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GC-FUSION FRAMES

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Abstract. In this paper we introduce the generalized continuous version of fusion frame, namely *gc*-fusion frame. Also we get some new results about Bessel mappings and perturbation in this case.

1. Introduction and preliminaries

Throughout this paper H will be a Hilbert space and $\mathbb{H}$ will be the collection of all closed subspace of H, respectively. Also, (X, μ) will be a measure space, and $v : X \to [0, +\infty)$ a measurable mapping such that $v \neq 0$ *a.e.* We shall denote the unit closed ball of H by H_1.

Frames was first introduced in the context of non-harmonic Fourier series [9]. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of [8]. Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has been growing rapidly, several new applications have been developed. For example, besides traditional application as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence to improve the robustness of data transmission on [6], and to design high-rate constellation with full diversity in multiple-antenna code design [10]. The intrusted reader can find the details of frames in the introductory book [7]. In [1, 2, 3] some applications have been developed.

The fusion frames were considered by Casazza, Kutyniok and Li in connection with distributed processing and are related to the construction of global frames [4, 5]. The fusion frame theory is in fact more delicate due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to the extreme sensitivity with respect to changes of the weights.

Definition 1.1. Let $\{f_i\}_{i\in I}$ be a sequence of members of H. We say that $\{f_i\}_{i\in I}$ is a frame for H if there exist $0 < A \leq B < \infty$ such that

$$A\|h\|^2 \leq \sum_{i\in I} |\langle f_i, h\rangle|^2 \leq B\|h\|^2 \tag{1.1}$$

for all $h \in H$.

The constants A and B are called frame bounds. If A, B can be chosen so that $A = B$, we call this frame an A-tight frame and if $A = B = 1$ it is called a Parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence.

If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded,

$$T : l^2(I) \to H, T(\{c_i\}) = \sum_{i\in I} c_i f_i, \tag{1.2}$$

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$$T^*: H \to l^2(I), T^*(f) = \{\langle f, f_i \rangle\}_{i\in I}, \tag{1.3}$$

$$Sf = TT^*f = \sum_{i\in I} \langle f, f_i \rangle f_i. \tag{1.4}$$

This operators are called synthesis operator, analysis operator and frame operator, respectively.

Definition 1.2. For a countable index set I, let $\{W_i\}_{i\in I}$ be a family of closed subspaces in H, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H if there exist $0 < C \leq D < \infty$ such that for all $h \in H$

$$C\|h\|^2 \leq \sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2 \leq D\|h\|^2, \tag{1.5}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i.

We call C and D the fusion frame bounds. The family $\{(W_i, v_i)\}_{i\in I}$ is called a C-tight fusion frame, if in (1.5) the constants C and D can be chosen so that $C = D$, a Parseval fusion frame provided $C = D = 1$ and an orthogonal fusion basis if $H = \bigoplus_{i\in I} W_i$. If $\{(W_i, v_i)\}_{i\in I}$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it is a Bessel fusion sequence with Bessel fusion bound D. The representation space employed in this setting is

$$\Big(\sum_{i\in I} \oplus W_i\Big)_{l_2} = \{\{f_i\}_{i\in I} | f_i \in W_i \quad \text{and} \quad \{||f_i||\}_{i\in I} \in l^2(I)\}.$$

Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for H. The synthesis operator, analysis operator and frame operator are defined by

$$T_W : \Big(\sum_{i\in I} \oplus W_i\Big)_{l_2} \to H \quad \text{with} \quad T_W(\{f_i\}) = \sum_{i\in I} v_i f_i,$$

$$T_W^* : H \to \Big(\sum_{i\in I} \oplus W_i\Big)_{l_2} \quad \text{with} \quad T_W^*(f) = \{v_i \pi_{W_i}(f)\}_{i\in I},$$

$$S_W(f) = T_W T_W^* = \sum_{i\in I} v_i^2 \pi_{W_i}(f).$$

By proposition 3.7 in [5], if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H with fusion frame bounds C and D then S_W is a positive and invertible operator on H with $CId \leq S_W \leq DId$.

