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ON DECOMPOSITIONS OF THE IDENTITY OPERATOR INTO A LINEAR COMBINATION OF ORTHOGONAL PROJECTIONS

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Abstract. In this paper we consider decompositions of the identity operator into a linear combination of $k \geq 5$ orthogonal projections with real coefficients. It is shown that if the sum A of the coefficients is closed to an integer number between 2 and $k-2$ then such a decomposition exists. If the coefficients are almost equal to each other, then the identity can be represented as a linear combination of orthogonal projections for $\frac{k-\sqrt{k^2-4k}}{2} < A < \frac{k+\sqrt{k^2-4k}}{2}$. In the case where some coefficients are sufficiently close to 1 we find necessary conditions for the existence of the decomposition.

A classical decomposition of the identity into an orthogonal sum of orthogonal projections is widely used in various mathematical applications. We consider here non-orthogonal decompositions of the identity into a linear combination of orthogonal projections with positive coefficients,

$$I = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + \alpha_k P_k, \tag{1}$$

where I is the identity operator and $P_i = P_i^*$, $i = 1, \ldots, k$, are orthogonal projections on a separable Hilbert space H. We shall suppose that $\alpha_i \leq 1$ for every i since otherwise $P_i = 0$ and we shall not find new decompositions in that case.

Let us denote by Ω^k the set of all vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in [0,1]^k \subset \mathbb{R}^k$ for which there exists the resolution of (1) in a separable Hilbert space. Then (see [4, 7]) $\Omega^1 = \{1\}$,

$$\begin{aligned}
\Omega^2 =&\{(1,\alpha),(\alpha,1) \mid \alpha \in [0,1]\} \cap \{(\alpha, 1-\alpha) \mid \alpha \in [0,1]\},\\
\Omega^3 =&[0,1]\times\Omega^2 \cup \Omega^2 \times [0,1] \cup \{(\alpha_1,\alpha_2,\alpha_3) \mid (\alpha_1,\alpha_3) \in \Omega^2, \alpha_2 \in [0,1]\}\\
&\cup 1 \times [0,1]^2 \cup [0,1]^2 \times 1 \cup \{(\alpha_1, 1, \alpha_3) \mid (\alpha_1,\alpha_3) \in [0,1]^2\}\\
&\cup \{(\alpha_1,\alpha_2,\alpha_3) \mid \alpha_i \in [0,1], \alpha_1+\alpha_2+\alpha_3 = 2 \text{ or } \alpha_1+\alpha_2+\alpha_3 = 1\}.
\end{aligned}$$

A description of the set Ω^4 was obtained by different approaches in [5, 9] where, in particular, the authors showed that Ω^4 does not contain a closed ball in $\mathbb{R}^4$ of nonzero radius.

The first statement about the existence of an open set in Ω^k for $k \geq 5$ was proved in [8], where the decompositions of the identity were constructed directly.

In this paper we find new subsets of Ω^k (sections 2–4) and show that only for integer sums $A = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ a solution of (1) exits for any $\vec{\alpha}$. Following [4], let $\mathcal{P}_{k,\vec{\alpha}}$ denote the $*$-algebra with the identity e,

$$\mathcal{P}_{k,\vec{\alpha}} = \mathbb{C} < p_1, p_2, \ldots, p_k \mid p_i^2 = p_i^* = p_i, \sum_{i=1}^{k} \alpha_i p_i = e > .$$

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By definition, a vector $\vec{\alpha} \in \Omega^k$ if and only if $\mathcal{P}_{k,\vec{\alpha}}$ has a $*$-representation in a separable Hilbert space. We shall show that for $A \in \mathbb{N}, 1 < A < k-1$ and $\alpha_i \neq 0, \alpha_i \neq 1$, the point $\vec{\alpha}$ is contained in in Ω^k with a sufficiently small neighborhood $\Omega_{\vec{\alpha}}$. Beside this, if $A = 2$ and $\vec{\beta} \in \Omega_{\vec{\alpha}}$, the algebra $\mathcal{P}_{k,\vec{\beta}}$ has an irreducible representation in the matrix algebra over the Cunts algebra $M_3(\mathcal{Q}_2)$ when $k \geq 6$ and with some additional assumptions on the coordinates of the vector $\vec{\alpha}$ when $k = 5$.

Also we shall show that for $|\alpha_i - \alpha_j| < \epsilon(A)$ with small $\epsilon(A) > 0$, the vector $\vec{\alpha} \in \Omega^k$ if

$$\frac{k - \sqrt{k^2 - 4k}}{2} < A < \frac{k + \sqrt{k^2 - 4k}}{2},$$

i.e., when every scalar operator γI with $|\gamma - A| < \epsilon(A)$ can be decomposed into a sum of k orthogonal projections (see [3]).

In the section 1 we investigate actions of Coxeter functors [1, 4], the transformations that give new decompositions of the identity by using constructions of an initial decomposition. Such decompositions have another coefficients in general and so the functors generate mappings on coefficients. We shall prove a theorem about a monotone property of the mappings related to the second iteration of Coxeter functors.

We find a relation between lower and upper bound of the spectrum of a sum of nonnegative operators in section 2. Using it we deduce that the existence of a solution of (1) for $\sum_1^m \alpha_i > m - 1$ and

$$\sum_1^m \alpha_i + \alpha_j > m, \quad j = m+1, \dots, k \tag{2}$$

implies the inequality $\sum_{m+1}^k \alpha_j \geq 1$. In a finite dimensional space the inequality (2) follows from a trace equality and Horn inequalities for a sum of Hermitian matrices [2]. Since an orthrogonal projection has at most two eigenvalues, they are having the following simple form:

$$\sum_1^k l_i \alpha_i \geq n(l_1, \dots, n_k),$$

where l_i and $n(l_1, \dots, n_k)$ are integer numbers that are dependant on the Horn inequality and on the ranks of the orthogonal projections $P_1, \dots, P_k$.

1. Monotone mappings related to the Coxeter functors

Let the decomposition (1) hold. By substituting every P_i with its complement $\tilde{P}_i = I - P_i$, we write down the equation $(\sum \alpha_i - 1)I = \alpha_1(I - P_1) + \alpha_2(I - P_2) + \cdots + \alpha_k(I - P_k)$ and, after reducing to the standard form, we obtain a new decomposition,

$$\tilde{I} = \frac{\alpha_1}{\sum \alpha_i - 1}\tilde{P}_1 + \frac{\alpha_2}{\sum \alpha_i - 1}\tilde{P}_2 + \cdots + \frac{\alpha_1}{\sum \alpha_i - 1}\tilde{P}_k. \tag{3}$$

Following [4], we call the described transformation the *linear reflection functor* T. Beside the linear functor there were found in [4] a *hyperbolic reflection functor* S, which transforms (1) into the following decomposition:

$$\hat{I} = (1 - \alpha_1)\hat{P}_1 + (1 - \alpha_2)\hat{P}_2 + \cdots + (1 - \alpha_k)\hat{P}_k. \tag{4}$$

Leaving apart the exact formula for $\hat{P}_i$, we remark that the application of T to (3) as well as the application of S to (4) give the equation (1).

