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DIMENSION STABILIZATION EFFECT FOR A BLOCK JACOBI-TYPE MATRIX OF A BOUNDED NORMAL OPERATOR WITH THE SPECTRUM ON AN ALGEBRAIC CURVE

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Abstract. Under some natural assumptions, any bounded normal operator in an appropriate basis has a three-diagonal block Jacobi-type matrix. Just as in the case of classical Jacobi matrices (e.g. of self-adjoint operators) such a structure can be effectively used.

There are two sources of difficulties: rapid growth of blocks in the Jacobi-type matrix of such operators (they act in $\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \cdots$) and potentially complicated spectra structure of the normal operators. The aim of this article is to show that these two aspects are closely connected: simple structure of the spectra can effectively bound the complexity of the matrix structure.

The main result of the article claims that if the spectra is concentrated on an algebraic curve the dimensions of Jacobi-type matrix blocks do not grow starting with some value.

1. Introduction

The Jacobi (three-diagonal) representation of a self-adjoint operator is well-known. As it was recently shown (see [4, 5]), the similar Jacobi structure is typical not only for the self-adjoint operators but also for arbitrary unitary and even for any bounded normal operators for which a cyclic vector exists. This leads to numerous applications of these objects just in the same way as it is for the classical Jacobi matrices, see e.g. the application to non-Abelian difference-differential lattices generated by Lax equation [7, 8].

It is much easier to use the normal operator if one knows that its matrix has well-defined three-diagonal block structure and acts in $\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^4 \oplus \cdots$. Things become even better if one knows that these blocks are interleaved with columns and rows of zeros that actually makes it act over a subspaces of constant dimensions, e.g. if a normal operator is in fact a unitary one then it acts in $\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots$ and if it is self-adjoint then it acts in $\mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \cdots$.

The main result of the article claims that if the spectrum of a normal bounded operator is concentrated on a curve $\{z \in \mathbb{C} : \sum_{k=0}^{n} \sum_{\alpha=0}^{k} \gamma_{k,\alpha} z^{k-\alpha} \bar{z}^{\alpha} = 0\}, \gamma_{k,\alpha} \in \mathbb{C}$ then the dimensions of the Jacobi matrix blocks do not grow starting from some value. We call this phenomenon the *dimension stabilization effect*. This article gives an overview of how the dimension stabilization arises, presents both obvious and non-trivial examples, contains the necessary theorems with proofs.

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2. Examples of dimension stabilization effect

Denote $\mathbb{N}_0 = \{0, 1, \dots\}$. Consider the space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0. \tag{1}$$

$\mathbf{l}_2$ is the Hilbert space with the natural scalar product. Let $\delta_{n,k}$ be the standard Kronecker symbol. We call the family $e_{n,\alpha} = ((\delta_{k,n}\delta_{\alpha,\beta})_{\beta=0}^{k})_{k=0}^{\infty} \in \mathbf{l}_2$, $n, k \in \mathbb{N}_0$, $\alpha = 0, \dots, n$; $\beta = 0, \dots, k$ the *standard orthonormal basis* of $\mathbf{l}_2$. Note that $e_{n,\alpha}, \alpha = 0, \dots, n$, are constructed from the classical orthonormal basis elements of the space $\mathcal{H}_n = \mathbb{C}^{n+1}, n \in \mathbb{N}$. The vector $f \in \mathbf{l}_2$ is a sequence $f = (f_n)_{n=0}^{\infty}$ where $f_n = (f_{n,\alpha})_{\alpha=0}^{n} \in \mathcal{H}_n$ and $\sum\limits_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} < \infty$.

Consider the three-diagonal block Jacobi-type matrix

$$J = \begin{pmatrix} b_0 & c_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & c_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & c_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \begin{aligned} a_n :\ & \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ b_n :\ & \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ c_n :\ & \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0, \end{aligned} \tag{2}$$

$$a_n = \left.\underbrace{\begin{bmatrix} a_{n;0,0} & a_{n;0,1} & a_{n;0,2} & \cdots & a_{n;0,n-1} & a_{n;0,n} \\ 0 & a_{n;1,1} & a_{n;1,2} & \cdots & a_{n;1,n-1} & a_{n;1,n} \\ 0 & 0 & a_{n;2,2} & \cdots & a_{n;2,n-1} & a_{n;2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n;n-1,n-1} & a_{n;n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & a_{n;n,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{n+1}\right\} n+2,$$

$$c_n = \left.\underbrace{\begin{bmatrix} c_{n;0,0} & c_{n;0,1} & 0 & \cdots & 0 & 0 \\ c_{n;1,0} & c_{n;1,1} & c_{n;1,2} & \cdots & 0 & 0 \\ c_{n;2,0} & c_{n;2,1} & c_{n;2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n;n,0} & c_{n;n,1} & c_{n;n,2} & \cdots & c_{n;n,n} & c_{n;n,n+1} \end{bmatrix}}_{n+2}\right\} n+1,$$

$$a_{n;0,0} > 0, a_{n;1,1} > 0, \dots, a_{n;n,n} > 0; \quad c_{n;0,1} > 0, c_{n;1,2} > 0, \dots, c_{n;n,n+1} > 0.$$

Entries of b_n are arbitrary.

This matrix generates a linear operator defined on finite vectors. Suppose its closure $\boldsymbol{J} : \mathbf{l}_2 \to \mathbf{l}_2$ is a bounded normal operator. Actually any bounded normal operator $\boldsymbol{J}$ for which a cyclic vector exists and all $z^{n-\alpha}\bar{z}^{\alpha}, n \in \mathbb{N}_0, \alpha = 0, \dots, n$, are linearly independent in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$ (where $\mathfrak{B}(\mathbb{C})$ is the Borel σ-algebra of subsets of $\mathbb{C}$ and ρ is the spectral measure of $\boldsymbol{J}$) has this structure in an appropriate basis, see [5, Theorem 5].

Introduce the linear order

$$z^0\bar{z}^0; \quad z^1\bar{z}^0, z^0\bar{z}^1; \quad z^2\bar{z}^0, z^1\bar{z}^1, z^0\bar{z}^2; \quad \cdots \tag{3}$$

Assume that all the elements of (3) are linearly independent in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. Let us orthogonalize (3) in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. Denote the resulting orthonormal basis by $P_{n,\alpha}(z)$, $n \in \mathbb{N}_0$, $\alpha = 0, \dots, n$. According to [5, Theorem 6] the following unitary map represents the Fourier transform $\hat{\ }$ that maps $\mathbf{l}_2$ onto $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$:

$$\mathbf{l}_2 \ni f = (f_n)_{n=0}^{\infty} \overset{\frown}{\longmapsto} \sum_{n=0}^{\infty} \sum_{\alpha=0}^{n} P_{n,\alpha}(z) f_{n,\alpha} \in L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho). \tag{4}$$

For the purpose of this article it is convenient to consider the inverse $\nu = \widehat{\cdot}^{\,-1}$:

$$L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho) \ni P_{n,\alpha} \overset{\nu}{\mapsto} e_{n,\alpha} \in \mathbf{l}_2. \tag{5}$$

The Fourier transform gives a possibility to map $\boldsymbol{J}$ into the operator $L = \nu^{-1} \circ \boldsymbol{J} \circ \nu$ of multiplication by independent variable z in the space $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$,

$$\begin{array}{ccc} L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho) & \xrightarrow{L} & L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho) \\ {\scriptstyle \nu = \widehat{\cdot}^{\,-1}} \big\downarrow & & \big\downarrow {\scriptstyle \nu = \widehat{\cdot}^{\,-1}} \\ \mathbf{l}_2 & \xrightarrow{\boldsymbol{J}} & \mathbf{l}_2 \end{array} \tag{6}$$

Note that the matrices of L in the basis $P_{n,\alpha}$ and $\boldsymbol{J}$ in $e_{n,\alpha}$ coincide.