The theory of frames has a continuous version as follows:

Definition 1.3. Let (X, μ) be a measure space. Let $f : X \to H$ be weakly measurable (i.e., for all $h \in H$, the mapping $x \to \langle f(x), h \rangle$ is measurable). Then f is called a continuous frame or c-frame for H if there exist $0 < A \leq B < \infty$ such that

$$A\|h\|^2 \leq \int_X |\langle f(x), h \rangle|^2 d\mu \leq B\|h\|^2. \tag{1.6}$$

for all $h \in H$.

The representation space employed in this setting is

$$L^2(X, \mu) = \Big\{\varphi : X \to H | \quad \varphi \quad \text{is measurable and} \quad \int_X ||\varphi(x)||^2 d\mu < \infty\Big\}.$$

The synthesis, analysis and frame operators are defined by

$$T_f : L^2(X, \mu) \to H$$

$$\langle T_f \varphi, h \rangle = \int_X \varphi(x) \langle f(x), h \rangle d\mu(x), \tag{1.7}$$

$$T_f^*: H \to L^2(X, \mu)$$

$$(T_f^* h)(x) = \langle h, f(x) \rangle, \quad x \in X, \tag{1.8}$$

$$S_f = T_f T_f^*. \tag{1.9}$$

Also by Theorem 2.5. in [12] S_f is positive, self-adjoint and invertible.

Theorem 1.4. *Let f be a continuous frame for H with a frame operator S_f and let $V : H \to K$ be a bounded and invertible operator. Then $V \circ f$ is a continuous frame for K with the frame operator VS_fV^*.*

Proof. See [12]. □

Now we introduce the generalized continuous version of fusion frames and we show some of its properties.

Definition 1.5. Let $F : X \to \mathbb{H}$ be such that for each $h \in H$, the mapping $x \mapsto \pi_{F(x)}(h)$ is measurable (i.e. is weakly measurable) and let $\{K_x\}_{x \in X}$ be a collection of Hilbert spaces. For each $x \in X$, suppose that $\Lambda_x \in B(F(x), K_x)$ and put

$$\Lambda = \{\Lambda_x \in B(F(x), K_x) : x \in X\}.$$

Then (Λ, F, v) is a gc-fusion frame for H if there exist $0 < A \leq B < \infty$ such that

$$A\|h\|^2 \leq \int_X v^2(x) \|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu \leq B\|h\|^2 \tag{1.10}$$

for all $h \in H$, where $\pi_{F(x)}$ is the orthogonal projection onto the subspace $F(x)$.

(Λ, F, v) is called a tight gc-fusion frame for H if $A = B$, and Parseval if $A = B = 1$. If we only have the upper bound, we call (Λ, F, v) is a Bessel gc-fusion mapping for H.

Let $K = \oplus_{x \in X} K_x$ and $L^2(X, K)$ be a collection of all measurable functions $\varphi : X \longrightarrow K$ such that for each $x \in X$, $\varphi(x) \in K_x$ and

$$\int_X ||\varphi(x)||^2 d\mu < \infty.$$

It can be verified that $L^2(X, K)$ is a Hilbert space with inner product defined by

$$\langle \varphi, \gamma \rangle = \int_X \langle \varphi(x), \gamma(x) \rangle d\mu$$

for $\varphi, \gamma \in L^2(X, K)$ and the representation space in this setting is $L^2(X, K)$.

Remark 1.6. Let (Λ, F, v) be a Bessel gc-fusion mapping with Bessel bound B, $\varphi \in L^2(X, K)$ and $h \in H$. Then

$$\begin{aligned}
\Big| \int_X v(x) \langle \Lambda_x^*(\varphi(x)), h \rangle d\mu \Big| &= \Big| \int_X v(x) \langle \Lambda_x^*(\varphi(x)), \pi_{F(x)}(h) \rangle d\mu \Big| \\
&= \Big| \int_X v(x) \langle \varphi(x), \Lambda_x(\pi_{F(x)}(h)) \rangle d\mu \Big| \\
&\leq \int_X v(x) \|\varphi(x)\|.\|\Lambda_x(\pi_{F(x)}(h))\| d\mu \\
&\leq \Big(\int_X \|\varphi(x)\|^2 d\mu \Big)^{1/2} \Big(\int_X v^2(x) \|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu \Big)^{1/2} \\
&\leq B^{1/2} \|h\| (\int_X \|\varphi(x)\|^2 d\mu)^{1/2}.
\end{aligned}$$