Let us denote by Φ^+ the subsequent actions of the transformations T and S and by Φ^- the action TS. The coefficient vector $\vec{\alpha}$ and the value of the sum of coefficients

$A = \sum_{i=1}^{k} \alpha_i$ are transformed by the formulas

$$\Phi^{+}(\vec{\alpha}, A) = \left((1,1,\ldots,1) - \vec{\alpha}/(A-1), k - \frac{A}{A-1}\right) \tag{5}$$

and it is correctly defined for $0 < \alpha_i < \max(1, A-1)$, $i = 1, \ldots, k$,

$$\Phi^{-}(\alpha_i, A) = \left(\frac{(1,1,\ldots,1) - \vec{\alpha}}{k-A-1}, \frac{k-A}{k-A-1}\right), \tag{6}$$

and it is correctly defined for $1 < A < n-1$, $0 < \alpha_i < 1$, $i = 1, \ldots, k$.

It will be convenient to use functions which correspond to mappings for sums of coefficients and every coefficient under Φ^- and Φ^+. Note that every coordinate of $\vec{\alpha}$, under the action of Φ^-, can be calculated by the same functional formula: $f_A^-(\alpha_i) = \frac{1-\alpha_i}{k-1-A}$. So in what follows $f_A^{-n}(\alpha_i)$ means the value of the i-th coefficient coordinate under the n-th subsequent action of the transformation Φ^- and $F^{-n}(A)$ means the value of the sum of the coefficients under the same action. Similarly the functions $f_A^+(\alpha_i) = 1 - \frac{\alpha_i}{A-1}$, $f_A^{+n}(\alpha_i)$ and $F^{+n}(A)$ mean the analogous values but under the transformation Φ^+.

In the next Lemmas we often use the number $\beta_k = \frac{k-\sqrt{k^2-4k}}{2}$ which is the main constant in the section.

Lemma 1. *For $1 < A < k-2$ and $0 < \alpha_i < 1$, the sequence $\alpha_i, f_A^-(\alpha_i), f_A^{-2}(\alpha_i), \ldots, f_A^{-n}(\alpha_i), \ldots$ tends to the number β_k/k.*

Proof. Let $0 < \alpha_i < 1$ and $0 < \alpha_j < 1$ be two different numbers. The distance $|\alpha_i - \alpha_j|$ under the transformation Φ^- becomes smaller for $k > 4$ and $A < k-2$,

$$|f_A^-(\alpha_i) - f_A^-(\alpha_j)| = |\frac{1-\alpha_i}{k-1-A} - \frac{1-\alpha_j}{k-1-A}| = \frac{1}{k-1-A}|\alpha_i - \alpha_j|.$$

Since $F^{-n}(A) \to \beta_k$ for $n \to \infty$ and $\lim_{n\to\infty} f_A^{-n}(\frac{A}{k}) = \beta_k/k$, we conclude that $f_A^{-n}(\alpha_i) \to \beta_k/k$. □

Remark 1. We have $f_A^{-2}(x) \neq f_A^-(f_A^-(x))$ in general.

Let us introduce the following function: $Z(A) = \frac{k-2-A}{k^2-3k-A(k-2)}$. It is monotone for $k > 4$, $1 < A < k-2$ and $\lim_{A\to\beta_k} Z(A) = \frac{\beta_k}{k}$. Besides, $f_A^{-2}(Z(A)) = Z(A)$.

Lemma 2. *Let $k > 4$, $\beta_k \leq A < k-2$ and $0 < \alpha < 1$.*

1. *If $\alpha_i \notin \left(Z(A), \frac{A}{k}\right)$, then $\{\alpha_i, f_A^{-2}(\alpha_i), \ldots, f_A^{-2n}(\alpha_i), \ldots\}$ is monotone.*
2. *If $\alpha_i \in \left(Z(A), \frac{A}{k}\right)$, then there exists a number $m \in \mathbb{N}$ such that $\{\alpha_i, f_A^{-2}(\alpha_i), \ldots, f_A^{-2m}(\alpha_i)\}$ is a monotone decreasing sequence and $\{f_A^{-2m}(\alpha_i), f_A^{-2(m+1)}(\alpha_i), \ldots\}$ is a monotone increasing sequence.*

Proof. The value $f_A^{-2}(\alpha_i)$ can be calculated by the formula $f_A^{-2}(\alpha_i) = \frac{k-2-A+\alpha}{k^2-3k+1-A(k-2)}$.
Case 1. Let $\alpha_i \notin \left(Z(A), \frac{A}{k}\right)$. It is easy to check that $1 > f_A^{-2}(\alpha_i) > \alpha_i$ for $\alpha_i < Z(A)$ and $\alpha_i > f_A^{-2}(\alpha_i) > 0$ for $\alpha_i > Z(A)$.

Note that for $0 < x < y < 1$, the inequality $f_A^{-2}(x) < f_A^{-2}(y)$ holds. Hence if $\alpha_i < Z(A)$, then $f_A^{-2}(\alpha_i) < f_A^{-2}(Z(A)) = Z(A)$. The sequence $A, F^-(A), F^{-2}(A), \ldots$ is decreasing. Whence $Z(A), Z(F^-(A)), Z(F^{-2}(A)), \ldots$ is increasing. By an induction argument, we obtain $\alpha_i < f_A^{-2}(\alpha_i) < f_A^{-4}(\alpha_i) < \ldots$ for $\alpha_i < Z(A)$. Also $Z(F^{-2}(A)) \leq F^{-2}(A)/k$. Therefore for $\alpha_i > A/k$, we have $\alpha_i > f_A^{-2}(\alpha_i)$, $f_A^{-2}(\alpha_i) > f_A^{-4}(\alpha_i)$ and so on.

Case 2. Let $\alpha_i \in \big(Z(A), \frac{A}{k}\big)$. To simplify the proof we consider at first $\alpha_i \in (Z(A), \frac{\beta_k}{k}]$. Since $Z(A), Z(f_A^{-2}(A)), \dots, Z(f_A^{-2n}(A)), \dots$ tends to $\frac{\beta_k}{k}$, there exists a number $n \in N$ such that $Z(f_A^{-2n}(A)) > f_A^{-2}(\frac{\beta_k}{k})$. Also $f_A^{-2}(\alpha_i) < f_A^{-2}(\frac{\beta_k}{k})$. So there exists the smallest number $m \le n$, such that $Z(F^{-2m}(A)) \ge f_A^{-2m}(\alpha_i)$. Then it follows directly from the previous case that $\{\alpha_i, f_A^{-2}(\alpha_i), \dots, f_A^{-2m}(\alpha_i)\}$ is a monotone decreasing sequence and $\{f_A^{-2m}(\alpha_i), f_A^{-2(m+1)}(\alpha_i), \dots\}$ is a monotone increasing sequence.

