ON THE NUMBER OF NEGATIVE EIGENVALUES OF A
SCHRÖDINGER OPERATOR WITH \( \delta \) INTERACTIONS

OSAMU OGURISU

Abstract. We give necessary and sufficient conditions for a one-dimensional Schrödinger operator to have the number of negative eigenvalues equal to the number of negative intensities in the case of \( \delta \) interactions.

1. Introduction and Main Theorem

In [3], S. Albeverio and L. Nizhnik gave necessary and sufficient conditions for a one-dimensional Schrödinger operator \( L_{X,\alpha} \) with point \( \delta \)-interactions to satisfy that the number of negative eigenvalues, \( N = N(L_{X,\alpha}) \), of \( L_{X,\alpha} \) equals the number of point interactions, \( n \), in the case where all the intensities are negative. Moreover, in [2], they gave an elegant ‘algorithm’ for determining \( N \). This yields the result obtained in [3] and gives necessary and sufficient conditions for \( L_{X,\alpha} \) not to have negative eigenvalues. In [5] N. I. Goloshchapova and L. L. Oridoroga proved that \( N \) is equal to the number of negative eigenvalues of a kind of finite Jacobi matrix using the method developed in [4]. This gives another characterization of Albeverio-Nizhnik’s algorithm. In the previous paper [8] the author gave a sufficient condition for \( L_{X,\alpha} \) to have at least \( m \) negative eigenvalues; we denote by \( m \) the number of negative intensities.

In this paper we prove that \( N \leq m \) and obtain necessary and sufficient conditions for \( L_{X,\alpha} \) to satisfy \( N = m \) and some extensions of Criteria in [3]. We use Albeverio-Nizhnik’s algorithm to obtain necessary condition and do [8, Lemma 1] to obtain sufficient condition. See Remark 4.

We begin by recalling the definition of \( L_{X,\alpha} \) in [2]; a Schrödinger operator \( L_{X,\alpha} \) with point \( \delta \)-interactions on a finite set \( X = (x_1, \ldots, x_n) \in \mathbb{R}^n \) of points, which are called ‘points of interaction’, and intensities \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \) is defined by the differential expression \(-\left( \frac{d^2}{dx^2} \right)\) on a function \( \psi(x) \) that belongs to the Sobolev space \( W^2_2(\mathbb{R}^1 \setminus X) \) and satisfy, in the points of the set \( X \), the following conjugation conditions:

\[
\psi(x_i + 0) = \psi(x_i - 0), \quad \psi'(x_i + 0) - \psi'(x_i - 0) = \alpha_i \psi(x_i).
\]

The operator \( L_{X,\alpha} \) has the following representation:

\[
L_{X,\alpha} \psi(x) = \left[ -\frac{d^2}{dx^2} + \sum_{i=1}^{n} \alpha_i \delta(x - x_i) \right] \psi(x),
\]

where \( \delta \) is the Dirac’s \( \delta \)-function. Without loss of generality we can assume that \( \alpha_i \neq 0 \) and \( x_1 < x_2 < \cdots < x_n \). Put \( d_i = x_{i+1} - x_i \) and \( m = \{|\alpha_i < 0; 1 \leq i \leq n\}|. \) It is well known that the operators \( L_{X,\alpha} \) are self-adjoint on \( L^2(\mathbb{R}^1) \). Their spectra contain the positive semiaxis, where they are absolutely continuous, and no more than \( n \) simple negative eigenvalues [1].

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Theorem 2. Our main theorem is the following.

\[
Q_{X,\alpha}(\varphi, \psi) = \langle \varphi', \psi' \rangle + \sum_{i=1}^{n} \alpha_i \overline{\varphi(x_i)} \psi(x_i), \quad D(Q_{X,\alpha}) = H^{2,1}(\mathbb{R})
\]

is densely defined, semibounded, and closed and the unique self-adjoint operator associated with \( Q_{X,\alpha} \) is given by \( L_{X,\alpha} \) (cf. \cite[§ II.2.1]{1}). Therefore, it holds that

\[
\langle \varphi, L_{X,\alpha} \varphi \rangle = \|\varphi'\|^2 + \sum_{i=1}^{n} \alpha_i |\varphi(x_i)|^2
\]

for \( \varphi \in D(L_{X,\alpha}) \). Using this, we can obtain the upper bound of \( N \).