So we may define:

Definition 1.7. Let (Λ, F, v) be a Bessel gc-fusion mapping for H. We define the gc-fusion pre-frame operator (synthesis operator) $T_{gf} : L^2(X,K) \to H$, by

$$\big\langle T_{gf}(\varphi), h\big\rangle = \int_X v(x)\big\langle \Lambda_x^*(\varphi(x)), h\big\rangle d\mu, \tag{1.11}$$

where $\varphi \in L^2(X,K)$ and $h \in H$. It is obvious that T_{gf} is linear and by Remark 1.6, T_{gf} is a bounded linear mapping. Its adjoint

$$T_{gf}^* : H \to L^2(X,K)$$

will be called gc-fusion analysis operator, and $S_{gf} = T_{gf} \circ T_{gf}^*$ will be called gc-fusion frame operator. For each $h \in H$ and $\varphi \in L^2(X,K)$, we have

$$\begin{aligned}
\big\langle T_{gf}^*(h), \varphi\big\rangle &= \big\langle h, T_{gf}(f)\big\rangle \\
&= \int_X v(x)\big\langle h, \Lambda_x^*(\varphi(x))\big\rangle d\mu \\
&= \int_X v(x)\big\langle \pi_{F(x)}(h), \Lambda_x^*(\varphi(x))\big\rangle d\mu \\
&= \int_X v(x)\big\langle \Lambda_x(\pi_{F(x)}(h)), \varphi(x)\big\rangle d\mu \\
&= \big\langle v\Lambda_{(.)}\pi_{F_{(.)}}(h), \varphi\big\rangle.
\end{aligned}$$

Hence for each $h \in H$,

$$T_{gf}^* = v\Lambda_{(.)}\pi_{F_{(.)}}. \tag{1.12}$$

2. Main Result

Definition 2.1. For each Bessel c-fusion mapping (F, v) for H, we shall denote

$$A_{\Lambda,v} = \inf_{h\in H_1} \|v\Lambda_{(.)}\pi_{F_{(.)}}(h)\|^2, \tag{2.1}$$

$$B_{\Lambda,v} = \sup_{h\in H_1} \|v\Lambda_{(.)}\pi_{F_{(.)}}(h)\|^2 = \|v\Lambda_{(.)}\pi_{F_{(.)}}\|^2. \tag{2.2}$$

Remark 2.2. Let (Λ, F, v) be a Bessel gc-fusion mapping for H. Since, for each $h \in H$

$$\big\langle T_{gf}T_{gf}^*(h), h\big\rangle = \|v\Lambda_{(.)}\pi_{F_{(.)}}(h)\|^2,$$

$A_{\Lambda,v}$ and $B_{\Lambda,v}$ are optimal scalars which satisfy

$$A_{\Lambda,v} \le T_{gf}T_{gf}^* \le B_{\Lambda,v}.$$

So (Λ, F, v) is a gc-fusion frame for H if and only if $A_{\Lambda,v} > 0$.

Proposition 2.3. *Let (Λ, F, v) be a Bessel gc-fusion mapping for H with bound D. The following conditions are equivalent.*
(i) (Λ, F, v) is a gc-fusion frame H with bounds C and D;
(ii) $CId \le S_{gf} \le DId$.
Moreover the optimal bounds are $\|S_{gf}\|$ and $\|S_{gf}^{-1}\|^{-1}$.

Proof. $(i) \Rightarrow (ii)$ is obvious. For $(ii) \Rightarrow (i)$, let T_{gf}^* denote the analysis operator of (Λ, F, v). Since $S_{gf} = T_{gf}T_{gf}^*$, for each $h \in H$, we have

$$\int_X v^2\|\Lambda_x(\pi_{F(x)}(h))\|^2\, d\mu = \|T_{gf}^*(h)\|^2 \le \|T_{gf}^*\|^2\|h\|^2 = \|S_{gf}\|\|h\|^2 \le D\|h\|^2.$$