Let now $\alpha_i \in (\beta_k/k, A/k)$. We are going to show that there exist a number $m \in N$ such that $f_A^{-2m}(\alpha_i) < \beta_k/k$. The following parameter characterizes the rate with which the mean A/k of the coefficients approach β_k/k,

$$\begin{aligned}\varpi_1 = \frac{f_A^{-2}(A/k) - \beta_k/k}{\dfrac{A}{k} - \dfrac{\beta_k}{k}} &= \frac{\dfrac{k(k-2) - A(k-1) - \beta_k(k^2-3k+1) + A\beta_k(k-2)}{k^2-3k+1-A(k-2)}}{A-\beta_k} \\ &= \frac{(k-2)(k-k\beta_k+\beta_k^2-\beta_k^2) + \beta_k(n-1) - A(k-1) + A\beta_k(k-2)}{(A-\beta_k)(k^2-3k+1-A(k-2))} \\ &= \frac{(A-\beta_k)(\beta_k(k-2)-n+1)}{(A-\beta_k)(k^2-3k+1-A(k-2))} \\ &= \frac{1}{(k^2-3k+1-A(k-2))(k^2-3k+1-\beta_k(k-2))}.\end{aligned}$$

Also we can find the rate with which the coefficient α_i approach the mean A/k,

$$\begin{aligned}\varpi_2 = \frac{f_A^{-2}(A/k) - f_A^{-2}(\alpha_i)}{\dfrac{A}{k} - \alpha_i} &= \frac{\dfrac{k-2-A\frac{k-1}{k}}{k^2-3k+1-A(k-2)} - \dfrac{k-2-A+\alpha_i}{k^2-3k+1-A(k-2)}}{\dfrac{A}{k}-\alpha_i} \\ &= \frac{\dfrac{A}{k}-\alpha_i}{\Big(\dfrac{A}{k}-\alpha_i\Big)(k^2-3k+1-A(k-2))} = \frac{1}{k^2-3k+1-A(k-2)}.\end{aligned}$$

This gives

$$\frac{f_A^{-2m}(A/k) - \beta_k/k}{f_A^{-2m}(A/k) - f_A^{-2}(\alpha_i)} = \left(\frac{\varpi_1}{\varpi_2}\right)^m \cdot \frac{A-\beta_k}{A-k\alpha_i}. \tag{7}$$

Since $\frac{\varpi_1}{\varpi_2} < 1$, there exists $m \in \mathbb{N}$ such that the right-hand side of (7) is less than the number 1. Whence $f_A^{-2m}(\alpha_i) < \frac{\beta_k}{k}$. This completes the proof. □

Since $f_A^{-(2m+1)}(\alpha_i) = f^{-}_{F^{-2m}(A)}(f_A{}^{-2m}(\alpha_i))$, we find that $f_A^{-}(\alpha_i)$, $f_A^{-3}(\alpha_i), \dots$ has a similar monotone property as α_i, $f_A^{-2}(\alpha_i), \dots$.

Let us introduce the function

$$W(A) = \frac{k-2-A(k-3)}{k-A(k-2)}.$$

It is monotone for $k > 4$, $2 < A < k-1$ and $\lim\limits_{A\to\beta_k} W(A) = 1 - \frac{\beta_k}{k}$.

Lemma 3. *Let $k > 4$, $2 \le A \le k - \beta_k$ and $0 < \alpha < 1$.*

(1) *If $\alpha_i \notin \big(\frac{A}{k}, W(A)\big)$, then $\{\alpha_i, f_A^{+2}(\alpha_i), \dots, f_A^{+2n}(\alpha_i), \dots\}$ is monotone.*
(2) *If $\alpha_i \in \big(\frac{A}{k}, W(A)\big)$, then there exists a number $m \in \mathbb{N}$ such that $\{\alpha_i, f_A^{+2}(\alpha_i), \dots, f_A^{+2m}(\alpha_i)\}$ is a monotone increasing sequence and $\{f_A^{+2m}(\alpha_i), f_A^{+2(m+1)}(\alpha_i), \dots\}$ is a monotone decreasing sequence.*

Proof. Note that
$$\Phi^{+m}(\vec{\alpha}, A) = S\Phi^{-m}S(\vec{\alpha}, A)$$
and $\beta_k \le k - A \le k - 2$, the statement of the Lemma 3 follows from Lemma 2. □

If under the action of T (or Φ^{-n} in general) we obtain a decomposition of the identity (3) with the i-th coefficient greater than 1, then $\tilde{P}_i$ has to be equal to zero. So in order to find (1) one can try at first to find a decomposition of the identity into a linear combination of $k-1$ orthogonal projections with the coefficients $\tilde{\alpha}_1 \dots, \tilde{\alpha}_{i-1}, \tilde{\alpha}_{i+1}, \dots, \tilde{\alpha}_k$. Then adding $\tilde{\alpha}_i \cdot 0$ to the new decomposition and acting by T (or Φ^{+n}), we can get (1). To use the described argument numerically we have to define functions $f_A^{\pm m}(x)$ for the argument $x \in \mathbb{R}$. They depend on the parameter A only. So let
$$f_A^{-n}(x) := f^-_{F^{-(n-1)}(A)}(f_A^{-(n-1)}(x))$$
and write a similar formula with "pluses" for $f_A^{+n}(x)$
$$f_A^{+n}(x) := f^+_{F^{+(n-1)}(A)}(f_A^{+(n-1)}(x)).$$
Invertability of the functions $f_A^-(x)$ and $f_A^+(x)$ and statements of Lemmas 2 and 3 lead to the following theorems.

Theorem 1. *Let* $\sum_{i=1}^{k} \alpha_i P_i = I$, $A \in (\frac{k-\sqrt{k^2-4k}}{2}, 2)$ *and for some* $m \in \mathbb{N}$ *the inequality* $2 \le F_A^{+m}(A) < k-2$ *holds. If* $f_A^{+m}(\alpha_i) \notin [0,1]$ *then for some* $n \le m$, *the functor* $T\Phi^{+n}$ *transforms (1) into a decomposition of the identity with the* i*-th coefficient greater than or equal to* 1.

Theorem 2. *Let* $\sum_{i=1}^{k} \alpha_i P_i = I$, $A \in (k-2, k-\beta_k)$ *and for some* $m \in \mathbb{N}$ *the inequality* $2 \le F_A^{-m}(A) < k-2$ *holds. If* $f_A^{-m}(\alpha_i) \notin [0,1]$ *then for some* $n \le m$, *the functor* Φ^{-n} *transforms (1) into the decomposition of the identity with the* i*-th coefficient greater than or equal to* 1.

