Methods of Functional Analysis and Topology
Vol. 16 (2010), no. 2, pp. 101–111

MAHARAM TRACES ON VON NEUMANN ALGEBRAS

V. I. CHILIN AND B. S. ZAKIROV

Abstract. Traces Φ on von Neumann algebras with values in complex order complete vector lattices are considered. The full description of these traces is given for the case when Φ is the Maharam trace. The version of Radon-Nikodym-type theorem for Maharam traces is established.

1. Introduction

The theory of integration for measures μ with values in order complete vector lattices has inspired the study of (bo)-complete lattice-normed spaces $L^p(\mu)$ (see, for example, [1], 6.1.8). The spaces $L^p(\mu)$ are the Banach-Kantorovich spaces if the measure μ possesses the Maharam property. In the proof of this fact, description of Maharam operators acting in order complete vector lattices plays an important role ([1], 3.4.3).

The existence of the center-valued traces in finite von Neumann algebras makes it natural to construct the theory of integration for traces with values in the complex order complete vector lattice $F_{\mathbb{C}} = F \oplus iF$. If the von Neumann algebra is commutative, then construction of $F_{\mathbb{C}}$-valued integration for it is the component part for the investigation of the properties of order continuous maps of vector lattices.

Let M be a non-commutative von Neumann algebra, let $F_{\mathbb{C}}$ be a von Neumann subalgebra in the center of M and let $\Phi : M \to F_{\mathbb{C}}$ be a trace with modularity property: $\Phi(zx) = z\Phi(x)$ for all $z \in F_{\mathbb{C}},\ x \in M$. It is known that the non-commutative L^p-space $L^p(M, \Phi)$ is a Banach-Kantorovich space [2], [3]. In addition, Φ possesses the Maharam property: if $0 \leq z \leq \Phi(x),\ z \in F_{\mathbb{C}},\ 0 \leq x \in M$, then there exists $0 \leq y \leq x$ such that $\Phi(y) = z$ (compare with [1], 3.4.1).

In the present article, we will study the faithful normal traces Φ on a von Neumann algebra M with values in an arbitrary complex order complete vector lattice. We give the full description of such traces in the case when Φ is a Maharam trace. With the help of the locally measure topology in the algebra $S(M)$ of all measurable operators we construct the Banach-Kantorovich space $L^1(M, \Phi) \subset S(M)$. We also state the version of Radon-Nikodym-type theorem for Maharam traces.

We use the terminology and results of the von Neumann algebras theory (see [4], [5]), measurable operators theory (see [6], [7]) and order complete vector lattices and Banach-Kantorovich spaces theory (see [1]).

2. Preliminaries

Let H be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and $\mathbf{1}$ be the identity operator on H. Let M be a von Neumann algebra acting on H, let $Z(M)$ be the center of M and $P(M)$ be the lattice of all projectors in M. We denote by $P_{fin}(M)$ the set of all finite projectors in M.

2000 *Mathematics Subject Classification.* Primary 28B15, 46L50.

Key words and phrases. Von Neumann algebra, measurable operator, vector-valued trace, order complete vector lattice, Radon-Nikodym-type theorem.

A densely-defined closed linear operator x (possibly unbounded) affiliated with M is said to be *measurable* if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathfrak{D}(x)$ and $p_n^{\perp} = \mathbf{1} - p_n \in P_{fin}(M)$ for every $n = 1, 2, \ldots$ (here $\mathfrak{D}(x)$ is the domain of x). Let us denote by $S(M)$ the set of all measurable operators.

Let x, y be measurable operators. Then $x + y$, xy and x^* are densely-defined and preclosed. Moreover, the closures $\overline{x+y}$ (strong sum), $\overline{xy}$ (strong product) and x^* are again measurable, and $S(M)$ is a $*$-algebra with respect to the strong sum, strong product, and the adjoint operation (see [6]). It is clear that M is a $*$-subalgebra in $S(M)$. For any subset $A \subset S(M)$, let $A_h = \{x \in A : x = x^*\}$, $A_+ = \{x \in A : (x\xi, \xi) \geq 0$ for all $\xi \in \mathfrak{D}(x)\}$.

Let $x \in S(M)$ and $x = u|x|$ be the polar decomposition, where $|x| = (x^*x)^{\frac{1}{2}}$, u is a partial isometry in $B(H)$. Then $u \in M$ and $|x| \in S(M)$. If $x \in S_h(M)$ and $\{E_\lambda(x)\}$ are the spectral projections of x, then $\{E_\lambda(x)\} \subset P(M)$.

Let M be a commutative von Neumann algebra. Then M admits a faithful semi-finite normal trace τ, and M is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all bounded complex measurable functions with the identification almost everywhere, where (Ω, Σ, μ) is a measurable space. In addition, $\mu(A) = \tau(\chi_A)$, $A \in \Sigma$. Moreover, $S(M) \cong L^0(\Omega, \Sigma, \mu)$, where $L^0(\Omega, \Sigma, \mu)$ is the $*$-algebra of all complex measurable functions with the identification almost everywhere [6].

The locally measure topology $t(M)$ on $L^0(\Omega, \Sigma, \mu)$ is by definition the linear (Hausdorff) topology whose fundamental system of neighborhoods around 0 is given by

$$W(B, \varepsilon, \delta) = \{f \in L^0(\Omega, \Sigma, \mu) : \text{ there exists a set } E \in \Sigma, \text{ such that}$$
$$E \subseteq B, \ \mu(B \setminus E) \leqslant \delta, \ f\chi_E \in L^\infty(\Omega, \Sigma, \mu), \ \|f\chi_E\|_{L_\infty(\Omega,\Sigma,\mu)} \leqslant \varepsilon\}.$$

Here ε, δ run over all strictly positive numbers and $B \in \Sigma$, $\mu(B) < \infty$. It is known that $(S(M), t(M))$ is a complete topological $*$-algebra.

It is clear that zero neighborhoods $W(B, \varepsilon, \delta)$ are closed and have the following property: if $f \in W(B, \varepsilon, \delta)$, $g \in L^\infty(\Omega, \Sigma, \mu)$, $\|g\|_{L\infty(\Omega,\Sigma,\mu)} \leq 1$, then $gf \in W(B, \varepsilon, \delta)$.

A net $\{f_\alpha\}$ converges to f locally in measure (notation: $f_\alpha \xrightarrow{t(M)} f$) if and only if $f_\alpha \chi_B$ converges to $f\chi_B$ in μ-measure for each $B \in \Sigma$ with $\mu(B) < \infty$. Thus $\{f_\alpha\}$ remains convergent to f if τ is replaced by another faithful semi-finite normal trace on M. If M is σ-finite, i.e. any family of nonzero mutually orthogonal projectors from $P(M)$ is at most countable, then there exists a faithful finite normal trace τ on M. In this case, the topology $t(M)$ is metrizable, and convergence of a sequence $f_n \xrightarrow{t(M)} f$ is equivalent to convergence of f_n to f in trace τ.

Let now M be an arbitrary finite von Neumann algebra, $\Phi_M : M \to Z(M)$ be a center-valued trace on M ([4], 7.11). Let $Z(M) \cong L^\infty(\Omega, \Sigma, \mu)$. The locally measure topology $t(M)$ on $S(M)$ is by definition the linear (Hausdorff) topology whose fundamental system of neighborhoods around 0 is given by

$$V(B, \varepsilon, \delta) = \{x \in S(M) : \text{ there exists } p \in P(M), z \in P(Z(M))$$
$$\text{such that } xp \in M, \|xp\|_M \leqslant \varepsilon, \ z^\perp \in W(B, \varepsilon, \delta), \ \Phi_M(zp^\perp) \leqslant \varepsilon z\},$$

where $\|\cdot\|_M$ is the C^*-norm in M. It is known that, $(S(M), t(M))$ is a complete topological $*$-algebra [8].