Also for all $h \in H$,

$$\|T_{gf}^*(h)\|^2 = \big\langle T_{gf}T_{gf}^*(h), h\big\rangle = \big\langle S_{gf}h, h\big\rangle = \big\langle S_{gf}^{\frac12}h, S_{gf}^{\frac12}h\big\rangle = \|S_{gf}^{\frac12}h\|^2 \ge C\|h\|^2.$$

Also

$$\|S_{gf}\| = \sup_{h\in H_1} \big\langle S_{gf}(h), h\big\rangle = \sup_{h\in H_1} \|v\Lambda_{(.)}\pi_{F_{(.)}}(h)\|^2 = B_{\Lambda,v}.$$

So the optimal upper bound is $\|S_{gf}\|$. For the optimal lower bound, if C be the lower bound we have

$$C\|h\|^2 \leq \big\langle S_{gf}^{1/2}(h), S_{gf}^{1/2}(h)\big\rangle \leq D\|h\|^2.$$

By putting $h = S_{gf}^{-1/2}(h)$, we have

$$C\|S_{gf}^{-1/2}(h)\|^2 \leq \big\langle h, h\big\rangle \leq D\|S_{gf}^{-1/2}(h)\|^2,$$

thus

$$\|S_{gf}^{-1}\| = \sup_{h\in H_1} \|S_{gf}^{-1/2}(h)\|^2 \leq C^{-1}.$$

We conclude that $A_{\Lambda,v} \leq \|S_{gf}^{-1}\|^{-1}$. For other implication we have

$$\|h\| \leq \|S_{gf}^{-1/2}\|\|S_{gf}^{1/2}(h)\|.$$

Hence

$$\inf_{h\in H_1} \|S_{gf}^{1/2}(h)\|^2 \geq \inf_{h\in H_1} \|h\|^2\|S_{gf}^{-1/2}\|^{-2} = \|S_{gf}^{-1}\|^{-1},$$

we conclude that $A_{\Lambda,v} \geq \|S_{gf}^{-1}\|^{-1}$. Finally $A_{\Lambda,v} = \|S_{gf}^{-1}\|^{-1}$. □

Corollary 2.4. *S_{gf} is a positive and invertible operator from H into H.*

Proof. It is results from the Proposition 2.3 . □

Like the perturbation of g-frames in [11], we can present the perturbation of gc-fusion frames.

Definition 2.5. Let $F : X \to \mathbb{H}$, $\tilde{F} : X \to \mathbb{H}$, $\Lambda = \{\Lambda_x \in B(F(x), K_x) : x \in X\}$ and $\tilde{\Lambda} = \{\tilde{\Lambda}_x \in B(\tilde{F}(x), K_x) : x \in X\}$. Let $0 \leq \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$. We say that $(\tilde{\Lambda}, \tilde{F}, v)$ is a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of (Λ, F, v) if for each $h \in H$ and $x \in X$

$$\|\Lambda_x(\pi_{F(x)}(h)) - \tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\| \leq \lambda_1\|\Lambda_x(\pi_{F(x)}(h))\| + \lambda_2\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\| + \varepsilon\|h\|.$$

Theorem 2.6. *Let (Λ, F, v) be a gc-fusion frame for H with bounds C and D, and let $v \in L^2(X)$. Choose $0 \leq \lambda_1 < 1$ and $\varepsilon > 0$ such that*

$$(1-\lambda_1)\sqrt{C} - \varepsilon\Big(\int_X v^2(x)d\mu\Big)^{1/2} > 0.$$

Let $\tilde{F} : X \to \mathbb{H}$ be weakly measurable. Further, if $(\tilde{\Lambda}, \tilde{F}, v)$ be a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of (Λ, F, v) for some $0 \leq \lambda_2 < 1$, then $(\tilde{\Lambda}, \tilde{F}, v)$ is a gc-fusion frame for H with bounds

$$\Big[\frac{(1-\lambda_1)\sqrt{C} - \varepsilon(\int_X v^2(x)d\mu)^{1/2}}{1+\lambda_2}\Big]^2$$

and

$$\Big[\frac{(1+\lambda_2)\sqrt{D} - \varepsilon(\int_X v^2(x)d\mu)^{1/2}}{1-\lambda_2}\Big]^2.$$

