Methods of Functional Analysis and Topology
Vol. 16 (2010), no. 1, pp. 83–100

POSITIVE DEFINITE KERNELS SATISFYING DIFFERENCE EQUATIONS

S. M. ZAGORODNYUK

Abstract. We study positive definite kernels $K = (K_{n,m})_{n,m\in A}$, $A = \mathbb{Z}$ or $A = \mathbb{Z}_+$, which satisfy a difference equation of the form $L_n K = \overline{L}_m K$, or of the form $L_n\overline{L}_m K = K$, where L is a linear difference operator (here the subscript n (m) means that L acts on columns (respectively rows) of K). In the first case, we give new proofs of Yu. M. Berezansky results about integral representations for K. In the second case, we obtain integral representations for K. The latter result is applied to strengthen one our result on abstract stochastic sequences. As an example, we consider the Hamburger moment problem and the corresponding positive matrix of moments. Classical results on the Hamburger moment problem are derived using an operator approach, without use of Jacobi matrices or orthogonal polynomials.

1. Introduction

The object of our present investigation will be a positive definite kernel

$$K = (K_{n,m})_{n,m\in A}$$

defined on a set of integers $A = \mathbb{Z}$, or on a set of non-negative integers $A = \mathbb{Z}_+$. By the kernel we mean a symmetric infinite matrix $(K_{n,m})_{n,m\in A}$, and the positive definiteness means that

$$\sum_{n,m\in A} K_{n,m}\xi_n\overline{\xi_m} \geq 0, \tag{1}$$

for finite sequences $(\xi_n)_{n\in A}$ of complex numbers, $A = \mathbb{Z}, \mathbb{Z}_+$, see [1].

Let us consider the following operator L:

$$(Lu)_n = \sum_{k=-r^-}^{r^+} \alpha_{n,k} u_{n+k}, \quad n \in \mathbb{Z}, \tag{2}$$

where $\alpha_{n,k} \in \mathbb{C}$, $\alpha_{n,-r-} \neq 0$, $\alpha_{n,r+} \neq 0$, $r^-, r^+ \in \mathbb{Z}_+$: $r^- + r^+ > 0$. It can be considered on finite complex sequences $(u_k)_{k\in\mathbb{Z}}$ from $l^2(\mathbb{Z})$, where $l^2(\mathbb{Z})$ is the standard space of square summable complex sequences $(u_k)_{k\in\mathbb{Z}}$. Notice that the operator L is a difference operator of order $r = r^- + r^+$. We also define an operator $\overline{L}$ as

$$(\overline{L}u)_n = \sum_{k=-r^-}^{r^+} \overline{\alpha_{n,k}} u_{n+k}, \quad n \in \mathbb{Z}. \tag{3}$$

Suppose that a positive definite kernel $K = (K_{n,m})_{n,m\in\mathbb{Z}}$ satisfies the relation

$$L_n K = \overline{L}_m K, \tag{4}$$

2000 *Mathematics Subject Classification.* Primary 42A82; Secondary 43A35.
Key words and phrases. Positive definite kernel, difference equation, integral representation.

where L_n means that L acts on each column of K, and $\overline{L}_m$ means that $\overline{L}$ acts on each row of K. In the coordinate form this relation takes the form

$$\sum_{k=-r^-}^{r^+} \alpha_{n,k} K_{n+k,m} = \sum_{l=-r^-}^{r^+} \overline{\alpha_{m,l}} K_{n,m+l}, \quad n,m \in \mathbb{Z}. \tag{5}$$

Necessary and sufficient conditions that an arbitrary positive definite kernel

$$K = (K_{n,m})_{n,m\in\mathbb{Z}}$$

satisfies relation (4) is that K admits the following integral representation, see [1, Ch. 8, Theorem 5.1]:

$$K_{n,m} = \int_{\mathbb{R}} \sum_{k,l=0}^{r-1} \chi_{k;n}(\lambda)\overline{\chi_{l;m}(\lambda)} d\sigma_{k,l}(\lambda), \quad n,m \in \mathbb{Z}, \tag{6}$$

where $\chi_{k;n}(\lambda)$ is a solution of the equation

$$Lu = \lambda u, \quad (\lambda \in \mathbb{R}), \tag{7}$$

with the initial conditions

$$\chi_{k;n}(\lambda) = \delta_{n,k+a-r^-}, \quad n = a - r^-, \dots, a + r^+ - 1, \quad k = 0, 1, \dots, r-1, \tag{8}$$

and a is a fixed integer. Here $(\sigma_{k,l}(\lambda))_{k,l=0}^{r-1}$ is a non-decreasing matrix-valued function on $\mathbb{R}$. This result was easily transferred to the case of $A = \mathbb{Z}_+$, see [1, Ch. 8, Theorem 5.2]. Proofs of these results were based on the theory of expansions by generalized eigenfunctions of self-adjoint operators developed by Yu. M. Berezansky.

Our first purpose is to give other proofs of the mentioned results. These proofs are based on standard facts from the extension theory of Hilbert space operators [2].

Our second purpose will be to obtain integral representations for positive definite kernels satisfying the following equation:

$$L_n \overline{L}_m K = K. \tag{9}$$

Finally, we apply our result to strengthen one our result about abstract stochastic sequences in [3].

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real, complex, positive integer, integer, non-negative integer numbers, respectively; $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$. If $\sigma(x)$ is a non-decreasing left-continuous function on $\mathbb{R}$, we denote by L^2_σ a space of (classes of equivalence) of complex-valued functions on $\mathbb{R}$ measurable with respect to the positive Borel measure σ generated by $\sigma(x)$, and such that $\|f(x)\|_\sigma := \left(\int_{\mathbb{R}} |f(x)|^2 d\sigma\right)^{\frac{1}{2}} < \infty$. The space L^2_σ is a Hilbert space with the scalar product $(f(x), g(x))_\sigma := \int_{\mathbb{R}} f(x)\overline{g(x)}\, d\sigma$, $f, g \in L^2_\sigma$.

For a separable Hilbert space H we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ the scalar product and the norm in H, respectively. The indices may be omitted in obvious cases. For a complex polynomial $p(\lambda) = \sum_{k=0}^n a_k \lambda^k$, $a_k \in \mathbb{C}$, $n \in \mathbb{Z}_+$, we set $\overline{p}(\lambda) = \sum_{k=0}^n \overline{a_k}\lambda^k$. For a linear operator A we denote by $D(A)$ its domain and by A^* we denote its adjoint if it exists. For a set of elements $\{x_n\}_{n\in A}$ in a separable Hilbert space H, we denote by $\mathrm{Lin}\{x_n\}_{n\in A}$ and $\mathrm{span}\{x_n\}_{n\in A}$ the linear span and the closed linear span (in the norm of H), respectively, $A = \mathbb{Z}$ or $A = \mathbb{Z}_+$. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M with respect to the norm of H. By E_H we denote the identity operator in H, i.e., $E_H x = x$, $x \in H$. If H_1 is a subspace of H, by $P_{H_1} = P^H_{H_1}$ we denote an operator of the orthogonal projection on H_1 in H.

2. Difference equations of a "self-adjoint" type

2.1. **Case $A = \mathbb{Z}$.** We will make use of the following important fact (e.g., [4, p. 215]).

Theorem 1. *Let $K = (K_{n,m})_{n,m\in A}$ be a positive definite kernel, $A = \mathbb{Z}$ or $A = \mathbb{Z}_+$. Then there exist a separable Hilbert space H with a scalar product $(\cdot,\cdot)$ and a sequence $\{x_n\}_{n\in A}$ in H, such that*

$$K_{n,m} = (x_n, x_m), \quad n, m \in A, \tag{10}$$

and $\operatorname{span}\{x_n\}_{n\in A} = H$.

Proof. Consider an arbitrary infinite-dimensional linear vector space V (for example a space of complex sequences $(u_n)_{n\in\mathbb{Z}_+}$, $u_n \in \mathbb{C}$). Let $X = \{x_n\}_{n\in A}$ be an arbitrary infinite sequence of linear independent elements in V. Let $L = \operatorname{Lin}\{x_n\}_{n\in A}$ be the linear span of elements of X. Introduce the following functional:

$$[x, y] = \sum_{n,m\in A} K_{n,m} a_n \overline{b_m}, \tag{11}$$

for $x, y \in L$,

$$x = \sum_{n\in A} a_n x_n, \quad y = \sum_{m\in A} b_m x_m, \quad a_n, b_m \in \mathbb{C}.$$

The space V with $[\cdot,\cdot]$ will be a pre-Hilbert space. Factorizing and making the completion we obtain the required space H (see [1, p. 10–11]). □

Let $K = (K_{n,m})_{n,m\in\mathbb{Z}}$ be a positive definite kernel which satisfies difference relation (5). Let H and $\{x_n\}_{n\in\mathbb{Z}}$ be the Hilbert space and the sequence provided by Theorem 1. Set

$$x_n' := \sum_{k=-r^-}^{r^+} \alpha_{n,k} x_{n+k}, \quad n \in \mathbb{Z}. \tag{12}$$

By virtue of (10) and (5) we get

$$(x_n', x_m) = (x_n, x_m'), \quad n, m \in \mathbb{Z}. \tag{13}$$

Suppose that $n, m \in \mathbb{Z}$ are such that $x_n = x_m$. In this case, using (13) we can write

$$(x_n', x_k) = (x_n, x_k') = (x_m, x_k') = (x_m', x_k),$$
$$(x_n' - x_m', x_k) = 0, \quad k \in \mathbb{Z}.$$

Since, by Theorem 1, $\operatorname{span}\{x_n\}_{n\in\mathbb{Z}} = H$, we conclude that $x_n' = x_m'$.