The net $\{x_\alpha\} \subset S(M)$ converges to $x \in S(M)$ in trace Φ_M (notation: $x_\alpha \xrightarrow{\Phi_M} x$) if $\Phi_M(E_\lambda^\perp(|x_\alpha - x|)) \xrightarrow{t(Z(M))} 0$ for all $\lambda > 0$.

Proposition 2.1. (see [7], § 3.5). *Let M be a finite von Neumann algebra, x_α, $x \in S(M)$. The following conditions are equivalent:*

(*i*) $x_\alpha \xrightarrow{t(M)} x$;

(*ii*) $x_\alpha \xrightarrow{\Phi_M} x$;

(iii) $E_\lambda^\perp(|x_\alpha - x|) \xrightarrow{t(M)} 0$ *for all* $\lambda > 0$.

Let τ be a faithful semi-finite normal trace on M. An operator $x \in S(M)$ is said to be τ-*measurable* if $\tau(E_\lambda^\perp(|x|)) < \infty$ for some $\lambda > 0$. The set $S(M,\tau)$ of all τ-measurable operators is the $*$-subalgebra in $S(M)$, in addition $M \subset S(M,\tau)$. If $\tau(\mathbf{1}) < \infty$, then $S(M,\tau) = S(M)$.

Denote by t_τ the locally measure topology in $S(M,\tau)$ generated by a trace τ (see, for example, [9]). If $x_\alpha, x \in S(M,\tau)$ and x_α converges to x in topology t_τ (notation: $x_\alpha \xrightarrow{\tau} x$), then $x_\alpha \xrightarrow{t(M)} x$ ([7], § 3.5). If τ is finite, then topologies $t(M)$ and t_τ coincide ([7], § 3.5). It is known that $x_\alpha \xrightarrow{\tau} x$ if and only if $\tau(E_\lambda^\perp(|x_\alpha - x|)) \to 0$ for all $\lambda > 0$ [10].

Denote by $T(M)$ the set of all nonzero finite normal traces on the finite von Neumann algebra M.

Proposition 2.2. *Let M be a finite von Neumann algebra, $x_\alpha, x \in S(M)$. Then*

(i) if $x_\alpha \xrightarrow{t(M)} x$, then $|x_\alpha| \xrightarrow{t(M)} |x|$ and $\tau(E_\lambda^\perp(|x_\alpha - x|)) \to 0$ for all $\lambda > 0$ and $\tau \in T(M)$;

(ii) if $T_1(M)$ is a separating subset of $T(M)$ and $\tau(E_\lambda^\perp(|x_\alpha - x|)) \to 0$ for all $\lambda > 0$, $\tau \in T_1(M)$, then $x_\alpha \xrightarrow{t(M)} x$.

Proof. (i) Let $\tau \in T(M)$ and $s(\tau)$ be the support of a trace τ. Then $s(\tau) \in P(Z(M))$ and $\tau(x) = \tau(xs(\tau))$ for all $x \in M$ ([4], 5.15, 7.13). Since $x_\alpha \xrightarrow{t(M)} x$, $x_\alpha s(\tau) \xrightarrow{t(M)} xs(\tau)$. The restriction of τ on $Ms(\tau)$ is a faithful finite normal trace. Therefore $\tau(E_\lambda^\perp(|x_\alpha - x|)) = \tau(E_\lambda^\perp(|x_\alpha s(\tau) - xs(\tau)|)) \to 0$ for all $\lambda > 0$.

If $|x_\alpha| \overset{t(M)}{\nrightarrow} |x|$, then there are $\lambda_0 > 0$, $\tau \in T(M)$ such that $\tau(E_{\lambda_0}^\perp(|\,|x_\alpha| - |x|\,|)) \nrightarrow 0$. The restriction τ_0 of the trace τ on $Ms(\tau)$ is a faithful finite normal trace. Therefore convergence $x_\alpha s(\tau) \xrightarrow{t(M)} xs(\tau)$ implies $x_\alpha s(\tau) \xrightarrow{\tau_0} xs(\tau)$. Using continuity of the operator function $\sqrt{y}$, $y \in S_+(Ms(\tau))$ [11], we obtain

$$|x_\alpha|s(\tau) = \sqrt{(x_\alpha s(\tau))^*(x_\alpha s(\tau))} \xrightarrow{\tau_0} \sqrt{(x^* s(\tau))(xs(\tau))} = |x|s(\tau).$$

Hence $\tau(E_{\lambda_0}^\perp(|\,|x_\alpha| - |x|\,|)) = \tau(E_{\lambda_0}^\perp(|\,|x_\alpha|s(\tau) - |x|s(\tau)\,|)) \to 0$, which is not the case.

(ii) Since $T_1(M)$ is the separating family traces on M, $\sup\limits_{\tau \in T_1(M)} s(\tau) = \mathbf{1}$. Hence there is a family $\{z_i\}_{i \in I}$ of nonzero mutually orthogonal central projectors such that $\sup\limits_{i \in I} z_i = \mathbf{1}$, and for any $i \in I$, there exists $\tau_i \in T_1(M)$ with $z_i \le s(\tau_i)$ ([12], chapter III, § 2). We defined the faithful semi-finite normal trace on M as $\tau(x) = \sum\limits_{i \in I} \tau_i(xz_i)$, $x \in M$. It is clear that restrictions τ and τ_i coincide on Mz_i. In addition, $\tau_i(E_\lambda^\perp(|x_\alpha z_i - xz_i|)) = \tau_i(E_\lambda^\perp(|x_\alpha - x|)) \to 0$ for all $\lambda > 0$, $i \in I$. Hence, $E_\lambda^\perp(|x_\alpha - x|)z_i \xrightarrow{\tau} 0$, and therefore $E_\lambda^\perp(|x_\alpha - x|)z_i \xrightarrow{t(M)} 0$.

For any finite subset $\gamma \subset I$, let $u_\gamma = \sum\limits_{i \in \gamma} z_i$. It is clear that $u_\gamma \uparrow \mathbf{1}$ and $\Phi_M(u_\gamma) \uparrow \Phi_M(\mathbf{1})$. Hence, $\Phi_M(u_\gamma^\perp) \xrightarrow{t(Z(M))} 0$, i.e. $u_\gamma^\perp \xrightarrow{t(M)} 0$.

Let U be an arbitrary neighborhood of 0 in $(S(M), t(M))$. We choose $V(B,\varepsilon,\delta)$ such that $V(B,\varepsilon,\delta) + V(B,\varepsilon,\delta) \subset U$. Fix γ_0 with $(\mathbf{1} - u_{\gamma_0}) \in V(B, \frac{\varepsilon}{4}, \delta)$. Since $E_\lambda^\perp(|x_\alpha - x|)u_{\gamma_0} \xrightarrow{t(M)} 0$, there is an α_0 such that $E_\lambda^\perp(|x_\alpha - x|)u_{\gamma_0} \in V(B,\varepsilon,\delta)$ as $\alpha \ge \alpha_0$. We have $aV(B, \frac{\varepsilon}{4}, \delta)b \subset V(B,\varepsilon,\delta)$, where $a, b \in M$, $\|a\|_M \leqslant 1$, $\|b\|_M \leqslant 1$ (see, for example, [7], § 3.5). Hence

$$\begin{aligned} E_\lambda^\perp(|x_\alpha - x|) &= E_\lambda^\perp(|x_\alpha - x|)u_{\gamma_0} + E_\lambda^\perp(|x_\alpha - x|)(\mathbf{1} - u_{\gamma_0}) \\ &\in V(B,\varepsilon,\delta) + V(B,\varepsilon,\delta) \subset U \end{aligned}$$

for all $\alpha \geq \alpha_0$. Therefore $E_\lambda^\perp(|x_\alpha - x|) \xrightarrow{t(M)} 0$ for all $\lambda > 0$. Proposition 2.1 implies that $x_\alpha \xrightarrow{t(M)} x$. □

3. Vector lattice-valued traces

Throughout the section, let M be an von Neumann algebra, let F be an order complete vector lattice, and let $F_\mathbb{C} = F \oplus iF$ be a complexification of F. If $z = \alpha + i\beta \in F_\mathbb{C},\ \alpha, \beta \in F$, then $\overline{z} := \alpha - i\beta$, and $|z| := \sup\{\mathrm{Re}(e^{i\theta}z) : 0 \leq \theta < 2\pi\}$ (see [1], 1.3.13).