**Theorem 1.** It holds that \( N \leq m \).

**Proof.** Assume that \( N > m \) and let \( \varphi_1, \varphi_2, \ldots, \varphi_{m+1} \) be linearly independent eigenfunctions of \( L_{X,\alpha} \), whose associated eigenvalues are negative. Let \( \varphi \) be a linear combination of \( \varphi_i \). Then, it holds that \( \langle \varphi, L_{X,\alpha} \varphi \rangle < 0 \). In addition, we can assume that \( \varphi(x) = 0 \) on \( X_- = \{ x_i; \alpha_i < 0 \} \), since \( |X_-| = m \). This implies that \( \langle \varphi, L_{X,\alpha} \varphi \rangle \geq 0 \) by (1). However, this is impossible.

To state our main theorem, we give some notations. We denote by the symbol, \([x_n, y_n, x_{n-1}, y_{n-1}, \ldots, x_2, y_2, x_1]\), the continued fraction with respect to \( \{x_i\}_{i=1}^{n} \) and \( \{y_i\}_{i=2}^{n} \) defined by

\[
[x_1] = x_1, \\
[x_n, y_n, x_{n-1}, y_{n-1}, \ldots, x_2, y_2, x_1] = x_n - \frac{y_n}{[x_{n-1}, y_{n-1}, \ldots, x_2, y_2, x_1]}.
\]

If some denominators are zero, such continued fraction does not be defined. We write \([x_n, y_n, \ldots, x_2, y_2, x_1] \gg 0 \) if all \([x_i, y_i, \ldots, x_2, y_2, x_1] \gg 0 \) for all \( i = 1, 2, \ldots, n \). Put

\[
c_i = \frac{1}{\alpha_i} + d_i + \frac{1}{\alpha_{i+1}}, \quad w_i = [c_i, \alpha_i^{-2}, \alpha_{i-1}^{-2}, \ldots, c_2, \alpha_2^{-2}, c_1].
\]

Our main theorem is the following.

**Theorem 2.** \( N = m \) if and only if \( w_{n-1} \gg 0 \).

We prove this theorem in the following sections; the plan of this paper is the following. We prove the sufficiency in Section 2 (Theorem 7) and the necessity in Section 3 (Theorem 14). We give some criteria for \( L_{X,\alpha} \) to satisfy \( N = m \) in Section 4.

**Remark 3.** In the case where all intensities are negative (i.e., \( m = n \)), it holds that \( N = m \) if and only if \( (|\alpha_1|, d_1, |\alpha_2|, d_2, \ldots, d_{n-1}, |\alpha_n|) \gg 0 \) by \cite[Theorem 2]{3}. Here, proper continued fraction, \( v_n = (a_n, a_{n-1}, \ldots, a_1) \), is defined by \( v_1 = a_1 \) and \( v_n = a_n - 1/v_{n-1} \), and we write \( v_n \gg 0 \) if \( v_i > 0 \) for all \( i = 1, 2, \ldots, n \).

**Remark 4.** One of the referees suggested to the author that N. Goloschapova and L. Oridoroga \cite{6} recently obtained necessary and sufficient conditions for \( L_{X,\alpha} \) to satisfy \( N = m \) in different forms using the concept of boundary triplets and the corresponding Weyl functions developed in \cite{7}. In these papers Schrödinger operators with finite and infinite number of point interactions were investigated.

**Remark 5.** For later use, we remark a fact on recurrence formula. Let \( A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \) and assume that \( b_n \neq 0, c_n \neq 0 \) and \( \det A_n \neq 0 \) for all \( n \in \mathbb{N} \). Let \((x_1, y_1)\) be given and
\{(x_n, y_n); n \in \mathbb{N}\} be defined by \(x_{n+1} = A_n x_n\) and \(y_{n+1} = A_n y_n\). Then, this sequence satisfies

\[
x_{n+1} = \left( a_n + \frac{b_n}{b_{n-1}} d_{n-1} \right) x_n - \frac{b_n}{b_{n-1}} \det A_{n-1} x_{n-1},
\]

\[
y_{n+1} = \left( d_n + \frac{c_n}{c_{n-1}} a_{n-1} \right) y_n - \frac{c_n}{c_{n-1}} \det A_{n-1} y_{n-1},
\]

and the converse holds.

2. Sufficiency

Let \(M_k(\lambda)\) be the real symmetric matrix defined by

\[
M_k(\lambda) = \left( \frac{2\lambda}{\alpha_i} \delta_{i,j} + e^{-\lambda|x_i-x_j|} \right)^k_{i,j=1}
\]

and \(D_k = \det M_k(\lambda)\). Since we already know that if \(M_n(\lambda)\) is positive definite for some positive \(\lambda\) then \(N = m\) by [8, Lemma 1] and Theorem 1, we examine the positive definiteness in the case where \(w_{n-1} \gg 0\). Since \(D_k\) is a leading principal minor of \(M_n(\lambda)\) with order \(k\), \(M_n(\lambda)\) is positive definite if and only if all \(D_1, D_2, \ldots, D_n\) are positive.

