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ON THE NUMBER OF NEGATIVE EIGENVALUES OF A SCHRÖDINGER OPERATOR WITH δ INTERACTIONS

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ABSTRACT. We give necessary and sufficient conditions for a one-dimensional Schrödinger operator to have the number of negative eigenvalues equal to the number of negative intensities in the case of δ interactions.

1. Introduction and Main Theorem

In [3], S. Albeverio and L. Nizhnik gave necessary and sufficient conditions for a one-dimensional Schrödinger operator $L_{X,\alpha}$ with point δ-interactions to satisfy that the number of negative eigenvalues, $N = N(L_{X,\alpha})$, of $L_{X,\alpha}$ equals the number of point interactions, n, in the case where all the intensities are negative. Moreover, in [2], they gave an elegant 'algorithm' for determining N. This yields the result obtained in [3] and gives necessary and sufficient conditions for $L_{X,\alpha}$ not to have negative eigenvalues. In [5] N. I. Goloshchapova and L. L. Oridoroga proved that N is equal to the number of negative eigenvalues of a kind of finite Jacobi matrix using the method developed in [4]. This gives another characterization of Albeverio-Nizhnik's algorithm. In the previous paper [8] the author gave a sufficient condition for $L_{X,\alpha}$ to have at least m negative eigenvalues; we denote by m the number of negative intensities.

In this paper we prove that $N \le m$ and obtain necessary and sufficient conditions for $L_{X,\alpha}$ to satisfy $N = m$ and some extensions of Criteria in [3]. We use Albeverio-Nizhnik's algorithm to obtain necessary condition and do [8, Lemma 1] to obtain sufficient condition. See Remark 4.

We begin by recalling the definition of $L_{X,\alpha}$ in [2]; a Schrödinger operator $L_{X,\alpha}$ with point δ-interactions on a finite set $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ of points, which are called 'points of interaction', and intensities $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ is defined by the differential expression $-(d^2/dx^2)$ on a function $\psi(x)$ that belongs to the Sobolev space $W_2^2(\mathbb{R}^1 \setminus X)$ and satisfy, in the points of the set X, the following conjugation conditions:

$$\psi(x_i + 0) = \psi(x_i - 0), \quad \psi'(x_i + 0) - \psi'(x_i - 0) = \alpha_i \psi(x_i).$$

The operator $L_{X,\alpha}$ has the following representation:

$$L_{X,\alpha}\psi(x) = \Big[-\frac{d^2}{dx^2} + \sum_{i=1}^{n} \alpha_i \delta(x - x_i) \Big] \psi(x),$$

where δ is the Dirac's δ-function. Without loss of generality we can assume that $\alpha_i \ne 0$ and $x_1 < x_2 < \cdots < x_n$. Put $d_i = x_{i+1} - x_i$ and $m = |\{\alpha_i < 0\,;\, 1 \le i \le n\}|$. It is well known that the operators $L_{X,\alpha}$ are self-adjoint on $L^2(\mathbb{R}^1)$. Their spectra contain the positive semiaxis, where they are absolutely continuous, and no more than n simple negative eigenvalues [1].

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We first prove that $L_{X,\alpha}$ has at most m negative eigenvalues. $L_{X,\alpha}$ can be obtained from the theory of quadratic forms: the form

$$Q_{X,\alpha}(\varphi,\psi) = \langle \varphi', \psi' \rangle + \sum_{i=1}^{n} \alpha_i \overline{\varphi(x_i)} \psi(x_i), \quad \mathcal{D}(Q_{X,\alpha}) = H^{2,1}(\mathbb{R})$$

is densely defined, semibounded, and closed and the unique self-adjoint operator associated with $Q_{X,\alpha}$ is given by $L_{X,\alpha}$ (cf. [1, § II.2.1]). Therefore, it holds that

$$\langle \varphi, L_{X,\alpha}\varphi \rangle = \|\varphi'\|^2 + \sum_{i=1}^{n} \alpha_i |\varphi(x_i)|^2 \tag{1}$$

for $\varphi \in D(L_{X,\alpha})$. Using this, we can obtain the upper bound of N.

Theorem 1. *It holds that* $N \le m$.

