

INVERSE THEOREMS IN THE THEORY OF APPROXIMATION OF VECTORS IN A BANACH SPACE WITH EXPONENTIAL TYPE ENTIRE VECTORS

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ABSTRACT. An arbitrary operator A on a Banach space \mathfrak{X} which is a generator of a C_0 -group with a certain growth condition at infinity is considered. A relationship between its exponential type entire vectors and its spectral subspaces is found. Inverse theorems on the connection between the degree of smoothness of a vector $x \in \mathfrak{X}$ with respect to the operator A , the rate of convergence to zero of the best approximation of x by exponential type entire vectors for operator A , and the k -module of continuity with respect to A are established. Also, a generalization of the Bernstein-type inequality is obtained. The results allow to obtain Bernstein-type inequalities in weighted L_p spaces.

1. INTRODUCTION

Direct and inverse theorems which establish the relationship between the degree of smoothness of a function with respect to a differentiation operator and the rate of convergence to zero of its best approximation by trigonometric polynomials are well known in the theory of approximation of periodic functions. Bernstein's and Jackson's inequalities are ones among such results.

N. P. Kuptsov proposed a generalized notion of the module of continuity, expanded onto C_0 -groups in a Banach space [1]. Using this notion, A. P. Terekhin [2] proved generalized Bernstein-type inequalities for the cases of a bounded group and an s -regular group. Recall that a group $\{U(t)\}_{t \in \mathbb{R}}$ is called s -regular if the resolvent of its generator A satisfies the following condition: $\exists \theta \in \mathbb{R} : \|R_\lambda(e^{i\theta} A^s)\| \leq \frac{C}{\operatorname{Im} \lambda}$.

M. L. Gorbachuk and V. I. Gorbachuk proposed to use entire vectors of some operator as basic approximation objects and constructed [3, 4] a general operator approach to direct and inverse theorems.

G. V. Radzievsky studied direct and inverse theorems [5, 6], using the notion of a K -functional (it should be noted that a K -functional has two-sided estimates with regard to the module of continuity at least for bounded C_0 -groups).

In [7], the authors investigated groups of unitary operators in Hilbert space and established Bernstein-type and Jackson-type inequalities. These inequalities are used to estimate the rate of convergence to zero of the best approximation of both finite and infinite smoothness vectors for the operator A by exponential type entire vectors.

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We consider C_0 -groups in the Banach space, generated by so-called *non-quasianalytic operators* [8], i.e., the groups satisfying

$$(1.1) \quad \int_{-\infty}^{\infty} \frac{\ln \|U(t)\|}{1+t^2} dt < \infty.$$

We recall that belonging of a group to the C_0 -class means that for every $x \in \mathfrak{X}$ the vector-valued function $U(t)x$ is continuous on \mathbb{R} with respect to the norm of the space \mathfrak{X} .

As it was shown in [3], the set of exponential type entire vectors for a non-quasianalytic operator A is dense in \mathfrak{X} , so the problem of approximation by exponential type entire vectors is correct. On the other hand, it was shown in [9] that condition (1.1) is close to the necessary one, that is, in the case when (1.1) doesn't hold, the class of entire vectors isn't necessarily dense in \mathfrak{X} , and the corresponding approximation problem loses its meaning.

In [10], generalized Jackson-type inequalities for approximation by entire vectors of exponential type of non-quasianalytic operators are established. The purpose of this paper is to obtain Bernstein-type inequalities and an analogue of an inverse theorem for such approximations, and to give some applications of these results to weighted L_p spaces. In order to do this, it is proved that the set of exponential type entire vectors of type not exceeding some $\sigma > 0$ coincides with some spectral subspace of a non-quasianalytic operator (constructed in [8]), and the well-developed technique for spectral subspaces is used. The last result (coincidence of the two sets of vectors) improves the embedding, established in [3].

2. PRELIMINARIES

Let A be a closed linear operator with dense domain of definition, $\mathcal{D}(A)$, on a Banach space $(\mathfrak{X}, \|\cdot\|)$ over the field of complex numbers.

Let $C^\infty(A)$ denote the set of all infinitely differentiable vectors of the operator A , i.e.,

$$C^\infty(A) = \bigcap_{n \in \mathbb{N}_0} \mathcal{D}(A^n), \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

For a number $\alpha > 0$ we set

$$\mathfrak{E}^\alpha(A) = \{x \in C^\infty(A) \mid \exists c = c(x) > 0 \forall k \in \mathbb{N}_0 \ \|A^k x\| \leq c\alpha^k\}.$$

The set $\mathfrak{E}^\alpha(A)$ is a Banach space with respect to the norm

$$\|x\|_{\mathfrak{E}^\alpha(A)} = \sup_{n \in \mathbb{N}_0} \frac{\|A^n x\|}{\alpha^n}.$$

Then $\mathfrak{E}(A) = \bigcup_{\alpha > 0} \mathfrak{E}^\alpha(A)$ is a linear locally convex space with respect to the topology of the inductive limit of the Banach spaces $\mathfrak{E}^\alpha(A)$,

$$\mathfrak{E}(A) = \lim_{\alpha \rightarrow \infty} \text{ind } \mathfrak{E}^\alpha(A).$$

Elements of the space $\mathfrak{E}(A)$ are called [11] exponential type entire vectors of the operator A . The type $\sigma(x, A)$ of a vector $x \in \mathfrak{E}(A)$ is defined as the number

$$\sigma(x, A) = \inf \{\alpha > 0 : x \in \mathfrak{E}^\alpha(A)\} = \limsup_{n \rightarrow \infty} \|A^n x\|^{\frac{1}{n}}.$$

Denote by $\Xi^\alpha(A)$ the following set:

$$(2.1) \quad \Xi^\alpha(A) = \{x \in \mathfrak{E}(A) \mid \sigma(x) \leq \alpha\}.$$

It is easy to see that

$$(2.2) \quad \mathfrak{E}^\alpha(A) \subset \Xi^\alpha(A) = \bigcap_{\epsilon > 0} \mathfrak{E}^{\alpha+\epsilon}(A).$$

Example 1. Let \mathfrak{X} be one of $L_p(2\pi)$ – spaces ($1 \leq p < \infty$) of p -th degree integrable in over $[0, 2\pi]$, 2π -periodic functions or the space $C(2\pi)$ of continuous 2π -periodic functions (the norm is defined in a standard way), and let A be the differentiation operator on the space \mathfrak{X} , $\mathcal{D}(A) = \{x \in \mathfrak{X} \cap AC(\mathbb{R}) : x' \in \mathfrak{X}\}$; $(Ax)(t) = \frac{dx}{dt}$, where $AC(\mathbb{R})$ denotes the space of absolutely continuous functions over \mathbb{R} . It can be proved that in such a case the space $\mathfrak{E}(A)$ coincides with the space of all trigonometric polynomials, and $\sigma(y, A) = \deg(y)$ for $y \in \mathfrak{E}(A)$, where $\deg(y)$ is the degree of the trigonometric polynomial y .