To prove this, let us establish the recurrence formula for \(D_k\). We put

\[
p_i = \frac{2\lambda}{\alpha_i} + 1 + e^{-2\lambda d_{i-1}} \left( \frac{2\lambda}{\alpha_{i-1}} - 1 \right), \quad q_i = e^{-2\lambda d_{i-1}} \left( \frac{2\lambda}{\alpha_{i-1}} \right)^2.
\]

**Proposition 6.** We have that \(D_i = p_i D_{i-1} - q_i D_{i-2}\) for \(i = 3, 4, \ldots, n\).

**Proof.** Let \(i \geq 2\), \(v_1 = (e^{-\lambda d_1}) \in \mathbb{R}^1\) and \(v_i = e^{-\lambda d_i} (v_{i-1}, 1) \in \mathbb{R}^2\). Since \(|x_i - x_j| = \sum_{k=i}^{j-1} d_k\) when \(i < j\), we have

\[
M_i = \begin{pmatrix} M_{i-1} & v_i^t \\ v_i & 1 \end{pmatrix},
\]

Put

\[
E_{i-1} = \det \begin{pmatrix} M_{i-1} & v_i^t \\ v_i & 0 \end{pmatrix}.
\]

Then, we have that

\[
D_i = \begin{vmatrix} M_{i-1} & v_i^t \\ 0 & 2\lambda/\alpha_i + 1 \end{vmatrix} + \begin{vmatrix} M_{i-1} & v_i^t \\ v_i & 0 \end{vmatrix} = \left( \frac{2\lambda}{\alpha_i} + 1 \right) D_{i-1} + E_{i-1}
\]

and

\[
E_i = e^{-2\lambda d_i} \begin{vmatrix} M_{i-1} & v_i^t \\ v_i & 0 \end{vmatrix} - \begin{vmatrix} M_{i-1} & v_i^t \\ v_i & 1 \end{vmatrix} = e^{-2\lambda d_i} \begin{vmatrix} M_{i-1} & v_i^t \\ v_i & 0 \end{vmatrix} - \begin{vmatrix} M_{i-1} & v_i^t \\ v_i & 1 \end{vmatrix} = e^{-2\lambda d_i} \begin{vmatrix} M_{i-1} & v_i^t \\ v_i & 0 \end{vmatrix} - e^{-2\lambda d_i} D_{i-1}.
\]

Thus, we obtain that

\[
\begin{pmatrix} D_i \\ E_i \end{pmatrix} = A_{i-1} \begin{pmatrix} D_{i-1} \\ E_{i-1} \end{pmatrix}
\]

with

\[
A_{i-1} = \begin{pmatrix} 2\lambda/\alpha_i + 1 & 1 \\ -e^{-2\lambda d_i} & e^{-2\lambda d_i} \left( \frac{2\lambda}{\alpha_i} - 1 \right) \end{pmatrix}.
\]
Since $\det A_{i-1} = e^{-2\lambda d_i}(2\lambda/\alpha_i)^2 \neq 0$, it holds that
\[ D_{i+1} = \left( \frac{2\lambda}{\alpha_{i+1}} + 1 + e^{-2\lambda d_i} \left( \frac{2\lambda}{\alpha_i} - 1 \right) \right) D_i - e^{-2\lambda d_i} \left( \frac{2\lambda}{\alpha_i} \right)^2 D_{i-1}. \]
This is the desired recurrence formula. \(\square\)

The following is the sufficient condition for $L_{X,\alpha}$ to satisfy $N = m$, which is stated in Theorem 2.

**Theorem 7.** If $w_n \rightarrow 0$, then $N = m$.

**Proof.** We assume that $\lambda$ is small enough and remark that $p_i = 2c_{i-1}/\alpha_0 + O(\lambda^2)$ and $q_k = 4\alpha_{i-1}^{-2} \lambda^2 + O(\lambda^3)$. Since $D_1 = 2\lambda/\alpha_1 + 1 = 1 + O(\lambda)$ and $D_2 = (2\lambda/\alpha_2 + 1)(2\lambda/\alpha_1 + 1) - e^{-2\lambda d_i} = 2w_1 + O(\lambda^2)$ are positive and it holds that
\[ D_2/D_1 = 2w_1 + O(\lambda^2). \]

Since $D_3/D_2 = [p_3, q_3, D_2/D_1]$ by Proposition 6, it holds that
\[ D_3/D_2 = 2[c_2, \alpha_2^{-2}, w_1] + O(\lambda^2) = 2w_2 + O(\lambda^2). \]

Therefore, $D_3$ is positive. By repeating similar calculations, we obtain that all $D_1, D_2, \ldots, D_n$ are positive. Consequently, $M_n(\lambda)$ is positive definite for some positive $\lambda$, and thus $N = m$ by [8, Lemma 1] and Theorem 1. \(\square\)

3. NECESSITY

In this section we obtain the necessary condition for $L_{X,\alpha}$ to satisfy $N = m$, which is stated in Theorem 2. We assume that $N = m$ throughout this section. We first give some notations and recall Albeverio-Nizhnik’s algorithm.