Corollary 1. *Let (1) hold for* $k \ge 5$. *If* $A = \beta_k$ *and* $\alpha_1 \ne \alpha_2$, *then by applying* S *and* T, *the decomposition (1) can be obtained from a decomposition of the identity into a linear combination of* k *orthogonal projections, where one of the orthogonal projections is zero or the identity operator.*

Proof. Since $F^+(\beta_k) = \beta_k$, we have that $f_A^{+n}(x) = f_A^+(f_A^+(\dots(f_A^+(x))\dots))$. Whence $f_A^{+n}(\alpha_1) - f_A^{+n}(\alpha_2) = \left(\frac{1}{1-\beta_k}\right)^n (\alpha_1 - \alpha_2)$. For great enough n, we have $f_A^{+n}(\alpha_1) \notin [0,1]$ or $f_A^{+n}(\alpha_2) \notin [0,1]$. So for $\vec{\alpha} \in (0,1)^k$, there exist $m \le n$ and $1 \le i \le k$ such that for every $j = 1, \dots, k$ and $s < m$, we have $f_A^{+s}(\alpha_j) \in (0,1)$ and $f_A^{+m}(\alpha_i)/(\beta_k - 1) \ge 1$. Therefore all the transformations $\Phi^+, \Phi^{+2}, \dots, \Phi^{+(m-1)}$ can be applied correctly and the orthogonal projection P_i, under the transformation Φ^{+m}, becomes the identity or the zero operator. Applying the transformation Φ^{-m} to this new decomposition we obtain (1). □

2. Necessary conditions

As was mentioned in the previous section, the transformations S and T map points of Ω_k into Ω_k. Therefore, if all the coefficients α_i coincide, then either there exists decomposition (1) or, after a number iterations of Φ^+ or Φ^-, the values of the coefficients will not be in the segment $[0,1]$, and so $\vec{\alpha} \notin \Omega_k$ [3].

The theory is more complicated for different coefficients. In [4] and [8] there were found necessary conditions for having "proper" inclusion $\vec{\alpha} \in \Omega^k$,

$$\alpha_j \leq \sum_1^k \alpha_i - 1$$

and, correspondingly,

$$\alpha_j \leq \sum_{i \neq j} \alpha_i.$$

For $\alpha_j \leq 1$, the action of T leads to $\alpha_j \to \alpha_j/(\sum_1^k \alpha_i - 1) > 1$, provided the mentioned conditions do not hold. Whence $\tilde{P}_j = 0$. It appears that there exist numbers $\alpha_1 \dots, \alpha_k$ for which $\vec{\alpha} \notin \Omega_k$ and $f_A^{\pm(n)}(\alpha_i) \in (0,1)$ for every n. We start with a technical lemma about the spectrum of a sum of nonnegative operators on a Hilbert space.

Lemma 4. *Let, for some positive $0 < \alpha_i < 1$, $i = 1, \dots, k$ and orthogonal projections $P_1, \dots, P_k$, the decomposition (1) hold. We denote by H_m the sum of the subspaces $\operatorname{Im} P_1 + \operatorname{Im} P_2 + \dots + \operatorname{Im} P_m$. If $\sum_1^m \alpha_i > m - 1$, then*

$$\alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_m P_m \geq \Big(\sum_1^m \alpha_i - m + 1 \Big) P_{H_m},$$

where P_{H_m} is the orthogonal projection onto H_m.

Proof. We carry out the proof by induction on the number m. For $m = 1$ the Lemma is true. Suppose it is true for $m = s \geq 1$. If now (1) holds and $\alpha_1 + \dots + \alpha_{s+1} > s$, then $\alpha_1 + \dots + \alpha_s > s - 1$ and

$$\alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_s P_s \geq \Big(\sum_1^s \alpha_i - s + 1 \Big) P_{H_s}.$$

Whence $(\sum_1^s \alpha_i - s + 1) P_{H_s} + \alpha_{s+1} P_{s+1} \leq I$. The spectrum of a linear combination of two orthogonal projections in a general position is symmetric around the mean value of the coefficients [6]. By this property, we have

$$\Big(\sum_1^s \alpha_i - s + 1 \Big) P_{H_s} + \alpha_{s+1} P_{s+1} \geq \Big(\sum_1^s \alpha_i - s + 1 + \alpha_{s+1} - 1 \Big) P_{H_{s+1}}.$$

Therefore,

$$\sum_1^{s+1} \alpha_i P_i \geq \Big(\sum_1^{s+1} \alpha_i - s \Big) P_{H_{s+1}}.$$

The proof is complete. □

Theorem 3. *Let $\vec{\alpha} \in \Omega_k$. If for some $m < k$ the inequality $\sum_1^m \alpha_i > m - 1$ holds and for every $j > m$, the sum $\sum\limits_1^m \alpha_i + \alpha_j > m$, then $\sum\limits_{i=m+1}^k \alpha_i \geq 1$.*

Proof. Note that the spectrum of the operator

$$\sum_{l=m+1}^k \alpha_l P_l = I - \sum_1^m \alpha_i P_i$$

is a subset of the set $\{[0, m - \sum_1^m \alpha_i], 1\}$. Since

$$\sum_{l=m+1}^k \alpha_l P_l \geq \alpha_j P_j \quad \text{and} \quad \alpha_j > m - \sum_1^m \alpha_i,$$

we see that 1 is in the spectrum of $\sum_{l=m+1}^k \alpha_l P_l$ and, hence, $\sum_{i=m+1}^k \alpha_i \geq 1$. □

Corollary 2. *For any number $0 < \epsilon < 1/9$, $\vec{\alpha} \notin \Omega_5$ where $\vec{\alpha} = (1-\epsilon, 1-\epsilon, 3\epsilon, 3\epsilon, 3\epsilon)$. Besides, $\Phi^{\pm(n)}(\alpha_i) \in (0,1)$ for every integer n.*

3. Particular points in Ω^k

In this section we consider cases where $\vec{\alpha} \in \Omega^k$ and the sum of the coefficients is an integer.

Theorem 4. *If $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in [0,1]^k$ and $\sum\limits_{i=1}^{k} \alpha_i = 2$, then $\vec{\alpha} \in \Omega^k$.*

Proof. It is sufficient to consider the case with $1 > \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$. Let us show that the numbers $\alpha_i, i = 1, \ldots, k$, can be grouped into three groups with the sum in each of them less than or equal to 1. At first we assume that $\alpha_3 + \alpha_4 + \ldots + \alpha_k \leq 1$. Whence we immediately obtain the three needed groups. If $\alpha_3 + \alpha_4 + \ldots + \alpha_k > 1$, then the equation $\sum\limits_{i=1}^{k} \alpha_i = 2$ implies the inequality $\alpha_1 + \alpha_2 < 1$. Denote $\acute{\alpha_1} = \alpha_1 + \alpha_2, \acute{\alpha_2} = \alpha_3, \ldots, \acute{\alpha_{k-1}} = \alpha_k$. For the numbers $\acute{\alpha_i}, i = 1, \ldots, k-1$, we apply the same arguments and then obtain the three needed groups or a new collection but with a fewer number of coefficients in it. At the end of this procedure we shall have three numbers less than 1 and with the sum equal to 2.