An $F_\mathbb{C}$*-valued trace* on the von Neumann algebra M is a linear mapping $\Phi : M \to F_\mathbb{C}$ given $\Phi(x^*x) = \Phi(xx^*) \geq 0$ for all $x \in M$. It is clear that $\Phi(M_h) \subset F,\ \Phi(M_+) \subset F_+ = \{a \in F : a \geq 0\}$. A trace Φ is said to be *faithful* if the equality $\Phi(x^*x) = 0$ implies $x = 0$, *normal* if $\Phi(x_\alpha) \uparrow \Phi(x)$ for every $x_\alpha, x \in M_h,\ x_\alpha \uparrow x$.

If M is a finite von Neumann algebra, then its center-valued trace $\Phi_M : M \to Z(M)$ is an example of a $Z(M)$-valued faithful normal trace.

Let Δ be a separating family of finite normal numerical traces on the von Neumann algebra M, $\mathbb{C}^\Delta = \prod\limits_{\tau \in \Delta} \mathbb{C}_\tau$, where $\mathbb{C}_\tau = \mathbb{C}$ for all $\tau \in \Delta$. Then $\Phi(x) = \{\tau(x)\}_{\tau \in \Delta}$ is also an example of an faithful normal $\mathbb{C}^\Delta$-valued trace on M.

Let us list some properties of the trace $\Phi : M \to F_\mathbb{C}$.

Proposition 3.1. (i) *Let $x, y, a, b \in M$. Then*
$\Phi(x^*) = \overline{\Phi(x)},\ \Phi(xy) = \Phi(yx),\ \Phi(|x^*|) = \Phi(|x|)$,
$|\Phi(axb)| \leq \|a\|_M \|b\|_M \Phi(|x|)$;
(ii) *If Φ is a faithful trace, then M is finite;*
(iii) *If $x_n, x \in M$ and $\|x_n - x\|_M \to 0$, then $|\Phi(x_n) - \Phi(x)|$ relative uniform converges to zero;*
(iv) *If M is a finite von Neumann algebra, then $\Phi(\Phi_M(x)) = \Phi(x)$ for all $x \in M$;*
(v) *$\Phi(|x + y|) \leq \Phi(|x|) + \Phi(|y|)$ for all $x, y \in M$.*

Proof. The proof of (i) and (ii) is the same as for numerical traces (see, for example, [5], chapter V, § 2).

The proof of (iii) follows from the inequality $|\Phi(x_n) - \Phi(x)| \leq \|x_n - x\|_M \Phi(\mathbf{1})$.

(iv) Let $U(M)$ be the set of all unitary operators in M. Then $\Phi_M(x)$ belongs to the closure of the convex hull $co\{uxu^* : u \in U(M)\}$ ([4], 7.11). Since $\Phi(uxu^*) = \Phi(u^*ux) = \Phi(x)$, we get $\Phi(y) = \Phi(x)$ for any $y \in co\{uxu^* : u \in U(M)\}$. Therefore, because of (iii), we have $\Phi(x) = \Phi(\Phi_M(x))$.

(v) Since $|x + y| \leq u|x|u^* + v|y|v^*$ for some partial isometries u, v in M (see [13]), we have, by virtue of (i)

$$\begin{aligned}\Phi(|x+y|) &\leq \Phi(u|x|u^*) + \Phi(v|y|v^*) = \Phi(u^*u|x|) + \Phi(v^*v|y|)\\ &\leq \Phi(|x|) + \Phi(|y|). \quad \square\end{aligned}$$

The trace $\Phi : M \to F_\mathbb{C}$ possesses *the Maharam property* if for any $x \in M_+,\ 0 \leq f \leq \Phi(x),\ f \in F$, there exists a positive $y \leq x$ such that $\Phi(y) = f$. A faithful normal $F_\mathbb{C}$-valued trace Φ with the Maharam property is called *a Maharam trace* (compare with [1], III, 3.4.1). Obviously, any faithful finite numerical trace on M is a $\mathbb{C}$-valued Maharam trace.

Let us give another examples of Maharam traces. Let M be a finite von Neumann algebra, let $\mathcal{A}$ be a von Neumann subalgebra in $Z(M)$, and let $T : Z(M) \to \mathcal{A}$ be an injective linear positive normal operator. If $f \in S(\mathcal{A})$ is a reversible positive element, then $\Phi(T, f)(x) = fT(\Phi_M(x))$ is an $S(\mathcal{A})$-valued faithful normal trace on M. In addition, if $T(ab) = aT(b)$ for all $a \in \mathcal{A}, b \in Z(M)$, then $\Phi(T, f)$ is a Maharam trace on M.

Note that if τ is a faithful normal finite numerical trace on M and $\dim(Z(M)) > 1$, then $\Phi(x) = \tau(x)\mathbf{1}$ is a $Z(M)$-valued faithful normal trace. In addition, Φ does not possess the Maharam property. In fact, if $p \in Z(M)$, $0 \neq p \neq \mathbf{1}$, then for all $y \in M_+$, $y \leq \mathbf{1}$ the relation $\Phi(y) = \tau(y)\mathbf{1} \neq \tau(y)p \leq \Phi(\mathbf{1})$ is valid.

Let F have an order unit $\mathbf{1}_F$. Denote by $B(F)$ the complete Boolean algebra of unitary elements with respect to $\mathbf{1}_F$, and by $s(a) := \sup\limits_{n\geq 1}\{\mathbf{1}_F \wedge n|a|\} \in B(F)$ the support of an element $a \in F$. Since $|\Phi(x)| \leq \|x\|_M \Phi(\mathbf{1})$ (Proposition 3.1(i)), the inequality $s(|\Phi(x)|) \leq s(\Phi(\mathbf{1}))$ holds for all $x \in M$. Let therefore $s(\Phi(\mathbf{1})) = \mathbf{1}_F$.

Let Q be the Stone representation space of the Boolean algebra $B(F)$. Let $C_\infty(Q)$ be the order complete vector lattice of all continuous functions $a : Q \to [-\infty, +\infty]$ such that $a^{-1}(\{\pm\infty\})$ is a nowhere dense subset of Q. We identify F with the order-dense ideal in $C_\infty(Q)$ containing algebra $C(Q)$ of all continuous real functions on Q. In addition, $\mathbf{1}_F$ is identified with the function equal to 1 identically on Q ([1], 1.4.4).

The next theorem gives the description of Maharam traces on von Neumann algebras.

Theorem 3.2. *Let Φ be an $F_{\mathbb{C}}$-valued Maharam trace on a von Neumann algebra M. Then there exists a von Neumann subalgebra $\mathcal{A}$ in $Z(M)$, a $*$-isomorphism ψ from $\mathcal{A}$ onto the $*$-algebra $C(Q)_{\mathbb{C}}$, an injective positive linear normal operator $\mathcal{E}$ from $Z(M)$ onto $\mathcal{A}$ with $\mathcal{E}(\mathbf{1}) = \mathbf{1}$, $\mathcal{E}^2 = \mathcal{E}$, such that*

1) $\Phi(x) = \Phi(\mathbf{1})\psi(\mathcal{E}(\Phi_M(x)))$ *for all* $x \in M$;
2) $\Phi(zy) = \Phi(z\mathcal{E}(y))$ *for all* $z, y \in Z(M)$;
3) $\Phi(zy) = \psi(z)\Phi(y)$ *for all* $z \in \mathcal{A}$, $y \in M$.

Proof. Since $s(\Phi(\mathbf{1})) = \mathbf{1}_F$, we get that $\Phi_1(x) = \Phi(\mathbf{1})^{-1}\Phi(x)$ is a $(C(Q))_{\mathbb{C}}$-valued Maharam trace on M. In addition, $\Phi_1(\mathbf{1}) = \mathbf{1}_F$.