Note that all previous definitions do not change if we replace the operator A by any operator of the form $e^{i\vartheta} A$, $\vartheta \in \mathbb{R}$. Moreover, the main results of this article, which are theorems 2 and 3, do not depend on which operator generates the group $U(t)$, either A or iA . So, in what follows, we always assume that the operator iA is a generator of the C_0 -group of linear continuous operators $\{U(t) : t \in \mathbb{R}\}$ [12] on \mathfrak{X} . Moreover, we suppose that the operator A is non-quasianalytic.

For $t \in \mathbb{R}_+$, we set

$$(2.3) \quad M_U(t) := \sup_{\tau \in \mathbb{R}, |\tau| \leq t} \|U(\tau)\|.$$

The estimation $\|U(t)\| \leq M e^{\omega t}$ for some $M, \omega \in \mathbb{R}$ implies $M_U(t) < \infty$ (for all $t \in \mathbb{R}_+$). It is easy to see that the function $M_U(\cdot)$ has the following properties:

- 1) $M_U(t) \geq 1, t \in \mathbb{R}_+$;
- 2) $M_U(\cdot)$ is monotonically non-decreasing on \mathbb{R}_+ ;
- 3) $M_U(t_1 + t_2) \leq M_U(t_1)M_U(t_2), t_1, t_2 \in \mathbb{R}_+$.

According to [1], for $x \in \mathfrak{X}, t \in \mathbb{R}_+$ and $k \in \mathbb{N}$, we set as a generalization of the module of smoothness,

$$(2.4) \quad \omega_k(t, x, A) = \sup_{0 \leq \tau \leq t} \|\Delta_\tau^k x\|, \quad \text{where}$$

$$(2.5) \quad \Delta_h^k = (U(h) - \mathbb{I})^k = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} U(jh), \quad k \in \mathbb{N}_0, \quad h \in \mathbb{R} \quad (\Delta_h^0 \equiv 1).$$

For an arbitrary $x \in \mathfrak{X}$, according to [4, 7], the best approximation by exponential type entire vectors y of an operator A for which $\sigma(y, A) \leq r$ is defined as

$$\mathcal{E}_r(x, A) = \inf_{y \in \Xi^r(A)} \|x - y\|, \quad r > 0.$$

For fixed x , $\mathcal{E}_r(x, A)$ does not increase and $\mathcal{E}_r(x, A) \rightarrow 0, r \rightarrow \infty$, for every $x \in \mathfrak{X}$ if and only if the set $\mathfrak{E}(A)$ of exponential type entire vectors is dense in \mathfrak{X} . Particularly, as indicated above, the set $\mathfrak{E}(A)$ is dense in \mathfrak{X} if the operator A generates a C_0 -group $U(t)$ and this group belongs to the non-quasianalytic class (that is, it satisfies (1.1)).

3. SPECTRAL SUBSPACES OF NON-QUASIANALYTIC OPERATORS

The main instrument for proving a generalized Bernstein inequality is the theory of spectral subspaces of non-quasianalytic operator A constructed in [8]. Recall that spectral subspaces (denoted by $\mathcal{L}(\Delta)$) are defined for all segments $\Delta \subset \mathbb{R}$ and are characterized by the following properties [8, p. 446]:

- 1) the operator A is defined on whole $\mathcal{L}(\Delta)$ and is bounded on it;
- 2) $\mathcal{L}(\Delta)$ is invariant with respect to A ;
- 3) the spectrum of $\mathcal{L}(\Delta)$ -induced part A_Δ of the operator A , consists of the intersection of the spectrum of A with the interior of the segment Δ and, perhaps, the endpoints of the segment Δ . And at that, if an endpoint of the segment Δ

does not belong to the spectrum of A , it does not belong to the spectrum of A_Δ either;

- 4) if there is some subspace \mathcal{L} on which the operator A is defined everywhere and is bounded, and this subspace is invariant with respect to A , and at the same time the spectrum of the \mathcal{L} -induced part of A is included in Δ , then $\mathcal{L} \subset \mathcal{L}(\Delta)$.

Now we describe the construction of spectral subspaces and their main properties, and later prove a relationship with entire vectors of exponential type. Let $\theta(t)$ ($-\infty < t < \infty$) be an entire function of order 1 with zeroes on the positive imaginary ray,

$$(3.1) \quad \theta(t) = C \prod_{k=1}^{\infty} \left(1 - \frac{t}{it_k}\right), \quad \text{where } 0 < t_1 \leq t_2 \leq \dots, \quad \sum_{k=1}^{\infty} \frac{1}{t_k} < \infty,$$

and C be a constant. Note that $|\theta(t)|$ satisfies the conditions $|\theta(t_1 + t_2)| \leq |\theta(t_1)| \cdot |\theta(t_2)|$, $t_1, t_2 \in \mathbb{R}$ and $\int_{-\infty}^{\infty} \frac{|\ln(\alpha(t))|}{1+t^2} dt < \infty$, i.e. it belongs to \mathfrak{Q} (for a definition of the class \mathfrak{Q} , see [10]).

Denote by $E_\theta^{(\infty)}$ the class of entire functions $\phi(t)$ of finite type and order 1 which satisfy, for all $m = 0, 1, \dots$ and for all $a > 0$, the condition

$$(3.2) \quad M_\theta^{(m,a)}(\phi) := \int_{-\infty}^{\infty} |t^m \theta(at) \phi(t)| dt < \infty.$$

As shown in [8, Lemma 1.1.1], the Fourier transform of a functions from $E_\theta^{(\infty)}$ is non-quasianalytic, that is, the following property takes place:

Proposition 1. *For any segment Δ of the real axis and for any open finite interval $I \supset \Delta$ there exists $\phi(t) \in E_\theta^{(\infty)}$ such that its Fourier transform equals one in Δ and equals zero outside I .*

Moreover, the class $E_\theta^{(\infty)}$ is linear and is closed under convolutions and differentiation.

The next step is a construction of finite functions of the operator A . For the C_0 -group with non-quasianalytic generator there exists [13] such an entire function $\theta(t)$ of order 1 with zeroes on the positive imaginary ray that

$$\|U(t)\| \leq |\theta(t)| \quad \forall t \in \mathbb{R}.$$

Let us consider an arbitrary $\phi(t) \in E_\theta^{(\infty)}$ and construct the linear operator

$$(3.3) \quad P_\phi = \int_{-\infty}^{\infty} \phi(t) U(t) dt.$$

The operator defined by (3.3) is bounded due to (3.2). Next, consider an arbitrary segment Δ of the real axis and denote by $E_\theta^{(\infty)}(\Delta)$ the set of such functions $\phi(t) \in E_\theta^{(\infty)}$ that the Fourier transform $\tilde{\phi}(\lambda) = 1$ in some interval containing Δ . Denote by $\mathcal{L}(\Delta)$ the subspace of vectors x such that

$$(3.4) \quad P_\phi x = x$$

for all $\phi(t) \in E_\theta^{(\infty)}(\Delta)$.