Define an operator A in the following way:

$$A x_n = x_n', \quad n \in \mathbb{Z}. \tag{14}$$

Let $L = \operatorname{Lin}\{x_n\}_{n\in\mathbb{Z}}$. Choose an arbitrary $x \in L$. Suppose that

$$x = \sum_{k\in\mathbb{Z}} \alpha_k x_k, \quad x = \sum_{j\in\mathbb{Z}} \beta_j x_j, \quad \alpha_k, \beta_j \in \mathbb{C}. \tag{15}$$

Then

$$\Big(\sum_{k\in\mathbb{Z}} \alpha_k x_k', x_m\Big) = \sum_{k\in\mathbb{Z}} \alpha_k (x_k', x_m) = \sum_{k\in\mathbb{Z}} \alpha_k (x_k, x_m') = (x, x_m'),$$
$$\Big(\sum_{j\in\mathbb{Z}} \beta_j x_j', x_m\Big) = \sum_{j\in\mathbb{Z}} \beta_j (x_j', x_m) = \sum_{j\in\mathbb{Z}} \beta_j (x_j, x_m') = (x, x_m'),$$

and therefore we get

$$\Big(\sum_{k\in\mathbb{Z}} \alpha_k x_k' - \sum_{j\in\mathbb{Z}} \beta_j x_j', x_m\Big) = 0, \quad m \in \mathbb{Z}.$$

Since $\operatorname{span}\{x_n\}_{n\in\mathbb{Z}} = H$, we obtain

$$\sum_{k\in\mathbb{Z}} \alpha_k x_k' = \sum_{j\in\mathbb{Z}} \beta_j x_j'. \tag{16}$$

Thus, we can correctly define an operator A on L in the following way:

$$Ax = \sum_{k\in\mathbb{Z}} \alpha_k x_k', \tag{17}$$

for

$$x \in L, \quad x = \sum_{k\in\mathbb{Z}} \alpha_k x_k, \quad \alpha_k \in \mathbb{C}.$$

For arbitrary

$$x = \sum_{k\in\mathbb{Z}} a_k x_k \in L, \quad y = \sum_{j\in\mathbb{Z}} b_j x_j \in L, \quad a_k, b_j \in \mathbb{C},$$

we have

$$\begin{aligned}(Ax, y) &= \Big(\sum_{k\in\mathbb{Z}} a_k x_k', \sum_{j\in\mathbb{Z}} b_j x_j\Big) = \sum_{k,j\in\mathbb{Z}} a_k \overline{b_j} (x_k', x_j) \\ &= \sum_{k,j\in\mathbb{Z}} a_k \overline{b_j} (x_k, x_j') = \Big(\sum_{k\in\mathbb{Z}} a_k x_k, \sum_{j\in\mathbb{Z}} b_j x_j'\Big) = (x, Ay).\end{aligned}$$

So, the operator A is symmetric. Its closure we denote by A'. There exists a self-adjoint extension $\widetilde{A} \supseteq A'$ in a space $\widetilde{H} \supseteq H$, see [2].

Choose an arbitrary $a \in \mathbb{Z}$ and let $\chi_{k;n}(\lambda)$ be a solution of (7), (8). From the definition of the operator A we see that

$$x_{n+r^+} = \frac{1}{\alpha_{n,r^+}} \Big(Ax_n - \sum_{l=-r^-}^{r^+-1} \alpha_{n,l} x_{n+l}\Big), \quad n = a, a+1, \dots, \tag{18}$$

$$x_{n-r^-} = \frac{1}{\alpha_{n,-r^-}} \Big(Ax_n - \sum_{l=-r^-+1}^{r^+} \alpha_{n,l} x_{n+l}\Big), \quad n = a-1, a-2, \dots. \tag{19}$$

On the other hand, $\chi_{k;n}(\lambda)$ satisfy the difference equations

$$u_{n+r^+} = \frac{1}{\alpha_{n,r^+}} \Big(\lambda u_n - \sum_{l=-r^-}^{r^+-1} \alpha_{n,l} u_{n+l}\Big), \quad n = a, a+1, \dots, \tag{20}$$

$$u_{n-r^-} = \frac{1}{\alpha_{n,-r^-}} \Big(\lambda u_n - \sum_{l=-r^-+1}^{r^+} \alpha_{n,l} u_{n+l}\Big), \quad n = a-1, a-2, \dots. \tag{21}$$

Notice that $AL \subseteq L$ and that $\chi_{k;n}(\lambda)$ are polynomials of λ. Set

$$x_n^{[k]} = \chi_{k;n}(A) x_k, \quad n \in \mathbb{Z}, \quad k = 0, 1, \dots, r-1. \tag{22}$$

From (20), (21) it follows that $\{x_n^{[k]}\}_{n\in\mathbb{Z}}$, $k = 0, 1, \dots, r-1$, satisfy relations (18), (19). Thus, the elements

$$\widetilde{x}_n := \sum_{k=0}^{r-1} x_n^{[k]}, \quad n \in \mathbb{Z}, \tag{23}$$

are also solutions of (18), (19). Since

$$\widetilde{x}_n = x_n, \quad n = a - r^-, a - r^- + 1, \dots, a + r^+ - 1,$$

using (18), (19) we get $\widetilde{x}_n = x_n$, $n \in \mathbb{Z}$. Thus, we get

$$x_n = \sum_{k=0}^{r-1} x_n^{[k]} = \sum_{k=0}^{r-1} \chi_{k;n}(A) x_k, \quad n \in \mathbb{Z}. \tag{24}$$

Let

$$\widetilde{A} = \int_{\mathbb{R}} \lambda \, dE_\lambda, \tag{25}$$

be the spectral decomposition of $\widetilde{A}$, where $\{E_\lambda\}$ is the resolution of unity of $\widetilde{A}$. From (10) and (24) we obtain

$$\begin{aligned} K_{n,m} = (x_n, x_m) &= \Big(\sum_{k=0}^{r-1} \chi_{k;n}(A) x_k, \sum_{l=0}^{r-1} \chi_{l;m}(A) x_l \Big) \\ &= \sum_{k,l=0}^{r-1} \Big(\chi_{k;n}(\widetilde{A}) x_k, \chi_{l;m}(\widetilde{A}) x_l \Big) = \sum_{k,l=0}^{r-1} \Big(\overline{\chi_{l;m}}(\widetilde{A}) \chi_{k;n}(\widetilde{A}) x_k, x_l \Big) \\ &= \sum_{k,l=0}^{r-1} \int_{\mathbb{R}} \chi_{k;n}(\lambda) \overline{\chi_{l;m}(\lambda)} \, d(E_\lambda x_k, x_l). \end{aligned}$$

If we set $\sigma_{k,l}(\lambda) := (E_\lambda x_k, x_l)$, we get relation (6). Since

$$\begin{aligned} |((E_\lambda - E_\mu) x_k, x_l)| &= |((E_\lambda - E_\mu) x_k, (E_\lambda - E_\mu) x_l)| \le \|(E_\lambda - E_\mu) x_k\| \|(E_\lambda - E_\mu) x_l\| \\ &= \sqrt{((E_\lambda - E_\mu) x_k, x_k)((E_\lambda - E_\mu) x_l, x_l)}, \quad \lambda \ge \mu, \end{aligned}$$

we can obtain that all main minors of the matrix $((E_\lambda - E_\mu) x_k, x_l)_{l=0}^{r-1}$ are non-negative. Thus, $((E_\lambda - E_\mu) x_k, x_l)_{l=0}^{r} \ge 0$.

2.2. **Case** $A = \mathbb{Z}_+$**.** Let us consider the following operator L:

$$(Lu)_n = \sum_{j=0}^{n+r^+} d_{n,j} u_j, \quad n \in \mathbb{Z}_+, \tag{26}$$

where $d_{n,j} \in \mathbb{C}$, $d_{n,n+r^+} \neq 0$, $r^+ \in \mathbb{N}$. This relation can be considered on finite complex sequences $(u_k)_{k \in \mathbb{Z}_+}$ from l^2, where l^2 is the standard space of square summable complex sequences $(u_k)_{k \in \mathbb{Z}_+}$. We define an operator $\overline{L}$ as

$$(\overline{L}u)_n = \sum_{j=0}^{n+r^+} \overline{d_{n,j}} u_j, \quad n \in \mathbb{Z}_+. \tag{27}$$

Suppose that a positive definite kernel $K = (K_{n,m})_{n,m \in \mathbb{Z}_+}$ satisfies the relation (4), which in the coordinate form is

$$\sum_{j=0}^{n+r^+} d_{n,j} K_{j,m} = \sum_{l=0}^{m+r^+} \overline{d_{m,l}} K_{n,l}, \quad n, m \in \mathbb{Z}_+. \tag{28}$$

