove the theorem for $i = 1$ (the proof for $i = 2$ is analogous). By Corollary 4.3, T attains its norm at x^+. Let $v \in L_\infty^+(A_1)$ be any nonzero element. We set $\beta = \frac{\|Tv\|}{\|T(x^+)\|}$. Since T attains its norm at v (cf. Theorem 4.2), one has, in particular, that $\beta > 0$. Theorem 4.2 implies also that T attains its norm at $w = \beta x^+ + v$. Since $\|w\| = \|\beta x^+\| + \|v\|$, using Lemma 4.1, we obtain that $\|Tw\| = \|T(\beta x^+)\| + \|Tv\|$. On the other hand, $\|T(\beta x^+)\| = \|Tv\|$ by definition of β. If we suppose $T(\beta x^+) \neq Tv$, then the strict convexity of Y gives

$$2\|Tv\| = \|T(\beta x^+)\| + \|Tv\| = \|Tw\| = \|T(\beta x^+) + Tv\| < 2\|Tv\|,$$

a contradiction. Thus, $Tv = \beta T(x^+) = \frac{\|Tv\|}{\|T(x^+)\|} T(x^+)$. Suppose now that $v \in L_\infty(A_1)$ be any element. Then

$$Tv = T(v^+) - T(v^-) = \frac{\|T(v^+)\| - \|T(v^-)\|}{\|T(x^+)\|} \, T(x^+).$$

□

A Banach space Y is called *locally uniformly convex*, provided for each $x, x_n \in Y$, $n \in \mathbb{N}$ the conditions $\|x_n\| \longrightarrow \|x\|$ and $\|x_n + x\| \longrightarrow 2\|x\|$ yield $\|x_n - x\| \longrightarrow 0$. It is easy to see that a locally uniformly convex Banach space is strictly convex. In 1959 M. I. Kadec proved [6] that in every separable Banach space there exists an equivalent locally uniformly convex (in particular, strictly convex) norm.

So, there exists a strictly convex Banach space Y, isomorphic to L_1. Let $T : L_1 \to Y$ be an isomorphism. Since T is one-to-one, T cannot be a rank one operator when being restricted to any infinite dimensional subspace.

Thus, Theorem 4.4 has the following consequence.

Corollary 4.5. *Let Y be a strictly convex Banach space and $T : L_1 \to Y$ be an injective operator. Then T is L_1-unattainable.*

Theorem 4.6. *There exists a Banach space Y and an isomorphism $T : L_1 \to Y$ which is L_1-unattainable but is not ℓ_1-unattainable.*

Proof. Let Y be a strictly convex Banach space isomorphic to L_1 and $T : L_1 \to Y$ be an isomorphism. By Corollary 4.5, T does not attain its norm on each subspace of L_1 isometric to L_1.

Fix any normalized disjoint sequence (x_n) in L_1 and set $X = [x_n]$. Choose $\delta \in \big(0, \frac{1}{\|T\|\|T^{-1}\|}\big)$ and $n_0 \in \mathbb{N}$ so that $(1+\delta)\|Tx_{n_0}\| \geq \sup\limits_m \|Tx_m\|$. Now define an operator $S \in \mathcal{L}(L_1, Y)$ by putting for each $x \in L_1$

$$Sx = Tx + \delta \Big(\int\limits_{\operatorname{supp} x_{n_0}} x \, d\lambda \Big) Tx_{n_0}.$$

Remark that S is an isomorphic embedding, because for each $x \in L_1$ one has

$$\|Sx\| \geq \|Tx\| - \delta \|x\| \|Tx_{n_0}\| \geq \frac{\|x\|}{\|T^{-1}\|} - \delta \|T\| \|x\| = \eta \|x\|,$$

where $\eta = \|T^{-1}\|^{-1} - \delta \|T\| > 0$ by the choice of δ. By Corollary 4.5, S is L_1-unattainable.

Now observe that $\|Sx_n\| = \|Tx_n\|$ if $n \neq n_0$ and

$$\|Sx_{n_0}\| = (1+\delta)\|Tx_{n_0}\| \geq \sup_m \|Tx_m\| = \|T|_X\|$$

by the choice of n_0. Thus, S attains its norm on X which is isometric to ℓ_1. □

5. Some open problems

Problem 5.1. *Do there exist a Banach space Y and an operator $T \in \mathcal{L}(L_1, Y)$ which nowhere attains its norm?*

We also do not know, what if one replace "isometric" with "isomorphic" in problem 2.4.

Problem 5.2. *Does for every Banach space Y and every operator $T \in \mathcal{L}(L_1, Y)$ there exists a subspace of L_1 isomorphic to ℓ_1 on which T attains its norm?*

Problem 5.3. *Does for every Banach space Y and every operator $T \in \mathcal{L}(L_1, Y)$ there exists a subspace of L_1 isomorphic to L_1 on which T attains its norm?*

Acknowledgments. The authors thank M. I. Ostrovskii for helpful discussions and the referee for valuable remarks, especially for communicating us Proposition 2.3.

References

1. M. D. Acosta, *Norm attaining operators into $L_1(\mu)$*, Contemp. Math. **232** (1999), 1–11.
2. E. Bishop, R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98.
3. J. Bourgain, *On dentability and the Bishop-Phelps property*, Israel J. Math. **28** (1977), 265–271.
4. A. Iwanik, *Norm attaining operators on Lebesgue spaces*, Pacific J. Math. **83** (1979), 381–386.
5. J. Johnson, J. Wolfe, *Norm attaining operators*, Studia Math. **65** (1979), 7–19.
6. M. I. Kadec, *On spaces isomorphic to locally uniformly convex spaces*, Izv. Vyssh. Uchebn. Zaved., Mat. **6** (1959), 51–57.
7. V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 855–873.

8. J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148.
9. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces.* I, Springer-Verlag, Berlin—Heidelberg—New York, 1977.
10. J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces.* II, Springer-Verlag, Berlin—Heidelberg—New York, 1979.
11. V. V. Mykhaylyuk, M. M. Popov, *Some geometrical aspects of operators acting from L_1*, Positivity **10** (2006), 431–466.
12. A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. **19** (1960), no. 2, 209–228.
13. W. Schachermayer, *Norm attaining operators on some classes of Banach spaces*, Pacific J. Math. **105** (1983), 427–438.

Department of Mathematics, Zaporizhzhya National University, 2 Zhukovs'koho, Zaporizhzhya, Ukraine

E-mail address: yudp@mail.ru

Department of Mathematics, Chernivtsi National University, 2 Kotsyubyns'koho, Chernivtsi, 58012, Ukraine

E-mail address: mathan@ukr.net

Departamento de Analisis Matematico, Facultad de Ciencias, Universidad de Granada, E-18071, Granada, Spain

E-mail address: misham.popov@gmail.com

Received 03/04/2009; Revised 24/04/2009