Proof. Assume that $N > m$ and let $\varphi_1, \varphi_2, \ldots, \varphi_{m+1}$ be linearly independent eigenfunctions of $L_{X,\alpha}$, whose associated eigenvalues are negative. Let φ be a linear combination of φ_i. Then, it holds that $\langle \varphi, L_{X,\alpha}\varphi \rangle < 0$. In addition, we can assume that $\varphi(x) = 0$ on $X_- = \{x_i;\, \alpha_i < 0\}$, since $|X_-| = m$. This implies that $\langle \varphi, L_{X,\alpha}\varphi \rangle \ge 0$ by (1). However, this is impossible. □

To state our main theorem, we give some notations. We denote by the symbol, $[x_n, y_n, x_{n-1}, y_{n-1}, \ldots, x_2, y_2, x_1]$, the continued fraction with respect to $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=2}^n$ defined by

$$[x_1] = x_1,$$
$$[x_n, y_n, x_{n-1}, y_{n-1}, \ldots, x_2, y_2, x_1] = x_n - \frac{y_n}{[x_{n-1}, y_{n-1}, \ldots, x_2, y_2, x_1]}.$$

If some denominators are zero, such continued fraction does not be defined. We write $[x_n, y_n, \ldots, x_2, y_2, x_1] \gg 0$ if all $[x_i, y_i, \ldots, x_2, y_2, x_1] > 0$ for all $i = 1, 2, \ldots, n$. Put

$$c_i = \frac{1}{\alpha_i} + d_i + \frac{1}{\alpha_{i+1}}, \quad w_i = [c_i, \alpha_i^{-2}, c_{i-1}, \alpha_{i-1}^{-2}, \ldots, c_2, \alpha_2^{-2}, c_1].$$

Our main theorem is the following.

Theorem 2. $N = m$ *if and only if* $w_{n-1} \gg 0$.

We prove this theorem in the following sections; the plan of this paper is the following. We prove the sufficiency in Section 2 (Theorem 7) and the necessity in Section 3 (Theorem 14). We give some criteria for $L_{X,\alpha}$ to satisfy $N = m$ in Section 4.

Remark 3. In the case where all intensities are negative (i.e., $m = n$), it holds that $N = m$ if and only if $(|\alpha_1|, d_1, |\alpha_2|, d_2, \ldots, d_{n-1}, |\alpha_n|) \gg 0$ by [3, Theorem 2]. Here, proper continued fraction, $v_n = (a_n, a_{n-1}, \ldots, a_1)$, is defined by $v_1 = a_1$ and $v_n = a_n - 1/v_{n-1}$, and we write $v_n \gg 0$ if $v_i > 0$ for all $i = 1, 2, \ldots, n$.

Remark 4. One of the referees suggested to the author that N. Goloschapova and L. Oridoroga [6] recently obtained necessary and sufficient conditions for $L_{X,\alpha}$ to satisfy $N = m$ in different forms using the concept of boundary triplets and the corresponding Weyl functions developed in [7]. In these papers Schrödinger operators with finite and infinite number of point interactions were investigated.

Remark 5. For later use, we remark a fact on recurrence formula. Let $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ and assume that $b_n \ne 0$, $c_n \ne 0$ and $\det A_n \ne 0$ for all $n \in \mathbb{N}$. Let (x_1, y_1) be given and

$\{(x_n, y_n); n \in \mathbb{N}\}$ be defined by $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A_n \begin{pmatrix} x_n \\ y_n \end{pmatrix}$. Then, this sequence satisfies that

$$x_{n+1} = \left(a_n + \frac{b_n}{b_{n-1}} d_{n-1} \right) x_n - \frac{b_n}{b_{n-1}} \det A_{n-1}\, x_{n-1},$$
$$y_{n+1} = \left(d_n + \frac{c_n}{c_{n-1}} a_{n-1} \right) y_n - \frac{c_n}{c_{n-1}} \det A_{n-1}\, y_{n-1},$$

and the converse holds.

2. Sufficiency

Let $M_k(\lambda)$ be the real symmetric matrix defined by

$$M_k(\lambda) = \left(\frac{2\lambda}{\alpha_i} \delta_{i,j} + e^{-\lambda |x_i - x_j|} \right)_{i,j=1}^{k} \tag{2}$$

and $D_k = \det M_k(\lambda)$. Since we already know that if $M_n(\lambda)$ is positive definite for some positive λ then $N = m$ by [8, Lemma 1] and Theorem 1, we examine the positive definiteness in the case where $w_{n-1} \gg 0$. Since D_k is a leading principal minor of $M_n(\lambda)$ with order k, $M_n(\lambda)$ is positive definite if and only if all $D_1, D_2, \ldots$, and D_n are positive. To prove this, let us establish the recurrence formula for D_k. We put

$$p_i = \frac{2\lambda}{\alpha_i} + 1 + e^{-2\lambda d_{i-1}} \left(\frac{2\lambda}{\alpha_{i-1}} - 1 \right), \quad q_i = e^{-2\lambda d_{i-1}} \left(\frac{2\lambda}{\alpha_{i-1}} \right)^2 .$$

Proposition 6. *We have that $D_i = p_i D_{i-1} - q_i D_{i-2}$ for $i = 3, 4, \ldots, n$.*