Let H and $\{x_k\}_{k \in \mathbb{Z}_+}$ be from Theorem 1. We set

$$x_n' = \sum_{j=0}^{n+r^+} d_{n,j} x_j, \quad n \in \mathbb{Z}_+. \tag{29}$$

From (10) and (28) we get

$$(x_n', x_m) = (x_n, x_m'), \quad n, m \in \mathbb{Z}_+. \tag{30}$$

Let $L = \mathrm{Lin}\{x_n\}_{n\in\mathbb{Z}_+}$. Choose an arbitrary $x \in L$. Suppose that

$$x = \sum_{k\in\mathbb{Z}_+} \alpha_k x_k, \quad x = \sum_{j\in\mathbb{Z}_+} \beta_j x_j, \quad \alpha_k, \beta_j \in \mathbb{C}. \tag{31}$$

Like in the previous case (see considerations after (15)) we get

$$\sum_{k\in\mathbb{Z}_+} \alpha_k x_k' = \sum_{j\in\mathbb{Z}_+} \beta_j x_j'. \tag{32}$$

Thus, we can correctly define an operator A on L in the following way:

$$Ax = \sum_{k\in\mathbb{Z}_+} \alpha_k x_k' \tag{33}$$

for

$$x \in L, \quad x = \sum_{k\in\mathbb{Z}_+} \alpha_k x_k, \quad \alpha_k \in \mathbb{C}.$$

For arbitrary

$$x = \sum_{k\in\mathbb{Z}_+} a_k x_k \in L, \quad y = \sum_{j\in\mathbb{Z}_+} b_j x_j \in L, \quad a_k, b_j \in \mathbb{C},$$

we have

$$(Ax, y) = \Big(\sum_{k\in\mathbb{Z}_+} a_k x_k', \sum_{j\in\mathbb{Z}_+} b_j x_j \Big) = \sum_{k,j\in\mathbb{Z}_+} a_k \overline{b_j} (x_k', x_j)$$
$$= \sum_{k,j\in\mathbb{Z}_+} a_k \overline{b_j} (x_k, x_j') = \Big(\sum_{k\in\mathbb{Z}_+} a_k x_k, \sum_{j\in\mathbb{Z}_+} b_j x_j' \Big) = (x, Ay).$$

So, the operator A is symmetric. There exists a self-adjoint extension $\widetilde{A} \supseteq A$ in a Hilbert space $\widetilde{H} \supseteq H$ with resolution (25).

Let $\chi_{k;n}(\lambda)$ be a solution of the equation

$$Lu = \lambda u, \quad (\lambda \in \mathbb{R}), \tag{34}$$

with the initial conditions

$$\chi_{k;n}(\lambda) = \delta_{k,n}, \quad n, k = 0, 1, \ldots, r^+ - 1. \tag{35}$$

From (29) it follows that

$$x_{n+r^+} = \frac{1}{d_{n,n+r^+}} \Big(Ax_n - \sum_{j=0}^{n+r^+-1} d_{n,j} x_j \Big), \quad n \in \mathbb{Z}_+. \tag{36}$$

The functions $\chi_{k;n}(\lambda)$ satisfy

$$\chi_{k,n+r^+} = \frac{1}{d_{n,n+r^+}} \Big(\lambda \chi_{k,n} - \sum_{j=0}^{n+r^+-1} d_{n,j} \chi_{k,j} \Big), \quad n \in \mathbb{Z}_+. \tag{37}$$

Notice that $AL \subseteq L$ and that $\chi_{k;n}(\lambda)$ are polynomials of λ. Thus we can define

$$x_n^{[k]} = \chi_{k;n}(A) x_k, \quad n \in \mathbb{Z}_+, \quad k = 0, 1, \ldots, r^+ - 1. \tag{38}$$

From (37) it follows that $\{x_n^{[k]}\}_{n\in\mathbb{Z}_+}$, $k = 0, 1, \ldots, r^+ - 1$, satisfy relations (36). So, the elements

$$\widetilde{x}_n := \sum_{k=0}^{r^+-1} x_n^{[k]}, \tag{39}$$

are also solutions of (36). Since

$$\widetilde{x}_n = x_n, \quad n = 0, 1, \dots, r^+ - 1,$$

using (36) we obtain $\widetilde{x}_n = x_n$, $n \in \mathbb{Z}_+$. Thus, we have

$$x_n = \sum_{k=0}^{r^+-1} x_n^{[k]} = \sum_{k=0}^{r^+-1} \chi_{k;n}(A) x_k, \quad n \in \mathbb{Z}_+. \tag{40}$$

From the latter relation we obtain that

$$\begin{aligned} K_{n,m} = (x_n, x_m) &= \Big(\sum_{k=0}^{r^+-1} \chi_{k;n}(A) x_k, \sum_{l=0}^{r^+-1} \chi_{l;m}(A) x_l \Big) \\ &= \sum_{k,l=0}^{r^+-1} \Big(\chi_{k;n}(\widetilde{A}) x_k, \chi_{l;m}(\widetilde{A}) x_l \Big) = \sum_{k,l=0}^{r^+-1} \Big(\overline{\chi_{l;m}}(\widetilde{A}) \chi_{k;n}(\widetilde{A}) x_k, x_l \Big) \\ &= \sum_{k,l=0}^{r^+-1} \int_{\mathbb{R}} \chi_{k;n}(\lambda) \overline{\chi_{l;m}(\lambda)}\, d(E_\lambda x_k, x_l). \end{aligned}$$

Thus, we get the following theorem.

Theorem 2. *Let $K = (K_{n,m})_{n,m \in \mathbb{Z}_+}$ be a positive definite kernel. It satisfies relation (28) if and only if there exists a representation*

$$K_{n,m} = \sum_{k,l=0}^{r^+-1} \int_{\mathbb{R}} \chi_{k;n}(\lambda) \chi_{l;m}(\lambda)\, d\sigma_{k,l}(\lambda), \tag{41}$$

where $\chi_{k;l}(\lambda)$ are solutions of (34), (35), and $(\sigma_{k,l}(\lambda))_{k,l=0}^{r^+-1}$ is a non-decreasing matrix-valued function on $\mathbb{R}$ the elements of which have bounded variation on $\mathbb{R}$. In (41) one understands the improper Riemann-Stieltjes integrals.

Proof. Necessity was shown above. Sufficiency follows from (34). □

In the case $d_{n,j} = 0$, for $j < n - r^-$, $n \in \mathbb{Z}_+$, with some $r^- \in \mathbb{Z}_+$, we obtain the well-known result, see. [1, Ch. 8, Theorem 5.2].

Example 2.1. Consider the classical Hamburger moment problem (see, e.g., [7]). The problem is to find a non-decreasing left-continuous bounded function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^k d\sigma(x) = s_k, \quad k \in \mathbb{Z}_+, \tag{42}$$

where $\{s_k\}_{k=0}^{\infty}$ is a given sequence of real numbers.

Sequences $\{s_k\}_{k=0}^{\infty}$ for which this problem has a solution are called *moment sequences.* Solutions of the Hamburger moment problem are said to be equal if they differ by a constant (notice that such solutions produce the same positive Borel measure on $\mathbb{R}$). We will seek for solutions such that $\sigma(0) = 0$. The Hamburger moment problem is said to be *determinate* if the solution is unique and *indeterminate* in the opposite case.

Let $\{s_k\}_{k=0}^{\infty}$ be a moment sequence. Consider $K = (K_{n,m})_{n,m \in \mathbb{Z}_+}$, with $K_{n,m} = s_{n+m}$. For an arbitrary complex polynomial $p(x) = \sum_{n=0}^{\infty} \xi_n x^n$, where $\xi_n \in \mathbb{C}$ (all but finite number of ξ_n are zero), we get

$$0 \le \int_{\mathbb{R}} |p(x)|^2 d\sigma(x) = \sum_{n,m=0}^{\infty} \xi_n \overline{\xi_m} \int_{\mathbb{R}} x^{n+m} d\sigma(x) = \sum_{n,m=0}^{\infty} s_{n+m} \xi_n \overline{\xi_m}.$$

Thus, the kernel K is positive definite.

On the other hand, consider an arbitrary sequence $\{s_k\}_{k=0}^{\infty}$. Suppose that the kernel $K = (s_{n+m})_{n,m\in\mathbb{Z}_+}$ is positive definite. In such a case, the corresponding sequence of moments is called *positive*. There exists a sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ in a Hilbert space H such that

$$(x_n, x_m) = K_{n,m}, \quad n, m \in \mathbb{Z}_+, \tag{43}$$

and $\operatorname{span}\{x_n\}_{n\in\mathbb{Z}_+} = H$. Let us define an operator A on $L := \operatorname{Lin}\{x_n\}_{n\in\mathbb{Z}_+}$ in the following way:

$$Ax = \sum_{k\in\mathbb{Z}_+} \alpha_k x_{k+1}, \tag{44}$$

for

$$x \in L, \quad x = \sum_{k\in\mathbb{Z}_+} \alpha_k x_k, \quad \alpha_k \in \mathbb{C}.$$

This definition is correct. If there exists another representation for x

$$x = \sum_{l\in\mathbb{Z}_+} \beta_l x_l, \quad \beta_l \in \mathbb{C},$$

then

$$\begin{aligned}\Big(\sum_{k\in\mathbb{Z}_+} \alpha_k x_{k+1}, x_m \Big) &= \sum_{k\in\mathbb{Z}_+} \alpha_k (x_{k+1}, x_m) = \sum_{k\in\mathbb{Z}_+} \alpha_k K_{k+1,m} \\ &= \sum_{k\in\mathbb{Z}_+} \alpha_k K_{k,m+1} = \sum_{k\in\mathbb{Z}_+} \alpha_k (x_k, x_{m+1}) = (x, x_{m+1}), \quad m \in \mathbb{Z}_+,\end{aligned}$$

and, analogously, we have

$$\Big(\sum_{l\in\mathbb{Z}_+} \beta_l x_{l+1}, x_m \Big) = (x, x_{m+1}), \quad m \in \mathbb{Z}_+.$$