For every such collection L_j we define the sum $\beta_j = \sum\limits_{i \in L_i} \alpha_i$, $j = 1,2,3$, $L_1 \cup L_2 \cup L_3 = \{1, \ldots, k\}$, $L_1 \cap L_2 = L_2 \cap L_3 = L_1 \cap L_3 = \emptyset$.

Let $x := \beta_1 + \beta_2 - 1$ and the orthogonal projections Q_1, Q_2, Q_3 be defined by the formulas

$$Q_1 = \frac{1}{\beta_1(1-x)} \begin{pmatrix} x(1-\beta_1) & \sqrt{x(1-\beta_1)(1-\beta_2)} \\ \sqrt{x(1-\beta_1)(1-\beta_2)} & 1-\beta_2 \end{pmatrix},$$
$$Q_2 = \frac{1}{\beta_2(1-x)} \begin{pmatrix} x(1-\beta_2) & -\sqrt{x(1-\beta_1)(1-\beta_2)} \\ -\sqrt{x(1-\beta_1)(1-\beta_2)} & 1-\beta_1 \end{pmatrix},$$
$$Q_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Putting $P_i = Q_j$ for $i \in L_j$, we obtain

$$I = \beta_1 Q_1 + \beta_2 Q_2 + \beta_3 Q_3 = \sum_{1}^{k} \alpha_i P_i.$$

$\square$

Theorem 5. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in [0,1)^k$ and $\sum\limits_{i=1}^{k} \alpha_i = m$, $m \in \mathbb{N}$, then $\vec{\alpha} \in \Omega^k$.*

Proof. If $m = 1$, then the statement of the theorem is obvious. The case $m = 2$ is proved in Theorem 7. Let now $m > 2$. We use the induction on k. Suppose Theorem 7 is true for every $m = 1, 2, \ldots, k-1$ with $k = s \geq 3$. Let $\alpha_1, \ldots, \alpha_{s+1} \in [0,1)$ and $\sum_1^{s+1} \alpha_i = A \in \mathbb{N}$. Using the transformation S, it is sufficient to prove the inclusion $\vec{\alpha} \in \Omega^{s+1}$ only for $A \leq (s+1)/2$. Under the last conditions there are two coefficients, say α_s and α_{s+1}, such that $\alpha_s + \alpha_{s+1} \leq 1$. If $\alpha_s + \alpha_{s+1} = 1$, then putting $P_s = P_{s+1} = I$ and the other projections are zero, we obtain the needed decomposition. We consider now the case $\alpha_s + \alpha_{s+1} < 1$. Defining $P_{s+1} = P_s$, we obtain a new problem

$$\sum_{1}^{s-1} \alpha_i P_i + (\alpha_s + \alpha_{s+1}) P_s = I,$$

but the number of orthogonal projections in the decomposition is equal to s. Since $A \leq (s+1)/2 \leq s-1$ for $s \geq 3$, by the induction assumption, such a problem has a solution. This completes the proof. $\square$

4. Open sets in Ω^k

In this section we present decompositions of the identity in the case where the sum of the coefficients is close to an integer or when they are almost the same.

In the following formulas, $\gamma = a+b-x$. Let us define two orthogonal projections

$$P(a,b,x) = \frac{1}{a(\gamma - x)} \begin{pmatrix} x(b-x) & \sqrt{x\gamma(a-x)(b-x)} \\ \sqrt{x\gamma(a-x)(b-x)} & \gamma(a-x) \end{pmatrix}, \tag{8}$$

$$Q(a,b,x) = \frac{1}{b(\gamma - x)} \begin{pmatrix} x(a-x) & -\sqrt{x\gamma(a-x)(b-x)} \\ -\sqrt{x\gamma(a-x)(b-x)} & \gamma(b-x) \end{pmatrix}. \tag{9}$$

Direct calculations show that

$$aP(a,b,x) + bQ(a,b,x) = \operatorname{diag}(x, a+b-x) \tag{10}$$

and, for $x \in [0, \min(a,b)] \cup [\max(a,b), a+b]$, the orthogonal projections $P(a,b,x)$ and $Q(a,b,x)$ are correctly defined.

Lemma 5. *Let a real number ϵ and a vector $\vec{\alpha} \in \mathbb{R}^5$ be such that $0 < \epsilon < \alpha_5 \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1 < 1-\epsilon$, $\alpha_1 + \alpha_2 > 1 - \epsilon$, and $\alpha_1 + \cdots + \alpha_5 = 2 + \epsilon$. Then the algebra $\mathcal{P}_{5,\vec{\alpha}}$ has an irreducible $*$-representation in the matrix algebra $M_3(\mathcal{Q}_2)$ over the Cunts algebra $\mathcal{Q}_2$.*

Proof. Let us denote by p_{ij} the entries of the matrix $P(\alpha_1, \alpha_2, 1-\epsilon)$ obtained by the formula (8). If S_1 and S_2 are standard generators in Cunts algebra $\mathcal{Q}_2$ satisfied the relations $S_1^* S_2 = 0$ and $S_1^* S_1 = I = S_2^* S_2 = S_1 S_1^* + S_2 S_2^*$, then the block matrix operators P_1 and P_2

$$P_1 = \begin{pmatrix} p_{11}I & 0 & p_{12}S_1^* \\ 0 & p_{11}I & p_{12}S_2^* \\ p_{21}S_1 & p_{21}S_2 & p_{22}I \end{pmatrix}, \quad P_2 = \begin{pmatrix} p_{11}I & 0 & -p_{12}S_1^* \\ 0 & p_{11}I & -p_{12}S_2^* \\ -p_{21}S_1 & p_{21}S_2 & p_{22}I \end{pmatrix}$$

are orthogonal projections and

$$\alpha_1 P_1 + \alpha_2 P_2 = \operatorname{diag}\left((1-\epsilon)I, (1-\epsilon)I, (\alpha_1 + \alpha_2 - 1 + \epsilon)I\right). \tag{11}$$

Beside, by (10),

$$\alpha_3 P(\alpha_3, \alpha_5, \epsilon) + \alpha_5 Q(\alpha_3, \alpha_5, \epsilon) = \operatorname{diag}(\epsilon, \alpha_3 + \alpha_5 - \epsilon) \tag{12}$$

and, for $\tilde{\alpha}_4 = \alpha_3 + \alpha_5 - \epsilon$,

$$\alpha_4 P(\alpha_4, \tilde{\alpha}_4, \epsilon) + \tilde{\alpha}_4 Q(\alpha_4, \tilde{\alpha}_4, \epsilon) = \operatorname{diag}\left(\epsilon, \sum_3^5 \alpha_i - 2\epsilon\right). \tag{13}$$