Proof. Let $i \geq 2$, $v_1 = (e^{-\lambda d_1}) \in \mathbb{R}^1$ and $v_i = e^{-\lambda d_i}(v_{i-1}, 1) \in \mathbb{R}^i$. Since $|x_i - x_j| = \sum_{k=i}^{j-1} d_k$ when $i < j$, we have

$$M_i = \begin{pmatrix} M_{i-1} & v_{i-1}^t \\ v_{i-1} & \frac{2\lambda}{\alpha_i} + 1 \end{pmatrix}.$$

Put

$$E_{i-1} = \det \begin{pmatrix} M_{i-1} & v_{i-1}^t \\ v_{i-1} & 0 \end{pmatrix}.$$

Then, we have that

$$D_i = \begin{vmatrix} M_{i-1} & v_{i-1}^t \\ 0 & \frac{2\lambda}{\alpha_i} + 1 \end{vmatrix} + \begin{vmatrix} M_{i-1} & v_{i-1}^t \\ v_{i-1} & 0 \end{vmatrix} = \left(\frac{2\lambda}{\alpha_i} + 1 \right) D_{i-1} + E_{i-1}$$

and

$$\begin{aligned} E_i &= e^{-2\lambda d_i} \begin{vmatrix} M_{i-1} & v_{i-1}^t & v_{i-1}^t \\ v_{i-1} & \frac{2\lambda}{\alpha_i} + 1 & 1 \\ v_{i-1} & 1 & 0 \end{vmatrix} = e^{-2\lambda d_i} \begin{vmatrix} M_{i-1} & v_{i-1}^t & v_{i-1}^t \\ 0 & \frac{2\lambda}{\alpha_i} & 1 \\ v_{i-1} & 1 & 0 \end{vmatrix} \\ &= e^{-2\lambda d_i} \left(\frac{2\lambda}{\alpha_i} \begin{vmatrix} M_{i-1} & v_{i-1}^t \\ v_{i-1} & 0 \end{vmatrix} - \begin{vmatrix} M_{i-1} & v_{i-1}^t \\ v_{i-1} & 1 \end{vmatrix} \right) \\ &= e^{-2\lambda d_i} \left(\frac{2\lambda}{\alpha_i} \begin{vmatrix} M_{i-1} & v_{i-1}^t \\ v_{i-1} & 0 \end{vmatrix} - \left(\begin{vmatrix} M_{i-1} & v_{i-1}^t \\ v_{i-1} & 0 \end{vmatrix} + \begin{vmatrix} M_{i-1} & v_{i-1}^t \\ 0 & 1 \end{vmatrix} \right) \right) \\ &= e^{-2\lambda d_i} \left(\frac{2\lambda}{\alpha_i} - 1 \right) E_{i-1} - e^{-2\lambda d_i} D_{i-1}. \end{aligned}$$

Thus, we obtain that

$$\begin{pmatrix} D_i \\ E_i \end{pmatrix} = A_{i-1} \begin{pmatrix} D_{i-1} \\ E_{i-1} \end{pmatrix} \quad \text{with} \quad A_{i-1} = \begin{pmatrix} \frac{2\lambda}{\alpha_i} + 1 & 1 \\ -e^{-2\lambda d_i} & e^{-2\lambda d_i} \left(\frac{2\lambda}{\alpha_i} - 1 \right) \end{pmatrix}.$$

Since $\det A_{i-1} = e^{-2\lambda d_i}(2\lambda/\alpha_i)^2 \neq 0$, it holds that

$$D_{i+1} = \left(\frac{2\lambda}{\alpha_{i+1}} + 1 + e^{-2\lambda d_i}\left(\frac{2\lambda}{\alpha_i} - 1\right)\right) D_i - e^{-2\lambda d_i}\left(\frac{2\lambda}{\alpha_i}\right)^2 D_{i-1}.$$

This is the desired recurrence formula. □

The following is the sufficient condition for $L_{X,\alpha}$ to satisfy $N = m$, which is stated in Theorem 2.

Theorem 7. *If $w_{n-1} \gg 0$, then $N = m$.*

Proof. We assume that λ is small enough and remark that $p_i = 2c_{i-1}\lambda + O(\lambda^2)$ and $q_i = 4\alpha_{i-1}^{-2}\lambda^2 + O(\lambda^3)$. Since $D_1 = 2\lambda/\alpha_1 + 1 = 1 + O(\lambda)$ and $D_2 = (2\lambda/\alpha_2 + 1)(2\lambda/\alpha_1 + 1) - e^{-2\lambda d_1} = 2w_1\lambda + O(\lambda^2)$ are positive and it holds that

$$D_2/D_1 = 2w_1\lambda + O(\lambda^2).$$

Since $D_3/D_2 = [p_3, q_3, D_2/D_1]$ by Proposition 6, it holds that

$$D_3/D_2 = 2[c_2, \alpha_2^{-2}, w_1]\lambda + O(\lambda^2) = 2w_2\lambda + O(\lambda^2).$$