The operators P_ϕ are useful for studying the vectors $A^n x$ and for proving a Bernstein-type inequality because of the properties (3.3), (3.4) and the property [8, p. 445]

$$(3.5) \quad AP_\phi = \overline{P_\phi A} = P_{-i\phi'},$$

which allows to deal with derivatives of some entire functions instead of Banach-space operators and vectors.

The following theorem shows a close relationship between spectral subspaces and entire vectors of exponential type.

Theorem 1. For all $\alpha > 0$,

$$\mathfrak{E}^\alpha(A) \subset \Xi^\alpha(A) = \mathcal{L}([- \alpha, \alpha]),$$

moreover, $\Xi^\alpha(A)$ is a closed subspace of \mathfrak{X} .

Proof. First we will prove the embedding $\Xi^\alpha(A) \subset \mathcal{L}([- \alpha, \alpha])$. To do this, the forth property of spectral subspaces (mentioned at the beginning of this section) will be used.

Obviously, $\mathfrak{E}^\alpha(A)$ is an invariant subspace of A , and so is $\Xi^\alpha(A)$. Denote the Ξ^α -part of A by A_α :

$$A_\alpha = A \upharpoonright \Xi^\alpha(A).$$

By the mentioned property of spectral subspaces, to finish the proof, it is enough to show that $\sigma(A_\alpha) \subset [- \alpha, \alpha]$ and that A is bounded on $\Xi^\alpha(A)$.

Let us show that $\sigma(A_\alpha) \subset [- \alpha, \alpha]$. For that we check that all points from $\mathbb{C} \setminus [- \alpha, \alpha]$ are regular.

Let $\lambda \in \mathbb{R} \setminus [- \alpha, \alpha]$, λ cannot be an eigenvalue, otherwise for some $x \in \Xi^\alpha(A)$ and for all $n \in \mathbb{N}$, $\|A_\alpha^n x\| = |\lambda|^n \|x\|$, which implies $x \notin \mathfrak{E}^{\alpha+\epsilon}(A)$ for some $\epsilon > 0$, a contradiction with (2.2). That is, λ is not an eigenvalue of A_α .

The equation

$$(3.6) \quad Ax - \lambda x = y$$

has a solution

$$x = - \sum_{n=0}^{\infty} \frac{A^n y}{\lambda^{n+1}}.$$

for any $\lambda \in \mathbb{R} \setminus [- \alpha, \alpha]$ and $y \in \Xi^\alpha(A)$, and this solution belongs to $\Xi^\alpha(A)$, so $\lambda \in \rho(A_\alpha)$.

Let $\text{Im } \lambda \neq 0$. Then, as shown in [8, p. 442], λ is not an eigenvalue of A (as well as A_α) and the resolvent $R_\lambda(A)$ is defined. We set, for all $y \in \Xi^\alpha(A)$, $x = R_\lambda y$. Then

$$\|A^n x\| = \|A^n R_\lambda y\| = \|R_\lambda A^n y\| \leq \|R_\lambda\| \cdot \|A^n y\|,$$

hence $x \in \Xi^\alpha(A)$ and (by definition of the resolvent) x, y satisfy the equation (3.6). So again $\lambda \in \rho(A_\alpha)$.

Thus it is shown that $\{\lambda \in \mathbb{R} \mid |\lambda| > \alpha\} \subset \rho(A_\alpha)$ and $\{\lambda \in \mathbb{C} \mid \text{Im } \lambda \neq 0\} \subset \rho(A_\alpha)$, therefore, $\sigma(A_\alpha) \subset [- \alpha, \alpha]$.

To prove boundedness of A on $\Xi^\alpha(A)$, consider the notion of S -operators [8, p. 452]¹, it results from the following facts (see [8, pp. 462–465 and Theorem 6.1]):

- If an operator A is non-quasianalytic, then it is an S -operator.
- There exists a bounded linear operator $\Phi_{\Delta}^-(A)$ defined on the whole \mathfrak{X} such that its kernel $K_{\Delta}^- := \text{Ker } \Phi_{\Delta}^-(A)$ is a spectral subspace $\mathcal{L}(\Delta)$ of the operator A .
- The operator A is defined and is bounded on whole K_{Δ}^- .
- If \mathcal{L} is an invariant subspace of A and if the spectrum of \mathcal{L} -induced part $A_{\mathcal{L}}$ of the operator A is included into segment Δ , then $\mathcal{L} \subset K_{\Delta}^-$.

Moreover, from these facts it follows that $\mathcal{L}(\Delta)$ is a closed subspace. This means that closedness of $\Xi^\alpha(A)$ would result from the first statement of the theorem ($\Xi^\alpha(A) = \mathcal{L}([- \alpha, \alpha])$).

Let us prove the embedding $\mathcal{L}([- \alpha, \alpha]) \subset \bigcap_{\epsilon > 0} \mathfrak{E}^{\alpha+\epsilon}(A) = \Xi^\alpha(A)$.² Denote $A_{\mathcal{L}} = A \upharpoonright \mathcal{L}([- \alpha, \alpha])$. The operator $A_{\mathcal{L}}$ is bounded (by the first property of $\mathcal{L}([- \alpha, \alpha])$), its spectrum is contained in $[- \alpha, \alpha]$, therefore, the spectral radius of $A_{\mathcal{L}}$ does not exceed α , i.e., $\lim_{n \rightarrow \infty} \sqrt[n]{\|A_{\mathcal{L}}^n\|} \leq \alpha$. From the latter relation, for all $x \in \mathcal{L}$ and for all $\epsilon > 0$ there exists a constant $c = c(x, \epsilon)$ such that $\|A^n x\| = \|A_{\mathcal{L}}^n x\| \leq c(\alpha + \epsilon)^n$, i.e., $x \in \Xi^\alpha(A)$.

¹A detailed definition of S -operators and a construction of spectral subspaces for them goes beyond the scope of this article, thus not cited. Only required properties are mentioned.

²This embedding improves the result of [3]: $\forall \alpha > 0 \exists r(\alpha) : \mathcal{L}([- \alpha, \alpha]) \subset \mathfrak{E}^{r(\alpha)}(A)$.