In this article we consider the case where the elements of (3) are linearly dependent in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. In this case it is necessary to construct the injective (but not surjective) map $\nu : L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho) \to \mathbf{l}_2$ explicitly.

We deliberately use the same notation ν for both mappings (the one that is the inverse to the Fourier transform and the one that will be constructed) for two reasons. If all elements of (3) are linearly independent then these two mappings are actually the same. And if they are not linearly independent then the corresponding Fourier transform $\nu^{-1} = \widehat{\cdot}$ can also be constructed but it acts not from the whole $\mathbf{l}_2$ but from its subspace $\tilde{\mathbf{l}}_2 = \operatorname{Ran} \nu \subset \mathbf{l}_2$.

Suppose (3) are total in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$, i.e., $\overline{\operatorname{span}}\{z^{n-\alpha}\bar{z}^{\alpha} |\ n \in \mathbb{N}_0,\ \alpha = 0, \dots, n\} = L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. Thus (3) can be used to build an orthonormal basis of $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. It is necessary to extract a linearly independent subsystem of (3) and apply the Gramm-Schmidt orthogonalization procedure. Consider this process in details.

Introduce an *orthogonalization procedure.* While performing the Gramm-Schmidt orthogonalization do as follows:

(1) if the next element $z^{n-\alpha}\bar{z}^{\alpha}$ of (3) occurs to be linearly independent on the previous elements, then denote $P_{n,\alpha}(z)$ the element of the orthonormal basis generated by $z^{n-\alpha}\bar{z}^{\alpha}$ according to the Gramm-Schmidt formula, and map $P_{n,\alpha}(z)$ to the corresponding basis element $e_{n,\alpha}$. Thus $\nu(P_{n,\alpha}) = e_{n,\alpha}$.
(2) if $z^{n-\alpha}\bar{z}^{\alpha}$ happens to be linearly dependent on the previous elements of (3) then skip it (the action of ν on such elements is already defined because they are linear combinations of the previous elements). For such n, α let $P_{n,\alpha}(z) \equiv 0$.

Thus ν is defined on the basis of $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. Continue it in an obvious way up to a map over the whole $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$.

Denote $\tilde{\mathbf{l}}_2 = \operatorname{Ran} \nu \subset \mathbf{l}_2$. Since ν is an isometry, we conclude that $\tilde{\mathbf{l}}_2$ is closed. Thus ν is a unitary map from the space $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$ to the subspace $\tilde{\mathbf{l}}_2$ of the space $\mathbf{l}_2$. Note the following properties.

(1) If all the elements of (3) are linearly independent in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$ then $\tilde{\mathbf{l}}_2 = \mathbf{l}_2$;
(2) The inverse operator $\nu^{-1} = \widehat{\cdot}$ is the Fourier transform that was mentioned above. For the case of self-adjoint $\boldsymbol{J}$ this result is classical (here $\tilde{\mathbf{l}}_2 = \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \cdots$). For the unitary case it can be found in [4] (here $\tilde{\mathbf{l}}_2 = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots$). For the general case of a normal operator we announce this result here.

Denote by $\mathcal{O}$ the natural embedding $\tilde{\mathbf{l}}_2 \hookrightarrow \mathbf{l}_2$. The diagram (7) demonstrates the role of the intermediate space $\tilde{\mathbf{l}}_2$,

$$
\begin{tikzcd}
L^2(\mathbb{C},\mathfrak{B}(\mathbb{C}),d\rho) \arrow[r, "L"] \arrow[d, "\nu=\widehat{\cdot}^{\,-1}"'] & L^2(\mathbb{C},\mathfrak{B}(\mathbb{C}),d\rho) \arrow[d, "\nu=\widehat{\cdot}^{\,-1}"] \\
\tilde{\mathbf{l}}_2 \arrow[r, "\tilde{\boldsymbol{J}}"] \arrow[d, "\mathcal{O}"'] & \tilde{\mathbf{l}}_2 \arrow[d, "\mathcal{O}"] \\
\mathbf{l}_2 \arrow[r, "\boldsymbol{J}"] & \mathbf{l}_2
\end{tikzcd} \tag{7}
$$

This diagram shows that the space $\tilde{\mathbf{l}}_2$ varies from $\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^1\oplus\cdots$ to $\mathbb{C}^1\oplus\mathbb{C}^2\oplus\mathbb{C}^3\oplus\cdots$ depending on the properties of $\operatorname{supp}\rho$. The idea is that the spaces $L^2(\mathbb{C},\mathfrak{B}(\mathbb{C}),d\rho)$ are parametrized by measures ρ, but $\mathbf{l}_2$ does not depend on ρ. It contains all possible $\tilde{\mathbf{l}}_2$ (that depend on ρ) and corresponds to the general case: when all the elements of (3) are linearly independent.

Now we are interested in the matrix structure of the operators $L, \tilde{\boldsymbol{J}}, \boldsymbol{J}$. We continue by defining $\mathcal{O}^{-1}(f)=0$ for $f\in\mathbf{l}_2\ominus\tilde{\mathbf{l}}_2$ and construct the inverse $(\mathcal{O}\circ\nu)^{-1}:\mathbf{l}_2\to L^2(\mathbb{C},d\rho)$ with the kernel $\mathbf{l}_2\ominus\tilde{\mathbf{l}}_2$. Consider two images of L,

$$\boldsymbol{J}=(\mathcal{O}\circ\nu)\circ L\circ(\mathcal{O}\circ\nu)^{-1}:\mathbf{l}_2\to\mathbf{l}_2,$$

$$\tilde{\boldsymbol{J}}=\nu\circ L\circ\nu^{-1}:\tilde{\mathbf{l}}_2\to\tilde{\mathbf{l}}_2.$$

Lemma 1. *The matrices of the operators $L,\tilde{\boldsymbol{J}},\boldsymbol{J}$ have the following properties:*

(1) *the multiplication operator $L: L^2(\mathbb{C},\mathfrak{B}(\mathbb{C}),d\rho)\to L^2(\mathbb{C},\mathfrak{B}(\mathbb{C}),d\rho)$ in the basis $P_{n,\alpha}$ and its image $\tilde{\boldsymbol{J}}=\nu\circ L\circ\nu^{-1}:\tilde{\mathbf{l}}_2\to\tilde{\mathbf{l}}_2$ in the basis $e_{n,\alpha}$ have the same numerical matrix $\tilde{J}$;*

(2) *matrix J of the operator $\boldsymbol{J}$ in the basis $e_{n,\alpha}$ will have the same numerical entries as the matrix $\tilde{J}$ but interleaved with zeros: rows and columns of zeros will correspond to the elements of (3) that occurred to be linearly dependent on the previous ones while doing the orthogonalization procedure.*

We call the matrix $\tilde{J}$ "shrinked" and the matrix J the "interleaved" one.