Let $j_1 < j_2 < \cdots < j_m$ be the indices of negative intensities, $\alpha_{j_k} < 0$, and put $y_k = x_{j_k}$ for $k = 1, 2, \ldots, m$. Let $\varphi$ be the special solution defined in [2], that is, $\varphi$ is the solution on the whole line of the following problem: $L_{X,\alpha}\varphi = 0$ and $\varphi(x) = 1$ if $x < x_1$. This $\varphi$ has exactly $m$ zeros by [2, Theorem 1]. Let $z_1 < z_2 < \cdots < z_m$ be the zeros of $\varphi$. We put
\[ \varphi'_i = \varphi'(x_i - 0), \quad \varphi_i = \varphi(x_i - 0) \]
for $i = 1, 2, \ldots, n$ and $\varphi'_{n+1} = \varphi'(x_n + 0)$. Since $\varphi$ is linear on $[x_{i-1}, x_i]$, it holds that
\[ \varphi'_i = \varphi'(x_{i-1} + 0) = (\varphi_i - \varphi_{i-1})/d_{i-1}. \]

These $\varphi'_i$ and $\varphi_i$ satisfy the following recurrence formula by the conjugation condition:
\[ \begin{align*}
\varphi'_1 &= 0, \quad \varphi'_{i+1} = \varphi'_i + \alpha_i \varphi_i, \\
\varphi_1 &= 1, \quad \varphi_{i+1} = \varphi_i + d_i \varphi'_{i+1} = d_i \varphi'_i + (1 + \alpha_i d_i) \varphi_i. 
\end{align*} \]
This implies the recurrence formulas for each $\varphi'_i$ and $\varphi_i$:
\[ \begin{align*}
\varphi'_{i+1} &= \left( 1 + \alpha_i d_{i-1} + \frac{\alpha_i}{\alpha_{i-1}} \right) \varphi'_i - \frac{\alpha_i}{\alpha_{i-1}} \varphi'_{i-1}, \\
\varphi_{i+1} &= \left( 1 + \alpha_i d_i + \frac{d_i}{d_{i-1}} \right) \varphi_i - \frac{d_i}{d_{i-1}} \varphi_{i-1}.
\end{align*} \]
Formula (5) already appears in [2].

Recall Albeverio-Nizhnik’s algorithm.

**Theorem 8** (Theorem 4 in [2]). $N$ equals the signature (the number of sign changes) of the sequence, $(\varphi_1, \varphi_2, \ldots, \varphi_n, (1 + \alpha_n d_{n-1}) \varphi_n - \varphi_{n-1})$. 

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In the following, we use the sequence, \((\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi'_{n+1})\), instead of the original one, since \((1 + \alpha_n d_{n-1})\varphi_n - \varphi_{n-1} = d_{n-1}\varphi'_{n+1} + 1\).

We summarize some simple and useful facts on \(\varphi\).

**Proposition 9.** The following hold:

(i) The function \(\varphi\) is continuous and piecewise linear on \(\mathbb{R}\). The derivative \(\varphi'\) and \(\varphi\) can not be simultaneously zero.

(ii) If \(\varphi_i = 0\) then \(\varphi_{i-1}\varphi_{i+1} < 0\). If \(\varphi'_i = 0\) then \(\varphi_i \neq 0\).

(iii) Assume that \(\alpha_i > 0\). If \(\varphi'_i \geq 0\) and \(\varphi_i > 0\), then \(\varphi'_{i+1} \geq \varphi'_i \geq 0\) and \(\varphi_{i+1} > \varphi_i \geq 0\). Similarly, if \(\varphi'_i \leq 0\) and \(\varphi_i < 0\), then \(\varphi'_{i+1} \leq \varphi'_i \leq 0\) and \(\varphi_{i+1} < \varphi_i < 0\).

(iv) \(\varphi'(z_i + \varepsilon)\) and \(\varphi(z_i + \varepsilon)\) have same sign for \(\varepsilon\) small enough. Thus, if \(x_{j-1} < z_i < x_j\), then \(\varphi'_j\) and \(\varphi_j\) have same sign.