Therefore, D_3 is positive. By repeating similar calculations, we obtain that all $D_1, D_2, \ldots, D_n$ are positive. Consequently, $M_n(\lambda)$ is positive definite for some positive λ, and thus $N = m$ by [8, Lemma 1] and Theorem 1. □

3. Necessity

In this section we obtain the necessary condition for $L_{X,\alpha}$ to satisfy $N = m$, which is stated in Theorem 2. We assume that $N = m$ throughout this section. We first give some notations and recall Albeverio-Nizhnik's algorithm.

Let $j_1 < j_2 < \cdots < j_m$ be the indices of negative intensities, $\alpha_{j_k} < 0$, and put $y_k = x_{j_k}$ for $k = 1, 2, \ldots, m$. Let φ be the special solution defined in [2], that is, φ is the solution on the whole line of the following problem: $L_{X,\alpha}\varphi = 0$ and $\varphi(x) = 1$ if $x < x_1$. This φ has exactly m zeros by [2, Theorem 1]. Let $z_1 < z_2 < \cdots < z_m$ be the zeros of φ. We put

$$\varphi_i' = \varphi'(x_i - 0), \quad \varphi_i = \varphi(x_i - 0)$$

for $i = 1, 2, \ldots, n$ and $\varphi_{n+1}' = \varphi'(x_n + 0)$. Since φ is linear on $[x_{i-1}, x_i]$, it holds that

$$\varphi_i' = \varphi'(x_{i-1} + 0) = (\varphi_i - \varphi_{i-1})/d_{i-1}.$$

These φ_i' and φ_i satisfy the following recurrence formula by the conjugation condition:

$$\begin{aligned} &\varphi_1' = 0, \quad \varphi_{i+1}' = \varphi_i' + \alpha_i\varphi_i, \\ &\varphi_1 = 1, \quad \varphi_{i+1} = \varphi_i + d_i\varphi_{i+1}' = d_i\varphi_i' + (1 + \alpha_i d_i)\varphi_i. \end{aligned} \tag{3}$$

This implies the recurrence formulas for each φ_i' and φ_i:

$$\varphi_{i+1}' = \left(1 + \alpha_i d_{i-1} + \frac{\alpha_i}{\alpha_{i-1}}\right)\varphi_i' - \frac{\alpha_i}{\alpha_{i-1}}\varphi_{i-1}', \tag{4}$$

$$\varphi_{i+1} = \left(1 + \alpha_i d_i + \frac{d_i}{d_{i-1}}\right)\varphi_i - \frac{d_i}{d_{i-1}}\varphi_{i-1}. \tag{5}$$

Formula (5) already appears in [2].

Recall Albeverio-Nizhnik's algorithm.

Theorem 8 (Theorem 4 in [2])**.** *N equals the signature (the number of sign changes) of the sequence, $(\varphi_1, \varphi_2, \ldots, \varphi_n, (1 + \alpha_n d_{n-1})\varphi_n - \varphi_{n-1})$.*

In the following, we use the sequence, $(\varphi_1, \varphi_2, \dots, \varphi_n, \varphi'_{n+1})$, instead of the original one, since $(1 + \alpha_n d_{n-1})\varphi_n - \varphi_{n-1} = d_{n-1}\varphi'_{n+1}$.

We summarize some simple and useful facts on φ.

Proposition 9. *The following hold:*

(i) *The function φ is continuous and piecewise linear on $\mathbb{R}$. The derivative φ' and φ can not be simultaneously zero.*
(ii) *If $\varphi_i = 0$ then $\varphi_{i-1}\varphi_{i+1} < 0$. If $\varphi'_i = 0$ then $\varphi_i \neq 0$.*
(iii) *Assume that $\alpha_i > 0$. If $\varphi'_i \geq 0$ and $\varphi_i > 0$, then $\varphi'_{i+1} > \varphi'_i \geq 0$ and $\varphi_{i+1} > \varphi_i > 0$. Similarly, If $\varphi'_i \leq 0$ and $\varphi_i < 0$, then $\varphi'_{i+1} < \varphi'_i \leq 0$ and $\varphi_{i+1} < \varphi_i < 0$.*
(iv) *$\varphi'(z_i + \varepsilon)$ and $\varphi(z_i + \varepsilon)$ have same sign for ε small enough. Thus, if $x_{j-1} < z_i < x_j$, then φ'_j and φ_j have same sign.*
(v) *If $\varphi'_i = 0$ with $x_i < z_j$, there is at least one y_k in $I = [x_i, z_j)$.*
(vi) *If $\varphi'_i = 0$ with $z_j < x_i$, there is at least one y_k in $I = (z_j, x_i)$.*

Proof. We can find (i) in [2]. Using (5), we can see (ii). (iii) and (iv) can be proved by straight forward calculations.