Therefore, we get $\sum_{k\in\mathbb{Z}_+} \alpha_k x_{k+1} = \sum_{l\in\mathbb{Z}_+} \beta_l x_{l+1}$.
For arbitrary

$$x = \sum_{k\in\mathbb{Z}_+} a_k x_k \in L, \quad y = \sum_{j\in\mathbb{Z}_+} b_j x_j \in L, \quad a_k, b_j \in \mathbb{C},$$

we have

$$\begin{aligned}(Ax, y) &= \Big(\sum_{k\in\mathbb{Z}_+} a_k x_{k+1}, \sum_{j\in\mathbb{Z}_+} b_j x_j \Big) = \sum_{k,j\in\mathbb{Z}_+} a_k \overline{b_j} (x_{k+1}, x_j) \\ &= \sum_{k,j\in\mathbb{Z}_+} a_k \overline{b_j} (x_k, x_{j+1}) = \Big(\sum_{k\in\mathbb{Z}_+} a_k x_k, \sum_{j\in\mathbb{Z}_+} b_j x_{j+1} \Big) = (x, Ay).\end{aligned}$$

Thus, the operator A is symmetric. There exists a self-adjoint extension $\widetilde{A} \supseteq A$ in a Hilbert space $\widetilde{H} \supseteq H$. Let $\widetilde{A} = \int_{\mathbb{R}} \lambda \, d\widetilde{E}_\lambda$, be the spectral decomposition of $\widetilde{A}$, where $\{\widetilde{E}_\lambda\}$ is the left-continuous orthogonal resolution of unity of $\widetilde{A}$. From the equality

$$Ax_n = x_{n+1}, \quad n \in \mathbb{Z}_+,$$

by induction we get

$$x_n = A^n x_0, \quad n \in \mathbb{Z}_+. \tag{45}$$

Since $AL \subseteq L$, by induction we obtain that

$$A^n x = \widetilde{A}^n x, \quad x \in L.$$

Therefore we get

$$x_n = \widetilde{A}^n x_0 = \int_{\mathbb{R}} \lambda^n d\widetilde{E}_\lambda x_0, \quad n \in \mathbb{Z}_+. \tag{46}$$

Consequently, we obtain that

$$K_{n,m} = (x_n, x_m) = \int_{\mathbb{R}} \lambda^{n+m} d(P_H^{\widetilde{H}} \widetilde{E}_\lambda x_0, x_0), \quad n, m \in \mathbb{Z}_+. \tag{47}$$

In particular, we can write

$$s_n = K_{n,0} = \int_{\mathbb{R}} \lambda^n d(P_H^{\widetilde{H}} \widetilde{E}_\lambda x_0, x_0), \quad n \in \mathbb{Z}_+. \tag{48}$$

That means that the moment problem has a solution $(P_H^{\widetilde{H}} \widetilde{E}_\lambda x_0, x_0)$. So, *the Hamburger moment problem has a solution if and only if the kernel* $K = (s_{n+m})_{n,m\in\mathbb{Z}_+}$ *is positive definite.*

Let $\sigma(\lambda)$ be an arbitrary solution of the Hamburger moment problem above. Consider the corresponding space L^2_σ. Let Q_σ be an operator of multiplication by an independent variable in L^2_σ. It is defined for $f(x) \in L^2_\sigma$ such that $xf(x) \in L^2_\sigma$. This operator is self-adjoint (see, e.g., [2, p. 158]). Denote by $\mathbb{P}_\sigma$ a set of all polynomials in L^2_σ (more precisely, it is a set of all classes of equivalence in L^2_σ, which contain at least one polynomial). The closure of $\mathbb{P}_\sigma$ we denote by $L^2_{\sigma,0}$. For $f(x) \in \mathbb{P}_\sigma$, $f(x) = \sum_{k=0}^\infty \alpha_k x^k$, $\alpha_k \in \mathbb{C}$, (all but finite number of α_k are zero), we set

$$Vf = \sum_{k=0}^\infty \alpha_k x_k. \tag{49}$$

If there are two polynomials in the same class of equivalence, that is $f(x) = \sum_{k=0}^\infty \alpha_k x^k$, $g(x) = \sum_{n=0}^\infty \beta_n x^n$, $\alpha_k, \beta_n \in \mathbb{C}$, and

$$0 = \int_{\mathbb{R}} \Big| \sum_{k=0}^\infty (\alpha_k - \beta_k) x^k \Big|^2 d\sigma(x) = \int_{\mathbb{R}} \sum_{k,n=0}^\infty (\alpha_k - \beta_k)\overline{(\alpha_n - \beta_n)} x^{k+n} d\sigma(x)$$
$$= \sum_{k,n=0}^\infty (\alpha_k - \beta_k)\overline{(\alpha_n - \beta_n)} s_{k+n} = \Big\| \sum_{k=0}^\infty (\alpha_k - \beta_k) x_k \Big\|_H,$$

we obtain $\sum_{k=0}^\infty \alpha_k x_k = \sum_{k=0}^\infty \beta_k x_k$. Thus, the operator V is a correctly defined linear operator from $\mathbb{P}_\sigma$ to H. From (49) it follows that V maps $\mathbb{P}_\sigma$ on the whole set $L = \mathrm{Lin}\{x_k\}_{k\in\mathbb{Z}_+}$. For arbitrary $f(x), g(x) \in \mathbb{P}_\sigma$, $f(x) = \sum_{k=0}^\infty \alpha_k x^k$, $g(x) = \sum_{n=0}^\infty \beta_n x^n$, $\alpha_k, \beta_n \in \mathbb{C}$, we can write

$$(f,g)_\sigma = \sum_{k,n=0}^\infty \alpha_k \overline{\beta_n} (x^k, x^n)_\sigma = \sum_{k,n=0}^\infty \alpha_k \overline{\beta_n} s_{k+n} = \sum_{k,n=0}^\infty \alpha_k \overline{\beta_n} (x_k, x_n)_H$$
$$= \Big(\sum_{k=0}^\infty \alpha_k x_k, \sum_{n=0}^\infty \beta_n x_n \Big) = (Vf, Vg)_H.$$

By continuity, we extend the operator V to an isometric operator from $L^2_{\sigma,0}$ on H. Let $L^2_{\sigma,1} := L^2_\sigma \ominus L^2_{\sigma,0}$. The operator $U := V \oplus E_{L^2_{\sigma,1}}$ maps isometrically $L^2_\sigma = L^2_{\sigma,0} \oplus L^2_{\sigma,1}$ on $\widehat{H} := H \oplus L^2_{\sigma,1}$.

Let us consider an operator $\widehat{A} := UQ_\sigma U^{-1}$. It is a self-adjoint operator in $\widehat{H}$ isomorphic to the operator Q_σ. Notice that $\widehat{A} \supseteq A$. In fact, $\widehat{A} x_k = UQ_\sigma U^{-1} x_k = UQ_\sigma x^k = Ux^{k+1} = x_{k+1}$, and by linearity we obtain the required result. Let $\{\widehat{E}_\lambda\}_{\lambda\in\mathbb{R}}$ be a left-continuous resolution of unity of the operator $\widehat{A}$. Notice that $E_{Q,\lambda} := U^{-1}\widehat{E}_\lambda U$, $\lambda \in \mathbb{R}$, is an orthogonal (left-continuous) resolution of unity of Q_σ. Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$ and write

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{\lambda - z}\, d(\widehat{E}_\lambda x_0, x_0)_{\widehat{H}} &= \left(\int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\widehat{E}_\lambda x_0, x_0 \right)_{\widehat{H}} \\ &= \left(U^{-1} \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\widehat{E}_\lambda x_0, U^{-1}x_0 \right)_\sigma = \left(\int_{\mathbb{R}} \frac{1}{\lambda - z}\, dU^{-1}\widehat{E}_\lambda U 1, 1 \right)_\sigma \\ &= \left(\int_{\mathbb{R}} \frac{1}{\lambda - z}\, dE_{Q,\lambda} 1, 1 \right)_\sigma = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d(E_{Q,\lambda}1, 1)_\sigma. \end{aligned} \tag{50}$$

By the Stieltjes-Perron inversion formula (see, e.g., [7]) we conclude that

$$(E_{Q,\lambda}1, 1)_\sigma = (\widehat{E}_\lambda x_0, x_0)_{\widehat{H}} = (P^{\widehat{H}}_H \widehat{E}_\lambda x_0, x_0)_H.$$

Notice that $E_{Q,\lambda} f(t) = \chi_{[-\infty,\lambda)}(t) f(t)$, $f \in L^2_\sigma$, where $\chi_{(-\infty,\lambda)}(t)$ is the characteristic function of an interval $[-\infty, \lambda)$, see, e.g, [2, p. 267]. Thus, we have

$$(E_{Q,\lambda}1, 1)_\sigma = \int_{\mathbb{R}} \chi_{[-\infty,\lambda)}(t)\, d\sigma(t) = \int_{-\infty}^{\lambda} d\sigma(t) = \sigma(\lambda),$$

and therefore

$$\sigma(\lambda) = (P^{\widehat{H}}_H \widehat{E}_\lambda x_0, x_0)_H. \tag{51}$$