(v) If \(\varphi'_i = 0\) with \(x_i < z_j\), there is at least one \(y_k\) in \(I = [x_i, z_j]\).

(vi) If \(\varphi'_i = 0\) with \(z_j < x_i\), there is at least one \(y_k\) in \(I = (z_j, x_i)\).

Proof. We can find (i) in [2]. Using (5), we can see (ii). (iii) and (iv) can be proved by straightforward calculations.

Consider (v) and assume that no such \(y_k\) exists. Since \(\varphi_i \neq 0\), \(|\varphi(x)|\) is monotonously increasing on \(I\), and thus \(\varphi(z_j) \neq 0\). This contradicts to the definition of \(z_j\).

Consider (vi) and assume that no such \(y_k\) exists. Since \(\varphi'(z_j + \varepsilon)\) and \(\varphi(z_j + \varepsilon)\) have same sign, \(|\varphi'(x)|\) is monotonously non-decreasing on \(I\), and thus \(\varphi'_i \neq 0\). This contradicts to the assumption. \(\square\)

We divide the proof of the necessity of Theorem 2 into a sequence of propositions and a lemma. We first prove that the points of interaction with negative intensities exactly interlace the zeros of \(\varphi\).

**Proposition 10.** If \(N = m\), then we have that
\[
x_1 \leq y_1 < z_1 < y_2 < z_2 < \cdots < z_{m-1} < y_m < z_m.
\]

Proof. Since \(\varphi(x) > 0 \) for \(x < z_1\), it holds that \(\varphi'(z_1 - 0) < 0\). Since \(\varphi'(x) \geq 0\) for \(x \leq y_1\), we have that \(x_1 \leq y_1 < z_1\). We next prove that at least one \(y_k\) exists in \((z_{j-1}, z_j)\); in both cases where \(x_{i-1} = z_{j-1}\) and \(x_{i-1} < z_{j-1} < x_i\), we have that \(\varphi_i \neq 0\), \(\varphi'_i\) and \(\varphi_i\) have same sign and \(x_i < z_j\). Therefore, if no \(y_{k}\) exists in \(I = (x_i, z_j)\), \(|\varphi(x)|\) is monotonously increasing on \(I\), and thus \(\varphi(z_j) \neq 0\). Hence, at least one \(y_{k}\) exists in \(I\). Consequently, each interval of \(x_i < z_1 < z_2 < \cdots < z_m\) contains at least one \(y_{k}\). Since the number of \(y_{k}\) is equal to \(m\), each interval contains exactly one \(y_{k}\). \(\square\)

This interlacing property implies that \(\varphi'_i \neq 0\).

**Proposition 11.** If \(N = m\), then \(\varphi'_i \neq 0\) for \(i = 2, 3, \ldots, n\).

Proof. The proof is by contradiction. Assume that \(\varphi'_i = 0\). Since no \(y_k > z_m\) exists and \(\varphi'(z_m + \varepsilon)\) and \(\varphi(z_m + \varepsilon)\) have same sign, we have that \(\varphi'(x) \neq 0\) for all \(x \geq z_m\). Thus, it holds that \(x_i < z_m\). If \(z_j < x_i < z_{j+1}\), then at least two \(y_k\) and \(y_{k'}\) are in \((z_j, z_{j+1})\) by Proposition 9. Since this contradicts to Proposition 10, it holds that \(x_1 < x_i < z_1\). However, this is impossible, too; since \(\varphi'_1 = 0\) and \(\varphi_1 = 1\), if no \(y_{k}\) exists in \((x_1, x_i)\), then \(\varphi'_i > 0\). Hence, at least one \(y_{k}\) exists in \((x_1, x_i)\). On the other hand, there is at least one \(y_{k'}\) in \([x_i, z_1]\) by Proposition 9. This contradicts to Proposition 10. Consequently, we have that \(\varphi'_i \neq 0\). \(\square\)

Using the fact that \(\varphi'_i \neq 0\), we establish the relation between \(w_i\) and \(\varphi'_i\).