Consider (v) and assume that no such y_k exists. Since $\varphi_i \neq 0$, $|\varphi(x)|$ is monotonously increasing on I, and thus $\varphi(z_j) \neq 0$. This contradicts to the definition of z_j.

Consider (vi) and assume that no such y_k exists. Since $\varphi'(z_j + \varepsilon)$ and $\varphi(z_j + \varepsilon)$ have same sign, $|\varphi'(x)|$ is monotonously non-decreasing on I, and thus $\varphi'_i \neq 0$. This contradicts to the assumption. ☐

We divide the proof of the necessity of Theorem 2 into a sequence of propositions and a lemma. We first prove that the points of interaction with negative intensities exactly interlace the zeros of φ.

Proposition 10. *If $N = m$, then we have that*

$$x_1 \leq y_1 < z_1 < y_2 < z_2 < \cdots < z_{m-1} < y_m < z_m.$$

Proof. Since $\varphi(x) > 0$ for $x < z_1$, it holds that $\varphi'(z_1 - 0) < 0$. Since $\varphi'(x) \geq 0$ for $x \leq y_1$, we have that $x_1 \leq y_1 < z_1$. We next prove that at least one y_k exists in (z_{j-1}, z_j); in both cases where $x_{i-1} = z_{j-1}$ and $x_{i-1} < z_{j-1} < x_i$, we have that $\varphi_i \neq 0$, φ'_i and φ_i have same sign and $x_i < z_j$. Therefore, if no y_k exists in $I = [x_i, z_j)$, $|\varphi(x)|$ is monotonously increasing on I, and thus $\varphi(z_j) \neq 0$. Hence, at least one y_k exists in I. Consequently, each m interval of $x_1 < z_1 < z_2 < \cdots < z_m$ contains at least one y_k. Since the number of y_k is equal to m, each interval contains exactly one y_k. ☐

This interlacing property implies that $\varphi'_i \neq 0$.

Proposition 11. *If $N = m$, then $\varphi'_i \neq 0$ for $i = 2, 3, \dots, n$.*

Proof. The proof is by contradiction. Assume that $\varphi'_i = 0$. Since no $y_k > z_m$ exists and $\varphi'(z_m + \varepsilon)$ and $\varphi(z_m + \varepsilon)$ have same sign, we have that $\varphi'(x) \neq 0$ for all $x \geq z_m$. Thus, it holds that $x_i < z_m$. If $z_j < x_i < z_{j+1}$, then at least two y_k and $y_{k'}$ are in (z_j, z_{j+1}) by Proposition 9. Since this contradicts to Proposition 10, it holds that $x_1 < x_i < z_1$. However, this is impossible, too; since $\varphi'_1 = 0$ and $\varphi_1 = 1$, if no y_k exists in (x_1, x_i), then $\varphi'_i > 0$. Hence, at least one y_k exists in (x_1, x_i). On the other hand, there is at least one $y_{k'}$ in $[x_i, z_1)$ by Proposition 9. This contradicts to Proposition 10. Consequently, we have that $\varphi'_i \neq 0$. ☐

Using the fact that $\varphi'_i \neq 0$, we establish the relation between w_i and φ'_i.

Proposition 12. *If $N = m$, then $\varphi'_{i+1}/\alpha_i\varphi'_i = w_{i-1}$ for $i = 2, 3, \dots, n$.*

Proof. We use induction on i. Let $i = 2$. Since $\varphi'_2 = \alpha_1$ and $\varphi'_3 = \alpha_1 + \alpha_2(1 + \alpha_1 d_1)$, we obtain that

$$\frac{\varphi'_3}{\alpha_2\varphi'_2} = \frac{\alpha_1 + \alpha_2(1 + \alpha_1 d_1)}{\alpha_2\alpha_1} = \frac{1}{\alpha_2} + \frac{1}{\alpha_1} + d_1 = c_1 = w_1.$$

Let $i \geq 3$ and assume that $\varphi'_i/\alpha_{i-1}\varphi'_{i-1} = w_{i-2}$. Since it holds that

$$\varphi'_{i+1} = c_{i-1}(\alpha_i\varphi'_i) - \frac{\alpha_i}{\alpha_{i-1}^2}(\alpha_{i-1}\varphi'_{i-1})$$

by (4), we obtain that

$$\frac{\varphi'_{i+1}}{\alpha_i\varphi'_i} = c_{i-1} - \frac{\alpha_{i-1}^{-2}}{\varphi'_i/\alpha_{i-1}\varphi'_{i-1}} = [c_{i-1}, \alpha_{i-1}^{-2}, w_{i-2}] = w_{i-1}.$$

This completes the induction. □

To state the following key lemma, we give some notations. Let $X' = X \setminus \{x_n\}$, $\alpha' = \alpha \setminus \{\alpha_n\}$, $N' = N(L_{X',\alpha'})$ and $m' = |\{\alpha_i < 0\,;\, 1 \leq i \leq n-1\}|$. Then it holds that that $N' \leq m'$.