The set $Z_h(M)$ is an order complete vector lattice with a strong unit $\mathbf{1}$ with respect to algebraic operations, and the partial order induced from M_h. Moreover, the Boolean algebra of all unitary elements in $Z_h(M)$ with respect to $\mathbf{1}$ coincides with $P(Z(M))$. Let T be a restriction of Φ_1 on $Z_h(M)$. Since $|\Phi_1(x)| \leq \|x\|_M$, $T(Z_h(M)) \subset C(Q)$. It is clear that T is an injective positive order continuous linear operator. If $x \in Z_+(M)$, $0 \leq a \leq Tx = \Phi_1(x)$, $a \in C(Q)$, then there exists $y \in M_+$ such that $y \leq x$ and $\Phi_1(y) = a$. By Proposition 3.1 (iv), we have $a = \Phi_1(y) = \Phi_1(\Phi_M(y)) = T(\Phi_M(y))$, moreover, $0 \leq \Phi_M(y) \leq \Phi_M(x) = x$. Hence, $T : Z_h(M) \to C(Q)$ is a Maharam operator ([1], 3.4.1). Theorem 3.4.3 from [1] guarantees the existence of a Boolean isomorphism φ from $B(F)$ onto a regular Boolean subalgebra B in $P(Z(M))$ such that $gT(x) = T(\varphi(g)x)$ for all $g \in B(F)$ and $x \in Z_h(M)$. We denote by $\mathcal{A}$ a commutative von Neumann subalgebra in $Z(M)$ generated by B, i.e. $\mathcal{A}$ coincides with the bicommutant of B. It is known that $\mathcal{A}_h = \{x \in Z_h(M) : E_\lambda(x) \in B \text{ for all } \lambda\}$ where $\{E_\lambda(x)\}$ are the spectral projections of x. The Boolean isomorphism φ is extended to the $*$-isomorphism $\widetilde{\varphi}$ from the $*$-algebra $C(Q)_{\mathbb{C}}$ onto the von Neumann algebra $\mathcal{A}$. If $a = \sum\limits_{i=1}^{n} \lambda_i e_i$ is a simple element, $\lambda_i \in \mathbb{R}$, $e_i \in B(F)$, $i = 1, \dots, n$, then

$$T(\widetilde{\varphi}(a)x) = \sum_{i=1}^{n} \lambda_i T(\varphi(e_i)x) = aT(x)$$

for all $x \in \mathcal{A}_h$. Furthermore, we note $T(\widetilde{\varphi}(a)x) = aT(x)$ for any $a \in C(Q)$, $x \in \mathcal{A}_h$. This is obtained by approximating the elements from $C(Q)$ by simple elements. Therefore, $\Phi_1(\widetilde{\varphi}(a)x) = a\Phi_1(x)$ for all $a \in C(Q)_{\mathbb{C}}$, $x \in \mathcal{A}$, in particular,

$$\Phi_1(\widetilde{\varphi}(a)) = a. \tag{1}$$

Hence the restriction T_0 of the operator T on $\mathcal{A}_h$ is a lattice isomorphism from $\mathcal{A}_h$ onto $C(Q)$. Therefore T_0 is a Maharam operator. By Theorem 4.2.9 from [14], there exists an operator of conditionally mathematical expectation $\mathcal{E} : Z_h(M) \to \mathcal{A}_h$ satisfying the following conditions:

$(E1)$ $\mathcal{E}$ is an injective positive order continuous linear operator, $\mathcal{E}^2 = \mathcal{E}$ and $\mathcal{E}(\mathbf{1}) = \mathbf{1}$;
$(E2)$ $T(xy) = T(x\mathcal{E}(y))$ for all $x, y \in Z_h(M)$;
$(E3)$ $\mathcal{E}(zy) = z\mathcal{E}(y)$ for all $z \in \mathcal{A}_h,\ y \in Z_h(M)$.

The operator $\mathcal{E}$ is extended to the operator $\widetilde{\mathcal{E}} : Z(M) \to \mathcal{A}$. It is clear that the condition $(E1)$ is satisfied for $\widetilde{\mathcal{E}}$, the condition $(E2)$ has the form $\Phi_1(xy) = \Phi_1(x\widetilde{\mathcal{E}}(y))$ for all $x, y \in Z(M)$, and the condition $(E3)$ is valid for all $z \in \mathcal{A},\ y \in Z(M)$. The condition $(E2)$ implies that

$$\Phi_1(y) = \Phi_1(\widetilde{\mathcal{E}}(y)) \text{ for all } y \in Z(M). \tag{2}$$

Using equalities (1), (2) and Proposition 3.1 (iv), we get

$$\Phi_1(x) = \Phi_1(\Phi_M(x)) = \Phi_1(\widetilde{\mathcal{E}}(\Phi_M(x)) = \widetilde{\varphi}^{-1}(\widetilde{\mathcal{E}}(\Phi_M(x))) \tag{3}$$

for any $x \in M$.

Taking in (3) $\psi = \widetilde{\varphi}^{-1}$ and letting $\widetilde{\mathcal{E}}$ as $\mathcal{E}$, we obtain the statement of Theorem 3.2. $\square$

Due to Theorem 3.2, the $*$-algebra $\mathcal{B} = C(Q)_{\mathbb{C}}$ is $*$-isomorphic to a von Neumann subalgebra in $Z(M)$. Therefore $\mathcal{B}$ is a commutative von Neumann algebra, and $*$-algebra $C_\infty(Q)_{\mathbb{C}}$ is identified with $*$-algebra $S(\mathcal{B})$. In particular, there exists a separating family of completely additive scalar-valued measures on $B(F)$, and therefore F is a Kantorovich-Pinsker space ([1], 1.4.10).

We claim that a version of Radon-Nikodym-type theorem is valid for a Maharam trace Φ. For this, we need the space $L^1(M, \Phi)$ of operators from $S(M)$ to be integrable with respect to Φ.

Let F be a Kantorovich-Pinsker space and let Φ be an $F_{\mathbb{C}}$-valued Maharam trace on the von Neumann algebra M. The net $\{x_\alpha\} \subset S(M)$ converges to $x \in S(M)$ with respect to the trace Φ (notation: $x_\alpha \xrightarrow{\Phi} x$) if $\Phi(E^\perp_\lambda(|x_\alpha - x|)) \xrightarrow{t(\mathcal{B})} 0$ for all $\lambda > 0$.

Proposition 3.3. *$x_\alpha \xrightarrow{\Phi} x$ iff $x_\alpha \xrightarrow{t(M)} x$.*

Proof. Let ν be a faithful normal semi-finite numerical trace on $\mathcal{B}$. Choose $\{e_i\}_{i\in I}$ to be a set of nonzero mutually orthogonal projections from $P(\mathcal{B})$ with $\sup\limits_{i\in I} e_i = \mathbf{1}_F$ and $\nu(e_i) < \infty, i \in I$. Set $\tau_i(x) = \nu(\Phi(x)\Phi(\mathbf{1})^{-1}e_i),\ x \in M,\ i \in I$. It is clear that $\{\tau_i\}_{i\in I}$ is a separating family of finite traces on M. Due to Proposition 2.2, $x_\alpha \xrightarrow{t(M)} x$ if and only if $\tau_i(E^\perp_\lambda(|x_\alpha - x|)) \to 0$ for all $\lambda > 0, i \in I$. The last convergence is equivalent to convergence $\Phi(E^\perp_\lambda(|x_\alpha - x|)) \xrightarrow{t(\mathcal{B})} 0$. $\square$

For each $x \in M$, let $\|x\|_\Phi = \Phi(|x|)$. Proposition 3.1 implies that $\|\cdot\|_\Phi$ is an F-valued norm on M. In addition, $\|x\|_\Phi = \|x^*\|_\Phi = \|\,|x|\,\|_\Phi$ and $\|axb\|_\Phi \le \|a\|_M\|b\|_M\|x\|_\Phi$ for all $x, a, b \in M$.