Proof. The first statement is obvious because ν is a unitary map from $L^2(\mathbb{C},\mathfrak{B}(\mathbb{C}),d\rho)$ to $\tilde{\mathbf{l}}_2$. To prove the second one, use the orthogonal decomposition

$$\mathbf{l}_2=\tilde{\mathbf{l}}_2\oplus\tilde{\mathbf{l}}_2^{\perp}=\tilde{\mathbf{l}}_2\oplus(\operatorname{Ran}\mathcal{O})^{\perp}. \tag{8}$$

Thus for any $e_{k,\beta}$ and for $e_{n,\alpha}\notin\tilde{\mathbf{l}}_2=\operatorname{Ran}\mathcal{O}$, we have

$$J_{n,\alpha;k,\beta}=(\mathcal{O}\circ\nu\circ L\circ\nu^{-1}\circ\mathcal{O}^{-1}e_{k,\beta},e_{n,\alpha})_{\mathbf{l}_2}=(u,e_{n,\alpha})_{\mathbf{l}_2}=0,\quad u\in\operatorname{Ran}\mathcal{O}.$$

Similarly for any $e_{k,\beta}\notin\tilde{\mathbf{l}}_2$ and arbitrary $e_{n,\alpha}$, we obtain

$$J_{n,\alpha;k,\beta}=(\mathcal{O}\circ\nu\circ L\circ\nu^{-1}\circ\mathcal{O}^{-1}e_{k,\beta},e_{n,\alpha})_{\mathbf{l}_2}=(\mathcal{O}\circ\nu\circ L\circ\nu^{-1}0,e_{n,\alpha})_{\mathbf{l}_2}=0.$$

Finally for any $e_{n,\alpha},e_{k,\beta}\in\tilde{\mathbf{l}}_2$, we see that

$$J_{n,\alpha;k,\beta}=(\mathcal{O}\circ\nu\circ L\circ\nu^{-1}\circ\mathcal{O}^{-1}e_{k,\beta},e_{n,\alpha})_{\mathbf{l}_2}=(\nu\circ L\circ\nu^{-1}e_{k,\beta},e_{n,\alpha})_{\mathbf{l}_2}=\tilde{J}_{n,\alpha;k,\beta}.$$

□

Consider three examples. Denote by $\mathcal{H}_n^{(k)}, k=0,\dots,n$, the k-th one-dimensional subspace of $\mathcal{H}_n=\mathbb{C}^{n+1}$. If L is self-adjoint, then $\operatorname{supp}\rho\subset\mathbb{R}=\{z\in\mathbb{C}:z-\bar{z}=0\}$. Thus only $\alpha = 0$ components survive in the ν construction procedure and $\tilde{\mathbf{I}}_2 = \bigoplus_{n=0}^{+\infty} \mathcal{H}_n^{(0)}$. The corresponding matrices are as follows (note the classical three-diagonal Jacobi structure):

$$\tilde{J} = \begin{pmatrix} * & + & 0 & 0 & 0 & 0 & \cdot \\ + & * & + & 0 & 0 & 0 & \cdot \\ 0 & + & * & + & 0 & 0 & \cdot \\ 0 & 0 & + & * & + & 0 & \cdot \\ 0 & 0 & 0 & + & * & + & \cdot \\ 0 & 0 & 0 & 0 & + & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad J = \left(\begin{array}{|c|cc|ccc|cccc|c} \hline * & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \hline + & * & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \hline 0 & + & 0 & * & 0 & 0 & + & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \hline 0 & 0 & 0 & + & 0 & 0 & * & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right).$$

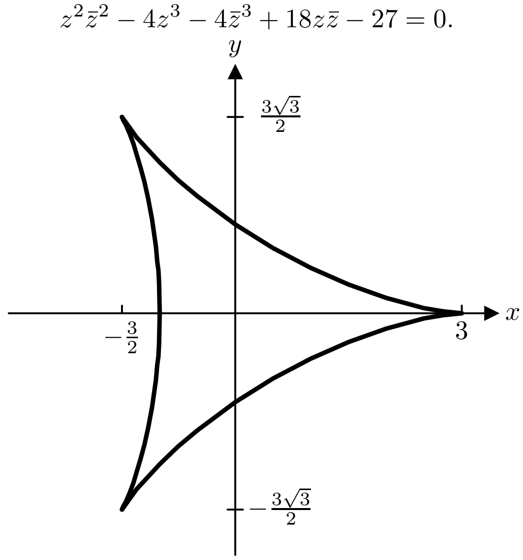
Here the plus sign denotes the entry that is real and positive. Asterisk corresponds to the element for which no information is available.

Similarly if the normal operator L occurs to be unitary, then $\operatorname{supp} \rho \subset \mathbb{T} = \{z \in \mathbb{C} : z \cdot \bar{z} - 1 = 0\}$ and only $\alpha = 0, \alpha = n$, components survive. The corresponding matrix is as follows (note CMV structure):

$$J = \left(\begin{array}{|c|cc|ccc|cccc|c} \hline * & * & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \hline + & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & * & * & * & 0 & + & 0 & 0 & 0 & 0 & \cdot \\ \hline 0 & + & * & * & 0 & * & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & * & 0 & * & * & 0 & 0 & + & \cdot \\ \hline 0 & 0 & 0 & + & 0 & * & * & 0 & 0 & * & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & * & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right). \tag{9}$$

Now let's take something completely different from the real line and the unit circle. Consider the hypocycloid

$$z^2\bar{z}^2 - 4z^3 - 4\bar{z}^3 + 18z\bar{z} - 27 = 0. \tag{10}$$



By an explicit calculation it can be found that the entries of (2) will have the following form:

$$a_n = \underbrace{\left[\begin{array}{cc|ccc|cc} + & \ast & 0 & \cdots & 0 & \ast & \ast \\ 0 & + & 0 & \cdots & 0 & \ast & \ast \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & + \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right]}_{n+1} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} n+2.$$

The vertical zero stripe is $n+1-4$ columns wide, the horizontal zero stripe is $n+2-4$ rows wide. At $n = 0,1,2,3$ the matrices are filled up as in (2).

$$c_n = \underbrace{\left[\begin{array}{cc|ccc|cc} \ast & + & 0 & \cdots & 0 & 0 & 0 \\ \ast & \ast & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline \ast & \ast & 0 & \cdots & 0 & + & 0 \\ \ast & \ast & 0 & \cdots & 0 & \ast & + \end{array}\right]}_{n+2} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} n+1.$$

Diagonal entries have a similar form,

$$b_n = \underbrace{\left[\begin{array}{cc|ccc|cc} \ast & \ast & 0 & \cdots & 0 & \ast & \ast \\ \ast & \ast & 0 & \cdots & 0 & \ast & \ast \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline \ast & \ast & 0 & \cdots & 0 & \ast & \ast \\ \ast & \ast & 0 & \cdots & 0 & \ast & \ast \end{array}\right]}_{n+1} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} n+1.$$

These examples demonstrate the way how the matrix structure simplification can arise and can be understood. If we know the interleaved matrix J in $\mathbf{l}_2$ it becomes quite natural to simplify it by eliminating the unnecessary zeros to obtain the shrinked matrix $\tilde{J}$. This is equivalent to using the simpler space $\tilde{\mathbf{l}}_2$ and embedding self-adjoint operators into the classical $\ell_2 \simeq \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \cdots$ (and work with ordinary Hermitian matrices), unitary operators – into the block space $\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \cdots$ (and use OPUC results here), normal operators with the spectrum on the hypocycloid – into the space $\mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \cdots$ and work with four-dimensional matrices.