Consequently, *all solutions of the Hamburger moment problem are generated by self-adjoint extensions of the corresponding operator A by formula (51), where $\{\widehat{E}_\lambda\}_{\lambda\in\mathbb{R}}$ is an orthogonal (left-continuous) resolution of unity of an extension $\widehat{A}$ in a Hilbert space $\widehat{H} \supseteq H$.*

For $x \in L$, $x = \sum_{k=0}^\infty c_k x_k$, $c_k \in \mathbb{C}$, we set

$$Jx := \sum_{k=0}^\infty \overline{c_k} x_k. \tag{52}$$

If there exists another representation $x = \sum_{k=0}^\infty d_k x_k$, $d_k \in \mathbb{C}$, then

$$\begin{aligned}\Big\| \sum_{k=0}^\infty \overline{c_k} x_k - \sum_{k=0}^\infty \overline{d_k} x_k \Big\|^2 &= \Big\| \sum_{k=0}^\infty \overline{(c_k - d_k)} x_k \Big\|^2 = \sum_{k,n=0}^\infty \overline{(c_k - d_k)}(c_n - d_n)(x_k, x_n) \\ &= \sum_{k,n=0}^\infty \overline{(c_k - d_k)}(c_n - d_n) s_{n+k} = \sum_{k,n=0}^\infty \overline{(c_k - d_k)}(c_n - d_n)(x_n, x_k) \\ &= \Big(\sum_{n=0}^\infty (c_n - d_n) x_n, \sum_{k=0}^\infty (c_k - d_k) x_k \Big) = \Big\| \sum_{k=0}^\infty c_k x_k - \sum_{k=0}^\infty d_k x_k \Big\|^2 = 0.\end{aligned}$$

Thus, J is a correctly defined antilinear operator on L. Notice that

$$J^2 u = u, \quad u \in L. \tag{53}$$

For arbitrary $u, v \in L$, $u = \sum_{k=0}^\infty c_k x_k$, $v = \sum_{n=0}^\infty d_n x_n$, $c_k, d_n \in \mathbb{C}$, we can write

$$(Ju, Jv) = \Big(\sum_{k=0}^\infty \overline{c_k} x_k, \sum_{n=0}^\infty \overline{d_n} x_n \Big) = \sum_{k,n=0}^\infty \overline{c_k} d_n (x_k, x_n) = \sum_{k,n=0}^\infty \overline{c_k} d_n s_{k+n} \tag{54}$$

$$= \sum_{k,n=0}^\infty \overline{c_k} d_n (x_n, x_k) = \Big(\sum_{n=0}^\infty d_n x_n, \sum_{k=0}^\infty c_k x_k \Big) = (v, u); \tag{55}$$

$$(Ju, Jv) = (v, u), \quad u, v \in L. \tag{56}$$

In particular, this means that $\|Ju\| = \|u\|$, $u \in L$. By continuity, the operator J can be extended to a bounded operator J in H. It is not hard to verify that it will be an antilinear operator in H and properties (54), (56) will be true on the whole H. Such an operator is called a conjugation (see [11]).

For an arbitrary $u \in L$, $u = \sum_{k=0}^{\infty} c_k x_k$, $c_k \in \mathbb{C}$, we can write

$$AJu = A \sum_{k=0}^{\infty} \overline{c_k} x_k = \sum_{k=0}^{\infty} \overline{c_k} x_{k+1},$$

$$JAu = J \sum_{k=0}^{\infty} c_k x_{k+1} = \sum_{k=0}^{\infty} \overline{c_k} x_{k+1},$$

and therefore A and J commute. In this case, the operator A is called real with respect to the conjugation J ([11]). Let $\overline{A}$ be the closure of a symmetric operator A. It is easy to check that $\overline{A}$ is real with respect to J (symmetric) operator. Consequently, defect numbers of $\overline{A}$ are equal (see [11, Theorem 9.14]).

Choose an arbitrary $u \in L$, $u = \sum_{k=0}^{\infty} c_k x_k$, $c_k \in \mathbb{C}$. Suppose that $c_k = 0$, $k > N$, for some $N \in \mathbb{N}$. Consider the following system of linear equations:

$$\begin{cases} -z d_0 = c_0, \\ d_{k-1} - z d_k = c_k, \quad k = 1, 2, 3, \dots, \end{cases} \tag{57}$$

where $\{d_k\}_{k \in \mathbb{Z}_+}$ are unknown complex numbers, $z \in \mathbb{C} \backslash \mathbb{R}$ is a fixed parameter. Set

$$\begin{aligned} d_k &= 0, \quad k \geq N, \\ d_{k-1} &= c_k + z d_k, \quad k = 1, 2, \dots, N. \end{aligned} \tag{58}$$

For such numbers $\{d_k\}_{k \in \mathbb{Z}_+}$, equations in (57) with $k \in \mathbb{N}$ are satisfied. Only the first equation is not satisfied. Set $v = \sum_{k=0}^{\infty} d_k x_k$, $v \in L$. Notice that

$$(A - zE_H)v = \sum_{k=0}^{\infty} (d_{k-1} - z d_k) x_k, \quad d_{-1} := 0.$$

By the construction of d_k we have

$$\begin{aligned} (A - zE_H)v - u &= \sum_{k=0}^{\infty} (d_{k-1} - z d_k) x_k - \sum_{k=0}^{\infty} c_k x_k \\ &= \sum_{k=0}^{N} (d_{k-1} - z d_k - c_k) x_k = (-z d_0 - c_0) x_0, \\ u = (A - zE_H)v + (c_0 + z d_0) x_0, &\quad u \in L. \end{aligned} \tag{59}$$

Set $H_z := \overline{(A - zE_H)L} = (\overline{A} - zE_H)D(\overline{A})$, and $H_0 := \mathrm{span}\{x_0\}$. If $H_z = H$, then the defect numbers of $\overline{A}$ are equal to 0.

If $H_z \neq H$, then we choose an arbitrary orthonormal basis in H_z: $\{\varepsilon_n\}_{n \in \mathbb{N}}$. Set $\varepsilon_0 := \frac{x_0 - P_{H_z} x_0}{\|x_0 - P_{H_z} x_0\|}$. From (59) it follows that $L \subseteq \mathrm{span}\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$, and therefore $H = \mathrm{span}\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$. Thus, $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ is an orthonormal basis in H. If $x \in H$, $x \perp H_z$, we obtain $x = \alpha \varepsilon_0$, $\alpha \in \mathbb{C}$. So, the defect numbers of $\overline{A}$ are equal to 1.

Let $\widehat{A}$ be a self-adjoint extension of A in a Hilbert space $\widehat{H}$. Let $R_z(\widehat{A})$ be the resolvent of $\widehat{A}$ and $\{\widehat{E}_\lambda\}_{\lambda \in \mathbb{R}}$ be an orthogonal left-continuous resolution of unity of $\widehat{A}$. Recall that the operator-valued function $\mathbf{R}_z = P_H^{\widehat{H}} R_z(\widehat{A})$ is called a generalized resolvent of A, $z \in \mathbb{C} \backslash \mathbb{R}$. The function $\mathbf{E}_\lambda = P_H^{\widehat{H}} \widehat{E}_\lambda$, $\lambda \in \mathbb{R}$, is called a spectral function of a symmetric operator A. There exists a one-to-one correspondence between generalized resolvents and spectral functions established by the following relation:

$$(\mathbf{R}_z f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, d(\mathbf{E}_\lambda f, g)_H, \quad f, g \in H, \quad z \in \mathbb{C} \backslash \mathbb{R}. \tag{60}$$

In the case $\widehat{H} = H$, the generalized resolvent is called orthogonal and the corresponding spectral function of A is called orthogonal.

Formula (51) shows that *solutions of the Hamburger moment problem are produced by spectral functions of the corresponding operator* A.

It is known that for a self-adjoint operator the spectral function is unique [2]. So, *in the case of the deficiency index* $(0,0)$ *the Hamburger moment problem is determinate.* The solution is the spectral function of the self-adjoint operator $\overline{A}$.