We have $\Phi(E^\perp_\lambda(|x_\alpha - x|)) \le \frac{1}{\lambda}\Phi(|x_\alpha - x|)$, $\lambda > 0$, $x_\alpha, x \in M$. Hence $\|x_\alpha - x\|_\Phi \xrightarrow{t(\mathcal{B})} 0$ implies $x_\alpha \xrightarrow{\Phi} x$, and therefore $x_\alpha \xrightarrow{t(M)} x$ (Proposition 3.3).

An operator $x \in S(M)$ is said to be *Φ-integrable* if there exists a sequence $\{x_n\} \subset M$ such that $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t_l(\mathcal{B})} 0$ as $n, m \to \infty$. Denote by $L^1(M, \Phi)$ the set of all Φ-integrable operators from $S(M)$. It is clear that $M \subset L^1(M, \Phi)$ and $L^1(M, \Phi)$ is a linear subset of $S(M)$. It follows from Proposition 3.1 and 3.3 that $ML^1(M, \Phi)M \subset L^1(M, \Phi)$ and $x^* \in L^1(M, \Phi)$ for all $x \in L^1(M, \Phi)$.

We now define an $S_h(\mathcal{B})$-valued L^1-norm on $L^1(M, \Phi)$.

Proposition 3.4. *If $x_n \in M,\ x_n \xrightarrow{\Phi} 0,\ \|x_n - x_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0,$ then $\Phi(|x_n|) \xrightarrow{t(\mathcal{B})} 0$.*

Proof. Since $|\,\|x_n\|_\Phi - \|x_m\|_\Phi\,| \le \|x_n - x_m\|_\Phi$, $\Phi(|x_n|) = \|x_n\|_\Phi$ is a Cauchy sequence in $(S(\mathcal{B}), t(\mathcal{B}))$. Because of the completeness of $*$-algebra $(S(\mathcal{B}), t(\mathcal{B}))$, there exists $f \in S_+(\mathcal{B})$ such that $\Phi(|x_n|) \xrightarrow{t(\mathcal{B})} f$. We claim that $f = 0$. First, we assume that algebra $\mathcal{B}$ is σ-finite. Then there exists a faithful normal finite numerical trace ν on $\mathcal{B}$. We have $\Phi(|x_n|) \xrightarrow{\nu} f$ and the sequence $\{\Phi(|x_n|)\}$ has an (o)-convergent subsequence. Therefore, as usual, we may and do assume that the sequence $\{\Phi(|x_n|)\}$ (o)-converges to f in $S_h(\mathcal{B})$ (notation: $\Phi(|x_n|) \xrightarrow{(o)} f$). Hence, there exists $g = \sup\limits_{n\ge1} \Phi(|x_n|)$ in $S_h(\mathcal{B})$. It is clear that

$$\tau(x) = \nu(\Phi(x)(\mathbf{1}_F + g + \Phi(\mathbf{1}))^{-1}) \tag{4}$$

is a faithful normal finite numerical trace on M. Since topologies t_ν and $t(\mathcal{B})$ coincide, $\Phi(|x_n - x_m|) \xrightarrow{\nu} 0$. Therefore inequalities $0 \le \Phi(|x_n - x_m|) \le 2g$, imply $\tau(|x_n - x_m|) \to 0$. It is known that $(L^1(M,\tau), \|\,\|_{1,\tau})$ is complete, where $\|x\|_{1,\tau} = \tau(|x|)$ [6]. Hence there exists $x \in L^1(M,\tau) \subset S(M)$ such that $\|x - x_n\|_{1,\tau} \to 0$ and therefore, $x_n \xrightarrow{\tau} x$ [10]. Because of the equality of topologies t_τ and $t(M)$, we have $x = 0$. This means that $\tau(|x_n|) \to 0$, i.e. $\Phi(|x_n|) \xrightarrow{\nu} 0$.

Now let $\mathcal{B}$ be a general (not necessarily σ-finite) von Neumann algebra. For each $0 \ne e \in P(\mathcal{B})$, we set $\Phi_e(x) = \Phi(x)e,\ x \in M$. It is clear that Φ_e is a normal $S_h(\mathcal{B}e)$-valued trace on M, which does not have, generally speaking, the faithfulness property. A projection $s(\Phi_e) = \mathbf{1} - \sup\{p \in P(M) : \Phi_e(p) = 0\}$ is called *the support* trace of Φ_e. As well as in the case of numerical traces (see, for example, [4], 5.15, 7.13), one can establish that $s(\Phi_e) \in P(Z(M))$ and $\Phi_e(x) = \Phi_e(xs(\Phi_e))$ is a faithful normal $S_h(e\mathcal{B})$-valued trace on $Ms(\Phi_e)$.

If $\Phi(|x_n|) \overset{t(\mathcal{B})}{\nrightarrow} 0$, then there is a nonzero σ-finite projection $e \in P(\mathcal{B})$ such that $\Phi(|x_n|)e \overset{\nu}{\nrightarrow} 0$ where ν is a faithful normal finite numerical trace on $\mathcal{B}e$. The last contradicts to what we proved above. $\square$

Let $x \in L^1(M,\Phi)$, $x_n \in M$, $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0$. The inequality $|\Phi(x_n) - \Phi(x_m)| \le \Phi(|x_n - x_m|)$ and completeness of the $*$-algebra $(S(\mathcal{B}), t(\mathcal{B}))$ guarantees the existence of $\widehat{\Phi}(x) \in S(\mathcal{B})$ such that $\Phi(x_n) \xrightarrow{t(\mathcal{B})} \widehat{\Phi}(x)$. Due to Proposition 3.4, $\widehat{\Phi}(x)$ does not depend on the choice of a sequence $\{x_n\} \subset M$, for which $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0$, in particular, $\widehat{\Phi}(x) = \Phi(x)$ for all $x \in M$. The element $\widehat{\Phi}(x)$ is called an $S(\mathcal{B})$*-valued integral* of $x \in L^1(M,\Phi)$ by a trace Φ.

It follows immediately from the definition of $\widehat{\Phi}$ and Proposition 3.1 that $\widehat{\Phi}$ is a linear mapping from $L^1(M,\Phi)$ into $S(\mathcal{B})$ and $\widehat{\Phi}(xy) = \widehat{\Phi}(yx)$ for any $x \in M, y \in L^1(M,\Phi)$. For each $x \in L^1(M,\Phi)$, we set $\|x\|_\Phi = \widehat{\Phi}(|x|)$.

Theorem 3.5. (i) *The mapping* $\|\cdot\|_\Phi$ *is an* $S_h(\mathcal{B})$*-valued norm on* $L^1(M,\Phi)$.
(ii) $(L^1(M,\Phi), \|\cdot\|_\Phi)$ *is a Banach-Kantorovich space.*

Proof. (i) Let $x \in L^1(M,\Phi),\ x_n \in M,\ x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0$. It follows from Propositions 2.2(i) and 3.3 that $|x_n| \xrightarrow{\Phi} |x|$. We claim that $\|\,|x_n| - |x_m|\,\|_\Phi \xrightarrow{t(\mathcal{B})} 0$.

First, we assume that algebra $\mathcal{B}$ is σ-finite. Using the same trick as in the proof of Proposition 3.4, we can show that $\Phi(x_n) \xrightarrow{(o)} \widehat{\Phi}(x)$ in $S_h(\mathcal{B})$. Therefore there exists $g = \sup\limits_{n\ge1} |\Phi(x_n)|$ in $S_h(\mathcal{B})$. Consider a faithful normal finite numerical trace τ on M defined by (4). Since $\tau(|x_n - x_m|) \to 0$ as $n, m \to \infty$ (see the proof of Proposition 3.4), there exists $y \in L^1(M,\tau)$ such that $\|y - x_n\|_{1,\tau} \to 0$. Then $x_n \xrightarrow{\tau} y$, and therefore $x = y$. Moreover, $|x_n| \xrightarrow{\tau} |x|$ (Proposition 2.2(i)) and $\|\,|x_n|\,\|_{1,\tau} = \|x_n\|_{1,\tau} \to \|x\|_{1,\tau}$. It follows from ([10], Theorem 3.7) that $\| |x|-|x_n| \|_{1,\tau} \to 0$, in particular, $\tau(| |x_n| - |x_m| |) \to 0$ as $n, m \to \infty$. Convergence $\Phi(| |x_n| - |x_m| |)(\mathbf{1}_F + g + \Phi(\mathbf{1}))^{-1} \xrightarrow{\nu} 0$ implies $\| |x_n| - |x_m| \|_\Phi \xrightarrow{t(\mathcal{B})} 0$.