3. Dimension stabilization effect

In this section we present necessary notations and formulate the main result.

Let T be an arbitrary set. Denote by $\mathbb{C}^T$ the linear space of functions $T \to \mathbb{C}$ over the field $\mathbb{C}$. The linear structure here is induced from the linear space $\mathbb{C}$ in the obvious way: $\forall f, g : T \to \mathbb{C}, \forall \alpha, \beta \in \mathbb{C}$ we have $(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t)$.

Take $T = \sigma(\boldsymbol{J}) = \operatorname{supp} \rho$. The functions $z^{n-\alpha}\bar{z}^{\alpha}, n \in \mathbb{N}_0, \alpha = 0, \ldots, n$ (see (3)), are the elements of $\mathbb{C}^T$. The case where all these functions are linearly independent in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$ is out of the scope of this article because all the matrices have the form (2) and there is nothing to say more about their structure simplification.

Suppose there is at least one linearly dependent in $\mathbb{C}^T$ system of (3): $\exists n \in \mathbb{N}_0$, $\exists \gamma_{k,\alpha} \in \mathbb{C}$, $k = 0, \dots, n$; $\alpha = 0, \dots, k$,

$$\sum_{k=0}^{n} \sum_{\alpha=0}^{k} |\gamma_{k,\alpha}| \neq 0, \quad \sum_{k=0}^{n} \sum_{\alpha=0}^{k} \gamma_{k,\alpha} z^{k-\alpha} \bar{z}^{\alpha} = 0 \quad \forall z \in \sigma(\boldsymbol{J}). \tag{11}$$

Note that this implies that (3) are linearly dependent in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$ either (the converse statement is not true).

Denote by $\mathbb{C}[x, y]$ the ring of polynomials with complex coefficients of two variables. We emphasize that for $p \in \mathbb{C}[x, y]$, the symbols x, y in $p(x, y)$ are formal substitution places. In our particular case we substitute z in place of x and $\bar{z}$ in place of y. Formula (11) can be re-written in this way:

$$\exists p \in \mathbb{C}[x, y] : \quad p \neq 0, \quad p(z, \bar{z}) = 0 \quad \forall z \in \sigma(\boldsymbol{J}). \tag{12}$$

Note that any *linear dependency* (11) can be attached to some polynomial $q \in \mathbb{C}[x, y]$ by the formula (12). Thus the polynomial q can be found in the following set:

$$\mathcal{I} = \{p \in \mathbb{C}[x, y] : \quad p(z, \bar{z}) = 0 \quad \forall z \in \sigma(\boldsymbol{J})\}. \tag{13}$$

Conversely, any $q \in \mathcal{I}, q \neq 0$ represents some linear dependency of the (3) elements in $\mathbb{C}^T$. Thus the set $\mathcal{I} \setminus \{0\}$ is exactly the set of *all* possible linear dependencies. From now on we use "$p \in \mathcal{I}, p \neq 0$" instead of "consider a linearly dependent subsystem of (3) elements in $\mathbb{C}^T$".

Note the following two things. *First*: $\mathcal{I} \neq \{0\}$ is guaranteed by the assumption that at least one linear dependency exists. *Second*: it is easy to see that $\mathcal{I}$ is an ideal in $\mathbb{C}[x, y]$, i.e., 0 belongs to $\mathcal{I}$, the sum and the difference of any two elements of $\mathcal{I}$ belongs to $\mathcal{I}$, and the product of an element from $\mathcal{I}$ by any $f \in \mathbb{C}[x, y]$ belongs to $\mathcal{I}$.

Let us make another observation. From (12) we conclude that

$$\sigma(\boldsymbol{J}) \subset \Gamma_p = \{z \in \mathbb{C} : p(z, \bar{z}) = 0\}. \tag{14}$$

We conclude that the spectra ought to lie on a curve Γ_p. The properties of the curve Γ_p influence the structure of the ideal $\mathcal{I}$. And the ideal $\mathcal{I}$ is connected with the matrix structure: using its generators we are able to calculate the indexes of rows and columns that consist of zeros. Thus geometric properties of $\sigma(\boldsymbol{J})$ influence the matrix structure.

Since Γ_p is a geometric object and $\mathcal{I}$ is an algebraic object, further investigation can be done by means of algebraic geometry. Another interesting thing that arises here is the set $\{p \in \mathbb{C}[x, y] : p(z, \bar{z}) = 0 \quad \forall z \in S\}$ for an arbitrary $S \subset \mathbb{C}$. These two aspects will be covered in future publications. Here we raise only one important question: how to find the minimal spectrum extension $\Gamma = \bigcap_{p \in \mathcal{I}} \Gamma_p \supset \sigma(\boldsymbol{J})$ by exploring the structure of the ideal $\mathcal{I}$ and how to use it to investigate the matrix structure.

Now we do the necessary preparations. Consider the linear order defined by (3):

$$z^0\bar{z}^0 \prec z^1\bar{z}^0 \prec z^0\bar{z}^1 \prec z^2\bar{z}^0 \prec z^1\bar{z}^1 \prec z^0\bar{z}^2 \prec \cdots \tag{15}$$

Just this order arises while solving the Direct and Inverse Spectral Problems (see [5]).

For $p \in \mathbb{C}[x, y]$ the expression $p(z, \bar{z})$ is a linear combination of the elements (3). Let us call the *leading monomial* the item $\gamma_{n,\alpha} x^{n-\alpha} y^{\alpha}$ that generates the highest $z^{n-\alpha}\bar{z}^{\alpha}$ according to the linear order (15). Denote it as $\operatorname{Lead} p$. The corresponding tuple $(n, \alpha) \in \mathbb{N}_0^2$ will be called the *degree* of the polynomial p. For example

$$\operatorname{Deg}(1 + x^2y^2 + xy^3) = (4, 3).$$

The linear order (15) induces the linear order for $x^n y^m$

$$x^0y^0 \prec x^1y^0 \prec x^0y^1 \prec x^2y^0 \prec x^1y^1 \prec x^0y^2 \prec \cdots \tag{16}$$

and the corresponding order for their degrees:

$$(0,0) \prec (1,0) \prec (1,1) \prec (2,0) \prec (2,1) \prec (2,2) \prec \cdots \tag{17}$$

The coefficient $\gamma_{n,\alpha}$ at the leading monomial $\gamma_{n,\alpha}x^{n-\alpha}y^\alpha$ is called the *leading coefficient*. We say that a polynomial $f \in \mathbb{C}[x,y]$ is *monic* if its leading coefficient is $\gamma_{n,\alpha} = 1$.

Rewrite (3) as the following table:

$$\begin{tikzcd}
z^0\bar z^0 \arrow[r, "1"] & z^1\bar z^0 \arrow[dl, "2"] & z^2\bar z^0 \arrow[dl, "3"] & z^3\bar z^0 \arrow[dl, "5"] & z^4\bar z^0 \arrow[dl, "8"] \\
z^0\bar z^1 & z^1\bar z^1 \arrow[dl, "4"] & z^2\bar z^1 \arrow[dl, "6"] & z^3\bar z^1 \arrow[dl, "9"] & \cdot \arrow[dl] \\
z^0\bar z^2 & z^1\bar z^2 \arrow[dl, "7"] & z^2\bar z^2 \arrow[dl, "10"] & \cdot \arrow[dl] & z^\alpha\bar z^{n-\alpha} \arrow[dl, "\cdots"] \\
z^0\bar z^3 & z^1\bar z^3 & \cdot & z^{\alpha-1}\bar z^{n-(\alpha-1)} &
\end{tikzcd} \tag{18}$$

It is convenient to treat the multiplication by z and $\bar z$ as a shift in this table. Multiplication by z (the operator L) is a right-shift and multiplication by $\bar z$ (the adjoint operator L^*) is a down-shift. These actions commute and are associative.