Consider the case of the deficiency index $(1,1)$. First, let us show that *in the case of the deficiency index* $(1,1)$ *the Hamburger moment problem is indeterminate.* Assume to the contrary that for any two self-adjoint extensions $A_j \supseteq A$, in Hilbert spaces $H_j \supseteq H$, we have

$$(P_H^{H_1} E_{1,\lambda} x_0, x_0)_H = (P_H^{H_2} E_{2,\lambda} x_0, x_0)_H, \quad \lambda \in \mathbb{R}, \tag{61}$$

where $\{E_{j,\lambda}\}_{\lambda\in\mathbb{R}}$ are orthogonal left-continuous resolutions of unity of operators A_j, $j = 1,2$. Denote by $R_{j,\lambda}$ the resolvent of A_j, and set $\mathbf{R}_{j,\lambda} := P_H^{H_j} R_{j,\lambda}$, $j = 1,2$. From (60), (61) it follows that

$$(\mathbf{R}_{1,\lambda} x_0, x_0)_H = (\mathbf{R}_{2,\lambda} x_0, x_0)_H, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{62}$$

Choose an arbitrary $z \in \mathbb{C}\backslash\mathbb{R}$ and consider the space H_z defined as above. Since

$$R_{j,z}(A - zE_H)x = (A_j - zE_{H_j})^{-1}(A_j - zE_{H_j})x = x, \quad x \in L = D(A),$$

we get

$$R_{1,z} y = R_{2,z} y \in H, \quad y \in H_z; \tag{63}$$

$$\mathbf{R}_{1,z} y = \mathbf{R}_{2,z} y, \quad y \in H_z, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{64}$$

We can write

$$\begin{gathered} (\mathbf{R}_{j,z} x_0, u)_H = (R_{j,z} x_0, u)_{H_j} = (x_0, R_{j,\overline{z}} u)_{H_j} = (x_0, \mathbf{R}_{j,\overline{z}} u)_H, \\ u \in H_{\overline{z}}, \quad j = 1,2, \end{gathered} \tag{65}$$

and therefore we get

$$(\mathbf{R}_{1,z} x_0, u)_H = (\mathbf{R}_{2,z} x_0, u)_H, \quad u \in H_{\overline{z}}. \tag{66}$$

By (59) an arbitrary element $x \in L$ can be represented as $x = x_{\overline{z}} + cx_0$, $x_{\overline{z}} \in H_{\overline{z}}$, $c \in \mathbb{C}$. Using (62) and (66) we get

$$(\mathbf{R}_{1,z} x_0, x)_H = (\mathbf{R}_{1,z} x_0, x_{\overline{z}} + cx_0)_H = (\mathbf{R}_{2,z} x_0, x_{\overline{z}} + cx_0)_H = (\mathbf{R}_{2,z} x_0, x)_H.$$

Since $\overline{L} = H$, we obtain

$$\mathbf{R}_{1,z} x_0 = \mathbf{R}_{2,z} x_0, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{67}$$

For an arbitrary $x \in L$, $x = x_z + cx_0$, $x_z \in H_z$, $c \in \mathbb{C}$, using relations (64), (67) we obtain

$$\mathbf{R}_{1,z} x = \mathbf{R}_{1,z}(x_z + cx_0) = \mathbf{R}_{2,z}(x_z + cx_0) = \mathbf{R}_{2,z} x, \quad x \in L, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{68}$$

and

$$\mathbf{R}_{1,z} x = \mathbf{R}_{2,z} x, \quad x \in H, \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{69}$$

On the other hand, using von Neumann's formulas we can choose two different extensions of A inside H. Relation (69) means that their resolvents must coincide. By (60) that means that their resolutions of unity coincide and $A_1 = A_2$. We obtained a contradiction.

Let us describe all solutions in the case of the deficiency index $(1,1)$. We can use the classical Krein's results on a description of all generalized resolvents of a symmetric operator A with equal and finite defect numbers. In particular, we have (see [2, p. 389])

$$(\mathbf{R}_z x_0, x_0)_H = \frac{p_0(z) + p_1(z)\tau(z)}{q_0(z) + q_1(z)\tau(z)}, \tag{70}$$

where p_0, p_1, q_0, q_1 are some known entire functions and $\tau(z) \in \mathcal{N}$. Here $\mathcal{N}$ is a class of analytic functions in $\mathbb{C}_+ = \{z \in \mathbb{C} :\ \operatorname{Im} z > 0\}$, with values in $\mathbb{C}'_+ = \{z \in \mathbb{C} :\ \operatorname{Im} z \geq 0\}$ (including a function $\tau(z) \equiv \infty$). From (60), (70) we get that *all solutions of the Hamburger moment problem in the case of the deficiency index* $(1,1)$ *are obtained from the following relation:*

$$\int_{\mathbb{R}} \frac{1}{x-z}\, d\sigma(x) = \frac{p_0(z) + p_1(z)\tau(z)}{q_0(z) + q_1(z)\tau(z)}, \tag{71}$$

where $\tau(z) \in \mathcal{N}$.

Let us give a sufficient condition for determinacy of the Hamburger moment problem. Recall some known facts on quasianalytic classes of functions (see [1], [5], [6]). Let $[a,b] \subset \mathbb{R}$ be a finite segment, $(m_n)_{n\in\mathbb{Z}_+}$ be a fixed sequence of positive numbers. By $C^\infty([a,b])$ we denote a linear space of complex-valued functions on $[a,b]$ which have derivatives of all orders on $[a,b]$. By $C(m_n)$ we denote a linear set of all functions $f(t) \in C^\infty([a,b])$ such that

$$|f^{(n)}(t)| \leq K_f^n m_n, \quad t \in [a,b], \quad n \in \mathbb{Z}_+, \tag{72}$$

where $K_f > 0$ is a constant depending on f. The class $C(m_n)$ is called *quasianalytic* if equalities

$$f^{(n)}(t_0) = 0, \quad n \in \mathbb{Z}_+,$$

which hold for a function $f \in C(m_n)$ in a point $t_0 \in [a,b]$, imply that $f(t) = 0$, $t \in [a,b]$.

Let B be an operator in a Hilbert space H. A vector $x \in \cap_{n=0}^\infty D(B^n)$ is called *quasianalytic* if the class $C(m_n)$ with $m_n = \|B^n x\|_H$ is quasianalytic. If B is symmetric, a vector $x \in \cap_{n=1}^\infty D(B^n)$ is quasianalytic if and only if (see [5, Chapter 13, Lemma 9.1])

$$\sum_{n=1}^\infty \|B^n x\|_H^{-\frac{1}{n}} = \infty. \tag{73}$$

If B is closed and symmetric, the necessary and sufficient condition for B to be self-adjoint is that in H there exists a set M of quasianalytic vectors such that $\operatorname{span} M = H$ (see [5, Chapter 13, Theorem 9.1]).

Let us apply these results to the operator A defined above for a positive sequence of moments $\{s_n\}_{n\in\mathbb{Z}_+}$. We shall show that *if the class* $C(s_{2n})$ *is quasianalytic then the Hamburger moment problem is determinate.* Suppose that the class $C(s_{2n})$ is quasianalytic (note that s_{2k} should be positive in that case, $k \in \mathbb{Z}_+$). Let us check that $x_k \in H$, $k \in \mathbb{Z}_+$, are quasianalytic vectors for the symmetric operator $\overline{A}$. Notice that $\widetilde{m}_n := \|\overline{A}^n x_k\|_H = \|x_{n+k}\|_H = \sqrt{s_{2n+2k}}$, $n \in \mathbb{Z}_+$. The quasianalyticity of $C(m_{n+k})$ and $C(m_n)$, $k \in \mathbb{Z}_+$ is equivalent, [6, p. 263]. Thus, classes $C(s_{2n+2k})$ are quasianalytic, and vectors $x_k \in H$, $k \in \mathbb{Z}_+$, are therefore quasianalytic. So, $\overline{A}$ is self-adjoint and the moment problem is determinate.

Notice that the quasianalyticity of $C(s_{2n})$ is equivalent to the quasianalyticity of x_0 for $\overline{A}$. By (73) it is equivalent to the condition

$$\infty = \sum_{n=1}^\infty \|\overline{A}^n x_0\|_H^{-\frac{1}{n}} = \sum_{n=1}^\infty \frac{1}{\sqrt[2n]{s_{2n}}}. \tag{74}$$

Thus, *if* $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty$, *then the moment problem is determinate (Carleman's condition).*

If there exists $C > 0$ *such that*

$$s_{2n} \leq C^n (n!)^2, \quad n \in \mathbb{Z}_+, \tag{75}$$

then the moment problem is determinate ([6]*).* In fact, in this case we can write

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{C}\sqrt[n]{(n!)}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{C}n} = \infty,$$

and therefore by Carleman's condition we obtain that the moment problem is determinate.

Let us study some density questions. Suppose that $\sigma(x)$ is a solution of the Hamburger moment problem, generated by a self-adjoint extension $\widetilde{A}$ of the operator A inside the space H or, in other words, by an orthogonal spectral function:

$$\sigma(\lambda) = (\widetilde{E}_\lambda x_0, x_0)_H, \quad \lambda \in \mathbb{R},$$

where $\{\widetilde{E}_\lambda\}_{\lambda \in \mathbb{R}}$ is a resolution of unity of $\widetilde{A}$.

Notice that $\mathrm{span}\{\widetilde{A}^n x_0\}_{n\in\mathbb{Z}_+} = \mathrm{span}\{x_n\}_{n\in\mathbb{Z}_+} = H$, and therefore the operator A has a simple spectrum and x_0 is a generating vector of A (see [2, p. 272]). By virtue of the canonical representation of a self-adjoint operator with a simple spectrum we obtain that there exists an isometric transformation V from L^2_σ on H such that A is isomorphic to Q_σ (see [2, p. 269]). Moreover, $V1 = x_0$. By induction we can see that $x^n = Vx_n$, $n \in \mathbb{Z}_+$. Thus, we obtain $VH = L^2_{\sigma,0}$. In other words, this means that polynomials are dense in L^2_σ.

On the other hand, suppose that $L^2_{\sigma,0} = L^2_\sigma$. In this case, as it was done above we can construct an isometric operator U from L^2_σ on H (in this case $L^2_{\sigma,1} = \{0\}$) and $\widehat{A} := UQ_\sigma U^{-1}$ will be a self-adjoint extension of A inside H. By (51) it follows that $\sigma(\lambda)$ is constructed by a spectral function corresponding to $\widehat{A}$. This spectral function is orthogonal.