Below another proof of the embedding $\mathcal{L}([- \alpha, \alpha]) \subset \bigcap_{\epsilon > 0} \mathfrak{E}^{\alpha + \epsilon}(A) = \Xi^\alpha(A)$ is proposed. This proof contains some uniform bounds on the constants $c = c(x, \alpha)$ in the definition of the sets $\mathfrak{E}^\alpha(A)$. According to [10, Lemma 3.1], for $|\theta(t)|$ there exist an even entire function $K_\theta(t)$ and constants $c_r = c_r(\theta) > 0$, $r > 0$, such that, for all $z \in \mathbb{C}$,

$$(3.7) \quad |K_\theta(rz)| \leq c_r \frac{e^{r|\operatorname{Im} z|}}{|\theta(|z|)|}.$$

Let us consider $\Delta = [-\alpha, \alpha]$ and $I = (-\alpha - 4\epsilon, \alpha + 4\epsilon) \supset \Delta$. According to the proof of [8, Lemma 1.1.1], the Fourier transform of the function

$$\phi(t) = \frac{K_\theta^2(-\epsilon t)e^{-(\alpha+2\epsilon)it} - K_\theta^2(\epsilon t)e^{(\alpha+2\epsilon)it}}{-2\pi it} = \frac{\alpha + 2\epsilon}{\pi} K_\theta^2(\epsilon t) \frac{\sin((\alpha + 2\epsilon)t)}{(\alpha + 2\epsilon)t}$$

equals one in Δ and equals zero outside I . Denote by

$$\phi_{r,\epsilon}(z) := K_\theta^2(\epsilon z) \frac{\sin rz}{rz}, \quad z \in \mathbb{C}, \quad r > 0, \quad \epsilon > 0,$$

and estimate the derivatives $\phi_{r,\epsilon}^{(n)}(t)$, $t \in \mathbb{R}$. Using the inequality

$$\left| \frac{\sin z}{z} \right| \leq \frac{\min(1, |z|)}{|z|} e^{|\operatorname{Im} z|} \leq e^{|\operatorname{Im} z|}$$

and (3.7), one can find

$$(3.8) \quad |\phi_{r,\epsilon}(z)| \leq \frac{c_\epsilon^2 e^{2\epsilon|\operatorname{Im} z|}}{|\theta^2(|z|)|} \cdot e^{r|\operatorname{Im} z|} = \frac{c_\epsilon^2 e^{(r+2\epsilon)|\operatorname{Im} z|}}{|\theta^2(|z|)|}.$$

Similarly to the proof of [10, Lemma 3.2], the Cauchy integral formula for $\gamma_{n,r}(t) := \left\{ \zeta \in \mathbb{C} : |\zeta - t| = \frac{n}{r+2\epsilon} \right\}$ and inequality (3.8) allow to obtain, for $t \in \mathbb{R}$ and $n \in \mathbb{N}$, that

$$\begin{aligned} |\phi_{r,\epsilon}^{(n)}(t)| &\leq \frac{n!}{2\pi} \oint_{\gamma_{n,r}(t)} \frac{|\phi_{r,\epsilon}(\xi)|}{|\xi - t|^{n+1}} |d\xi| = \frac{n!}{2\pi} \frac{(r+2\epsilon)^{n+1}}{n^{n+1}} \oint_{\gamma_{n,r}(t)} |\phi_{r,\epsilon}(\xi)| |d\xi| \\ &\leq \frac{c^{(1)} c_\epsilon^2 e^{-n} (r+2\epsilon)^{n+1}}{\sqrt{2\pi n}} \oint_{\gamma_{n,r}(t)} \frac{e^{(r+2\epsilon)|\operatorname{Im} \xi - t|}}{|\theta^2(|\xi|)|} |d\xi|, \\ &\quad \text{where } c^{(1)} = \sup_{k \in \mathbb{N}} \frac{k!}{\sqrt{2\pi k}} \left(\frac{e}{k} \right)^k. \end{aligned}$$

Using $|\theta(t+s)| \leq |\theta(t)| \cdot |\theta(s)|$, it follows from the last inequality that

$$\begin{aligned} |\phi_{r,\epsilon}^{(n)}(t)| &\leq \frac{c^{(1)} c_\epsilon^2 e^{-n} (r+2\epsilon)^{n+1}}{\sqrt{2\pi n} |\theta^2(t)|} \oint_{\gamma_{n,r}(t)} \frac{e^{(r+2\epsilon)|\operatorname{Im} \xi - t|} |\theta^2(|(t-\xi) + \xi|)|}{|\theta^2(|\xi|)|} |d\xi| \\ &\leq c^{(1)} c_\epsilon^2 \sqrt{2\pi n} (r+2\epsilon)^n \left| \frac{\theta\left(\frac{n}{r+2\epsilon}\right)}{\theta(t)} \right|^2. \end{aligned}$$

Returning to the function $\phi(t)$ one can get

$$(3.9) \quad |\phi^{(n)}(t)| = \frac{\alpha + 2\epsilon}{\pi} |\phi_{\alpha+2\epsilon,\epsilon}^{(n)}(t)| \leq \frac{c^{(1)} c_\epsilon^2 \sqrt{2\pi n}}{\pi} (\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \left| \frac{\theta\left(\frac{n}{\alpha+4\epsilon}\right)}{\theta(t)} \right|^2.$$

Let $x \in \mathcal{L}([- \alpha, \alpha])$. By the construction, $\phi \in E_\theta^{(\infty)}(\Delta)$, thus $P_\phi x = x$ and, accordingly to (3.5),

$$\|A^n x\| = \|A^n P_\phi x\| = \|P_{(-i)^n \phi^{(n)}} x\|.$$

Using (3.3) and (3.9), the following estimate for the latter expression can be found:

$$\begin{aligned} \|P_{(-i)^n \phi^{(n)}} x\| &\leq \int_{-\infty}^{\infty} |\phi^{(n)}(t)\theta(t)| dt \cdot \|x\| \\ &\leq \frac{c^{(!)} c_\epsilon^2 \sqrt{2\pi n}}{\pi} (\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \|x\| \left| \theta^2 \left(\frac{n}{\alpha + 4\epsilon} \right) \right| \int_{-\infty}^{\infty} \frac{dt}{|\theta(t)|}. \end{aligned}$$

It follows from (3.1) that

$$\int_{-\infty}^{\infty} \frac{dt}{|\theta(t)|} = c_\theta < \infty,$$

so there exists $c > 0$ such that

$$(3.10) \quad \|A^n x\| \leq c\sqrt{n}(\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \left| \theta^2 \left(\frac{n}{\alpha + 4\epsilon} \right) \right| \|x\|, \quad \alpha > 0, \quad \epsilon > 0, \quad n \in \mathbb{N}.$$

The following relation holds:

$$(3.11) \quad \lim_{n \rightarrow \infty} (c\sqrt{n}(\alpha + 2\epsilon)(\alpha + 4\epsilon)^n)^{1/n} = \alpha + 4\epsilon, \quad \alpha, \epsilon \in \mathbb{R}_+.$$

As noted in the proof of [10, Theorem 3.1], for the function $|\theta(t)|$ we have

$$(3.12) \quad \lim_{n \rightarrow \infty} \left(\left| \theta^2 \left(\frac{n}{\alpha} \right) \right| \right)^{1/n} = 1, \quad \alpha \in \mathbb{R}_+,$$

therefore from (3.10), (3.11) and (3.12) one can get

$$\sigma(x, A) = \limsup_{n \rightarrow \infty} \|A^n x\|^{\frac{1}{n}} \leq \alpha + 4\epsilon,$$

that is $\forall \epsilon' > 0 \ x \in \mathfrak{E}^{\alpha+4\epsilon+\epsilon'}(A)$. Due to arbitrariness of ϵ ,

$$x \in \bigcap_{\epsilon > 0} \mathfrak{E}^{\alpha+\epsilon}(A),$$

this is what was to be proved. □

4. GENERALIZED BERNSTEIN-TYPE INEQUALITY

One of the well-known inequalities in approximation theory is the Bernstein inequality. If $f(x)$ is an entire function of exponential type $\sigma > 0$, and

$$|f(x)| \leq M, \quad -\infty < x < \infty,$$

then

$$|f'(x)| \leq \sigma M, \quad -\infty < x < \infty.$$

In this section some generalization of the Bernstein inequality for exponential type entire vectors is proved.