Hence, $|x| \in L^1(M,\Phi)$ and $\Phi(|x_n|) \xrightarrow{t(\mathcal{B})} \widehat{\Phi}(|x|)$. In particular, $\|x\|_\Phi = \widehat{\Phi}(|x|) \geq 0$ for all $x \in L^1(M,\Phi)$. If $\widehat{\Phi}(|x|) = 0$, then $0 \leq \|x_n\|_\Phi = \Phi(|x_n|) \xrightarrow{t(\mathcal{B})} 0$. Hence, $x_n \xrightarrow{\Phi} 0$, and therefore $x = 0$.

Let now $\mathcal{B}$ be not a σ-finite algebra. Let $\{e_i\}_{i\in I}$ be a family of nonzero mutually orthogonal σ-finite projections in $\mathcal{B}$ with $\sup\limits_{i\in I} e_i = \mathbf{1}_F$. Since $\sup\limits_{i\in I} s(\Phi_{e_i}) = \mathbf{1}$ and $\widehat{\Phi}(|x|)e_i = \widehat{\Phi}_{e_i}(|x|s(\Phi_{e_i})) \geq 0$ for all $i \in I$, we get $\widehat{\Phi}(|x|) \geq 0$. Similarly, the equality $\Phi(|x|) = 0$ implies $\widehat{\Phi}_{e_i}(|x|s(\Phi_{e_i})) = 0$, and therefore $|x|s(\Phi_{e_i}) = 0$ for all $i \in I$. Hence, $x = 0$.

Finally, we have

$$\|x+y\|_\Phi \leq \|x\|_\Phi + \|y\|_\Phi, \ x, y \in L^1(M,\Phi),$$

due to the inequality $|x+y| \leq u|x|u^* + v|y|v^*$, $x, y \in S(M)$ (see [7], § 2.4) and the trick in Proposition 3.1 (v).

(ii) Let $x \in L^1(M,\Phi)$, $x_n \in M$, $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0$. Fix m and set $y_{nm} = x_n - x_m$ for $n \geq m$. We have $y_{nm} \xrightarrow{\Phi} x - x_m$ and $\|y_{nm} - y_{km}\|_\Phi \xrightarrow{t(\mathcal{B})} 0$ as $n, k \to \infty$. It follows from the proof of (i) that $\Phi(|y_{nm}|) \xrightarrow{t(\mathcal{B})} \widehat{\Phi}(|x - x_m|) = \|x - x_m\|_\Phi$. Since $\Phi(|y_{nm}|) \xrightarrow{t(\mathcal{B})} 0$ as $n, m \to \infty$, $\|x - x_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0$.

Let us now show that any (bo)-Cauchy sequence in $(L^1(M,\Phi), \|\cdot\|_\Phi)$ (bo)-converges.

First, we assume that $\mathcal{B}$ is a σ-finite von Neumann algebra. Let $\{x_n\} \subset L^1(M,\Phi)$ and $\|x_n - x_m\|_\Phi \xrightarrow{(o)} 0$. Since $\widehat{\Phi}$ is a positive mapping (see the proof of item (i)), the inequality $\widehat{\Phi}(E_\lambda^\perp(|x_n - x_m|)) \leq \frac{1}{\lambda}\widehat{\Phi}(|x_n - x_m|)$, $\lambda > 0$ is valid. Hence, $\{x_n\}$ is a Cauchy sequence in $(S(M), t(M))$ and therefore there exists $x \in S(M)$ such that $x_n \xrightarrow{t(M)} x$. Choose a system $\{U_n\}$ of closed neighborhoods of 0 in $(S(\mathcal{B}), t(\mathcal{B}))$ with $U_{n+1} + U_{n+1} \subset U_n$, $n = 1, 2, \ldots$. Due to what we proved above, for any $x_n \in L^1(M,\Phi)$, there exists $y_n \in M$ such that $\|x_n - y_n\|_\Phi \in U_n$. Since $\sum\limits_{n=k+1}^{m} \|x_n - y_n\|_\Phi \in U_k$ for all $m \geq k+1$, the series $\sum\limits_{n=k+1}^{\infty} \|x_n - y_n\|_\Phi$ converges in $(S(\mathcal{B}), t(M))$. Hence, $\|x_n - y_n\|_\Phi \xrightarrow{(o)} 0$, and therefore $\|y_n - y_m\|_\Phi \xrightarrow{(o)} 0$. Also, by Proposition 3.3, we get $x_n - y_n \xrightarrow{\Phi} 0$, and consequently $y_n \xrightarrow{\Phi} x$. This means that $x \in L^1(M,\Phi)$, in addition, $\|x - y_n\|_\Phi \xrightarrow{t(\mathcal{B})} 0$ and $\|y_n - y_m\|_\Phi \xrightarrow{t(\mathcal{B})} \|x - y_m\|_\Phi$ as $n \to \infty$.

Since $\|x - y_m\|_\Phi \leq \sup\limits_{n\geq m} \|y_n - y_m\|_\Phi \downarrow 0$, we get $\|x - y_m\|_\Phi \xrightarrow{(o)} 0$ and therefore $\|x - x_n\|_\Phi \xrightarrow{(o)} 0$.

Now let $\{x_\alpha\}_{\alpha\in A}$ be an arbitrary (bo)-Cauchy net in $L^1(M,\Phi)$, i.e. $\sup\limits_{\alpha,\beta\geq\gamma} \|x_\alpha - x_\beta\|_\Phi \downarrow 0$. We choose a sequence of indices $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots$ in A such that $\sup\limits_{\beta\geq\alpha_n} \|x_\beta - x_{\alpha_n}\|_\Phi \in U_n$. Then $\sup\limits_{n,m\geq k} \|x_{\alpha_n} - x_{\alpha_m}\|_\Phi \in U_k$, and therefore $\{x_{\alpha_n}\}$ is a (bo)-Cauchy sequence in $L^1(M,\Phi)$. It follows from what we proved above that there exists $x \in L^1(M,\Phi)$ such that $\|x - x_{\alpha_n}\|_\Phi \xrightarrow{(o)} 0$. Let us claim that $\|x - x_\alpha\|_\Phi \xrightarrow{(o)} 0$, i.e. $(\sup\limits_{\alpha\geq\beta} \|x - x_\alpha\|_\Phi) \downarrow 0$. Fix $\beta \in A$ and consider the net $\{x_\alpha\}_{\alpha\geq\beta}$. We construct a sequence of indices $\beta \leq \beta_1 \leq \beta_2 \leq \cdots$ such that $\alpha_n \leq \beta_n$. Then $\|x_{\beta_n} - x_{\alpha_n}\|_\Phi \in U_n$, and therefore

$\|x_{\beta_n}-x_{\alpha_n}\|_\Phi \xrightarrow{(o)} 0$. Hence, $\|x-x_{\beta_n}\|_\Phi \xrightarrow{(o)} 0$ and $\|x_{\beta_n}-x_\beta\|_\Phi \xrightarrow{(o)} \|x-x_\beta\|_\Phi$ as $n\to\infty$. Thus, $\|x-x_\beta\|_\Phi \le \sup\limits_{n\ge 1}\|x_{\beta_n}-x_\beta\|_\Phi \le \sup\limits_{\alpha\ge\beta}\|x_\alpha-x_\beta\|_\Phi$ and $\|x-x_\beta\|_\Phi \xrightarrow{(o)} 0$.