Thus, *polynomials are dense in* L^2_σ *if and only if* σ *can be generated by an orthogonal spectral function of the corresponding operator* A. The orthogonal resolvents are known to correspond to constants $\tau(z) = t$, $t \in \mathbb{R} \cup \{\infty\}$ in the formula (70). So, such solutions $\sigma(\lambda)$ correspond to some constant functions $\tau(z)$ (including $\tau(z) = \infty$) in (71).

Let $\sigma(x)$ is a non-decreasing left-continuous bounded function on $\mathbb{R}$ and L^2_σ contains polynomials. We set $s_n := \int_{\mathbb{R}} x^n d\sigma(x)$, $n \in \mathbb{Z}_+$. The sequence $\{s_n\}_{n\in\mathbb{Z}_+}$ is positive. *Polynomials are dense in* L^2_σ *if and only if* $\tau(z) = c$ *is a solution of (71) for some* $c \in \mathbb{R} \cup \{\infty\}$.

Remark. An operator approach was used to study the Hamburger moment problem using the theory of Jacobi matrices in [8], [9], and in [7]. In [10] an operator approach which used the orthogonal polynomials was given. In [6] it was presented an operator approach based on the theory of expansions of operators by their generalized eigenvectors.

3. Difference equations of a "unitary" type

3.1. **Case** $A = \mathbb{Z}$**.** We shall consider the operator L from (2). Suppose that a positive definite kernel $K = (K_{n,m})_{n,m\in\mathbb{Z}}$ satisfies the relation

$$L_n \overline{L}_m K = K. \tag{76}$$

In the coordinate form this relation takes the form

$$\sum_{k,j=-r^-}^{r^+} \alpha_{n,k}\overline{\alpha_{m,j}} K_{n+k,m+j} = K_{n,m}, \quad n,m \in \mathbb{Z}. \tag{77}$$

Let H and $\{x_n\}_{n\in\mathbb{Z}}$ be the Hilbert space and the sequence provided by Theorem 1 for K. Define $\{x_n'\}_{n\in\mathbb{Z}}$ as in (12). Using (10) and (77) we get

$$(x_n', x_m') = (x_n, x_m), \quad n,m \in \mathbb{Z}. \tag{78}$$

Suppose that $n,m \in \mathbb{Z}$ are such that $x_n = x_m$. In this case, using (78) we can write

$$\begin{aligned} 0 = \|x_n - x_m\|^2 &= (x_n,x_n) - (x_n,x_m) - (x_m,x_n) + (x_m,x_m) \\ &= (x_n',x_n') - (x_n',x_m') - (x_m',x_n') + (x_m',x_m') = \|x_n' - x_m'\|^2. \end{aligned}$$

Thus, we get $x_n' = x_m'$. Define an operator A as in (14). Let $L = \mathrm{Lin}\{x_n\}_{n\in\mathbb{Z}}$. Choose an arbitrary $x \in L$. Suppose that

$$x = \sum_{k\in\mathbb{Z}} \alpha_k x_k, \quad x = \sum_{j\in\mathbb{Z}} \beta_j x_j, \quad \alpha_k, \beta_j \in \mathbb{C}. \tag{79}$$

Then

$$\begin{aligned} 0 &= \Big\| \sum_{k\in\mathbb{Z}} \alpha_k x_k - \sum_{j\in\mathbb{Z}} \beta_j x_j \Big\|^2 = \Big(\sum_{k\in\mathbb{Z}} \alpha_k x_k - \sum_{j\in\mathbb{Z}} \beta_j x_j, \sum_{l\in\mathbb{Z}} \alpha_l x_l - \sum_{r\in\mathbb{Z}} \beta_r x_r \Big) \\ &= \sum_{k,l\in\mathbb{Z}} \alpha_k\overline{\alpha_l}(x_k,x_l) - \sum_{k,r\in\mathbb{Z}} \alpha_k\overline{\beta_r}(x_k,x_r) - \sum_{j,l\in\mathbb{Z}} \beta_j\overline{\alpha_l}(x_j,x_l) + \sum_{j,r\in\mathbb{Z}} \beta_j\overline{\beta_r}(x_j,x_r) \\ &= \sum_{k,l\in\mathbb{Z}} \alpha_k\overline{\alpha_l}(x_k',x_l') - \sum_{k,r\in\mathbb{Z}} \alpha_k\overline{\beta_r}(x_k',x_r') - \sum_{j,l\in\mathbb{Z}} \beta_j\overline{\alpha_l}(x_j',x_l') + \sum_{j,r\in\mathbb{Z}} \beta_j\overline{\beta_r}(x_j',x_r') \\ &= \Big(\sum_{k\in\mathbb{Z}} \alpha_k x_k' - \sum_{j\in\mathbb{Z}} \beta_j x_j', \sum_{l\in\mathbb{Z}} \alpha_l x_l' - \sum_{r\in\mathbb{Z}} \beta_r x_r' \Big) = \Big\| \sum_{k\in\mathbb{Z}} \alpha_k x_k' - \sum_{j\in\mathbb{Z}} \beta_j x_j' \Big\|^2, \end{aligned}$$

and we get

$$\sum_{k\in\mathbb{Z}} \alpha_k x_k' = \sum_{j\in\mathbb{Z}} \beta_j x_j'.$$

So, we can correctly define the operator A on L as in (17). For arbitrary

$$x = \sum_{k\in\mathbb{Z}} a_k x_k \in L, \quad y = \sum_{j\in\mathbb{Z}} b_j x_j \in L, \quad a_k, b_j \in \mathbb{C},$$

we have

$$\begin{aligned} (Ax, Ay) &= \Big(\sum_{k\in\mathbb{Z}} a_k x_k', \sum_{j\in\mathbb{Z}} b_j x_j' \Big) = \sum_{k,j\in\mathbb{Z}} a_k\overline{b_j}(x_k', x_j') \\ &= \sum_{k,j\in\mathbb{Z}} a_k\overline{b_j}(x_k, x_j) = \Big(\sum_{k\in\mathbb{Z}} a_k x_k, \sum_{j\in\mathbb{Z}} b_j x_j \Big) = (x,y). \end{aligned}$$

So, the operator A is isometric. Thus, there exists a unitary extension $\widetilde{A} \supseteq A$ in a space $\widetilde{H} \supseteq H$, see [2].

Choose an arbitrary $a \in \mathbb{Z}$ and let $\chi_{k;n}(\lambda)$ be a solution of (7),(8). Repeating considerations after formula (18) we get

$$x_n = \sum_{k=0}^{r-1} \chi_{k;n}(A) x_k, \quad n \in \mathbb{Z}. \tag{80}$$

Let

$$\widetilde{A} = \int_0^{2\pi} e^{i\theta} dF_\theta, \tag{81}$$

be the spectral decomposition of $\widetilde{A}$, where $\{F_\theta\}$ is the resolution of unity of $\widetilde{A}$. From (10) and (80) we obtain

$$K_{n,m} = (x_n, x_m) = \Big(\sum_{k=0}^{r-1} \chi_{k;n}(A) x_k, \sum_{l=0}^{r-1} \chi_{l;m}(A) x_l \Big)$$
$$= \sum_{k,l=0}^{r-1} \Big(\chi_{k;n}(\widetilde{A}) x_k, \chi_{l;m}(\widetilde{A}) x_l \Big) = \sum_{k,l=0}^{r-1} \Big(\Big(\chi_{l;m}(\widetilde{A}) \Big)^* \chi_{k;n}(\widetilde{A}) x_k, x_l \Big)$$
$$= \sum_{k,l=0}^{r-1} \int_0^{2\pi} \chi_{k;n}(e^{i\theta}) \overline{\chi_{l;m}(e^{i\theta})}\, d(F_\theta x_k, x_l).$$

The following theorem is true.

Theorem 3. *Let $K = (K_{n,m})_{n,m\in\mathbb{Z}}$ be a positive definite kernel. It satisfies relation (77) if and only if there exists a representation*

$$K_{n,m} = \sum_{k,l=0}^{r-1} \int_0^{2\pi} \chi_{k;n}(e^{i\theta}) \overline{\chi_{l;m}(e^{i\theta})}\, d\sigma_{k,l}(\theta), \tag{82}$$

where $\chi_{k;l}(\cdot)$ are solutions of (7),(8), and $(\sigma_{k,l}(\theta))_{k,l=0}^{r-1}$ is a non-decreasing matrix function on $[0, 2\pi]$ which elements have a bounded variation on $[0, 2\pi]$.