Lemma 13. *If $N = m$, then $N' = m'$ and $w_{n-1} > 0$.*

Proof. We denote by $\mathrm{sig}(k)$ the signature of the sequence, $(\varphi_1, \ldots, \varphi_{k-1}, \varphi'_k)$, in this proof. We have that $N = \mathrm{sig}(n+1)$ and $N' = \mathrm{sig}(n)$ by Albeverio-Nizhnik's algorithm.

(i) Consider the case where $\alpha_n > 0$. In this case, $m' = m$. Assume that $\varphi_{n-1} \geq 0$. In Table 1, we list up all combinations of signs of φ_{n-1}, φ_n, φ'_n, and φ'_{n+1}. We remark that φ_{n-1} and φ_n can not be simultaneously zero. The combinations indicated by the marks (*n) are impossible:

(*1) If $\mathrm{sig}(n) - \mathrm{sig}(n+1) = 1$, then $N' > m'$.
(*2) $\varphi'_{n+1} = \varphi'_n + \alpha_n\varphi_n > 0$, but negative.
(*3) $\varphi'_{n+1} = \varphi'_n + \alpha_n\varphi_n < 0$, but positive.
(*4) $\varphi'_n = (\varphi_n - \varphi_{n-1})/d_{n-1} > 0$, but negative.
(*5) $\varphi'_n = (\varphi_n - \varphi_{n-1})/d_{n-1} < 0$, but positive.

Since it holds that $\mathrm{sig}(n) = \mathrm{sig}(n+1)$ and $\varphi'_{n+1}/\varphi'_n > 0$ for the other combinations, we have that $N' = m'$ and $w_{n-1} > 0$.

By exchanging the signs, $+$ and $-$, each other in Table 1, we can treat the case where $\varphi_{n-1} < 0$ in the same way. Under this exchange, all of the marks is still true with trivial modifications on above (*n). Consequently, we obtain that $N' = m'$ and $w_{n-1} > 0$.

(ii) Consider the case where $\alpha_n < 0$. In this case, $m' = m - 1$. In Table 2, we list up all combinations of signs of $\varphi_{n-1} \geq 0$, φ_n, φ'_n, and φ'_{n+1}:

(*6) If $\mathrm{sig}(n) - \mathrm{sig}(n+1) \geq 0$, then $N' > m'$.

In a similar way as in the proof of (i), we can prove that $N' = m'$ and $w_{n-1} > 0$. Consequently, we have obtained this lemma. □

The following is the necessary condition for $L_{X,\alpha}$ to satisfy $N = m$, which is stated in Theorem 2.

Theorem 14. *If $N = m$, then $w_{n-1} \gg 0$.*

Proof. We have that $N' = m'$ and $w_{n-1} > 0$ by Lemma 13. Therefore, we can inductively obtain that $w_i > 0$ for all $i = 1, 2, \ldots, n-1$. Thus, we derive that $w_{n-1} \gg 0$. □

φ_{n-1}	φ_n	φ'_n	φ'_{n+1}	$\mathrm{sig}(n) - \mathrm{sig}(n+1)$	comment
$+$	$+$	$+$	$+$	0	
$+$	$+$	$+$	$-$		(*2)
$+$	$+$	$-$	$+$	1	(*1)
$+$	$+$	$-$	$-$	0	
$+$	$0, -$	$+$	$\pm$		(*5)
$+$	$0, -$	$-$	$+$		(*3)
$+$	$0, -$	$-$	$-$	0	
0	$+$	$+$	$+$	0	
0	$+$	$+$	$-$		(*2)
0	$+$	$-$	$\pm$		(*4)
0	$-$	$+$	$\pm$		(*5)
0	$-$	$-$	$+$		(*3)
0	$-$	$-$	$-$	0	

Table 1. $\alpha_n > 0$

φ_{n-1}	φ_n	φ'_n	φ'_{n+1}	$\mathrm{sig}(n) - \mathrm{sig}(n+1)$	comment
$+$	$+$	$+$	$+$	0	(*6)
$+$	$+$	$+$	$-$	-1	
$+$	$+$	$-$	$+$	1	(*6), (*3)
$+$	$+$	$-$	$-$	0	(*6)
$+$	0	$+$	$\pm$		(*5)
$+$	0	$-$	$+$		(*3)
$+$	0	$-$	$-$	0	(*6)
$+, 0$	$-$	$+$	$\pm$		(*5)
$+, 0$	$-$	$-$	$+$	-1	
$+, 0$	$-$	$-$	$-$	0	(*6)
0	$+$	$+$	$+$	0	(*6)
0	$+$	$+$	$-$	-1	
0	$+$	$-$	$\pm$		(*4)