Let now $\mathcal{B}$ be not a σ-finite algebra and let $\{x_\alpha\}$ be a (bo)-Cauchy net in $L^1(M,\Phi)$. Due to the completeness of $(S(M),t(M))$, there is $x\in S(M)$ such that $x_\alpha \xrightarrow{\Phi} x$. Let $\{e_i\}_{i\in I}$ be the same family of projections in $\mathcal{B}$, as in the proof of (i). It is clear that $\{x_\alpha s(\Phi_{e_i})\}$ is a (bo)-Cauchy net in $L^1(Ms(\Phi_{e_i}),\Phi_{e_i})$, and therefore, by virtue of what we proved above, there exists $x_i\in L^1(Ms(\Phi_{e_i}),\Phi_{e_i})$ such that $\|x_i-x_\alpha s(\Phi_{e_i})\|_{\Phi_{e_i}} \xrightarrow{(o)} 0$. Convergence $x_\alpha s(\Phi_{e_i}) \xrightarrow{\Phi} xs(\Phi_{e_i})$ implies $x_i = xs(\Phi_{e_i})$ for all $i\in I$. Thus, $\widehat{\Phi}(|x-x_\alpha|)e_i = \widehat{\Phi}_{e_i}(|x_i-x_\alpha|s(\Phi_{e_i}))e_i \xrightarrow{(o)} 0$ and $\|x-x_\alpha\|_\Phi \xrightarrow{(o)} 0$.

Hence, $(L^1(M,\Phi),\|\cdot\|_\Phi)$ is a (bo)-complete lattice-normed space.

Now let us show that $(L^1(M,\Phi),\|\cdot\|_\Phi)$ is a Banach-Kantorovich space, i.e. for any element $x\in L^1(M,\Phi)$ and any decomposition $\|x\|_\Phi = f_1+f_2$, $f_1,f_2\in S_+(\mathcal{B})$, $f_1\wedge f_2=0$, there exist $x_1,x_2\in L^1(M,\Phi)$ such that $x=x_1+x_2$ and $\|x_i\|_\Phi = f_i$, $i=1,2$.

Set $e_i=s(f_i)$. It is clear that $e_i\in P(\mathcal{B})$, $e_1e_2=0$, $e_1+e_2=s(\|x\|_\Phi)$. Since Φ is a Maharam trace, we have $\Phi(y)=\Phi(\mathbf{1})\psi(\mathcal{E}(\Phi_M(y)))$, $y\in M$ (see Theorem 3.2). Let $p_i=\psi^{-1}(e_i)$, $x_i=xp_i$. Since $p_i\in P(\mathcal{A})\subset P(Z(M))$, $|x_i|=|x|p_i\in L^1(M,\Phi)$. We choose $y_n\in M$ such that $y_n\xrightarrow{\Phi} x$ and $\|y_n-y_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0$. Then $|y_n|\xrightarrow{\Phi}|x|$, $\|\,|y_n|-|y_m|\,\|_\Phi \xrightarrow{t(\mathcal{B})} 0$ and $\Phi(|y_n|)\xrightarrow{t(\mathcal{B})}\widehat{\Phi}(|x|)$ (see the proof of (i)). Set $y_n^{(i)}=y_np_i$, $i=1,2$. We have $|y_n^{(i)}|\xrightarrow{\Phi}|x_i|$ and $\|\,|y_n^{(i)}|-|y_m^{(i)}|\,\|_\Phi \le \|\,|y_n|-|y_m|\,\|_\Phi$. Hence, $\Phi(|y_n^{(i)}|)\xrightarrow{t(\mathcal{B})}\widehat{\Phi}(|x_i|)$. Due to the property 3) from Theorem 3.2, we have $\Phi(|y_n^{(i)}|)=\psi(p_i)\Phi(|y_n|)=e_i\Phi(|y_n|)$.

Thus, $\|x_i\|_\Phi=\widehat{\Phi}(|x_i|)=e_i\Phi(|x|)=f_i$, in addition $x_1+x_2=x(p_1+p_2)$ $=x\psi^{-1}(s(\|x\|_\Phi))$. As well as above, one can establish that $q\widehat{\Phi}(|x|)=\widehat{\Phi}(|x|\psi^{-1}(q))$ for all $q\in P(\mathcal{B})$. Taking $q=\mathbf{1}_F-s(\|x\|_\Phi)$, we get $\widehat{\Phi}(|x|)(\mathbf{1}-\psi^{-1}(s(\|x\|_\Phi)))=0$. Hence, $|x|=|x|\psi^{-1}(s(\|x\|_\Phi))$. Using the polar decomposition $x=u|x|$, we obtain $x=x\psi^{-1}(s(\|x\|_\Phi))=x_1+x_2$. $\square$

Note another useful properties of mapping $\widehat{\Phi}$.

Let Φ, M, Q, Φ_M, $\mathcal{A}$, ψ be the same as in Theorem 3.2, $\mathcal{B}=C(Q)_{\mathbb{C}}$. It is clear that the $*$-isomorphism ψ from $\mathcal{A}$ onto $\mathcal{B}$ can be extended to the $*$-isomorphism from $S(\mathcal{A})$ onto $S(\mathcal{B})$. We denote this mapping also by ψ.

Proposition 3.6. *$S(\mathcal{A})L^1(M,\Phi)\subset L^1(M,\Phi)$, in particular, $S(\mathcal{A})\subset L^1(M,\Phi)$, in addition, $\widehat{\Phi}(zx)=\psi(z)\widehat{\Phi}(x)$ and $\widehat{\Phi}(\widehat{\Phi}_M(zx))=\widehat{\Phi}(zx)$ for all $z\in S(\mathcal{A})$, $x\in L^1(M,\Phi)$.*

Proof. It is sufficient to show that $x\in L^1_+(M,\Phi)$, $z\in S_+(\mathcal{A})$ implies $zx\in L^1(M,\Phi)$ and $\widehat{\Phi}(zx)=\psi(z)\widehat{\Phi}(x)$, $\widehat{\Phi}(\widehat{\Phi}_M(zx))=\widehat{\Phi}(zx)$.

Let $z_n=E_n(z)z$. It is clear that $z_n\in\mathcal{A}_+$, $z_n\uparrow z$, $z_nx\in L^1_+(M,\Phi)$. Since $z_nx=\sqrt{x}z_n\sqrt{x}\uparrow\sqrt{x}z\sqrt{x}=zx$, we get

$$\psi(z_n)\widehat{\Phi}(x)=\widehat{\Phi}(z_nx)\le\widehat{\Phi}(z_{n+1}x)=\psi(z_{n+1})\widehat{\Phi}(x)\uparrow\psi(z)\widehat{\Phi}(x).$$

Hence,

$$\sup_{n\ge m}\|z_nx-z_mx\|_\Phi=\sup_{n\ge m}|\widehat{\Phi}(z_nx)-\widehat{\Phi}(z_mx)|\downarrow 0,$$

i.e. $\{z_nx\}$ is a (bo)-Cauchy sequence. By Theorem 3.5, there exists $y\in L^1(M,\Phi)$ such that $\|z_nx-y\|_\Phi\xrightarrow{(o)}0$. The inequality $\Phi(E_\lambda^\perp(|z_nx-y|)\le\frac{1}{\lambda}\Phi(|z_nx-y|)$ implies $z_nx\xrightarrow{\Phi}y$. Therefore $y=zx$, i.e. $zx\in L^1(M,\Phi)$. In addition, $\psi(z_n)\widehat{\Phi}(x)=\widehat{\Phi}(z_nx)=\|z_nx\|_\Phi\xrightarrow{t(\mathcal{B})}\|zx\|=\widehat{\Phi}(zx)$. Hence, $\widehat{\Phi}(zx)=\psi(z)\widehat{\Phi}(x)$.