Observe that n in $z^{n-\alpha}\bar z^\alpha$ is the number of the diagonal, α is the *shift* position along this diagonal. Fourier transform (4) maps $\mathcal{H}_n = \mathbb{C}^{n+1}$ into $\mathrm{span}\{z^{n-\alpha}\bar z^\alpha, \quad \alpha = 0,\dots,n\}$ that corresponds to the n-th diagonal. Note also that for $z^{n-\alpha}\bar z^\alpha$ we have $(n-\alpha)+\alpha = n$ that is equal to the number of the diagonal. Thus $z^n\bar z^m$ and $z^k\bar z^r$ lie on the same diagonal iff $n+m = k+r =$the number of the diagonal. All these conclusions can be re-written in terms of the variables x, y for $p \in \mathbb{C}[x,y]$.

The next Lemma proves that $\mathrm{Deg} : \mathbb{C}[x,y] \to \mathbb{N}_0^2$ is a homomorphism. As it was already noted the degree of a polynomial $p \in \mathcal{I}$ gives us the index of zero column and row. This lemma allows to perform simple addition in $\mathbb{N}_0^2$ instead of multiplying polynomials, extracting the leading coefficient of the product and finding its degree while analyzing the matrix structure.

Lemma 2. $\forall p, q \in \mathbb{C}[x,y]$

$$\mathrm{Deg}(pq) = \mathrm{Deg}(\mathrm{Lead}\, p \cdot \mathrm{Lead}\, q) = \mathrm{Deg}\,\mathrm{Lead}\, p + \mathrm{Deg}\,\mathrm{Lead}\, q = \mathrm{Deg}\, p + \mathrm{Deg}\, q.$$

This lemma claims that we can ignore the lower coefficients while calculating where zero rows and columns are situated.

Proof. The second equality is obvious. Let $\mathrm{Lead}\, p = \gamma_{n,\alpha}x^{n-\alpha}y^\alpha$, $\mathrm{Lead}\, q = \lambda_{k,\beta}x^{k-\beta}y^\beta$. Since $(x^{n-\alpha}y^\alpha)(x^{k-\beta}y^\beta) = x^{(n+k)-(\alpha+\beta)}y^{\alpha+\beta}$ (which corresponds to $(z^{n-\alpha}\bar z^\alpha)(z^{k-\beta}\bar z^\beta) = z^{(n+k)-(\alpha+\beta)}\bar z^{\alpha+\beta}$), we have

$$\begin{aligned}
\mathrm{Deg}(\mathrm{Lead}\, p \cdot \mathrm{Lead}\, q) &= \mathrm{Deg}((\gamma_{n,\alpha}x^{n-\alpha}y^\alpha)\cdot(\lambda_{k,\beta}x^{k-\beta}y^\beta)) \\
&= \mathrm{Deg}(\gamma_{n,\alpha}\lambda_{k,\beta}x^{(n+k)-(\alpha+\beta)}y^{\alpha+\beta}) = (n+k, \alpha+\beta) = (n,\alpha) + (k,\beta) \\
&= \mathrm{Deg}(\gamma_{n,\alpha}x^{n-\alpha}y^\alpha) + \mathrm{Deg}(\lambda_{k,\beta}x^{k-\beta}y^\beta) = \mathrm{Deg}\,\mathrm{Lead}\, p + \mathrm{Deg}\,\mathrm{Lead}\, q.
\end{aligned}$$

To prove the first equality we use the table (18). Let

$$p(x,y)=\sum_{i=0}^{n-1}\sum_{\mu=0}^{i}\gamma_{i,\mu}x^{i-\mu}y^{\mu}+\sum_{\mu=0}^{\alpha}\gamma_{n,\mu}x^{n-\mu}y^{\mu},$$

$$q(x,y)=\sum_{j=0}^{m-1}\sum_{\nu=0}^{j}\lambda_{j,\nu}x^{j-\nu}y^{\nu}+\sum_{\nu=0}^{\beta}\lambda_{m,\nu}x^{m-\nu}y^{\nu}.$$

According to the definition of $\operatorname{Deg} p$,

$$\operatorname{Deg} pq=\operatorname{Deg}\operatorname{Lead}(pq). \tag{19}$$

Thus it is sufficient to find $\operatorname{Lead}(pq)$,

$$\begin{aligned}p(x,y)q(x,y)&=\sum_{i=0}^{n-1}\sum_{\mu=0}^{i}\sum_{j=0}^{m-1}\sum_{\nu=0}^{j}\lambda_{j,\nu}\gamma_{i,\mu}x^{(i+j)-(\mu+\nu)}y^{\mu+\nu}\\&+\sum_{\mu=0}^{\alpha}\sum_{\nu=0}^{\beta}\lambda_{m,\nu}\gamma_{n,\mu}x^{(n+m)-(\mu+\nu)}y^{\mu+\nu}\\&+\sum_{i=0}^{n-1}\sum_{\mu=0}^{i}\sum_{\nu=0}^{\beta}\lambda_{m,\nu}\gamma_{i,\mu}x^{(m+i)-(\mu+\nu)}y^{\mu+\nu}\\&+\sum_{j=0}^{m-1}\sum_{\nu=0}^{j}\sum_{\mu=0}^{\alpha}\gamma_{n,\mu}\lambda_{j,\nu}x^{(n+j)-(\nu+\mu)}y^{\nu+\mu}.\end{aligned}$$

Recall that x^ny^m is situated on the $n+m$ diagonal at the m-th position (is shifted to the m-th position). The leading coefficient is situated on the diagonal with the largest number and has the largest shift along this diagonal. From the formula written above we see that the largest possible diagonal, where the monomials of the resulting polynomial pq are situated, is the $n+m$-th diagonal. Thus it is sufficient to analyze only the item

$$\sum_{\mu=0}^{\alpha}\sum_{\nu=0}^{\beta}\lambda_{m,\nu}\gamma_{n,\mu}x^{(n+m)-(\mu+\nu)}y^{\mu+\nu}.$$

Similarly we conclude that the largest possible shift is $\alpha+\beta$. Thus we have

$$\operatorname{Lead}(pq)=\gamma_{n,\alpha}\lambda_{m,\beta}x^{(n+m)-(\alpha+\beta)}y^{\alpha+\beta}=(\operatorname{Lead}p)\cdot(\operatorname{Lead}q).$$

This equality together with (19) finishes the proof. □

Now we are able to formulate the main result. Denote by $\operatorname{span} L$ the set of linear combinations constructed from the elements of the set L and $\overline{\operatorname{span}}L$ the closure of $\operatorname{span} L$.