Note that a more detailed examination of (3.10) allows to obtain a Bernstein-type inequality. Consider the relation (3.12). Note that it holds uniformly for all $\alpha \geq \alpha_0 > 0$. Therefore for all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$(4.1) \quad c\sqrt{n} \left| \theta^2 \left(\frac{n}{\alpha + 4\epsilon} \right) \right| \leq c_\epsilon (1 + \epsilon)^n, \quad \forall n \in \mathbb{N}, \quad \forall \alpha \in \mathbb{R}_+.$$

Inequalities (3.10) and (4.1) allow to prove the following.

Proposition 2. *For every $\epsilon > 0$ there exists $c_\epsilon > 0$, independent of α and of n , such that for all $\alpha > 0$*

$$(4.2) \quad \|A^n x\| \leq c_\epsilon (1 + \epsilon)^n (\alpha + 2\epsilon)(\alpha + 4\epsilon)^n \|x\|, \quad x \in \Xi^\alpha(A) \quad \text{or} \quad x \in \mathfrak{E}^\alpha(A).$$

But in contrast with the classic Bernstein inequality, the type α of the vector appears in (4.2) in the degree $n + 1$. Let us show that an analogous inequality with the degree n holds.

Theorem 2 (Generalized Bernstein-type inequality). *For all vectors $x \in \mathfrak{E}(A)$, of type, not exceeding some $\alpha \geq 1$, the following inequality holds*

$$(4.3) \quad \|A^n x\| \leq c_n \alpha^n \|x\|,$$

where the constants $c_n > 0$ do not depend on x and on α .

Proof. Let us consider the majorant $\theta(t)$ for the function $\|U(t)\|$, constructed in [13]³. Remark that $\theta(t)$ is of the form (3.1). Similarly to the proof of theorem 1 and as in [10, Lemma 3.1] from the function $\theta(t)$ one can construct the entire function $K(t)$ of exponential type.

Let us consider such a function $\phi_\alpha(t)$ that its Fourier transform equals 1 in $[-\alpha, \alpha]$ and equals 0 outside $(-3\alpha, 3\alpha)$. According to [8, Lemma 1.1.1], one can take $\phi_\alpha(t)$ to be the function

$$\phi_\alpha(t) = \frac{K^2\left(\frac{\alpha}{2}t\right) \sin 2\alpha t}{\pi t}.$$

Denote by

$$\phi(t) := \frac{K^2\left(\frac{t}{2}\right) \sin 2t}{\pi t}.$$

Then $\phi_\alpha(t) = \alpha\phi(\alpha t)$. As it follows from (3.3) and (3.5), it is enough to estimate the quantity

$$\int_{-\infty}^{\infty} |\phi_\alpha^{(n)}(t)\theta(t)| dt$$

to prove the theorem. For $\alpha \geq 1$ we have $|\theta(t)| \leq |\theta(\alpha t)|$ and

$$\int_{-\infty}^{\infty} |\phi_\alpha^{(n)}(t)\theta(t)| dt \leq \int_{-\infty}^{\infty} |\phi^{(n)}(\alpha t)\theta(\alpha t)| \alpha dt.$$

The change of variables $\tau = \alpha \cdot t$ gives

$$\frac{d^n \phi(\alpha t)}{dt^n} = \frac{d^n \phi(\tau)}{d\tau^n} \cdot \alpha^n,$$

thus

$$\int_{-\infty}^{\infty} |\phi^{(n)}(\alpha t)\theta(\alpha t)| \alpha dt = \alpha^n \cdot \int_{-\infty}^{\infty} |\phi^{(n)}(\tau)\theta(\tau)| d\tau.$$

It is easy to see that the last integral exists and does not depend on α . Let it equal $c_n > 0$. Then

$$\|A^n x\| = \|P_{(-i)^n \phi^{(n)}} x\| \leq c_n \alpha^n \|x\|,$$

which was to be proved. \square

As a consequence of theorem 2 we get the following estimate for the operator Δ_h^k .

Corollary 1. *Let $x \in \mathfrak{E}(A)$ and $\sigma(x) \leq \alpha$, $\alpha \geq 1$. Then for all $k \in \mathbb{N}$*

$$\|\Delta_h^k x\| \leq c_k (h\alpha)^k M_U(kh) \|x\|,$$

where the constant c_k is the same as in the Theorem 2, and the function $M_U(t)$ is defined by (2.3).

Proof. The following holds for Δ_h^k :

$$\Delta_h^k x = (U(t) - \mathbb{I})^k x = \int_0^t \cdots \int_0^t U(\xi_1 + \dots + \xi_k) A^k x d\xi_1 \dots d\xi_k.$$

By Theorem 2,

$$\|A^k x\| \leq c_k \alpha^k \|x\|,$$

³The majorant is denoted in [13] by $\omega(t)$, but in this article it is denoted by $\theta(t)$ in order not to confuse it with the module of continuity

and $\|U(\xi_1 + \dots + \xi_k)\| \leq M_U(kt)$ by the definition. Therefore,

$$\|\Delta_h^k x\| \leq \int_0^t \dots \int_0^t \|U(\xi_1 + \dots + \xi_k)\| \cdot \|A^k x\| d\xi_1 \dots d\xi_k \leq c_k h^k M_U(kh) \alpha^k \|x\|.$$

□

5. INVERSE THEOREM OF APPROXIMATION

The following results generalize the classical Bernstein theorem (also known as the inverse theorem).

Theorem 3. *Let $\omega(t)$ be a function of type of module of continuity for which the following conditions are satisfied:*

- (1) $\omega(t)$ is continuous and nondecreasing for $t \in \mathbb{R}_+$.
- (2) $\omega(0) = 0$.
- (3) $\exists c > 0 \forall t \in [0, 1] \quad \omega(2t) \leq c\omega(t)$.
- (4) $\int_0^1 \frac{\omega(t)}{t} dt < \infty$.