Table 2. $\alpha_n < 0$

4. Discussions

We say that the point interactions $V_k(x) = \sum_{i=1}^{k} \alpha_i \delta(x - x_i)$ are *internally balanced* if $\varphi(x_k) = \varphi(x_{k+1})$, that is, $\varphi'(x_k + 0) = \varphi'_{k+1} = 0$ (cf. [2]). In this case, we have that

$$N = N\Big(-\frac{d^2}{dx^2} + \sum_{i=1}^{k} \alpha_i \delta(x - x_i) \Big) + N\Big(-\frac{d^2}{dx^2} + \sum_{i=k+1}^{n} \alpha_i \delta(x - x_i) \Big) \tag{6}$$

as in [2, Remark 5]. If $N = m$, then $V_k(x)$ are not internally balanced by Proposition 11, however we can easily see that (6) holds. Using Lemma 13 repeatedly, we can obtain

$$N = \sum_{k=1}^{n} N_k \quad \text{with} \quad N_k = N\Big(-\frac{d^2}{dx^2} + \alpha_k \delta(x - x_k) \Big).$$

This is trivial, since $N_k = 1$ as $\alpha_k < 0$ and $N_k = 0$ as $\alpha_k > 0$.

In the rest of this paper, we give some criteria for $L_{X,\alpha}$ to satisfy $N = m$.

Example 15. Let $n = 2$. Then, $N = m$ if and only if $w_1 = c_1 = 1/\alpha_1 + d_1 + 1/\alpha_2 > 0$. In the case where $m = n = 2$, this is Criterion 1 in [3].

Example 16. Let $n = 3$. Then, $N = m$ if and only if

$$c_1 = \frac{1}{\alpha_1} + d_1 + \frac{1}{\alpha_2} > 0, \quad c_2 = \frac{1}{\alpha_2} + d_2 + \frac{1}{\alpha_3} > 0 \quad \text{and} \quad c_1 c_2 > \frac{1}{\alpha_2^2}.$$

In the case where $m = n = 3$, this is equivalent to Criterion 2 in [3].

Corollary 17. *(i) If $N = m$, then all $c_1, c_2, \ldots, c_{n-1}$ are positive. (ii) If $d_i > 2(1/|\alpha_i| + 1/|\alpha_{i+1}|)$ for all $i = 1, 2, \ldots, n-1$, then $N = m$.*

Proof. If $N = m$, then it holds that $w_{n-1} \gg 0$. This implies that $c_1 = w_1 > 0$ and $c_i = w_i + 1/\alpha_{i-1}^2 w_{i-1} > 0$. Thus, we obtain (i). We prove (ii); the assumption implies that

$$c_i = d_i + 1/\alpha_i + 1/\alpha_{i+1} \geq d_i - (1/|\alpha_i| + 1/|\alpha_{i+1}|) > 1/|\alpha_i| + 1/|\alpha_{i+1}|.$$

In particular, $w_1 = c_1 > 1/|\alpha_2|$. If $w_{i-1} > 1/|\alpha_i|$, then it holds that

$$w_i = c_i - 1/|\alpha_i|^2 w_{i-1} > c_i - 1/|\alpha_i| > 1/|\alpha_{i+1}|.$$

Thus, we obtain that $w_{n-1} \gg 0$ by induction. □

In the case where $m = n$, Corollary 17 is Criterion 4 in [3]. We remark that the coefficient, 2, in (ii) is best possible; consider the case where $n = m = 3$. Fix ε with $0 < \varepsilon < 2$ and let $\alpha_1 = \alpha_3 = -1$, $\alpha_2 = -1/t$ and $d_1 = d_2 = (1 + \varepsilon/2)(t + 1) > 0$. Then, it holds that $d_i > \varepsilon(1/|\alpha_i| + 1/|\alpha_{i+1}|) = \varepsilon(t + 1)$ for both $i = 1$ and 2. However, if t is large enough, it holds that $c_1 c_2 = (\varepsilon/2)^2 (t + 1)^2 < t^2 = \alpha_2^{-2}$. Therefore, we obtain that $N < m$ by Example 16.

We need the following fact from the theory of continued fractions to derive Corollary 19.

Proposition 18. *Assume that all y_i are positive. The following three conditions are equivalent.*

(i) It holds that $[x_n, y_n, \ldots, x_2, y_2, x_1] \gg 0$.