Since $Q(\alpha_4, \tilde{\alpha}_4, \epsilon)$ is an orthogonal projection, there exists a unitary 2×2 matrix U such that $U^* Q(\alpha_4, \tilde{\alpha}_4, \epsilon) U = \operatorname{diag}(1,0)$. Let us define now the orthogonal projections P_3, P_4 and P_5 as follows:

$$\begin{aligned} P_3 &:= \left(\operatorname{diag}(1,U)\operatorname{diag}(P(\alpha_3,\alpha_5,\epsilon),1)\operatorname{diag}(1,U^*)\right) \otimes I, \\ P_4 &:= \operatorname{diag}(1, P(\alpha_4, \tilde{\alpha}_4, \epsilon)) \otimes I, \\ P_5 &:= \left(\operatorname{diag}(1,U)\operatorname{diag}(1,Q(\alpha_3,\alpha_5,\epsilon))\operatorname{diag}(1,U^*)\right) \otimes I, \end{aligned} \tag{14}$$

Thus we have the equality

$$\alpha_3 P_3 + \alpha_4 P_4 + \alpha_5 P_5 = \operatorname{diag}\left(\epsilon I, \epsilon I, \Big(\sum_3^5 \alpha_i - 2\epsilon\Big) I\right).$$

In view of (11), we obtain (1) for $k = 5$ and the orthogonal projections $P_i \in M_3(\mathcal{Q}_2)$. The irreducibility of the such constructed representation of the algebra $\mathcal{P}_{5,\vec{\alpha}}$ is followed from the irreducibility of the triple P_3, P_4 and P_5. □

Lemma 6. *Let a real number ϵ and a vector $\vec{\alpha} \in \mathbb{R}^5$ be such that $0 < \epsilon < \alpha_5 \le \alpha_4 \le \alpha_3 \le \alpha_2 \le \alpha_1 < 1 - \epsilon$, $\alpha_1 + \alpha_2 \le 1 - \epsilon$ and $\alpha_1 + \dots + \alpha_5 = 2 + \epsilon$. Then $\vec{\alpha} \in \Omega^5$.*

Proof. Let us define four sequences of nonnegative numbers x_i, y_i, s_i and p_i by the rule

$$\begin{aligned}
&x_1 = 1, \\
&x_i = 1 + y_{i-1} - \alpha_3 - \alpha_4, \quad y_i = x_i + 1 - \alpha_1 - \alpha_2 - (s_i + p_i)\alpha_5, \\
&s_i = \begin{cases} 1, \text{ if } \max(1 - \alpha_5 + \epsilon, \alpha_2 + \alpha_5) < x_i, \\ 0, \text{ otherwise}, \end{cases} \\
&p_i = \begin{cases} 1, \text{ if } 1 - \alpha_5 + \epsilon < x_i \le \alpha_2 + \alpha_5, \\ 0, \text{ otherwise}. \end{cases}
\end{aligned} \tag{15}$$

We consider now operators P_i, $i = 1, \dots, 5$, of the following form:

$$\begin{aligned}
P_1 &= \operatorname{diag}\left(P(\alpha_1, \alpha_2 + s_1\alpha_5, x_1), P(\alpha_1, \alpha_2 + s_2\alpha_5, x_2), \dots\right), \\
P_2 &= \operatorname{diag}\left(Q(\alpha_1, \alpha_2 + s_1\alpha_5, x_1), Q(\alpha_1, \alpha_2 + s_2\alpha_5, x_2), \dots\right), \\
P_3 &= \operatorname{diag}\left(0, P(\alpha_3, \alpha_4, y_1), P(\alpha_3, \alpha_4, y_2), \dots\right), \\
P_4 &= \operatorname{diag}\left(0, Q(\alpha_3, \alpha_4, y_1), Q(\alpha_3, \alpha_4, y_2), \dots\right), \\
P_5 &= P_2 \operatorname{diag}\left(s_1, s_1, s_2, s_2, s_3, s_3, \dots\right) + \operatorname{diag}\left(0, p_1, 0, p_2, \dots\right).
\end{aligned}$$

By (11), we obtain the linear combination

$$\begin{aligned}
\alpha_1 P_1 + \alpha_2 P_2 + \alpha_5 P_5 = {} & \operatorname{diag}\left(1, \alpha_1 + \alpha_2 + (s_1 + p_1)\alpha_5 - x_1\right) \\
& \oplus \operatorname{diag}\left(x_2, \alpha_1 + \alpha_2 + (s_2 + p_2)\alpha_5 - x_2, \dots\right)
\end{aligned}$$

and the linear combination

$$\alpha_3 P_3 + \alpha_4 P_4 = \operatorname{diag}\left(0, y_1, \alpha_3 + \alpha_4 - y_1, y_2, \alpha_3 + \alpha_4 - y_2, \dots\right).$$

Let us prove that orthogonal projections $P_1, \dots, P_5$ are correctly defined. In order to show this, it is sufficient to prove the inequality

$$1 - \alpha_5 \le x_i \le 1 \tag{16}$$

and then we will have

$$\alpha_3 + \alpha_4 \ge y_i = x_i + 1 - \alpha_1 - \alpha_2 \ge x_i + \epsilon \ge \alpha_3$$

for $x_i \le 1 - \alpha_5 + \epsilon$ or

$$\alpha_3 + \alpha_4 \ge y_i = x_i + 1 - \alpha_1 - \alpha_2 - \alpha_5 \ge 2 + \epsilon - (\alpha_1 + \dots + \alpha_5) + \alpha_3 \ge \alpha_3$$

for $x_i > 1 - \alpha_5 + \epsilon$.

By definition, y_{i-1} is a function of x_{i-1} and we can substitute it, instead of y_{i-1}, into the expression (15) for x_i. Whence we obtain a recurrence relation for the numbers x_1, x_2, $x_3, \dots$,

$$x_i = 1 + x_{i-1} + 1 - \alpha_1 - \alpha_2 - (s_{i-1} + p_{i-1})\alpha_5 - \alpha_3 - \alpha_4 = x_{i-1} - \epsilon + (1 - s_{i-1} - p_{i-1})\alpha_5.$$

So we have another expression for x_i,

$$x_i = \begin{cases} x_{i-1} - \epsilon, & \text{if} \quad x_{i-1} > 1 - \alpha_5 + \epsilon, \\ x_{i-1} - \epsilon + \alpha_5, & \text{if} \quad x_{i-1} \le 1 - \alpha_5 + \epsilon. \end{cases}$$

It is easy to see now that x_i satisfies (16) for $x_{i-1} \in [1 - \alpha_5, 1]$. □

Lemma 7. *Let $k > 4$, $0 < \alpha_i < 1$, $i = 1, 2, \dots, k$ and $\sum_1^k \alpha_i = 2$. There exists $\epsilon > 0$ such that for every k and real numbers $\beta_1, \dots, \beta_k$ satisfying the inequality $|\alpha_i - \beta_i| < \epsilon$, we have $\vec{\beta} \in \Omega^k$. Besides, for $k > 5$ the algebra $\mathcal{P}_{k,\vec{\alpha}}$ has an irreducible $*$-representation in the matrix algebra $M_3(\mathcal{Q}_2)$.*

Proof. We put $\gamma = \frac{1}{2k}\min(\alpha_1, 1-\alpha_1, \dots, \alpha_k, 1-\alpha_k)$.