If, for $x \in \mathfrak{X}$, there exist $n \in \mathbb{N}$ and $m > 0$ such that

$$(5.1) \quad \mathcal{E}_r(x, A) \leq \frac{m}{r^n} \omega\left(\frac{1}{r}\right), \quad r \geq 1,$$

then $x \in \mathcal{D}(A^n)$ and for every $k \in \mathbb{N}$ there exists a constant $m_k > 0$ such that

$$\omega_k(t, A^n x, A) \leq m_k \left(t^k \int_t^1 \frac{\omega(u)}{u^{k+1}} du + \int_0^t \frac{\omega(u)}{u} du \right), \quad 0 < t \leq 1/2.$$

The following lemma is used for the proof of theorem.

Lemma 1. *Suppose that the function $\omega(t)$ satisfies Conditions 1–3 of Theorem 3. If, for $x \in \mathfrak{X}$, there exists $m > 0$ such that*

$$\mathcal{E}_r(x, A) \leq m\omega\left(\frac{1}{r}\right), \quad r \geq 1,$$

then, for every $k \in \mathbb{N}$ there exists a constant $\tilde{c}_k > 0$ such that

$$\omega_k(t, x, A) \leq \tilde{c}_k t^k \int_k^1 \frac{\omega(\tau)}{\tau^{k+1}} d\tau, \quad 0 < t \leq 1/2.$$

Remark 1. As would follow from the proof, the lemma remains true under somewhat weaker conditions than those formulated in the theorem, namely, it is sufficient that for an element $x \in \mathfrak{X}$ there exist at least one sequence $\{u_j\}_{j=1}^\infty \subset \mathfrak{E}(A)$ such that $\sigma(u_j, A) \leq 2^j$ and for all $j \in \mathbb{N}$

$$\|x - u_j\| \leq m \cdot \omega\left(\frac{1}{2^j}\right).$$

Proof of theorem 3. As shown in Theorem 1, the subspaces $\Xi^r(A)$ are closed, therefore it follows from the definition and from (5.1) that there exists a sequence of vectors $\{u_j\}_{j=0}^\infty \subset \mathfrak{E}(A)$ such that $\sigma(u_j, A) \leq 2^j$ and

$$(5.2) \quad \|x - u_j\| \leq \frac{m}{2^{nj}} \omega\left(\frac{1}{2^j}\right).$$

From the inequality (5.2) and Conditions 1, 2 one can get $\|x - u_j\| \rightarrow 0, j \rightarrow \infty$, and so the vector x has the representation

$$x = u_0 + \sum_{j=1}^\infty (u_j - u_{j-1}).$$

Due to $\sigma(u_j - u_{j-1}, A) \leq 2^j$, $j \in \mathbb{N}$, one can find from (4.3) that

$$\begin{aligned} \|A^n u_j - A^n u_{j-1}\| &\leq c_n 2^{jn} \|u_j - u_{j-1}\| \leq c_n 2^{jn} (\|x - u_j\| + \|x - u_{j-1}\|) \\ &\leq c_n 2^{jn} \left(\frac{m}{2^{nj}} \cdot \omega\left(\frac{1}{2^j}\right) + \frac{m}{2^{n(j-1)}} \cdot \omega\left(\frac{1}{2^{j-1}}\right) \right) \\ &\leq \frac{2m c_n 2^{jn}}{2^{n(j-1)}} \cdot \omega\left(\frac{1}{2^{j-1}}\right) \\ &\leq 2^{n+1} c c_n m \cdot \omega\left(\frac{1}{2^j}\right) \leq \frac{2^{n+1} c c_n m}{\ln 2} \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(u)}{u} du. \end{aligned}$$

Hence, $\sum_{j=1}^{\infty} (A^n u_j - A^n u_{j-1})$ is convergent. By virtue of closedness of the operator A^n , $x \in \mathcal{D}(A^n)$ and

$$A^n x = A^n u_0 + \sum_{j=1}^{\infty} (A^n u_j - A^n u_{j-1}),$$

therefore,

$$\begin{aligned} \|A^n x - A^n u_{j_0}\| &\leq \sum_{j=j_0+1}^{\infty} \|A^n u_j - A^n u_{j-1}\| \leq \frac{2^{n+1} c c_n m}{\ln 2} \sum_{j=j_0+1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(u)}{u} du \\ &= \frac{2^{n+1} c c_n m}{\ln 2} \int_0^{2^{-j_0}} \frac{\omega(u)}{u} du =: \tilde{c} \Omega(2^{-j_0}), \quad j_0 \in \mathbb{N}, \end{aligned}$$

where $\tilde{c} = \frac{2^{n+1} c c_n m}{\ln 2}$,

$$\Omega(t) := \int_0^t \frac{\omega(u)}{u} du.$$

It is easy to see that the function $\Omega(t)$ has the following properties:

- (1) $\Omega(t)$ is continuous and monotonically nondecreasing;
- (2) $\Omega(0) = 0$;
- (3) for $t \in [0, 1]$, the following relation is true:

$$\Omega(2t) = \int_0^{2t} \frac{\omega(u)}{u} du = \int_0^t \frac{\omega(2u)}{u} du \leq c \int_0^t \frac{\omega(u)}{u} du = c \Omega(t).$$

Therefore, setting $\omega(t) = \Omega(t)$ in Lemma 1 and taking remark into account, we get

$$\begin{aligned} \omega_k(t, A^n x, A) &\leq \tilde{c}_k t^k \int_t^1 \frac{\Omega(u)}{u^{k+1}} du = \frac{\tilde{c}_k t^k}{k} \left(\Omega(u) \frac{1}{u^k} \Big|_t^1 + \int_t^1 \frac{\omega(u)}{u^{k+1}} du \right) \\ &\leq m_k \left(t^k \int_t^1 \frac{\omega(u)}{u^{k+1}} du + \int_0^t \frac{\omega(u)}{u} du \right). \end{aligned}$$

The theorem is proved. \square

Proof of lemma 1. By the analogy with the proof of Theorem 3, it follows from (5.1) that there exists a sequence of vectors $\{u_j\}_{j=0}^{\infty} \subset \mathfrak{E}(A)$ such that $\sigma(u_j, A) \leq 2^j$ and

$$(5.3) \quad \|x - u_j\| \leq m \omega\left(\frac{1}{2^j}\right).$$

Let us take an arbitrary $h \in (0, 1/2]$ and choose a number N in such a way that $\frac{1}{2^{N+1}} < h \leq \frac{1}{2^N}$. Inequality (5.3) yields

$$(5.4) \quad \begin{aligned} \|u_j - u_{j-1}\| &\leq \|u_j - x\| + \|x - u_{j-1}\| \\ &\leq m \omega(2^{-j}) + m \omega(2^{-j+1}) \leq 2m \omega(2^{-j+1}) \leq 2cm \omega(2^{-j}). \end{aligned}$$

By virtue of monotonicity of $\omega(t)$,

$$(5.5) \quad \begin{aligned} 2^k \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(u)}{u^{k+1}} du &\geq 2^k \omega(2^{-j}) \int_{2^{-j}}^{2^{-j+1}} \frac{1}{u^{k+1}} du \\ &= \frac{2^{kj}(2^k - 1)}{k} \omega(2^{-j}) \geq 2^{kj} \omega(2^{-j}). \end{aligned}$$