Proof. We first prove the Upper bound. For any $h \in H$, we get

$$\Big[\int_X v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu\Big]^{1/2}$$
$$\leq \Big[\int_X v^2(x)(\|\Lambda_x(\pi_{F(x)}(h)) - \tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\| + \|\Lambda_x(\pi_{F(x)}(h))\|)^2 d\mu\Big]^{1/2}$$
$$\leq \Big[\int_X v^2(x)(\|\Lambda_x(\pi_{F(x)}(h))\| + \lambda_1\|\Lambda_x(\pi_{F(x)}(h))\|$$
$$+ \lambda_2\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\| + \varepsilon\|h\|)^2 d\mu\Big]^{1/2}$$
$$= \Big[\int_X ((1+\lambda_1)v(x)\|\Lambda_x(\pi_{F(x)}(h))\| + \lambda_2 v(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|$$
$$+ \varepsilon v(x)\|h\|)^2 d\mu\Big]^{1/2}$$
$$\leq \Big[(1+\lambda_1)^2\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu\Big]^{1/2}$$
$$+ \Big[\lambda_2^2\int_X v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu\Big]^{1/2} + \Big[\varepsilon^2\int_X v^2(x)\|h\|^2 d\mu\Big]^{1/2}.$$

Thus

$$(1-\lambda_2)\Big[\int_X v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu\Big]^{1/2}$$
$$\leq (1+\lambda_1)\Big[\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu\Big]^{1/2} + \varepsilon\|h\|\Big[\int_X v^2(x) d\mu\Big]^{1/2}$$
$$\leq \Big[(1+\lambda_1)\sqrt{D} + \varepsilon\Big(\int_X v^2(x) d\mu\Big)^{1/2}\Big]\|h\|.$$

Hence

$$\int_X v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu \leq \Big[\frac{(1+\lambda_1)\sqrt{D} + \varepsilon(\int_X v^2(x)d\mu)^{1/2}}{1-\lambda_2}\Big]^2\|h\|^2.$$

To prove the lower bound, for all $h \in H$ we have

$$\Big[\int_X v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu\Big]^2$$
$$\geq \Big[\int_X v^2(x)(\|\Lambda_x(\pi_{F(x)}(h))\| - \|\Lambda_x(\pi_{F(x)}(h)) - \tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|)^2 d\mu\Big]^{1/2}$$
$$\geq \Big[\int_X v^2(x)(\|\Lambda_x(\pi_{F(x)}(h))\| - \lambda_1\|\Lambda_x(\pi_{F(x)}(h))\|$$
$$- \lambda_2\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\| - \varepsilon\|h\|)^2 d\mu\Big]^{1/2}$$
$$= \Big[\int_X ((1-\lambda_1)v(x)\|\Lambda_x(\pi_{F(x)}(h))\| - \lambda_2 v(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|$$
$$- \varepsilon v(x)\|h\|)^2 d\mu\Big]^{1/2}$$
$$\geq \Big[\int_X (1-\lambda_1)^2 v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu\Big]^{1/2}$$
$$- \Big[\int_X \lambda_2^2 v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu\Big]^{1/2} - \Big[\int_X \varepsilon^2 v^2(x)\|h\|^2 d\mu\Big]^{1/2}.$$

Thus

$$\begin{aligned}(1+\lambda_2)\Big[\int_X v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu\Big]^{1/2} \\ \geq (1-\lambda_1)\Big[\int_X v^2(x)\|\Lambda_x(\pi_{F(x)}(h))\|^2 d\mu\Big]^{1/2} - \varepsilon\Big[\int_X v^2(x)d\mu\Big]^{1/2}\|h\| \\ = \Big[(1-\lambda_1)\sqrt{C} - \varepsilon\Big[\int_X v^2(x)d\mu\Big]^{1/2}\Big]\|h\|.\end{aligned}$$

Hence

$$\int_X v^2(x)\|\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))\|^2 d\mu \geq \left[\frac{(1-\lambda_1)\sqrt{C}+\varepsilon(\int_X v^2(x)d\mu)^{1/2}}{1+\lambda_2}\right]^2\|h\|^2.$$