Let $k = 5$. Suppose $\vec{\beta}$ satisfies the conditions of the lemma for $\epsilon = \gamma$ and $B = \beta_1 + \dots + \beta_5 > 2$. Then

$$B - 2 \le |\alpha_1 - \beta_1| + \dots + |\alpha_5 - \beta_5| < 5\epsilon < \frac{1}{2}\min(\alpha_i)$$

and, by Lemmas 5 and 6, the needed decomposition exists.

Let now $\vec{\beta}$ satisfies the conditions of the lemma for $\epsilon = \gamma/11$ and $B < 2$. We note that $\tilde{\beta} = (\beta_1/(B-1), \beta_2/(B-1), \dots, \beta_k/(B-1)) \in \Omega^k$, because $\sum_1^k \tilde{\beta}_i = B/(B-1) > 2$ and

$$|\alpha_i - \tilde{\beta}_i| = |(\alpha_i - \beta_i) + (\beta_i - \tilde{\beta}_i)| < \epsilon + \frac{1}{1-5\epsilon} < \gamma,$$

and this case has been proved in the previous paragraph. So there exists the decomposition

$$\tilde{I} = \tilde{\beta}_1\tilde{P}_1 + \tilde{\beta}_2\tilde{P}_2 + \dots + \tilde{\beta}_5\tilde{P}_5. \tag{17}$$

Applying the transformation T to (17) we conclude that $\vec{\beta} \in \Omega^5$.

Let $k > 5$. Using the transformation T it is sufficient to consider the case $A > 2$. Let $\epsilon = \gamma$ and $\vec{\beta}$ satisfy the conditions of the lemma. Without lost of generality one can suppose that β_i are arranged as follows: $\beta_1 \ge \beta_2 \ge \dots \ge \beta_k$. We are going to find a decomposition of the matrix identity,

$$\text{diag}\,(I, I, I) = \beta_1 Q_1 + \beta_2 Q_2 + \dots + \beta_k Q_k, \tag{18}$$

directly using the construction from Lemma 5. Let $\hat{\beta} = (\beta_1 + \beta_2 + \dots + \beta_{k-3})/2$. For the vector $\vec{\beta'} = (\hat{\beta}, \hat{\beta}, \beta_{k-2}, \beta_{k-1}, \beta_k)$, there exists a representation of $\mathcal{P}_{5,\vec{\beta'}}$ in $M_3(\mathcal{Q}_2)$. We set $Q_{k-2} = P_3$, $Q_{k-1} = P_4$ $Q_k = P_5$, where P_i are calculated according to (14) for the vector $\vec{\beta'}$.

There exists the greatest number m, which is less than $k-2$, such that $\sum_1^m \beta_i < 1-\epsilon$. Assume at first that $\sum_1^{m+1} \beta_i > 1-\epsilon$. Putting $P_1 = P_2 = \dots = P_{m-1} = \text{diag}\,(1,0)$,

$$P_m = P(\beta_m, \beta_{m+1}, x), \quad P_{m+1} = Q(\beta_m, \beta_{m+1}, x), \tag{19}$$

where $x = 1 - \epsilon - \sum_1^{m-1} \beta_i$ and $P_{m+2} = P_{m+3} = \dots = P_{k-3} = \text{diag}\,(0,1)$, we obtain

$$\sum_1^{m-3} \beta_i P_i = \text{diag}\left(1-\epsilon, \sum_1^{m-3} \beta_i - 1 + \epsilon\right) \tag{20}$$

As in Lemma 5 the orthogonal projections $Q_1, Q_2, \dots$ are constructed in the block matrix form,

$$Q_s = \begin{pmatrix} p_{11}^s I & 0 & p_{12}^s S_1^* \\ 0 & p_{11}^s I & p_{12}^s S_2^* \\ p_{21}^s S_1 & p_{21}^s S_2 & p_{22}^s I \end{pmatrix}, \tag{21}$$

where p_{ij}^s is the entry of P_s and $s = 1, \dots, k-3$. For the matrices Q_i, the decomposition (18) holds.

Assume secondly that $\sum_1^{m+1} \beta_i = 1 - \epsilon$. Let $0 < \delta << \epsilon$ and $\tilde{\beta}_1 = \sum_1^m \beta_i$. There exists a unitary matrix U such that

$$U^* P(1-\epsilon-2\delta, \beta_{m+2}, 1-\epsilon)U = \text{diag}\,(1,0)\,.$$

We define P_i by the equalities

$$P_i = UP(\tilde{\beta}_1, \beta_{m+1}, 1-\epsilon-\delta)U^*, \quad P_{m+1} = UQ(\tilde{\beta}_1, \beta_{m+1}, 1-\epsilon-\delta)U^*,$$
$$P_{m+2} = Q(1-\epsilon-2\delta, \beta_{m+2}, 1-\epsilon) \quad \text{and} \quad P_{m+3} = \dots = P_{k-3} = \text{diag}\,(0,1)\,.$$

The equation (20) is verified and, hence, using formula (21) we obtain the decomposition (18).

We remark that for both constructions, the matrix P_{m+1} is not a diagonal matrix. Therefore irreducibility of the such constructed representations follows from irreducibility of the triple Q_{k-2}, Q_{k-1} and Q_k. $\square$

Lemma 8. *Let $\alpha_i \in [2.49, 2.51]$, $i = 1, \dots, 5$. Then $\vec{\alpha} \in \Omega^5$.*

Proof. Let $A = \alpha_1 + \dots + \alpha_5 \leq 2.5$. We assume that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_5$. If $\alpha_1 = \alpha_5$, then $\vec{\alpha} \in \Omega^5$, since the scalar operator I/α_1 is a sum of five orthogonal projections [3].

So let $\alpha_1 > \alpha_2$. There are two cases for the sum $B = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $B < 2$ and $B \geq 2$.

1. $B < 2$. In this case the inequality $A - 2 < \alpha_5$ holds. So by Lemmas 5 and 6, $\vec{\alpha} \in \Omega^5$.