Theorem 1. *Let $T\subset\mathbb{C}$. Suppose the functions $z^{n-\alpha}\bar{z}^{\alpha}\in\mathbb{C}^T, n\in\mathbb{N}_0,\alpha=0,\dots,n$, are linearly dependent in $\mathbb{C}^T$ and $p\in\mathbb{C}[x,y], p\neq 0$ describes this linear dependence, $p(z,\bar z)=0\ \forall z\in T$. Let $\operatorname{Deg}p=(M,\gamma)$. Consider the finite-dimensional linear subsets $\mathcal{K}_n=\operatorname{span}\{z^{n-\alpha}\bar z^{\alpha}:\alpha=0,\dots,n\}\subset\mathbb{C}^T$, $n\in\mathbb{N}_0$. Then*

1. *there exists $N\in\mathbb{N}$ such that $\dim\mathcal{K}_n\leqslant n+1\quad\forall n<N$ and $\dim\mathcal{K}_n\leqslant N\quad\forall n\geqslant N$;*
2. *$N\leqslant M$;*
3. *N can be found by the following procedure.*

Consider the ideal $\mathcal{I}$ (see (13)*) of all possible linear dependencies in $\mathbb{C}^T$. It contains a unique monic polynomial $q\in\mathcal{I}$ of the smallest degree $\operatorname{Deg}q=(k,\beta)$ in terms of the linear order* (15) *(we call it the minimal polynomial). Then $N=k$.*

Remark 1. This theorem gives only an upper bound for the stabilization rate. This upper bound is obtained using the existence of a minimal polynomial in the ideal $\mathcal{I}$. We announce here that all the dimensions $\dim \mathcal{K}_n$ can be explicitly calculated. It can be done using an algebraic geometry technique.

The next statement is the direct corollary from Theorem 1. Here we take the concrete set $T = \operatorname{supp} \rho$ and use the Gramm-Schmidt orthogonalization procedure in $L^2(\mathbb{C}, S, d\rho)$ to find the rate of stabilization N explicitly.

Corollary 1. *Let $S \subset 2^{\mathbb{C}}$ be a σ-algebra, $\rho : S \to [0, +\infty]$ be a measure. Suppose that $z^{n-\alpha}\bar{z}^{\alpha}, n \in \mathbb{N}_0, \alpha = 0, \dots, n$ are total in $L^2(\mathbb{C}, S, d\rho)$. Then it is possible to build the following embedding (injective map)*

$$L^2(\mathbb{C}, S, d\rho) \hookrightarrow \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^{N-1} \oplus \mathbb{C}^N \oplus \mathbb{C}^N \oplus \cdots$$

such that each $\mathcal{K}_n$ will be mapped into the corresponding n-th block $\mathbb{C}^{k_n} \subset \mathcal{H}_n = \mathbb{C}^{n+1}$, $n \in \mathbb{N}_0$, $k_n = 0, \dots, n$.

The rate of stabilization N is the same as in Theorem 1. The minimal polynomial (that gives N) mentioned in Theorem 1 can be explicitly found as the first linear dependency that arises during the Gramm-Schmidt orthogonalization procedure of $z^{n-\alpha}\bar{z}^{\alpha}$, $n \in \mathbb{N}_0$, $\alpha = 0, \dots, n$ in $L^2(\mathbb{C}, S, d\rho)$.

Remark 2. Similar to the previous remark we announce here that it is possible to find all the dimensions $\dim \mathcal{K}_n$ explicitly and build an isometric isomorphism (Fourier transform) instead of injection here.

The next statement concretizes Corollary 1 for the case of a specific probability measure ρ. It describes the matrix structure simplification that was shown earlier in examples.

Theorem 2. *Let ρ be a Borel probability measure with a compact infinite support. Consider the multiplication operator*

$$L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho) \ni f(z) \overset{L}{\mapsto} z f(z) \in L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho).$$

This operator has the spectral measure ρ. There exists $N \in \mathbb{N}$ such that the following embedding takes place:

$$L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho) \hookrightarrow \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^{N-1} \oplus \mathbb{C}^N \oplus \mathbb{C}^N \oplus \cdots$$

As in Corollary 1 each $\mathcal{K}_n$ is mapped into the corresponding n-th block $\mathbb{C}^{k_n}$. The subspaces $\mathcal{K}_n$ are the same as in Theorem 1 and have the properties 1, 2, 3 listed in this theorem.

The matrix J of the operator $\boldsymbol{J}$ (see diagram (7)) is the block Jacobi matrix that has the following construction. Take the matrix (2) and erase those $\frac{n(n+1)}{2} + \alpha$ rows and columns for which the corresponding $z^{n-\alpha}\bar{z}^{\alpha}$ occurs to be linearly dependent on the previous ones while doing the Gramm-Schmidt orthogonalization in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. All other properties (positive elements and the placement of zeros) are preserved.

Remark 3. To "erase" is the convenient informal notion. It means to consider the operator in a subspace and re-write its matrix in the basis of this subspace that is a subset of the basis of the whole space. Such a process will actually erase the corresponding rows and columns preserving the rest of the matrix.

Remark 4. Let $\boldsymbol{J}$ be a linear operator for which a spectral measure ρ exists. Let $z^{n-\alpha}\bar{z}^{\alpha}, n \in \mathbb{N}, \alpha = 0, \dots, n$ be linearly dependent in $L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$. Then

$$\exists p \in \mathbb{C}[x, y] : \operatorname{supp} \rho \subset \{z \in \mathbb{C} : p(z, \bar{z}) = 0\}.$$

The statement is obvious. It says that the spectrum of $\boldsymbol{J}$ lies on a curve. The remark is the essential point for the article because here we deal with normal bounded operators of this type.

4. The proofs

Before giving the proofs of the theorem and corollaries we demonstrate why the statement of the Theorem 1 is obvious and then give some technical reasoning.

Let $p \in \mathcal{I}, p \neq 0$ be a monic polynomial with $\operatorname{Deg} p = (n, \alpha)$. According to Lemma 1 the matrix J has zeros at the $\frac{n(n+1)}{2} + \alpha$ row and column (counting from zero).

$\mathcal{I}$ is ideal. Thus $p\mathbb{C}[x, y] = \{p(x, y)q(x, y) | q \in \mathbb{C}[x, y]\} \subset \mathcal{I}$ gives a series of zero columns and rows at the positions $\operatorname{Deg}(pq) = (n + m, \alpha + \beta)$ where $\operatorname{Deg} q = (m, \beta)$ (here we used Lemma 2). Looking at the table (18) we see that $\{z^{(n+m)-(\alpha+\beta)}\bar{z}^{\alpha+\beta} | \ m \in \mathbb{N}_0$, $\beta = 0, \dots, m\}$ is the square sub-table infinite to the right and down with the left-top corner $z^{n-\alpha}\bar{z}^{\alpha}$ obtained by all possible right and down shifts of $z^{n-\alpha}\bar{z}^{\alpha}$.

This observation is sufficient to grasp the idea of how to prove the main result of the article: the sum of the dimensions of rows $n-\alpha$ and columns α in the stripes that are out of this sub-table is always the same: $(n - \alpha) + \alpha = n$ and is equal to the number of the diagonal where the leading coefficient is situated. That's just the dimension stabilization effect we need. Note that $\operatorname{Deg} p = (n, \alpha)$ thus the first number in the degree of the polynomial gives the dimension that will stabilize.

Demonstrate these reasonings using the real line and the hypocycloid examples shown in section 2 (see the matrix (10)).