This completes the proof. □

Theorem 2.7. *Let (Λ, F, v) and $(\tilde{\Lambda}, \tilde{F}, v)$ be two Bessel gc-fusion mappings for H and consider the operator $S_{F,\tilde{F}} : H \to H$ by*

$$S_{F,\tilde{F}}(h) = T_{gf}T^*_{g\tilde{f}}(h).$$

Then
(i) $S_{F,\tilde{F}}$ is bounded and $S^*_{F,\tilde{F}} = S_{\tilde{F},F}$.
(ii) Let there exists $\lambda_1 < 1$ and $\lambda_2 > -1$ such that

$$\|h - S_{F,\tilde{F}}(h)\| \leq \lambda_1||h|| + \lambda_2||S_{F,\tilde{F}}(h)||,$$

for each $h \in H$. Then $(\tilde{\Lambda}, \tilde{F}, v)$ is a gc-fusion frame for H and for each $h \in H$ we have

$$\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2\frac{1}{B_{F,v}}\|h\|^2 \leq \int_X v^2(x)||\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))||^2 d\mu.$$

Proof. (i) For any $h, k \in H$ we have

$$\begin{aligned}\langle S_{F,\tilde{F}}(h), k\rangle = \langle T_{gf}T^*_{g\tilde{f}}(h), k\rangle &= \int_X v(x)\langle \Lambda^*_x(T^*_{g\tilde{f}}(h)(x)), k\rangle d\mu \\ &= \int_X v(x)\langle v(x)\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h)), \Lambda_x(\pi_{F(x)}(k))\rangle d\mu \\ &= \int_X v^2(x)\langle \tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h)), \Lambda_x(\pi_{F(x)}(k))\rangle d\mu.\end{aligned}$$

Thus

$$\begin{aligned}|\langle S_{F,\tilde{F}}(h), k\rangle|^2 &\leq \Big(\int_X v^2(x)||\Lambda_x(\pi_{F(x)}(k))||^2 d\mu\Big)\Big(\int_X v^2(x)||\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))||^2 d\mu\Big) \\ &\leq B_{\Lambda,v}B_{\tilde{\Lambda},v}||h||^2||k||^2.\end{aligned}$$

Hence $S_{F,\tilde{F}}$ is a bounded operator with

$$||S_{F,\tilde{F}}|| \leq B^{1/2}_{\Lambda,v}B^{1/2}_{\tilde{\Lambda},v}.$$

Also $S^*_{F,\tilde{F}}$ is bounded and we have

$$S^*_{F,\tilde{F}} = (T_{gf}T^*_{g\tilde{f}})^* = T_{g\tilde{f}}T^*_{gf} = S_{\tilde{F},F}.$$

(ii) Since

$$||h - S_{F,\tilde{F}}(h)|| \leq \lambda_1||h|| + \lambda_2||S_{F,\tilde{F}}(h)||,$$

thus

$$\lambda_1||h|| + \lambda_2||S_{F,\tilde{F}}|| \geq ||h|| - ||S_{F,\tilde{F}}(h)||.$$

Hence

$$||S_{F,\tilde{F}}(h)|| \geq \frac{1-\lambda_1}{1+\lambda_2}||h||.$$

By the above inequalities for each $h \in H$ we have

$$|\langle S_{F,\tilde{F}}(h), h\rangle|^2 \leq (B_{\Lambda,v}\|h\|^2)\Big(\int_X v^2||\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))||^2 d\mu\Big).$$

Therefore

$$\int_X v^2||\tilde{\Lambda}_x(\pi_{\tilde{F}(x)}(h))||^2 d\mu) \geq \frac{1}{B_{\Lambda,v}}||S_{F,\tilde{F}}(h)||^2 \geq \frac{1}{B_{\Lambda,v}}\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2||h||^2.$$

□

Definition 2.8. Two Bessel *gc*-fusion mappings (Λ, F, v) and $(\tilde{\Lambda}, \tilde{F}, v)$ for H are called a dual pair if

$$T_{gf}T^*_{g\tilde{f}} = I.$$

Corollary 2.9. *If (Λ, F, v) and $(\tilde{\Lambda}, \tilde{F}, v)$ is a dual pair of gc-fusion Bessel mappings for H then both of (Λ, F, v) and $(\tilde{\Lambda}, \tilde{F}, v)$ are gc-fusion frames for H.*

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