(ii) It holds that $[x_1, y_2, x_2, \ldots, y_n, x_n] \gg 0$.

(iii) It holds that $[x_{k-1}, y_{k-1}, \ldots, x_2, y_2, x_1] \gg 0$, $[x_{k+1}, y_{k+2}, x_{k+2}, \ldots, y_n, x_n] \gg 0$, and

$$x_k > \frac{y_k}{[x_{k-1}, y_{k-1}, \ldots, x_2, y_2, x_1]} + \frac{y_{k+1}}{[x_{k+1}, y_{k+2}, x_{k+2}, \ldots, y_n, x_n]}.$$

Proof. In this proof, we denote by $[x_i : x_1] = [x_i, y_i, \ldots, x_2, y_2, x_1]$ and by $[x_i : x_n] = [x_i, y_{i+1}, x_{i+1}, \ldots, y_n, x_n]$ for brevity.

We prove that (i) implies (ii). Since $x_n > y_n/[x_{n-1} : x_1] > 0$, we have that

$$[x_{n-1} : x_n] = x_{n-1} - \frac{y_n}{x_n} > x_{n-1} - [x_{n-1} : x_1] = \frac{y_{n-1}}{[x_{n-2} : x_1]} > 0.$$

Using this, we have that

$$[x_{n-2} : x_n] = x_{n-2} - \frac{y_{n-1}}{[x_{n-1} : x_n]} > x_{n-2} - [x_{n-2} : x_1] = \frac{y_{n-2}}{[x_{n-3} : x_1]} > 0.$$

By repeating this procedure, we obtain (ii). We can prove the converse in the same way.

We prove that (iii) implies (i) and (ii). By the assumption, we have that

$$[x_k : x_1] = x_k - \frac{y_k}{[x_{k-1} : x_1]} > \frac{y_{k+1}}{[x_{k+1} : x_n]} > 0.$$

Using this, we have that

$$[x_{k+1} : x_1] = x_{k+1} - \frac{y_{k+1}}{[x_k : x_1]} > x_{k+1} - [x_{k+1} : x_n] = \frac{y_{k+2}}{[x_{k+2} : x_n]} > 0.$$

If $k \geq n/2$, then we obtain (i) by repeating this procedure. If $k \leq n/2$, then we can obtain (ii) in the same way.

We prove that (i) implies (iii) by contradiction. Assume that (iii) does not hold. Then we have that

$$0 < [x_k : x_1] = x_k - \frac{y_k}{[x_{k-1} : x_1]} \leq \frac{y_{k+1}}{[x_{k+1} : x_n]}.$$

Using this, we have that

$$0 < [x_{k+1} : x_1] = x_{k+1} - \frac{y_{k+1}}{[x_k : x_1]} \leq x_{k+1} - [x_{k+1} : x_n] = \frac{y_{k+2}}{[x_{k+2} : x_n]}$$

However, by repeating this procedure, we have that

$$0 < [x_n : x_1] \leq x_n - [x_n] = 0.$$

This is impossible, thus (iii) holds. □

Corollary 19. *Let the points of interactions, $X = \{x_i\}_{i=1}^n$ of $L_{X,\alpha}$ be partitioned into two groups, $X_1 = \{x_i\}_{i=1}^k$ and $X_2 = \{x_i\}_{i=k+1}^n$. Since $x_i < x_{i+1}$, all points of X_2 lie on the right of all points of X_1. Assume that the Schrödinger operators $L_{X_1,\alpha} = -\frac{d^2}{dx^2} + \sum_{i=1}^k \alpha_i \delta(x - x_i)$ and $L_{X_2,\alpha} = -\frac{d^2}{dx^2} + \sum_{i=k+1}^n \alpha_i \delta(x - x_i)$ satisfy that*

$$N(L_{X_1,\alpha}) = |\{\alpha_i < 0\,;\, x_i \in X_1\}| = m_1,$$
$$N(L_{X_2,\alpha}) = |\{\alpha_i < 0\,;\, x_i \in X_2\}| = m_2.$$

For $N(L_{X,\alpha}) = m = m_1 + m_2$, it is necessary and sufficient that it holds that

$$c_k > \frac{\alpha_k^{-2}}{[c_{k-1}, \alpha_{k-1}^{-2}, c_{k-2}, \ldots, \alpha_2^{-2}, c_1]} + \frac{\alpha_{k+1}^{-2}}{[c_{k+1}, \alpha_{k+2}^{-2}, c_{k+2}, \ldots, \alpha_{n-1}^{-2}, c_{n-1}]}.$$

Corollary 19 immediately follows from Theorem 2 and Proposition 18 and is an extension of Criterion 5 in [3].

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