**Proposition 12.** If \(N = m\), then \(\varphi'_{i+1}/\alpha_i \varphi'_i = w_{i-1}\) for \(i = 2, 3, \ldots, n\).
Proof. We use induction on $i$. Let $i = 2$. Since $\varphi_2' = \alpha_1$ and $\varphi_3' = \alpha_1 + \alpha_2(1 + \alpha_1 d_1)$, we obtain that
\[
\frac{\varphi_3'}{\alpha_2 \varphi_2'} = \frac{\alpha_1 + \alpha_2(1 + \alpha_1 d_1)}{\alpha_2 \alpha_1} = \frac{1}{\alpha_2} + \frac{1}{\alpha_1} + d_1 = c_1 = w_1.
\]
Let $i \geq 3$ and assume that $\varphi_i'/\alpha_{i-1} \varphi_{i-1}' = w_{i-2}$. Since it holds that
\[
\varphi_{i+1}' = c_{i-1}(\alpha_i \varphi_i') - \frac{\alpha_i}{\alpha_{i-1}}(\alpha_{i-1} \varphi_{i-1}')
\]
by (4), we obtain that
\[
\frac{\varphi_{i+1}'}{\alpha_i \varphi_i'} = \frac{c_{i-1} - \frac{\alpha_i}{\alpha_{i-1}}}{\varphi_i'/\alpha_{i-1} \varphi_{i-1}'} = \frac{c_{i-1} - \alpha_i}{\alpha_{i-1}} = w_{i-1}.
\]
This completes the induction. \qed

To state the following key lemma, we give some notations. Let $X' = X \setminus \{x_n\}$, $\alpha' = \alpha \setminus \{\alpha_n\}$, $N' = N(L_{X', \alpha'})$ and $m' = \{(\alpha_i < 0 : 1 \leq i \leq n - 1)\}$. Then it holds that
that $N' \leq m'$.

Lemma 13. If $N = m$, then $N' = m'$ and $w_{n-1} > 0$.

Proof. We denote by $\text{sig}(k)$ the signature of the sequence, $(\varphi_1, \ldots, \varphi_k, \varphi_k')$, in this proof. We have that $N = \text{sig}(n + 1)$ and $N' = \text{sig}(n)$ by Albeverio-Nizhnik’s algorithm.

(i) Consider the case where $\alpha_n > 0$. In this case, $m' = m$. Assume that $\varphi_{n-1} > 0$. In Table 1, we list up all combinations of signs of $\varphi_{n-1}$, $\varphi_n$, $\varphi_n'$, and $\varphi_n'$. We remark that $\varphi_{n-1}$ and $\varphi_n$ can not be simultaneously zero. The combinations indicated by the marks (*n) are impossible:
\[
\begin{align*}
(*1) \text{ If } \text{sig}(n) - \text{sig}(n + 1) = 1, \text{ then } N' > m'. \\
(*2) \varphi_{n+1}' = \varphi_n' + \alpha_n \varphi_n > 0, \text{ but negative.} \\
(*3) \varphi_{n+1}' = \varphi_n' + \alpha_n \varphi_n < 0, \text{ but positive.} \\
(*4) \varphi_n' = (\varphi_n - \varphi_{n-1})/d_{n-1} > 0, \text{ but negative.} \\
(*5) \varphi_n' = (\varphi_n - \varphi_{n-1})/d_{n-1} < 0, \text{ but positive.}
\end{align*}
\]
Since it holds that $\text{sig}(n) = \text{sig}(n + 1)$ and $\varphi_{n+1}'/\varphi_n' > 0$ for the other combinations, we have that $N' = m'$ and $w_{n-1} > 0$.

By exchanging the signs, + and −, each other in Table 1, we can treat the case where $\varphi_{n-1} < 0$ in the same way. Under this exchange, all of the marks is still true with trivial modifications on above (*n). Consequently, we obtain that $N' = m'$ and $w_{n-1} > 0$.

(ii) Consider the case where $\alpha_n < 0$. In this case, $m' = m - 1$. In Table 2, we list up all combinations of signs of $\varphi_{n-1} \geq 0$, $\varphi_n$, $\varphi_n'$, and $\varphi_{n+1}':$
\[
(*6) \text{ If } \text{sig}(n) - \text{sig}(n + 1) \geq 0, \text{ then } N' > m'.
\]
In a similar way as in the proof of (i), we can prove that $N' = m'$ and $w_{n-1} > 0$. Consequently, we have obtained this lemma. \qed

The following is the necessary condition for $L_{X, \alpha}$ to satisfy $N = m$, which is stated in Theorem 2.

Theorem 14. If $N = m$, then $w_{n-1} \gg 0$.