Since $\sigma(u_j - u_{j-1}, A) \leq 2^j$ and $\sigma(u_0, A) \leq 1$, according to Corollary 1,

$$\begin{aligned} \|\Delta_h^k u_0\| &\leq c_k h^k M_U(kh) \|u_0\|, \\ \|\Delta_h^k(u_j - u_{j-1})\| &\leq c_k h^k (2^j)^k M_U(kh) \|u_j - u_{j-1}\|, \quad j \geq 1. \end{aligned}$$

Relations (5.3)–(5.5) yield

$$\|\Delta_h^k(u_j - u_{j-1})\| \leq 2\tilde{c}h^k (2^j)^k \omega(2^{-j}) \leq 2^{k+1}\tilde{c}h^k \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(u)}{u^{k+1}} du,$$

where $\tilde{c} = cc_k m M_U(kh)$, and

$$\begin{aligned} \|\Delta(x - u_N)\| &\leq \|(U(h) - \mathbb{I})^k\| \|x - u_N\| \\ &\leq (M_U(h) + 1)^k \|x - u_N\| \leq (M_U(h) + 1)^k m \omega(2^{-N}). \end{aligned}$$

Using these inequalities, we obtain

$$\begin{aligned} \|\Delta_h^k x\| &= \left\| \Delta_h^k u_0 + \sum_{j=1}^N \Delta_h^k(u_j - u_{j-1}) + \Delta_h^k(x - u_N) \right\| \\ &\leq c_k M_U(kh) h^k \|u_0\| + 2^{k+1}\tilde{c}h^k \sum_{j=1}^N \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(u)}{u^{k+1}} du + (M_U(h) + 1)^k m \omega(2^{-N}) \\ &\leq h^k \left[c_k M_U(kh) \|u_0\| + 2^{k+1}\tilde{c} \int_h^1 \frac{\omega(u)}{u^{k+1}} du \right. \\ &\quad \left. + (M_U(h) + 1)^k cm \frac{k}{1-h^k} \int_h^1 \frac{\omega(h)}{u^{k+1}} du \right] \\ &\leq \tilde{c}_k h^k \int_h^1 \frac{\omega(u)}{u^{k+1}} du, \quad \text{where} \\ \tilde{c}_k &:= \frac{\|u_0\| c_k M_U(k/2)}{\int_{1/2}^1 \frac{\omega(u)}{u^{k+1}} du} + 2^{k+1} cc_k m M_U(k/2) + (M_U(1/2) + 1)^k \frac{cmk}{1 - (1/2)^k}. \end{aligned}$$

The last inequality holds for all $0 < h \leq 1/2$. Taking into account the definition of the module of continuity (2.4), this inequality finishes the proof. \square

6. EXAMPLES OF APPLICATION OF ABSTRACT DIRECT AND INVERSE THEOREMS IN PARTICULAR SPACES

In this section we discuss an application of the presented theory — the approximation of continuous functions by entire functions in the weighted $L_p(\mathbb{R}, \mu^p)$ space with a growing at the infinity weight (for example, $L_1(\mathbb{R}, x^n)$ spaces). Similar problems studied in several papers (see the review [14] and references therein).

Let us consider a real-valued function $\mu(t)$ satisfying the following conditions:

- 1) $\mu(t) \geq 1$, $t \in \mathbb{R}$;
- 2) $\mu(t)$ is even, monotonically non-decreasing when $t > 0$;
- 3) $\mu(t)$ satisfies the condition $\mu(t+s) \leq \mu(t) \cdot \mu(s)$, $s, t \in \mathbb{R}$.
- 4) $\int_{-\infty}^{\infty} \frac{\ln \mu(t)}{1+t^2} dt < \infty$,

or, alternatively, instead of 4), an equivalent condition holds,

$$4') \sum_{k=1}^{\infty} \frac{\ln \mu(k)}{k^2} < \infty.$$

Below are several important classes of functions satisfying conditions 1)–4) (see [10] for details).

1. The constant function $\mu(t) \equiv 1$, $t \in \mathbb{R}$.

2. Functions with polynomial order of growth at infinity. For such functions the following estimate holds: $\exists k \in \mathbb{N}$, $\exists M \geq 1$

$$\mu(t) \leq M(1 + |t|)^k, \quad t \in \mathbb{R}.$$

3. Functions of the form

$$\mu(t) = e^{|t|^\beta}, \quad 0 < \beta < 1, \quad t \in \mathbb{R}.$$

4. $\mu(t)$ represented as a power series for $t > 0$. I.e.,

$$\mu(t) = \sum_{n=0}^{\infty} \frac{|t|^n}{m_n},$$

where $\{m_n\}_{n \in \mathbb{N}}$ is a sequence of positive real numbers satisfying three conditions:

- $m_0 = 1$, $m_n^2 \leq m_{n-1} \cdot m_{n+1}$, $n \in \mathbb{N}$;
- for all $k, l \in \mathbb{N}$ $\frac{(k+l)!}{m_{k+l}} \leq \frac{k!}{m_k} \frac{l!}{m_l}$;
- $\sum_{n=1}^{\infty} \left(\frac{1}{m_n}\right)^{1/n} < \infty$.

5. $\mu(t)$ as a module of an entire function with zeroes on the imaginary axis. Let's consider

$$\omega(t) = C \prod_{k=1}^{\infty} \left(1 - \frac{t}{it_k}\right), \quad t \in \mathbb{R},$$

where $C \geq 1$, $0 < t_1 \leq t_2 \leq \dots$, $\sum_{k=1}^{\infty} \frac{1}{t_k} < \infty$, and set $\mu(t) := |\omega(t)|$.

Let us consider the space $L_p(\mathbb{R}, \mu^p)$, $1 \leq p \leq \infty$, of functions $x(s)$, $s \in \mathbb{R}$, integrable in p -th degree with the weight μ^p :

$$\|x\|_{L_p(\mathbb{R}, \mu^p)}^p = \int_{-\infty}^{\infty} |x(s)|^p \mu^p(s) ds.$$