If L is self-adjoint then $\sigma(L) = \operatorname{supp} \rho \subset \mathbb{R}$. Real line is generated by the following linear dependency of (3) elements: $\bar{z} - z = 0$. This equation can be re-written as $p(z, \bar{z}) = 0$ for $p(x, y) = y - x \in \mathbb{C}[x, y]$. Here $\operatorname{Lead} p = y$ which corresponds to $z^0\bar{z}^1$ that is the left-upper corner of the sub-table. Note also that $\operatorname{Deg} p = (1, 1)$. The sub-table of (18) is situated as follows (marked by lines):

$$\begin{array}{|ccccc}
\multicolumn{1}{c}{z^0\bar{z}^0} & z^1\bar{z}^0 & z^2\bar{z}^0 & z^3\bar{z}^0 & z^4\bar{z}^0 \\
\hline
\mathbf{z^0\bar{z}^1} & z^1\bar{z}^1 & z^2\bar{z}^1 & z^3\bar{z}^1 & z^4\bar{z}^1 \\
z^0\bar{z}^2 & z^1\bar{z}^2 & z^2\bar{z}^2 & z^3\bar{z}^2 & z^4\bar{z}^2 \\
z^0\bar{z}^3 & z^1\bar{z}^3 & z^2\bar{z}^3 & z^3\bar{z}^3 & z^4\bar{z}^3
\end{array} \tag{20}$$

We see that $n = 1 = (1-1)+1$: only one upper row is out of the sub-table, see(20). This is exactly the dimension at which the stabilization effect is being observed: $L^2(\mathbb{R}, d\rho) \hookrightarrow \ell_2 = \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \mathbb{C}^1 \oplus \cdots$.

Another example uses the equation of hypocycloid (10). Linear dependency can be re-written as $p(z, \bar{z}) = 0$ for $p(x, y) = x^2y^2 - 4x^3 - 4y^3 + 18xy - 27$. Here $\operatorname{Lead} p = x^2y^2$ and $\operatorname{Deg} p = (4, 2)$. Thus we have the following sub-table:

$$\begin{array}{cc|ccc}
z^0\bar{z}^0 & z^1\bar{z}^0 & \multicolumn{1}{c}{z^2\bar{z}^0} & z^3\bar{z}^0 & z^4\bar{z}^0 \\
z^0\bar{z}^1 & z^1\bar{z}^1 & \multicolumn{1}{c}{z^2\bar{z}^1} & z^3\bar{z}^1 & z^4\bar{z}^1 \\
\cline{3-5}
z^0\bar{z}^2 & z^1\bar{z}^2 & \mathbf{z^2\bar{z}^2} & z^3\bar{z}^2 & z^4\bar{z}^2 \\
z^0\bar{z}^3 & z^1\bar{z}^3 & z^2\bar{z}^3 & z^3\bar{z}^3 & z^4\bar{z}^3 \\
z^0\bar{z}^4 & z^1\bar{z}^4 & z^2\bar{z}^3 & z^4\bar{z}^3 & z^4\bar{z}^4
\end{array}$$

Here $n = 4 = (4 - 2) + 2$: two rows and two columns are out of the sub-table. Thus four-dimensional subspace $\mathbb{C}^4$ will stabilize

$$L^2(\mathbb{C}, d\rho) \hookrightarrow \ell_2 = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \cdots$$

Now we give the proof of the Theorem 1. First we prove the statement concerning the existence and uniqueness of the minimal polynomial.

Lemma 3. *Any non-trivial ideal $I \subset \mathbb{C}[x, y]$ (in particular $I = \mathcal{I}$) contains a unique monic polynomial of the smallest degree (according to the linear order* (17)*).*

Proof. Degree sequence is bounded by the element $(0, 0)$, see (17). Thus at least one polynomial with the necessary property exists.

Assume there are two different monic polynomials $p_1, p_2 \in I \subset \mathbb{C}[x,y]$ of the same smallest degree $\operatorname{Deg} p_1 = \operatorname{Deg} p_2 = (n, \alpha)$

$$p_1(x,y) = x^{n-\alpha}y^\alpha + f_1(x,y), \quad p_2(x,y) = x^{n-\alpha}y^\alpha + f_2(x,y), \quad \operatorname{Deg} f_1 < \operatorname{Deg} f_2.$$

Consider the difference

$$f = p_1 - p_2 = f_1 - f_2 \in I.$$

Since $\operatorname{Deg} f < \operatorname{Deg} p_1, \quad f \in I$ we have the contradiction: $\operatorname{Deg} p_1$ is not the smallest possible degree. Thus the monic polynomial with the smallest degree exists in I and is unique. □

Conclude that $\mathcal{I}$ contains a unique minimal polynomial.

Lemma 4. *Let* $p \in \mathcal{I}, \quad p \neq 0, \quad \operatorname{Deg} p = (n, \alpha)$. *Then* $\dim \mathcal{K}_k \leqslant n \quad \forall k \in \mathbb{N}_0$.

This lemma claims that if there is at least one linear dependent subsystem of (3) (described by either minimal or non-minimal polynomial) this leads to the stabilization effect at least at the rate of n-dimensional subspaces $\mathbb{C}^n$.

Proof. Polynomial p describes a linear dependency of $\operatorname{Lead} p$ on the previous elements of (3). If we multiply p by any $q \in \mathbb{C}[x,y], \operatorname{Deg} q = (m, \beta)$ then we obtain the new linear dependency $pq \in \mathcal{I}$. According to Lemma 2 $\operatorname{Deg}(pq) = \operatorname{Deg} p + \operatorname{Deg} q = (n+m, \alpha+\beta)$. As it was described earlier, the set $\{z^{(n+m)-(\alpha+\beta)}\bar{z}^{\alpha+\beta} | m \in \mathbb{N}_0, \beta = 0, \ldots, m\}$ is the sub-table of (18) infinite to the right and down with the left-top corner $z^{n-\alpha}\bar{z}^\alpha$. The k-th diagonal corresponds to the linear set $\mathcal{K}_k, k \in \mathbb{N}_0$. To find the upper bound for $\dim \mathcal{K}_k$ it is sufficient to count the number of elements in the sub-table that lie on the k-th diagonal and subtract this number from the length $k+1$ of the k-th diagonal.

Start with the left-upper corner $z^{n-\alpha}\bar{z}^\alpha$ of the sub-table. It lies on the n-th diagonal. Thus $\dim \mathcal{K}_n \leqslant (n+1) - 1 = n$. Note that $z^{n-\alpha}\bar{z}^\alpha = (z^{n-\alpha}\bar{z}^\alpha)(z^0\bar{z}^0)$.

Similarly there are exactly two elements $(z^{n-\alpha}\bar{z}^\alpha)(z^1\bar{z}^0)$ and $(z^{n-\alpha}\bar{z}^\alpha)(z^0\bar{z}^1)$ in the sub-table on the $n+1$-th diagonal. Thus $\dim \mathcal{K}_{n+1} \leqslant (n+2) - 2 = n$.