Set $x_k = E_k(x)x$. Then $0 \leq x_k \uparrow x,\ x_k \in M$. By virtue of Proposition 3.1(iv), $\Phi(z_n x_k) = \Phi(\Phi_M(z_n x_k)) = \Phi(z_n \Phi_M(x_k))$. Since $(z_n x_k) \uparrow (z_n x)$ as $k \to \infty$, we have $\Phi(z_n x_k) \uparrow \widehat{\Phi}(z_n x)$ and $\Phi(\Phi_M(z_n x_k)) \uparrow \widehat{\Phi}(\widehat{\Phi}_M(z_n x))$. Therefore $\widehat{\Phi}(z_n x) = \widehat{\Phi}(\widehat{\Phi}_M(z_n x))$ for all $n = 1, 2, \ldots$. After switching to the limit as $n \to \infty$, we obtain $\widehat{\Phi}(zx) = \widehat{\Phi}(\widehat{\Phi}_M(zx))$. □

Let Φ be an $F_{\mathbb{C}}$-valued Maharam trace on M and let Ψ be a normal $F_{\mathbb{C}}$-valued trace on M. A trace Ψ is called *absolutely continuous with respect to* Φ (notation $\Psi \ll \Phi$) if $s(\Psi(p)) \leq s(\Phi(p))$ for all $p \in P(M)$. The last condition is equivalent to inclusion $\Psi(p) \in \{\Phi(p)\}^{\perp\perp} = s(\Phi(p))S_h(\mathcal{B})$, $p \in P(M)$ where $B^{\perp} := \{x \in S_h(\mathcal{B}) : (\forall y \in B)|x| \wedge |y| = 0\}$ for a nonempty subset $B \subset S_h(\mathcal{B})$ (compare with [1], 6.1.11).

The next theorem is a non-commutative version of the Radon-Nikodym-type theorem for Maharam traces.

Theorem 3.7. *Let Φ be an $F_{\mathbb{C}}$-valued Maharam trace on the von Neumann algebra M. If Ψ is a normal $F_{\mathbb{C}}$-valued trace on M absolutely continuous with respect to Φ, then there exists an operator $y \in L^1_+(M, \Phi) \cap S(Z(M))$ such that*

$$\Psi(x) = \widehat{\Phi}(yx)$$

for all $x \in M$.

Proof. Let l be the restriction of Ψ on the complete Boolean algebra $P(Z(M))$, and let m be the restriction of Φ on $P(Z(M))$. Obviously, l and m are $S_h(\mathcal{B})$-valued completely additive measures on $P(Z(M))$. In addition, $m(ze) = \psi(z)m(e)$ for all $z \in P(\mathcal{A}),\ e \in P(Z(M))$ (see Theorem 3.2). Hence, m is a ψ-modular measure on $P(Z(M))$ (see [1], 6.1.9). Since the measure l is absolutely continuous with respect to m, by the Radon-Nikodym-type theorem from ([1], 6.1.11), there exists $y \in L^1_+(Z(M), m) = L^1_+(Z(M), \Phi)$ such that $l(e) = \widehat{\Phi}(ye)$ for all $e \in P(Z(M))$.

If $a = \sum\limits_{i=1}^{n} \lambda_i e_i$ is a simple element from $Z(M)$, where $\lambda_i \in \mathbb{C},\ e_i \in P(Z(M)),\ i = 1, \ldots, n$, then $\Psi(a) = \sum\limits_{i=1}^{n} \lambda_i \Psi(e_i) = \sum\limits_{i=1}^{n} \lambda_i \widehat{\Phi}(ye_i) = \widehat{\Phi}(ya)$. Let $a \in Z_+(M)$ and $\{a_n\}$ be a sequence of simple elements from $Z_+(M)$ with $a_n \uparrow a$. Then $\Psi(a_n) \uparrow \Psi(a),\ ya_n \uparrow ya$, and $\widehat{\Phi}(ya_n) \uparrow \widehat{\Phi}(ya)$ (see the proof of Proposition 3.6). Hence, $\Psi(a) = \widehat{\Phi}(ya)$ for all $a \in Z_+(M)$. Now using the linearity of traces Ψ and Φ, we obtain $\Psi(a) = \widehat{\Phi}(ya)$ for all $a \in M$.

Furthermore, due to Propositions 3.1(iv) and 3.6 we get

$$\Psi(x) = \Psi(\Phi_M(x)) = \widehat{\Phi}(y\Phi_M(x)) = \widehat{\Phi}(\widehat{\Phi}_M(yx)) = \widehat{\Phi}(yx)$$

for all $x \in M$. □

Remark 3.8. *If Ψ is a normal $F_{\mathbb{C}}$-valued trace on M and $\Psi \ll \Phi$, then Ψ possesses the Maharam property.*

In fact, by Theorem 3.7, $\Psi(x) = \widehat{\Phi}(yx)$ for all $x \in M$ where $y \in L^1_+(M, \Phi) \cap S(Z(M))$. Let $0 \neq x \in M_+,\ f \leq \Psi(x),\ f \in S_+(\mathcal{B}),\ g \in S_+(\mathcal{B}),\ g\Psi(x) = s(\Psi(x)$. Set $h = gf,\ z = \psi^{-1}(h),\ a = zx$. Then $0 \leq h \leq g\Psi(x) = s(\Psi(x)) \leq \mathbf{1}_F,\ 0 \leq z \leq \mathbf{1},\ 0 \leq a \leq x$ and

$$\Psi(a) = \widehat{\Phi}(ya) = \widehat{\Phi}(zyx) = \psi(z)\widehat{\Phi}(yx) = h\Psi(x) = fs(\Psi(x)) = f.$$

References

1. A. G. Kusraev, *Dominanted Operators*, Mathematics and its Applications, vol. 519, Kluwer Academic Publishers, Dordrecht, 2000.

2. I. G. Ganiev, V. I. Chilin, *Measurable bundles of non-commutative L^p-spaces associated with center-valued trace*, Mat. Trudy **4** (2001), no. 2, 27–41. (Russian)
3. V. I. Chilin, A. A. Katz, *On abstract characterization of non-commutative L^p-spaces associated with center-valued trace*, Methods Funct. Anal. Topology **11** (2005), no. 4, 346–355.
4. S. Stratila, L. Zsido, *Lectures on von Neumann Algebras*, England Abacus Press, 1975.
5. M. Takesaki, *Theory of Operator Algebras* I, Springer, New York, 1979.
6. I. E. Segal, *A non-commutative extension of abstract integration*, Ann. Math. (1953), no. 57, 401–457.
7. M. A. Muratov, V. I. Chilin, *Algebras of Measurable and Locally Measurable Operators*, Pratsi In-ty matematiki NAN Ukraini, vol. 69, Kyiv, 2007. (Russian)
8. F. J. Yeadon, *Convergence of measurable operators*, Proc. Camb. Phil. Soc. **74** (1974), 257–268.
9. E. Nelson, *Notes on non commutative integration*, J. Funct. Anal. **15** (1974), 103–116.
10. T. Fack, H. Kosaki, *Generalised s-numbers of τ-measurable operators*, Pacific J. Math. **123** (1986), 269–300.
11. O.Y. Tikhonov, *Continuity of operator functions on a von Neumann algebra with respect to topology of convergence in measure*, Izv. Vyssh. Uchebn. Zaved., Mat. (1987), no. 1, 77–79. (Russian)
12. D. A. Vladimirov, *Boolean Algebras*, Nauka, Moscow, 1969. (Russian)
13. C. A. Akemann, T. Andersen, G. K. Pedersen, *Triangle inequalities in operator algebras*, Linear Multilinear Algebra **11** (1982), 167–178.
14. A. G. Kusraev, *Vector Duality and its Applications*, Nauka, Novosibirsk, 1985. (Russian)

Department of Mathematics, National University of Uzbekistan, Vuzgorodok, Tashkent, 100174, Uzbekistan
E-mail address: `chilin@ucd.uz`

Tashkent Railway Engineering Institute, 1 Odilhodjaev str., Tashkent, 100167, Uzbekistan
E-mail address: `botirzakirov@list.ru`

Received 21/05/2009