2. $B \geq 2$. If $B = 2$, then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Omega^4$ by Theorem 4. Whence $\vec{\alpha} \in \Omega^5$. Let now $B > 2$. We define two sequences of real numbers x_i and p_i by $x_1 = 0$, $x_i = x_{i-1} + p_{i-1}(\alpha_5 - \alpha_1) + B - 2$, $p_i = \begin{cases} 1, & \text{if } x_i > 1/5, \\ 0, & \text{otherwise.} \end{cases}$

The needed operators P_i, $i = 1, \dots, 5$ have the following form:

$$P_1 = R \operatorname{diag}\left(1 - s_1, 1 - s_1, 1 - s_2, 1 - s_2, 1 - s_3, 1 - s_3, \dots\right),$$
$$P_5 = R \operatorname{diag}\left(s_1, s_1, s_2, s_2, s_3, s_3, \right),$$

where

$$R = \operatorname{diag}\left(P(\alpha_1 + p_1(\alpha_5 - \alpha_1), \alpha_2, 1 - x_1), P(\alpha_1 + p_2(\alpha_5 - \alpha_1), \alpha_2, 1 - x_2), \dots\right),$$
$$P_2 = \operatorname{diag}\left(Q(\alpha_1 + p_1(\alpha_5 - \alpha_1), \alpha_2, 1 - x_1), Q(\alpha_1 + p_2(\alpha_5 - \alpha_1), \alpha_2, 1 - x_2), \dots\right),$$
$$P_3 = 0 \oplus P(\alpha_3, \alpha_4, 2 - \alpha_1 - \alpha_2 - p_1(\alpha_5 - \alpha_1) - x_1)$$
$$\oplus P(\alpha_3, \alpha_4, 2 - \alpha_1 - \alpha_2 - p_2(\alpha_5 - \alpha_1) - x_2) \oplus \cdots,$$
$$P_4 = 0 \oplus Q(\alpha_3, \alpha_4, 2 - \alpha_1 - \alpha_2 - p_1(\alpha_5 - \alpha_1) - x_1)$$
$$\oplus Q(\alpha_3, \alpha_4, 2 - \alpha_1 - \alpha_2 - p_2(\alpha_5 - \alpha_1) - x_2) \oplus \cdots.$$

A direct calculation shows that (1) holds. Note that

$$0 \leq x_i \leq 1/5 + (B - 2) \leq 0.2 + 0.04 = 0.24.$$

So the orthogonal projections $P_1, \dots, P_5$ are correctly defined and $\vec{\alpha} \in \Omega^5$.

To complete the proof it remains to consider linear combinations with the sum $\alpha_1 + \dots + \alpha_5 > 2.5$. Since by the first part of the proof, $(1 - \alpha_1, \dots, 1 - \alpha_5) \in \Omega^5$, it follows that there exists a decomposition of identity into a linear combination of orthogonal projections, say, $I = (1 - \alpha_1)R_1 + (1 - \alpha_1)R_2 + (1 - \alpha_1)R_5$. Using the transformation S to it, we obtain a new decomposition with the coefficients $(\alpha_1, \dots, \alpha_5)$. $\square$

We now can use the ideas from the proof of Theorem 5 in order to show that neighborhoods of vectors with the sum of coordinates equal to an integer are lying in Ω^k.

Theorem 6. *Let $k > 4$, $m \in \mathbb{N}$, $m \in [2, k-2]$, $\alpha_i \in (0, 1)$, $i = 1, 2, \dots, k$, and $\sum_1^k \alpha_i = m$. There exists $\epsilon > 0$, which depends on $\vec{\alpha}$, such that every vector $\vec{\beta}$ with the differences $|\alpha_i - \beta_i| < \epsilon$ lies in Ω^k.*

Proof. For $k = 5$ the statement of the theorem follows from Lemmas 5, 6. Suppose that the theorem is true for every $k \leq k_0$, k_0 being fixed, and for every $m \in [2, k-2]$. For any real numbers $\alpha_1, \dots, \alpha_{k_0+1}$ from the interval $(0, 1)$ with the sum $\sum_1^{k_0+1} \alpha_i = s \in \mathbb{N}$, it suffices to prove the theorem only for the values of s satisfying $s \leq (k+1)/2$. Then either all the coefficients are equal to $1/2$, and the theorem is true for them due to Lemma 8, or the sum of two of them, say $\alpha_k + \alpha_{k+1}$ is less than 1. In the latter case, for the new coefficients $\vec{\alpha}' = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k + \alpha_{k+1})$, the statement of the Theorem 6 is valid by assumption, i.e., there exists $\epsilon(\vec{\alpha}') > 0$ such that for every vector $\vec{\gamma} \in \mathbb{R}^k$ with $|\alpha_i - \gamma_i| < \epsilon(\vec{\alpha}')$, $i = 1, \ldots, k-1$, $|\alpha_k + \alpha_{k+1} - \gamma_k| < \epsilon(\vec{\alpha}')$, we have $\vec{\gamma} \in \Omega^k$. Putting $\epsilon(\vec{\alpha}) = \epsilon(\vec{\alpha}')/2$, we conclude that the decomposition

$$I = \beta_1 P_1 + \beta_2 P_2 + \cdots + \beta_{k+1} P_{k+1}$$

exists for every $\vec{\beta}$ satisfying the inequality $|\alpha_i - \beta_i| < \epsilon(\vec{\alpha})$, $i = 1, \ldots, k+1$ even under the additional restriction that $P_k = P_{k+1}$. $\square$

Theorem 7. *For every $A \in (\beta_k, k - \beta_k)$, $k \geq 5$, there exists $\varepsilon > 0$ such that every vector $\vec{\alpha}$ with $|\alpha_i - A/k| < \varepsilon$, $i = 1, \ldots, k$, is lying in Ω^k.*

Proof. If we prove the theorem for $A \in [2, k/2]$, then by applying the transformations S, Φ^- and Φ^+ we prove the theorm for every $A \in (\beta_k, k - \beta_k)$.

So let $A \in [2, k/2]$ and $k = 5$. The cases $A = 2$ and $A = 2.5$ were proved in Lemma 7 and Lemma 8, correspondingly. For $2 < A < 2.5$, we put $\epsilon = \min(1/100, (2.5 - A)/10, (A-2)/10)$. Let $\vec{\alpha} \in \mathbb{R}^5$ and $|\alpha_i - A/k| < \varepsilon$, $i = 1, \ldots, 5$. Without loss of generality we can assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Then $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 5A/4 + 4\epsilon < 2$. So by Lemma 5 and Lemma 6, $\alpha \in \Omega^5$.

Let now $k > 5$ and $A \in [2, k/2]$. If $A = k/2$, then putting $\epsilon = 1/100$, we obtain the following condition on α_i: $|\alpha_i - 1/2| < 1/100$. Using Lemma 8, we conclude that $\alpha \in \Omega^k$.

Let $A < k/2$. There exists an integer number m such that $(m-1)A/k \leq 2 < mA/k$. Let $\varepsilon = \min(A/100k, (A - k/2)/100k, (mA/k - 2)/2k)$. We assume that $\vec{\alpha} \in \mathbb{R}^k$, $|\alpha_i - A/k| < \varepsilon$ and the following arrangement of the coordinates holds: $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. The vector $\vec{\alpha}' = (\alpha_1, \ldots, \alpha_m)$ is in Ω^m by Lemma 7 because $2 < mA/k - m\varepsilon < \sum_i^m \alpha_i < mA/k + m\varepsilon < 2 + \alpha_m$. So $\alpha \in \Omega^k$ in this case too. $\square$

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