Proof. We have that $N' = m'$ and $w_{n-1} > 0$ by Lemma 13. Therefore, we can inductively obtain that $w_i > 0$ for all $i = 1, 2, \ldots, n - 1$. Thus, we derive that $w_{n-1} \gg 0$. \qed
4. Discussions

We say that the point interactions \( V_k(x) = \sum_{i=1}^{k} \alpha_i \delta(x - x_i) \) are \textit{internally balanced} if \( \varphi(x_k) = \varphi(x_{k+1}) \), that is, \( \varphi'(x_k + 0) = \varphi'_k + 1 = 0 \) (cf. \[2\]). In this case, we have that

\[
(6) \quad N = N \left( - \frac{d^2}{dx^2} \right) + \sum_{i=1}^{k} \alpha_i \delta(x - x_i) \right) + N \left( - \frac{d^2}{dx^2} + \sum_{i=k+1}^{n} \alpha_i \delta(x - x_i) \right)
\]

as in \[2, \text{Remark 5}\]. If \( N = m \), then \( V_k(x) \) are not internally balanced by Proposition 11, however we can easily see that (6) holds. Using Lemma 13 repeatedly, we can obtain

\[
N = \sum_{k=1}^{n} N_k \quad \text{with} \quad N_k = N \left( - \frac{d^2}{dx^2} + \alpha_k \delta(x - x_k) \right).
\]

This is trivial, since \( N_k = 1 \) as \( \alpha_k < 0 \) and \( N_k = 0 \) as \( \alpha_k > 0 \).

In the rest of this paper, we give some criteria for \( L_{X, \alpha} \) to satisfy \( N = m \).
Example 15. Let \( n = 2 \). Then, \( N = m \) if and only if \( w_1 = c_1 = \frac{1}{\alpha_1} + d_1 = 1/\alpha_1 + 1/\alpha_2 > 0 \).

Example 16. Let \( n = 3 \). Then, \( N = m \) if and only if
\[
\begin{align*}
 c_1 &= \frac{1}{\alpha_1} + d_1 + \frac{1}{\alpha_2} > 0, \\
 c_2 &= \frac{1}{\alpha_2} + d_2 + \frac{1}{\alpha_3} > 0 \quad \text{and} \quad c_1c_2 > \frac{1}{\alpha_2}.
\end{align*}
\]
In the case where \( m = n = 3 \), this is equivalent to Criterion 2 in [3].

Corollary 17. (i) If \( N = m \), then all \( c_1, c_2, \ldots, c_{n-1} \) are positive. (ii) If \( d_i > 2(1/|\alpha_i| + 1/|\alpha_{i+1}|) \) for all \( i = 1, 2, \ldots, n-1 \), then \( N = m \).

Proof. If \( N = m \), then it holds that \( w_{n-1} > 0 \). This implies that \( c_1 = w_1 > 0 \) and \( c_i = w_i + 1/\alpha_{i-1}^2w_{i-1} > 0 \). Thus, we obtain (i). We prove (ii): the assumption implies that
\[
\begin{align*}
 c_i &= d_i + 1/\alpha_i + 1/\alpha_{i+1} \geq d_i - (1/|\alpha_i| + 1/|\alpha_{i+1}|) > 1/|\alpha_i| + 1/|\alpha_{i+1}|.
\end{align*}
\]
In particular, \( w_1 = c_1 > 1/|\alpha_2| \). If \( w_{n-1} > 1/|\alpha_i| \), then it holds that \( w_i = c_i - 1/|\alpha_i|w_{i-1} > c_i - 1/|\alpha_i| > 1/|\alpha_{i+1}| \).

Thus, we obtain that \( w_{n-1} > 0 \) by induction. \( \square \)

In the case where \( m = n \), Corollary 17 is Criterion 4 in [3]. We remark that the coefficient, 2, in (ii) is best possible; consider the case where \( n = m = 3 \). Fix \( \varepsilon \) with \( 0 < \varepsilon < 2 \) and let \( \alpha_1 = \alpha_3 = -1 \), \( \alpha_2 = -1/\varepsilon \) and \( d_1 = d_2 = (1 + \varepsilon/2)(t + 1) > 0 \). Then, it holds that \( d_i > \varepsilon(1/|\alpha_i| + 1/|\alpha_{i+1}|) = \varepsilon(t + 1) \) for both \( i = 1 \) and 2. However, if \( t \) is large enough, it holds that \( c_1c_2 = (\varepsilon/2)^2(t + 1)^2 < t^2 = \alpha_2^{-2} \). Therefore, we obtain that \( N < m \) by Example 16.

We need the following fact from the theory of continued fractions to derive Corollary 19.

Proposition 18. Assume that all \( y_i \) are positive. The following three conditions are equivalent.

(i) It holds that \( [x_n, y_n, \ldots, x_2, y_2, x_1] \gg 0 \).

(ii) It holds that \( [x_1, y_2, x_2, \ldots, y_{n-1}, x_n] \gg 0 \).