By doing the same steps we conclude that all the elements $(z^{n-\alpha}\bar{z}^\alpha)(z^{k-\beta}\bar{z}^\beta)$, $k \in \mathbb{N}_0$, $\beta = 0, \ldots, k$ belong to the sub-table. Thus $\dim \mathcal{K}_{n+k} \leqslant (n+k+1) - (k+1) = n$. The proof of Lemma 4 and the proof of Theorem 1 are finished. □

To prove Corollary 1 we need the following auxiliary lemma. We use the notation of the extended (compactified) complex plane: $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Lemma 5. *Let* (X, S_X, ρ) *be an arbitrary measurable space with measure. Let* $p \geqslant 1$, $p \in \mathbb{R}$. *Then* $L_p(X, S_X, d\rho)$ *can be embedded (e.g. injectively mapped) into* $\mathbb{C}^X$. *In other words in every factor-class* $\tilde{f} \in L_p(X, S_X, d\rho)$ *there exists a representative* $F \in \tilde{f}$ *that possesses only finite values (and becomes the image of* $\tilde{f}$ *in* $\mathbb{C}^X$*).*

Proof. It is necessary to build an injective function that maps each factor-class $\tilde{f} \in L_p(X, S_X, d\rho)$ into some $F : X \to \mathbb{C}$. Note that a representative $f \in \tilde{f}, f : X \to \overline{\mathbb{C}}$ can possess the value $f(x_0) = \infty$ for some $x_0 \in X$ but F cannot: since $F \in \mathbb{C}^X$ it can possess only finite values.

Take a factor-class $\tilde{f} \in L_p(X, S_X, d\rho)$. According to the definition of $L_p(X, S_X, d\rho)$ this factor-class consists of $(X, S_X) \to (\overline{\mathbb{C}}, \mathfrak{B}(\overline{\mathbb{C}}))$-measurable functions. Take an arbitrary representative $f \in \tilde{f}, f : X \to \overline{\mathbb{C}}$.

Since $\overline{\mathbb{C}}, \mathbb{C} \in \mathfrak{B}(\overline{\mathbb{C}})$ we conclude that $\{\infty\} = \overline{\mathbb{C}} \setminus \mathbb{C} \in \mathfrak{B}(\overline{\mathbb{C}})$. By the definition of measurable function we have $f^{-1}(\{\infty\}) \in S_X$. Thus ρ must be defined on the set $f^{-1}(\{\infty\})$. There are two possibilities: (1) $\rho(f^{-1}(\{\infty\})) = 0$ and (2) $\rho(f^{-1}(\{\infty\})) \neq 0$. The second option is impossible because if it would then $\tilde{f} \notin L_p(X, S_X, d\rho)$

$$\int\limits_X |f(x)|^p d\rho(x) \geqslant \int\limits_{f^{-1}(\{\infty\})} |f(x)|^p d\rho(x) = \infty \cdot \rho(f^{-1}(\{\infty\})) = \infty.$$

Thus only the first option is true and we are able to choose the necessary finite-valued representative that is equivalent to f w.r.t. ρ

$$F(x) = \begin{cases} f(x), & x \notin f^{-1}(\{\infty\}), \\ 0, & x \in f^{-1}(\{\infty\}). \end{cases}$$

Note that $F \in \mathbb{C}^X$. The map $\tilde{f} \mapsto F$ is injective because factor-classes do not intersect and $F \in \tilde{f}$ (e.g. $f = F(\operatorname{mod} \rho)$). $\square$

This lemma is essential for the article because we shall use the linearity of $\mathbb{C}^X$. It is easy to map $L_p(X, S_X, d\rho)$ into $\overline{\mathbb{C}}^X$ but this set is not linear.

Apply Theorem 1 and Lemma 5 to prove Corollary 1. Let $T = \operatorname{supp} \rho$. Using Lemma 5 we embed $L^2(\mathbb{C}, S, d\rho)$ into $\mathbb{C}^T$. Consider the images of $\mathcal{K}_n \subset L^2(\mathbb{C}, S, d\rho), n \in \mathbb{N}_0$ in $\mathbb{C}^T$ (we preserve the notation $\mathcal{K}_n \subset \mathbb{C}^T$). Using Theorem 1 map each $\mathcal{K}_n, n \in \mathbb{N}_0$ into the corresponding block $\mathcal{H}_n = \mathbb{C}^{n+1}$ by the injection ν. According to Theorem 1 each subspace $\mathcal{K}_{n+k}$ is mapped into the n-dimensional subspace of $\mathcal{H}_{n+k}$. Thus the necessary dimension stabilization effect takes place

$$L^2(\mathbb{C}, S, d\rho) \hookrightarrow \mathbb{C}^1 \oplus \cdots \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n \oplus \cdots \tag{21}$$

As it was proved in Lemma 3 the ideal $\mathcal{I}$ contains a unique minimal polynomial. The next lemma shows how to find it explicitly.

Lemma 6. *The minimal polynomial in $\mathcal{I}$ (that exists and is unique due to Lemma 3) describes the first linear dependency that arises while making Gramm-Schmidt orthogonalization of* (3).

The statement is obvious because we supply the powers $z^{n-\alpha}\bar{z}^\alpha$ to the Gramm-Schmidt algorithm just in the same order as it is defined for degrees. Compare formulae (15), (16) and (17): we deliberately built these orders to be synchronized to have the effect that is necessary for this Lemma to be true. Lemma 6 finishes the proof of the Corollary 1.

The last thing left unproved is Theorem 2. Here we explain the term "to erase the corresponding columns and rows" in details and demonstrate the matrix simplification mechanism.

Let the conditions of the Theorem 2 hold true. The structure of the interleaved matrix J is already known (see (8) and the explanations hereafter). It is only necessary to show how to shrink the matrix by eliminating the unnecessary zero columns and rows.

Denote by $\tilde{\mathcal{H}}_n = \nu(\mathcal{K}_n) \subset \mathcal{H}_n = \mathbb{C}^{n+1}$. Then $\tilde{\mathbf{l}}_2 = \bigoplus\limits_{n=0}^{\infty} \tilde{\mathcal{H}}_n$. It is sufficient to show how to build one block of the shrinked Jacobi matrix $\tilde{J}$. Consider the described above modified Gramm-Schmidt orthogonalization procedure. Let $z^{n-\alpha}\bar{z}^\alpha, \alpha \in I_n \subset \{0, \ldots, n\}$ are the elements that were not skipped. ν maps $P_{n,\alpha}$ into $e_{n,\alpha}$ iff $\alpha \in I_n$. Consider the orthonormal basis $e_{n,\alpha}, \alpha \in I_n$ of the space $\tilde{\mathcal{H}}_n$. Re-enumerate it (preserving the order) as $\varepsilon_{n,\beta}, \beta = 0, \ldots, |I_n|$. Denote by F_n the re-enumeration rule (this is not the map: it's only the "renaming" rule). Thus $\varepsilon_{n,F_n(\alpha)} = e_{n,\alpha}, \alpha \in I_n$. Then the matrix $\tilde{J}$ of operator $\tilde{\boldsymbol{J}} : \tilde{\mathbf{l}}_2 \to \tilde{\mathbf{l}}_2$ in the basis $\varepsilon_{n,F_n(\alpha)}$ (that coincides with the matrix of $L : L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho) \to L^2(\mathbb{C}, \mathfrak{B}(\mathbb{C}), d\rho)$ in the basis $P_{n,F_n(\alpha)}$) is just what we need.

Note that the matrix structure can be obtained by the direct calculation of its elements. Such a proof is similar to the one given in [4] for the case of the unitary L. Here only one possible linear dependency was considered: $1 - z\bar{z} = 0$ (the spectrum of a unitary operator lies on the unit circle). The function $F_n(\alpha)$ is defined in this case as follows: $F_n(\alpha) = \delta_{n,\alpha}$. In this article we consider an arbitrary normal operators and any possible linear dependencies. The direct calculation is unnecessary because they were already done in [5] for the general case.

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