Proof. Necessity was already shown above. Sufficiency follows from (7). □

3.2. **Case $A = \mathbb{Z}_+$.** We shall consider the operator L from (26). Suppose that a positive definite kernel $K = (K_{n,m})_{n,m\in\mathbb{Z}_+}$ satisfies the relation

$$L_n \overline{L}_m K = K. \tag{83}$$

In the coordinate form this relation takes the form

$$\sum_{j=0}^{n+r^+} \sum_{l=0}^{m+r^+} d_{n,j} \overline{d_{m,l}} K_{j,l} = K_{n,m}, \quad n, m \in \mathbb{Z}_+. \tag{84}$$

Let H and $\{x_n\}_{n\in\mathbb{Z}_+}$ be the Hilbert space and the sequence provided by Theorem 1 for K. Define $\{x'_n\}_{n\in\mathbb{Z}_+}$ as in (29). By virtue of (10) and (84) we get

$$(x'_n, x'_m) = (x_n, x_m), \quad n, m \in \mathbb{Z}_+. \tag{85}$$

Let $L = \mathrm{Lin}\{x_n\}_{n\in\mathbb{Z}_+}$. Choose an arbitrary $x \in L$. Suppose that

$$x = \sum_{k\in\mathbb{Z}_+} \alpha_k x_k, \quad x = \sum_{j\in\mathbb{Z}_+} \beta_j x_j, \quad \alpha_k, \beta_j \in \mathbb{C}. \tag{86}$$

Like it was done in the previous case after formula (79), we can get

$$\sum_{k\in\mathbb{Z}_+} \alpha_k x'_k = \sum_{j\in\mathbb{Z}_+} \beta_j x'_j.$$

So, we can correctly define an operator A on L as in (33). For arbitrary

$$x = \sum_{k\in\mathbb{Z}_+} a_k x_k \in L, \quad y = \sum_{j\in\mathbb{Z}_+} b_j x_j \in L, \quad a_k, b_j \in \mathbb{C},$$

we have

$$(Ax, Ay) = \Big(\sum_{k\in\mathbb{Z}_+} a_k x'_k, \sum_{j\in\mathbb{Z}_+} b_j x'_j \Big) = \sum_{k,j\in\mathbb{Z}_+} a_k \overline{b_j} (x'_k, x'_j)$$
$$= \sum_{k,j\in\mathbb{Z}_+} a_k \overline{b_j} (x_k, x_j) = \Big(\sum_{k\in\mathbb{Z}_+} a_k x_k, \sum_{j\in\mathbb{Z}_+} b_j x_j \Big) = (x, y).$$

So, the operator A is isometric. There exists a unitary extension $\widetilde{A} \supseteq A$ in a space $\widetilde{H} \supseteq H$.

Let $\chi_{k;n}(\lambda)$ be a solution of the equation (34) with the initial conditions (35). Repeating the arguments after formula (36) we obtain

$$x_n = \sum_{k=0}^{r^+-1} \chi_{k;n}(A)x_k, \quad n \in \mathbb{Z}_+. \tag{87}$$

From the last relation we obtain that

$$K_{n,m} = (x_n, x_m) = \sum_{k,l=0}^{r^+-1} (\chi_{k;n}(A)x_k, \chi_{l;m}(A)x_l) = \sum_{k,l=0}^{r^+-1} \Big(\chi_{k;n}(\widetilde{A})x_k, \chi_{l;m}(\widetilde{A})x_l\Big)$$

$$= \sum_{k,l=0}^{r^+-1} \Big(\Big(\chi_{l;m}(\widetilde{A})\Big)^* \chi_{k;n}(\widetilde{A})x_k, x_l\Big) = \sum_{k,l=0}^{r^+-1} \int_0^{2\pi} \chi_{k;n}(e^{i\theta})\overline{\chi_{l;m}(e^{i\theta})}\, d(F_\theta x_k, x_l).$$

Thus, we obtain the following theorem.

Theorem 4. *Let $K = (K_{n,m})_{n,m\in\mathbb{Z}_+}$ be a positive definite kernel. It satisfies relation (84) if and only if there exists a representation*

$$K_{n,m} = \sum_{k,l=0}^{r^+-1} \int_0^{2\pi} \chi_{k;n}(e^{i\theta})\overline{\chi_{l;m}(e^{i\theta})}\, d\sigma_{k,l}(\theta), \tag{88}$$

where $\chi_{k;l}(\cdot)$ are solutions of (34), (35), and $(\sigma_{k,l}(\theta))_{k,l=0}^{r^+-1}$ is a non-decreasing matrix function on $[0, 2\pi]$ which elements have a bounded variation on $[0, 2\pi]$.

Proof. Necessity was shown above and sufficiency follows from (34). □

3.3. **Stochastic sequences.** Recently, in [3] we study different classes of sequences $\{x_n\}_{n\in\mathbb{Z}_+}$ in a separable Hilbert space H. The function $K_{n,m} = (x_n, x_m)$, $n, m \in \mathbb{Z}_+$, is called a correlation function. Recall the following definition ([3]):

Definition 1. *A sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ of elements of a Hilbert space H is called* **P-stationary**, *if it admits a representation*

$$x_n = p_n(U)x_0 = \int_0^{2\pi} p_n(e^{i\theta})\, dF_\theta x_0, \quad n \in \mathbb{Z}_+, \tag{89}$$

where $\{p_n(\cdot)\}_{n\in\mathbb{Z}_+}$ is a system of orthogonal polynomials on the unit circle $\mathbb{T}$, U is a unitary operator in H and $\{F_\theta\}_{\theta\in[0,2\pi]}$ is its orthogonal resolution of unity (not necessarily left or right continuous).

Recall that a set of polynomials $\{p_n(z)\}_{n\in\mathbb{Z}_+}$ ($\deg p_n = n$ and p_n has a positive leading coefficient) is a system of orthogonal polynomials on $\mathbb{T}$ if

$$\int_0^{2\pi} p_n(e^{i\theta})\overline{p_m(e^{i\theta})}\, d\sigma(\theta) = A_n\delta_{n,m}, \quad A_n > 0, \quad n, m \in \mathbb{Z}_+, \tag{90}$$

where $\sigma(\theta)$ is a non-decreasing function on $[0, 2\pi]$, such that $\int_0^{2\pi} d\sigma = 1$. If $A_n = 1$, $n \in \mathbb{Z}_+$, the polynomials are called orthonormal. Orthonormal polynomials p_n satisfy a recurrence relation [12]

$$zp_n(z) = \sum_{j=0}^{n+1} d_{n,j}p_j(z), \tag{91}$$

where $d_{n,n+1} = \frac{\kappa_n}{\kappa_{n+1}}$, $d_{n,j} = -\frac{\kappa_j}{\kappa_n}\overline{a_j}a_{n+1}$, $a_n = \frac{p_n(0)}{\kappa_n}$, and κ_j is the leading coefficient of p_j.

The correlation function of a P-stationary sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ with orthonormal polynomials satisfies relation (84) with $r^+ = 1$, see [3, Theorem 5]. Now we can strengthen Theorem 6 in [3]. The following theorem is true.

Theorem 5. *Let a sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ in a Hilbert space H be given. If its correlation function $K_{n,m}$ satisfies relation (84) with $r^+ = 1$ and $d_{n,j}$ from (91) then it is P-stationary with orthonormal polynomials in a Hilbert space $\widetilde{H} \supseteq H$.*

Proof. The proof is the same as in [3] if we take into account that the operator V in (85) in [3] is correctly defined in our case (see our considerations above). □

Acknowledgments. The author is grateful to Yu. M. Berezansky for suggestions and comments, especially about possibility of Example 2.1.

References

1. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
2. N. I. Akhiezer, I. M. Glazman, *Theory of Linear Operators in a Hilbert Space*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow—Leningrad, 1950. (Russian)
3. S. M. Zagorodnyuk, L. Klotz, *An application of A. N. Kolmogorov's approach in a study of stochastic sequences related to orthogonal polynomials*, Visn. Khark. Univ., Ser. Mat. Prykl. Mat. Mekh. **826** (2008), no. 58, 3–37. (Russian)
4. V. S. Korolyuk, N. I. Portenko, A. V. Skorokhod, and A. F. Turbin, *Handbook on the Theory of Probability and Mathematical Statistics*, Naukova Dumka, Kiev, 1978. (Russian)
5. Yu. M. Berezansky, Z. G. Sheftel, G. F. Us, *Functional Analysis*, Vols. 1, 2, Birkhäuser Verlag, Basel—Boston—Berlin, 1996. (Russian edition: Vyshcha Shkola, Kiev, 1990)
6. Yu. M. Berezansky, *Some generalizations of the classical moment problem*, Integr. Equ. Oper. Theory **44** (2002), 255–289.
7. N. I. Akhiezer, *The Classical Moment Problem*, Fizmatgiz, Moscow, 1961. (Russian)
8. M. A. Neumark, *Spectral functions of a symmetric operator*, Izv. Akad. Nauk SSSR, Ser. Mat. **4** (1940), no. 3, 277–318. (Russian)
9. M. A. Neumark, *On spectral functions of a symmetric operator*, Izv. Akad. Nauk SSSR, Ser. Mat. **7** (1943), no. 6, 285–296. (Russian)
10. M. G. Krein, M. A. Krasnoselskiy, *Basic theorems on an extension of Hermitian operators and some their applications to the theory of orthogonal polynomials and the moment problem*, Uspekhi Mat. Nauk **3(19)** (1947), 60–106. (Russian)
11. M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, Amer. Math. Soc., Colloquium Publications, Providence, RI, 1932.
12. M. J. Cantero, P. Ferrer, L. Moral, L. Velázquez, *Functional analysis methods to study zeros and measures of orthogonal polynomials on the unit circle*, manuscript, 2001.

School of Mathematics and Mechanics, Karazin Kharkiv National University, 4 Svobody sq., Kharkiv, 61077, Ukraine
E-mail address: Sergey.M.Zagorodnyuk@univer.kharkov.ua

Received 09/04/2009; Revised 11.09.2009