(iii) It holds that \( [x_{k-1}, y_{k-1}, \ldots, x_2, y_2, x_1] \gg 0 \),
\[
[x_{k+1}, y_{k+1}, x_{k+2}, \ldots, y_n, x_n] \gg 0 \quad \text{and}
\]
\[
x_k > \frac{y_k}{[x_{k-1}, y_{k-1}, \ldots, x_2, y_2, x_1]} + \frac{y_{k+1}}{[x_{k+1}, y_{k+1}, x_{k+2}, \ldots, y_n, x_n]}.
\]

Proof. In this proof, we denote by \( [x_i : x_1] = [x_i, y_i, \ldots, x_2, y_2, x_1] \) and by \( [x_i, y_i, x_{i+1}, \ldots, y_{n-1}, x_n] \) for brevity.

We prove that (i) implies (ii). Since \( x_n > y_n/[x_{n-1} : x_1] > 0 \), we have that
\[
[x_{n-1} : x_n] = x_{n-1} - \frac{y_n}{x_n} > x_{n-1} - [x_{n-1} : x_1] = \frac{y_n}{x_{n-2} : x_1} > 0.
\]
Using this, we have that
\[
[x_{n-2} : x_n] = x_{n-2} - \frac{y_n}{x_{n-1} : x_n} > x_{n-2} - [x_{n-2} : x_1] = \frac{y_n}{x_{n-3} : x_1} > 0.
\]
By repeating this procedure, we obtain (ii). We can prove the converse in the same way.

We prove that (iii) implies (i) and (ii). By the assumption, we have that
\[
[x_k : x_1] = x_k - \frac{y_k}{[x_{k-1} : x_1]} > \frac{y_{k+1}}{[x_{k+1} : x_n]} > 0.
\]
Using this, we have that
\[
[x_{k+1} : x_1] = x_{k+1} - \frac{y_{k+1}}{[x_{k} : x_1]} > x_{k+1} - [x_{k+1} : x_n] = \frac{y_{k+2}}{[x_{k+2} : x_n]} > 0.
\]
If \( k \geq n/2 \), then we obtain (i) by repeating this procedure. If \( k \leq n/2 \), then we can obtain (ii) in the same way.

We prove that (i) implies (iii) by contradiction. Assume that (iii) does not hold. Then we have that

\[
0 < \left[ x_k : x_1 \right] = x_k - \frac{y_k}{x_k-1 : x_1} \leq \frac{y_k+1}{x_k+1 : x_n}.
\]

Using this, we have that

\[
0 < \left[ x_{k+1} : x_1 \right] = x_{k+1} - \frac{y_{k+1}}{x_k + 1 : x_1} \leq x_{k+1} - \left[ x_{k+1} : x_n \right] = \frac{y_{k+2}}{x_k+2 : x_n}.
\]

However, by repeating this procedure, we have that

\[
0 < \left[ x_n : x_1 \right] \leq x_n - \left[ x_n \right] = 0.
\]

This is impossible, thus (iii) holds. \( \square \)

**Corollary 19.** Let the points of interactions, \( X = \{x_i\}_{i=1}^n \) of \( L_{X,\alpha} \) be partitioned into two groups, \( X_1 = \{x_i\}_{i=1}^k \) and \( X_2 = \{x_i\}_{i=k+1}^n \). Since \( x_i < x_{i+1} \), all points of \( X_2 \) lie on the right of all points of \( X_1 \). Assume that the Schrödinger operators \( L_{X_1,\alpha} = -\frac{d^2}{dx^2} + \sum_{i=1}^k \alpha_i \delta(x - x_i) \) and \( L_{X_2,\alpha} = -\frac{d^2}{dx^2} + \sum_{i=k+1}^n \alpha_i \delta(x - x_i) \) satisfy that

\[
N(L_{X_1,\alpha}) = \left| \{\alpha_i < 0 ; x_i \in X_1 \} \right| = m_1,
\]

\[
N(L_{X_2,\alpha}) = \left| \{\alpha_i < 0 ; x_i \in X_2 \} \right| = m_2.
\]

For \( N(L_{X,\alpha}) = m = m_1 + m_2 \), it is necessary and sufficient that it holds that

\[
\alpha_k > \frac{\alpha_k^{-2}}{\left[ \alpha_{k-1}^{-2}, \alpha_{k-2}^{-2}, \ldots, \alpha_2^{-2} \right]} + \frac{\alpha_k^{-2}}{\left[ \alpha_{k+1}^{-2}, \alpha_{k+2}^{-2}, \ldots, \alpha_{n-1}^{-2} \right]}.
\]

Corollary 19 immediately follows from Theorem 2 and Proposition 18 and is an extension of Criterion 5 in [3].

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**References**


Division of Mathematical and Physical Sciences, Kanazawa University, Kanazawa, Ishikawa, 920-1192, Japan

E-mail address: ogurisu@kanazawa-u.ac.jp

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