$L_p(\mathbb{R}, \mu^p)$ is the Banach space. The differential operator

$$(Ax)(t) = \frac{dx}{dt}, \quad \mathcal{D}(A) = \{x \in L_p(\mathbb{R}, \mu^p) \cap AC(\mathbb{R}) : x' \in L_p(\mathbb{R}, \mu^p)\}.$$

generates the group of shifts $\{U(t)\}_{t \in \mathbb{R}}$ in the space $L_p(\mathbb{R}, \mu^p)$. This group is not bounded. As shown in [10],

$$\|U(t)\|_{L_p(\mathbb{R}, \mu^p)} \leq \mu(|t|), \quad t \in \mathbb{R}.$$

To apply the constructed theory, we need to determine how the space $\mathfrak{E}(A)$ and the space of exponential type entire functions are connected. Denote by B_σ the set of exponential functions of entire type σ . We show that the following embedding holds:

$$(6.1) \quad \Xi^\sigma(A) \subset B_\sigma \cap L_p(\mathbb{R}, \mu^p).$$

Let $f \in \Xi^\sigma(A)$. Obviously, $f \in L_p(\mathbb{R}, \mu^p)$. We prove that $f \in B_\sigma$. Due to $\mu(t) \geq 1$ we have

$$\|f\|_{L_p(\mathbb{R})} \leq \|f\|_{L_p(\mathbb{R}, \mu^p)},$$

thus for all $n \in \mathbb{N}$ and for any $\epsilon > 0$

$$(6.2) \quad \|A^n f\|_{L_p(\mathbb{R})} \leq \|A^n f\|_{L_p(\mathbb{R}, \mu^p)} \leq c_\epsilon(f)(\sigma + \epsilon)^n,$$

and so we can construct a continuation of $U(t)$ onto \mathbb{C} by

$$U(z) = \sum_{n=0}^{\infty} \frac{A^n f}{n!} z^n, \quad z \in \mathbb{C}.$$

Moreover, (6.2) ensures for all $\epsilon > 0$

$$\|f(x+z)\|_{L_p(\mathbb{R})} = \left\| \sum_{n=0}^{\infty} \frac{A^n f}{n!} z^n \right\| \leq c_\epsilon(f) \cdot \|f\| e^{(\sigma+\epsilon)|z|},$$

which means $f \in B_\sigma$, which was required.

By virtue of the classical Bernstein inequality, the embedding inverse to (6.1) holds for all bounded weights $\mu(t)$. We show that it holds for all functions $\mu(t)$ satisfying

$$\mu(t) \geq 1 + R|t|^q$$

for some $q > 1 - \frac{1}{p}$, $R > 0$ and for all $t > t_0 \geq 0$. The condition on $\mu(t)$ gives us $f \in L_1(\mathbb{R})$. $f \in B_\sigma$, thus it is infinitely differentiable and by the Paley-Wiener theorem the support of its Fourier transform is contained in $[-\sigma, \sigma]$. Let's prove that $f \in \Xi^\sigma(A)$ by using Theorem 1. Let us consider the majorant $\theta(t)$ for the function $\mu(t)$, constructed as in the proof of Theorem 2. We need to show that for all $\phi \in E_\theta^{(\infty)}([-\sigma, \sigma])$

$$f = P_\phi f = \int_{-\infty}^{\infty} \phi(t) U(t) f dt.$$

Since ϕ is arbitrary, we can consider $\phi_1(t) = \phi(-t) \in E_\theta^{(\infty)}([-\sigma, \sigma])$. Note that

$$\int_{-\infty}^{\infty} \phi_1(t) U(t) f(x) dt = \int_{-\infty}^{\infty} \phi(t) f(x-t) dt = \phi * f.$$

The Fourier transform of $\phi * f$ equals to

$$\widetilde{\phi * f} = \tilde{\phi} \cdot \tilde{f} = \tilde{f},$$

because $\text{supp } f \subset [-\sigma, \sigma]$, and by the definition of $E_\theta^{(\infty)}([-\sigma, \sigma])$ we have $\tilde{\phi} = 1$ on $[-\sigma, \sigma]$. Thus,

$$P_\phi f = f \quad \forall \phi \in E_\theta^{(\infty)}([-\sigma, \sigma]),$$

so $f \in \mathcal{L}([-\sigma, \sigma])$ and by means of Theorem 1 $f \in \Xi^\sigma(A)$.

We have shown that $\Xi^\sigma(A)$ coincides with $B_\sigma \cap L_p(\mathbb{R}, \mu^p)$. Note that $\|f - g_\sigma\|_{L_p(\mathbb{R}, \mu^p)}$ is defined only for those functions that belongs to $L_p(\mathbb{R}, \mu^p)$ (because of $\|g_\sigma\|_{L_p(\mathbb{R}, \mu^p)} \leq \|f - g_\sigma\|_{L_p(\mathbb{R}, \mu^p)} + \|f\|_{L_p(\mathbb{R}, \mu^p)}$), thus the best approximation by exponential type entire vectors is the same as the best approximation by entire functions of exponential type.

By applying Theorems 2 and 3 we get several results for the approximation theory in $L_p(\mathbb{R}, \mu^p)$ spaces. First two results are direct theorems (from [10]) for spaces $L_p(\mathbb{R}, \mu^p)$.

Corollary 2 ([10]). *For every $k \in \mathbb{N}$ there exists a constant $\mathbf{m}_k(p, \mu) > 0$ such that for all $f \in L_p(\mathbb{R}, \mu^p)$*

$$\mathcal{E}_r(f) \leq \mathbf{m}_k \cdot \tilde{\omega}_k \left(\frac{1}{r}, f \right), \quad r \geq 1.$$

Corollary 3 ([10]). *Let $f \in W_p^m(\mathbb{R}, \mu^p)$, $m \in \mathbb{N}_0$. Then for all $k \in \mathbb{N}_0$*

$$\mathcal{E}_r(f) \leq \mathbf{m}_{k+m} \frac{\mu \left(\frac{m}{r} \right)}{r^m} \tilde{\omega}_k \left(\frac{1}{r}, f^{(m)} \right), \quad r \geq 1,$$

where constants \mathbf{m}_n ($n \in \mathbb{N}$) are the same as in the Corollary 2.

Corollary 4. *Let $f \in L_p(\mathbb{R}, \mu^p) \cap B_\sigma$, $\sigma \geq 1$. Then for all $n \in \mathbb{N}$ there exist such constants $c_n > 0$, not depending on σ and on f , that*

$$\|f^{(n)}\|_{L_p(\mathbb{R}, \mu^p)} \leq c_n \sigma^n \|f\|_{L_p(\mathbb{R}, \mu^p)}.$$

Corollary 5. *Let $\omega(t)$ be a function of type of module of continuity for which the following conditions are satisfied:*

- (1) $\omega(t)$ is continuous and nondecreasing for $t \in \mathbb{R}_+$.

- (2) $\omega(0) = 0$.
 (3) $\exists c > 0 \forall t \in [0, 1] \quad \omega(2t) \leq c\omega(t)$.
 (4) $\int_0^1 \frac{\omega(t)}{t} dt < \infty$.

If, for $f \in L_p(\mathbb{R}, \mu^p)$, there exist such $n \in \mathbb{N}$ and $m > 0$ that

$$\mathcal{E}_r(f) \leq \frac{m}{r^n} \omega\left(\frac{1}{r}\right), \quad r \geq 1,$$

then $f \in W_p^n(\mathbb{R}, \mu^p)$ and for every $k \in \mathbb{N}$ there exists such $m_k > 0$ that

$$\omega_k(t, f^{(n)}) \leq m_k \left(t^k \int_t^1 \frac{\omega(u)}{u^{k+1}} du + \int_0^t \frac{\omega(u)}{u} du \right), \quad 0 < t \leq 1/2.$$

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